Electronic Journal of Differential Equations, Vol. 2021 (2021), No. 59, pp. 1–22.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

T-COERCIVITY FOR THE ASYMPTOTIC ANALYSIS OF SCALAR PROBLEMS WITH SIGN-CHANGING COEFFICIENTS IN THIN PERIODIC DOMAINS

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Abstract. We study a scalar problem in thin periodic composite media formed by two materials, a classical one and a metamaterial (also known as negative material). By applying T-coercivity methods and homogenization techniques specific to the thin periodic domains under consideration, for two geometric settings, we derive the homogenized limit problems, which both exhibit dimension-reduction effects.

1. Introduction

Metamaterials (also known as negative or left-handed materials) are artificial composite materials exhibiting negative dielectric permittivity and magnetic permeability for some frequencies, and hence behaving as negative refractive index materials (see the review papers by Shamonina and Solymar [50] and Smith, Pendry and Wiltshire [51]). This property leads to new super lens effects and explains the growing interest for them in the last decade. From a mathematical point a view, Bouchitté et al. proposed a rigorous derivation and a mathematical justification of negative materials in electromagnetics using homogenization techniques [12, 13, 14] and similar results have been established by Ávila et al. [5] in phononics. The well-posedness of scalar problems involving both classical dielectric materials and metamaterials (and, hence, leading to differential operators with sign-changing coefficients) has been studied by Bonnet-Ben Dhia et al. [8, 10], Chesnel and Ciarlet Jr. [24], Chung and Ciarlet Jr. [25], Nicaise and Venel [47], Nguyen [44, 45]. The case of Maxwell's system with sign changing coefficients has also been investigated by Bonnet-Ben Dhia, Chesnel and Ciarlet Jr. [7], Fernandes and Raffetto [30], Oliveri and Raffetto [48] and Nguyen and Sil [46]. More recently, homogenization for composite materials involving both positive and negative materials has been considered in [11, 16, 17].

Our goal in this article is to study a scalar problem in specific periodic thin domains, namely composite materials formed by two constituents, a classical one and a metamaterial, occupying a thin three-dimensional region denoted Ω^ε. The results are valid in $\mathbb{R}^d\,(d \geqslant 2)$, but the three-dimensional case is physically more relevant, if we think, for instance, to the practical applications mentioned below.

2010 *Mathematics Subject Classification.* 35B27, 78M40, 35Q60, 35J20.
Key words and phrases. Sign-changing coefficients; T-coercivity; thin domains; asymptotic analysis.

Submitted October 29, 2020. Published June 26, 2021.

The domain Ω^ε is divided in two open subdomains, denoted by Ω_1^ε and Ω_2^ε. The subdomain Ω_2^ε, assumed to be disconnected, is formed by ε-periodically distributed inclusions which do not touch the boundary of Ω^ε. We shall consider two distinct geometrical settings, which both correspond to important physical applications, like demultiplexer devices and antennas [49]. In the first geometrical setting (see Figure 1), the inclusions are ε-periodically distributed in an horizontal layer of height ε, while in the second one (see Figure 2), the inclusions are ε-periodically distributed at the interior of a thin vertical rod with thickness of order ε.

For both of these geometries, we analyze the well-posedness of the corresponding microscopic problems (see (2.4) and (5.2)) and the asymptotic behavior of their solutions as the small parameter ε tends to zero. The main difference with respect to the classical scalar case is the presence of two materials with conductivities of different sign. Consequently, classical methods based on ellipticity arguments to prove well-posedness and obtain energy estimates do not apply for the problems under study. To cope with this difficulty, we apply T-coercivity technics (see Definition 3.1; we refer the reader to the papers of Bonnet-Ben Dhia et al. [8, 9] for more details), which have been already successfully used in the context of homogenization by two of the authors in [17, 16]. The first difference from these last two references is the degenerate geometry considered in this work. The second difference concerns the construction of the T-coercive operators. More precisely, like in [17], the T-coercive operators involved here are constructed using suitably chosen extension operators from one subdomain (the one occupied by the positive or negative material) into the other. Defining the contrast κ between the two materials as in (2.3), we need to distinguish, throughout the whole paper, between the cases of large contrasts and small contrasts, each one involving a particular extension operator (and correspondingly a particular T-coercive operator). We propose in this paper a similar and quite direct proof for both cases, whereas [17] was limited to the case of large contrasts, while the proof proposed in [16] was more involved and indirect. The use of alternative approaches to study homogenization of these problems with sign-changing coefficients, like the analysis of the spectrum of the Neumann-Poincaré operator developed in Bonnetier et al. [11], is also probably possible.

Once the well-posedness proved, by using the two-scale convergence method (see Nguetseng [43] and Allaire [1]) adapted to thin periodic domains (see, for instance, Gahn and Neuss-Radu [32] and Jerez-Hanckes et al. [39]), we derive for both geometries depicted in Figures 1 and 2 the associated homogenized problems (see Theorem 4.3 and Theorem 5.3). Let us emphasize that they both exhibit dimension reduction effects, $3D \to 2D$ and $3D \to 1D$, respectively. Nevertheless, the values of the homogenized coefficients keep track of the *lost* variables, at a local scale, through the cell problems (see Remarks 4.4 and 5.4).

For mathematical studies of diffusion problems in thin periodic media, we refer, for instance, to [6, 18, 19, 28, 29, 31, 32, 33, 34, 38, 39, 40, 42] and the references therein. For elasticity problems in related thin periodic domains, we refer to [20, 21, 22, 27, 36, 37]. For flow problems in thin porous media, we refer, for example, to [2, 3, 4, 15, 35].

This article is organized as follows: in Section 2, we introduce the microscopic problem (2.4) stated in the thin periodic layer and we fix the notation. In Section 3, we prove the well-posedness of problem (2.4), using appropriate extension operators and the T-coercivity technics. The homogenization result for the case of a thin periodic layer is stated and proved in Section 4. Finally, in Section 5, we address the case of a three-dimensional thin periodic rod.

2. Setting of the problem in the case of a thin periodic layer

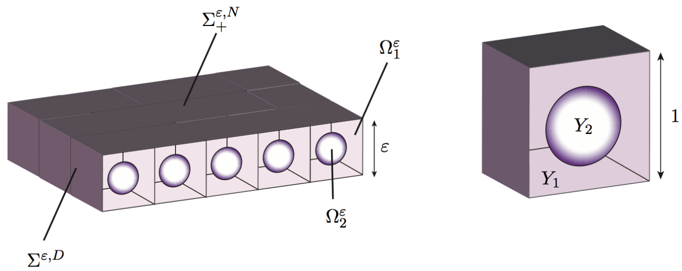

Figure 1. Example of a thin periodic layer and the corresponding reference cell Y.

We start by describing more precisely the geometry of the domain Ω^ε, which represents a two-phase thin periodic composite medium (see Figure 1). Let ω be a smooth and bounded domain in $\mathbb{R}^2$. We denote the independent variable $x \in \mathbb{R}^3$ by $x = (x_1, x_2, x_3) = (\bar{x}, x_3)$ and we define

$$\Omega^\varepsilon = \omega \times (0, \varepsilon) = \{x = (\bar{x}, x_3) \in \mathbb{R}^3 : \bar{x} \in \omega,\, 0 < x_3 < \varepsilon\}. \tag{2.1}$$

Here, $\varepsilon \in (0,1)$ is a sequence of strictly positive numbers such that $\frac{1}{\varepsilon} \in \mathbb{N}^*$. This small parameter is related to the characteristic dimension of our domain. Thus, Ω^ε is a thin heterogeneous layer, its thickness, the periodicity of its heterogeneities and their size being of order ε. More precisely, the microscopic structure of Ω^ε consists of an exact number of replicated unit cells $Y = (0,1)^3$, rescaled with ε. The reference cell is given by $Y = Y_1 \cup \overline{Y}_2$, where Y_1 and Y_2 are two non-empty disjoint connected open subsets of Y such that $\overline{Y}_2 \subset Y$. We assume that the boundary ∂Y_2 is Lipschitz continuous. For each $\mathbf{k} \in \mathbb{Z}^3$, we define the shifted cells $Y_1^{\mathbf{k}} = \mathbf{k} + Y_1$ and $Y_2^{\mathbf{k}} = \mathbf{k} + Y_2$. We also define, for each ε, the set of indexes $\mathbb{Z}_\varepsilon = \left\{\mathbf{k} \in \mathbb{Z}^3 : \varepsilon \overline{Y}_2^{\mathbf{k}} \subset \Omega^\varepsilon\right\}$. Finally, we set $\Omega_2^\varepsilon = \cup_{\mathbf{k} \in \mathbb{Z}_\varepsilon} \left(\varepsilon Y_2^{\mathbf{k}}\right)$ and $\Omega_1^\varepsilon = \Omega \setminus \overline{\Omega}_2^\varepsilon$. We denote by n^ε the unit outward normal to Ω_2^ε. The boundary of the domain Ω^ε is split into three parts: the lateral boundary $\Sigma^{\varepsilon,D} = \{x \in \mathbb{R}^3 : \bar{x} \in \partial\omega,\, 0 < x_3 < \varepsilon\}$, the top boundary $\Sigma_+^{\varepsilon,N} = \{x \in \mathbb{R}^3 : \bar{x} \in \omega,\, x_3 = \varepsilon\}$ and the bottom boundary $\Sigma_-^{\varepsilon,N} = \{x \in \mathbb{R}^3 : \bar{x} \in \omega,\, x_3 = 0\}$.

Given two real constants a_1 and a_2 such that $a_1 a_2 < 0$, let $a \in L^\infty(\mathbb{R}^3)$ denote the 1-periodic function in the variables y_1 and y_2 defined by

$$a(y) = a_1 \mathbf{1}_{Y_1}(y) + a_2 \mathbf{1}_{Y_2}(y).$$

For simplicity and without loss of generality, we assume that

$$a_1 > 0, \quad a_2 < 0, \tag{2.2}$$

and we define the contrast κ as the positive number

$$\kappa = \left|\frac{a_1}{a_2}\right| = \frac{a_1}{|a_2|}. \tag{2.3}$$

Setting

$$a^\varepsilon(x) = a\Big(\frac{x}{\varepsilon}\Big), \quad \forall x \in \Omega^\varepsilon,$$

our goal is to analyze the asymptotic behavior, as $\varepsilon \to 0$, of the solution u^ε of the problem

$$\begin{gathered} -\operatorname{div}(a^\varepsilon \nabla u^\varepsilon) = f \quad \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \Sigma^{\varepsilon,D}, \\ \frac{\partial u^\varepsilon}{\partial \nu^\varepsilon_\pm} = 0 \quad \text{on } \Sigma^{\varepsilon,N}_\pm, \end{gathered} \tag{2.4}$$

where the function $f \in L^2(\omega)$ is given and $\nu^\varepsilon_\pm$ is the unit outward normal to $\Sigma^{\varepsilon,N}_\pm$. Let us emphasize that this particular choice of the function f is classical in the framework of asymptotic analysis in thin domains. Nevertheless, more general right-hand sides can be considered, see, for instance, [41, Section 2].

It is worth noticing that the above scalar problem is nothing but the problem obtained from the homogenization problem

$$\begin{gathered} -\operatorname{div}(\mathbf{a}^\varepsilon \nabla u^\varepsilon) = f \quad \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \Sigma^{\varepsilon,D}, \\ \mathbf{a}^\varepsilon \nabla u^\varepsilon \cdot \nu^\varepsilon_\pm = 0 \quad \text{on } \Sigma^{\varepsilon,N}_\pm, \end{gathered}$$

in the special case where the 3x3 matrix $\mathbf{a} = (a_{ij})_{1\leqslant i,j\leqslant 3}$ is given by

$$\mathbf{a}^\varepsilon(x) = \mathbf{a}\Big(\frac{x}{\varepsilon}\Big), \quad \mathbf{a}(y) = a(y) I_3. \tag{2.5}$$

To write the variational formulation of problem (2.4), for every positive $\varepsilon < 1$, we introduce the Hilbert space

$$V^\varepsilon = \big\{ v \in H^1(\Omega^\varepsilon) : v = 0 \quad \text{on } \Sigma^{\varepsilon,D} \big\}, \tag{2.6}$$

endowed with the norm $\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega^\varepsilon)}$, for any $v \in V^\varepsilon$.

The variational formulation of problem (2.4) is as follows: find $u^\varepsilon \in V^\varepsilon$ such that

$$\mathcal{A}^\varepsilon(u^\varepsilon, v) = \ell^\varepsilon(v), \quad \forall v \in V^\varepsilon, \tag{2.7}$$

where the bilinear form $\mathcal{A}^\varepsilon : V^\varepsilon \times V^\varepsilon \to \mathbb{R}$ and the linear form $\ell^\varepsilon : V^\varepsilon \to \mathbb{R}$ are given by

$$\begin{gathered} \mathcal{A}^\varepsilon(u,v) = \int_{\Omega^\varepsilon} a^\varepsilon \nabla u \cdot \nabla v \,\mathrm{d}x = a_1 \int_{\Omega^\varepsilon_1} \nabla u \cdot \nabla v \,\mathrm{d}x + a_2 \int_{\Omega^\varepsilon_{2D}} \nabla u \cdot \nabla v \,\mathrm{d}x, \\ \ell^\varepsilon(v) = \int_{\Omega^\varepsilon} f v \,\mathrm{d}x. \end{gathered}$$

3. Well-posedness of the microscopic problem in a thin periodic layer via T-coercivity technics

Since the bilinear form involved in the above weak formulation is not coercive, the well-posedness of (2.7) is far from being obvious. To overcome this difficulty, an issue is to make use of T-coercivity technics (see, for instance, [8, 9]). In the framework of homogenization, this method has already been successfully used for scalar problems set in a fixed domain with periodic geometry; see [17] for the case of a Dirichlet scalar problem (case of large contrasts only) and [16] for the one of Dirichlet and Neumann scalar problems and vector Maxwell's equations for extreme contrasts, large or small. It is worth mentioning reference [11], in which homogenization of scalar problems with sign-changing coefficients is achieved using another approach, namely the analysis of the spectrum of the Neumann-Poincaré operator.

3.1. Background on T-coercivity.

We start by recalling the definition of T-coercivity.

Definition 3.1. Let V be a Hilbert space endowed with the norm $\|\cdot\|$ and let $\mathbf{T} \in \mathcal{L}(V)$ be a bounded linear operators on V. A bilinear form $a(\cdot,\cdot)$ defined on $V \times V$ is called T-coercive if there exists $\gamma > 0$ such that

$$|a(u, \mathbf{T}u)| \geqslant \gamma \|u\|^2, \quad \forall u \in V.$$

The T-coercivity result used throughout the paper is detailed below (Theorem 3.2). Note that this result is stated in the special case of symmetric bilinear forms (only such bilinear forms are involved in this work) depending on a small parameter. In particular, it is slightly different from the result given in [17, Theorem 2.1]) which holds for arbitrary bilinear forms. The T-coercivity operators needed to prove well-posedness for the three problems with sign -changing coefficients involved in our study (namely the microscopic problem, the cell problems and the two-scale limit problem) are constructed using suitable extension operators (from the background medium into the inclusions or vice-versa). It is worth noticing that such extension operators are generally used in homogenization theory to obtain compactness results, while they are used here to obtain well-posedness and energy estimates for the solutions of our problems. We also emphasize that the construction of suitable extension operators from the inclusions into the exterior domain is more technical than the construction of extension operators from the outer domain into the inclusions. Indeed, this requires to adapt an idea used in [23] to obtain uniform energy estimates of gradient type for these operators (see (3.13) and (3.21)).

Theorem 3.2. *Let V be a Hilbert space equipped with the norm $\|\cdot\|$ and let $\mathcal{A}^\varepsilon(\cdot,\cdot)$ be a bilinear form on V satisfying the following conditions:*

1. *$\mathcal{A}^\varepsilon(\cdot,\cdot)$ is symmetric: $\mathcal{A}^\varepsilon(u,v) = \mathcal{A}^\varepsilon(v,u)$.*
2. *$\mathcal{A}^\varepsilon(\cdot,\cdot)$ is uniformly continuous: there exists $M > 0$ such that*
$$|\mathcal{A}^\varepsilon(u,v)| \leqslant M\|u\|\|v\|, \quad \forall u, v \in V. \tag{3.1}$$
3. *There exists a family $(\mathbf{T}^\varepsilon)_{\varepsilon>0}$ of uniformly bounded linear operators on V and $\gamma > 0$ such that*
$$|\mathcal{A}^\varepsilon(u, \mathbf{T}^\varepsilon u)| \geqslant \gamma\|u\|^2, \quad \forall u \in V. \tag{3.2}$$

Then, given a family $(\ell^\varepsilon)_{\varepsilon>0}$ *in* V'*, the space of linear forms on* V*, the variational problem:*

$$\text{Find } u^\varepsilon \in V \text{ such that } \mathcal{A}^\varepsilon(u^\varepsilon, v) = \ell^\varepsilon(v) \text{ for all } v \in V \tag{3.3}$$

admits a unique solution $u^\varepsilon \in V$ *for all* $\varepsilon > 0$ *and there exists* $C > 0$ *independent of* ε *such that*

$$\|u^\varepsilon\| \leqslant C\|\ell^\varepsilon\|_{V'}. \tag{3.4}$$

Proof. First of all, we prove that $(\mathbf{T}^\varepsilon)_{\varepsilon>0}$ is a family of uniformly boundedly invertible operators, *i.e.* that $\mathbf{T}^\varepsilon \in \mathcal{L}(V)$ is an isomorphism and there exists a constant $C > 0$ independent of ε such that $\|(\mathbf{T}^\varepsilon)^{-1}\| \leqslant C$. Let us introduce the bounded operator $\mathbf{A}^\varepsilon \in \mathcal{L}(V)$ associated to the bilinear form $\mathcal{A}^\varepsilon(\cdot,\cdot)$,

$$(\mathbf{A}^\varepsilon u, v) = \mathcal{A}^\varepsilon(u, v), \quad \forall u, v \in V.$$

Equation (3.1) implies that

$$\|\mathbf{A}^\varepsilon u\| = \sup_{v\neq 0} \frac{(\mathbf{A}^\varepsilon u, v)}{\|v\|} = \sup_{v\neq 0} \frac{\mathcal{A}^\varepsilon(u, v)}{\|v\|} \leqslant M\|u\|.$$

Consequently, $(\mathbf{A}^\varepsilon)_{\varepsilon>0}$ is a family of uniformly bounded operators (with $\|\mathbf{A}^\varepsilon\| \leqslant M$). Since

$$\|(\mathbf{T}^\varepsilon)^*\mathbf{A}^\varepsilon\| \leqslant \|(\mathbf{T}^\varepsilon)^*\|\|\mathbf{A}^\varepsilon\| = \|\mathbf{T}^\varepsilon\|\|\mathbf{A}^\varepsilon\|,$$

the family $\left((\mathbf{T}^\varepsilon)^*\mathbf{A}^\varepsilon\right)_{\varepsilon>0}$ is also uniformly bounded (recall that $(\mathbf{T}^\varepsilon)_{\varepsilon>0}$ is uniformly bounded by assumption). Moreover, thanks to (3.2), the Lax-Milgram theorem implies that $(\mathbf{T}^\varepsilon)^*\mathbf{A}^\varepsilon$ is an isomorphism, and the estimates

$$\gamma\|u\|^2 \leqslant |\mathcal{A}^\varepsilon(u, \mathbf{T}^\varepsilon u)| = |\left(\mathbf{A}^\varepsilon u, \mathbf{T}^\varepsilon u\right)| = |\left((\mathbf{T}^\varepsilon)^*\mathbf{A}^\varepsilon u, u\right)| \leqslant \|(\mathbf{T}^\varepsilon)^*\mathbf{A}^\varepsilon u\|\|u\|,$$

show that $\|\left((\mathbf{T}^\varepsilon)^*\mathbf{A}^\varepsilon\right)^{-1}\| \leqslant 1/\gamma$. The family $\left((\mathbf{T}^\varepsilon)^*\mathbf{A}^\varepsilon\right)_{\varepsilon>0}$ is thus uniformly boundedly invertible. As a consequence, $\mathbf{A}^\varepsilon$ is injective and has closed range. On the other hand, since $\mathcal{A}^\varepsilon(\cdot,\cdot)$ is symmetric, (3.2) shows that $\mathbf{A}^\varepsilon\mathbf{T}^\varepsilon$ is uniformly boundedly invertible, as well. Hence, $\mathbf{A}^\varepsilon$ is onto. Consequently, $\mathbf{A}^\varepsilon$ and $\mathbf{T}^\varepsilon$ define two isomorphisms on V and are uniformly boundedly invertible, as can it be seen from the identities $(\mathbf{A}^\varepsilon)^{-1} = \mathbf{T}^\varepsilon(\mathbf{A}^\varepsilon\mathbf{T}^\varepsilon)^{-1}$ and $(\mathbf{T}^\varepsilon)^{-1} = (\mathbf{A}^\varepsilon\mathbf{T}^\varepsilon)^{-1}\mathbf{A}^\varepsilon$.

Let us prove now that the variational problem (3.3) is well-posed. Since $\mathbf{T}^\varepsilon$ is invertible, (3.3) is equivalent to the variational problem

$$\text{Find } u^\varepsilon \in V \text{ such that } \mathcal{A}^\varepsilon(u^\varepsilon, \mathbf{T}^\varepsilon v) = \ell^\varepsilon(\mathbf{T}^\varepsilon v) \text{ for all } v \in V. \tag{3.5}$$

By assumption, the bilinear form $\widetilde{\mathcal{A}}^\varepsilon(\cdot,\cdot) := \mathcal{A}^\varepsilon(\cdot, \mathbf{T}^\varepsilon\cdot)$ satisfies

$$\widetilde{\mathcal{A}}^\varepsilon(u, u)| \geqslant \gamma\|u\|^2, \qquad \forall u \in V.$$

Applying the Lax-Milgram theorem to the coercive bilinear form $\widetilde{\mathcal{A}}^\varepsilon(\cdot,\cdot)$ and the linear form $\widetilde{\ell}^\varepsilon(\cdot) := \ell^\varepsilon(\mathbf{T}^\varepsilon\cdot)$, we obtain the existence of a unique solution $u^\varepsilon \in V$ for (3.5). Moreover, since $(\mathbf{T}^\varepsilon)_{\varepsilon>0}$ is uniformly bounded, the estimate (3.4) holds. □

The result above is used in the next two sections to establish the well-posedness of a scalar problem set in the reference cell (namely the cell problem (4.10) involved in the definition of the homogenized matrix) and of problem (2.4) (or equivalently (2.7)) for extreme contrasts.

3.2. **Well-posedness in the reference cell.** Let $H^1_{\overline{\mathrm{per}}}(Y)$ denote the subset of $H^1(Y)$ constituted of functions being 1-periodic with respect to the variables y_1 and y_2:

$$H^1_{\overline{\mathrm{per}}}(Y) = \{v \in H^1(Y) \mid v \text{ is 1-periodic in } y_1 \text{ and } y_2\}.$$

Setting

$$\mathcal{M}(u) = \frac{1}{|Y|}\int_Y u\,\mathrm{d}y = \fint_Y u\,\mathrm{d}y,$$

let $W_{\overline{\mathrm{per}}}(Y)$ be the Hilbert space

$$W_{\overline{\mathrm{per}}}(Y) = \big\{u \in H^1_{\overline{\mathrm{per}}}(Y) : \mathcal{M}(u) = 0\big\}, \tag{3.6}$$

endowed with the norm $\|u\|_{W_{\overline{\mathrm{per}}}(Y)} := \|\nabla u\|_{L^2(Y)}$.

Given $f \in L^2(Y)$, we investigate in this section the well-posedness of the following cell-problem.

$$\text{Find } u \in W_{\overline{\mathrm{per}}}(Y) \text{ such that } -\operatorname{div}(a\nabla u) = f, \text{ in } Y, \tag{3.7}$$

whose variational formulation reads

$$\text{Find } u \in W_{\overline{\mathrm{per}}}(Y) \text{ such that } \mathcal{A}(u,v) = \int_Y fv\,\mathrm{d}y \text{ for all } v \in W_{\overline{\mathrm{per}}}(Y), \tag{3.8}$$

where $A(\cdot,\cdot)$ denotes the symmetric bilinear form defined on $H^1(Y)$ by

$$\mathcal{A}(u,v) = \int_Y a(y)\nabla u\cdot\nabla v\,\mathrm{d}y = a_1\int_{Y_1}\nabla u\cdot\nabla v\,\mathrm{d}y + a_2\int_{Y_2}\nabla u\cdot\nabla v\,\mathrm{d}y, \tag{3.9}$$

for all $u, v \in H^1(Y)$.

In the sequel, we denote the restrictions to Y_1 and Y_2 of a given function $u \in H^1(Y)$ by $u_1 := u_{|Y_1}$ and $u_2 := u_{|Y_2}$.

To prove the well-posedness of (3.7) (or, equivalently, (3.8)), we need to distinguish between the case of large contrasts (the positive number κ defined by (2.3) is large enough) and the one of small contrasts (κ is small enough), as each one is treated using a specific T-coercivity operator.

Large contrasts. Let $\mathbf{P}$ denote the harmonic extension operator from Y_1 to Y_2. In other words, given $u \in H^1(Y_1)$, $\mathbf{P}u \in H^1(Y)$ is defined by setting $\mathbf{P}u = u$ in Y_1, while in Y_2, $\mathbf{P}u \in H^1(Y_2)$ is the unique solution of the Dirichlet boundary value problem

$$\begin{aligned} -\Delta(\mathbf{P}u) &= 0 \quad \text{in } Y_2,\\ \mathbf{P}u &= u \quad \text{on } \partial Y_2. \end{aligned}$$

Clearly, there exists $\kappa_Y > 0$ such that

$$\|\nabla(\mathbf{P}u)\|^2_{L^2(Y)} \leqslant \kappa_Y\|\nabla u\|^2_{L^2(Y_1)}, \quad \forall u \in H^1(Y_1). \tag{3.10}$$

Proposition 3.3. *For $u \in H^1(Y)$, let*

$$\mathbf{T}_Y u := \widetilde{\mathbf{T}}_Y u - \mathcal{M}\big(\widetilde{\mathbf{T}}_Y u\big), \tag{3.11}$$

with

$$\widetilde{\mathbf{T}}_Y u = \begin{cases} u_1 & \text{in } Y_1,\\ -u_2 + 2\mathbf{P}u_1 & \text{in } Y_2. \end{cases}$$

Then, the following assertions hold:

(1) $\mathbf{T}_Y \in \mathcal{L}(W_{\overline{\mathrm{per}}}(Y))$.

(2) $\mathbf{T}_Y y = y - \mathcal{M}(y)$.
(3) *For $\kappa > \kappa_Y$, where κ is defined in* (2.3), *there exists $\gamma > 0$ such that for all $u \in W_{\overline{\text{per}}}(Y)$:*

$$\mathcal{A}(u, \mathbf{T}_Y u) = \int_Y a(y)\nabla u(y) \cdot \nabla(\mathbf{T}_Y u)(y)\, \mathrm{d}y \geqslant \gamma \|\nabla u\|^2_{L^2(Y)}. \tag{3.12}$$

Proof. 1. For every $u \in W_{\overline{\text{per}}}(Y)$, $\mathbf{T}_Y u$ clearly defines a function of $H^1(Y)$ (since it is continuous across the interface ∂Y_2) satisfying $\mathcal{M}\big(\mathbf{T}_Y u\big) = 0$. Moreover, we have $\mathbf{T}_Y u = u - \mathcal{M}\big(\widetilde{\mathbf{T}}_Y u\big)$ on the boundary ∂Y, and hence $\mathbf{T}_Y u$ is 1-periodic in y_1 and y_2. Consequently, $\mathbf{T}_Y u \in W_{\overline{\text{per}}}(Y)$ for every $u \in W_{\overline{\text{per}}}(Y)$. The continuity of the map $\mathbf{T}_Y$ is straightforward.

2. Since affine functions are harmonic, they are invariant by $\mathbf{P}$ and hence $\widetilde{\mathbf{T}}_Y y = y$.

3. For all $u \in W_{\overline{\text{per}}}(Y)$ and for all $\eta > 0$, we have by Young's inequality

$$\begin{aligned}
&\mathcal{A}(u, \mathbf{T}_Y u)\\
&= a_1 \int_{Y_1} |\nabla u|^2\, \mathrm{d}y + |a_2| \int_{Y_2} |\nabla u|^2\, \mathrm{d}y + 2a_2 \int_{Y_2} \nabla u \cdot \nabla(\mathbf{P}u)\, \mathrm{d}y\\
&\geqslant a_1 \|\nabla u\|^2_{L^2(Y_1)} + |a_2| \|\nabla u\|^2_{L^2(Y_2)} - |a_2|\eta \|\nabla u\|^2_{L^2(Y_2)} - \frac{|a_2|}{\eta} \|\nabla(\mathbf{P}u)\|^2_{L^2(Y_2)}.
\end{aligned}$$

Consequently, using (3.10), we obtain that

$$\mathcal{A}(u, \mathbf{T}_Y u) \geqslant |a_2| \big\{ \big(\kappa - \frac{\kappa_Y}{\eta}\big) \|\nabla u\|^2_{L^2(Y_1)} + (1 - \eta) \|\nabla u\|^2_{L^2(Y_2)} \big\}.$$

Thus, if the contrast satisfies $\kappa > \kappa_Y$, we can choose $\eta \in]\kappa_Y/\kappa, 1[$ and get the existence of a constant $\gamma > 0$ such that for all $u \in W_{\overline{\text{per}}}(Y)$,

$$\mathcal{A}(u, \mathbf{T}_Y u) \geqslant \gamma \|\nabla u\|^2_{L^2(Y)} = \gamma \|u\|^2_{W_{\overline{\text{per}}}(Y)}.$$

$\square$

Remark 3.4. Let us emphasize that the results of the above proposition have been already obtained in [17, Theorem 2.5]. However, the authors omitted to subtract the average term (the validity of the result was not affected).

Small contrasts. We denote by $H^1_{0,\Sigma}(Y)$ the space of functions in $H^1(Y)$ which are zero on the lateral boundary Σ^D of the domain Y (the top and bottom boundaries are denoted $\Sigma^N_{\pm}$). Let $\mathbf{Q}$ denote the harmonic extension operator from $H^1(Y_2)$ onto $H^1_{0,\Sigma}(Y)$. In other words, given $u \in H^1(Y_2)$, $\mathbf{Q}u \in H^1_{0,\Sigma}(Y)$ is defined by setting $\mathbf{Q}u = u$ in Y_2, while in Y_1, $\mathbf{Q}u$ is the unique solution of the boundary value problem

$$\begin{gathered}
-\Delta(\mathbf{Q}u) = 0 \quad \text{in } Y_1,\\
\mathbf{Q}u = u \quad \text{on } \partial Y_2,\\
\mathbf{Q}u = 0 \quad \text{on } \Sigma^D,\\
\frac{\partial(\mathbf{Q}u)}{\partial \nu} = 0 \quad \text{on } \Sigma^N_{\pm}.
\end{gathered}$$

Setting

$$\mathcal{M}_2(u) = \frac{1}{|Y_2|} \int_{Y_2} u\, \mathrm{d}y,$$

let

$$H^1_{\mathrm{mean}}(Y_2) = \{u \in H^1(Y_2) \mid \mathcal{M}_2(u) = 0\}.$$

It can be easily seen that there exists $\kappa'_Y > 0$ such that

$$\|\nabla(\mathbf{Q}u)\|^2_{L^2(Y)} \leqslant \kappa'_Y \|\nabla u\|^2_{L^2(Y_2)}, \quad \forall u \in H^1_{\mathrm{mean}}(Y_2). \tag{3.13}$$

Proposition 3.5. *For $u \in H^1(Y)$, set*

$$\mathbf{T}_Y u := \widetilde{\mathbf{T}}_Y u - \mathcal{M}\big(\widetilde{\mathbf{T}}_Y u\big), \tag{3.14}$$

where

$$\widetilde{\mathbf{T}}_Y u = \begin{cases} u_1 - 2\mathbf{Q}(u_2 - \mathcal{M}_2(u_2)) & \text{in } Y_1, \\ -u_2 + 2\mathcal{M}_2(u_2) & \text{in } Y_2. \end{cases} \tag{3.15}$$

Then, the following assertions hold:

1. $\mathbf{T}_Y \in \mathcal{L}(W_{\overline{\mathrm{per}}}(Y))$.
2. *For $\kappa < 1/\kappa'_Y$, where κ'_Y is defined in* (3.13) *there exists $\gamma' > 0$ such that for all $u \in W_{\overline{\mathrm{per}}}(Y)$:*

$$\mathcal{A}(u, \mathbf{T}_Y u) = \int_Y a(y)\nabla u(y) \cdot \nabla(\mathbf{T}_Y u)(y)\,\mathrm{d}y \geqslant \gamma' \|\nabla u\|^2_{L^2(Y)}. \tag{3.16}$$

Proof. 1. For every $u \in W_{\overline{\mathrm{per}}}(Y)$, $\mathbf{T}_Y u$ clearly defines a function of $H^1(Y)$ (since it is continuous across the interface ∂Y_2) satisfying by construction $\mathcal{M}(\mathbf{T}_Y u) = 0$. Moreover, we have $\mathbf{T}_Y u = u - \mathcal{M}\big(\widetilde{\mathbf{T}}_Y u\big)$ on the boundary ∂Y, and hence $\mathbf{T}_Y u$ is 1-periodic in y_1 and y_2. Consequently, $\mathbf{T}_Y u \in W_{\overline{\mathrm{per}}}(Y)$ for every $u \in W_{\overline{\mathrm{per}}}(Y)$. The continuity of the map $\mathbf{T}_Y$ is straightforward.

2. For all $u \in W_{\overline{\mathrm{per}}}(Y)$ and for all $\eta > 0$, we have by Young's inequality

$$\begin{aligned}
&\mathcal{A}(u, \mathbf{T}_Y u) \\
&= \mathcal{A}(u, \widetilde{\mathbf{T}}_Y u) \\
&= a_1 \int_{Y_1} |\nabla u|^2\,\mathrm{d}y + |a_2| \int_{Y_2} |\nabla u|^2\,\mathrm{d}y - 2a_1 \int_{Y_2} \nabla u \cdot \nabla(\mathbf{Q}(u_2 - \mathcal{M}_2(u_2)))\,\mathrm{d}y \\
&\geqslant a_1 \|\nabla u\|^2_{L^2(Y_1)} + |a_2| \|\nabla u\|^2_{L^2(Y_2)} - a_1\eta \|\nabla u\|^2_{L^2(Y_1)} \\
&\quad - \frac{a_1}{\eta} \|\nabla(\mathbf{Q}(u_2 - \mathcal{M}_2(u_2)))\|^2_{L^2(Y_1)}.
\end{aligned}$$

Consequently, using (3.13), we obtain that

$$\mathcal{A}(u, \mathbf{T}_Y u) \geqslant a_1 \|\nabla u\|^2_{L^2(Y_1)} + |a_2| \|\nabla u\|^2_{L^2(Y_2)} - a_1\eta \|\nabla u\|^2_{L^2(Y_1)} - \frac{a_1 \kappa'_Y}{\eta} \|\nabla u\|^2_{L^2(Y_2)},$$

that is,

$$\mathcal{A}(u, \mathbf{T}_Y u) \geqslant a_1 \Big\{(1 - \eta)\|\nabla u\|^2_{L^2(Y_1)} + \Big(\frac{1}{\kappa} - \frac{\kappa'_Y}{\eta}\Big)\|\nabla u\|^2_{L^2(Y_1)}\Big\}.$$

Thus, if the contrast satisfies $\kappa < 1/\kappa'_Y$, we can choose $\eta \in]\kappa\kappa'_Y, 1[$ and get the existence of a constant $\gamma' > 0$ such that for all $u \in W_{\overline{\mathrm{per}}}(Y)$:

$$\mathcal{A}(u, \mathbf{T}_Y u) \geqslant \gamma \|\nabla u\|^2_{L^2(Y)} = \gamma' \|u\|^2_{W_{\overline{\mathrm{per}}}(Y)}.$$

□

We can now state a well-posedness result for the cell problem (3.7) for extreme contrasts.

Theorem 3.6. *Let κ_Y and κ'_Y denote the constants in* (3.10) *and* (3.13), *respectively. If the contrast $\kappa := \left|\frac{a_1}{a_2}\right|$ satisfies*

$$\kappa > \kappa_Y \quad \text{or} \quad \kappa < 1/\kappa'_Y, \tag{3.17}$$

then, for every given $f \in L^2(Y)$, problem (3.7) *admits a unique solution and there exists a constant $C > 0$ such that*

$$\|u\|_{W_{\overline{\mathrm{per}}}(Y)} = \|\nabla u\|_{L^2(Y)} \leqslant C\|f\|_{L^2(Y)}. \tag{3.18}$$

Proof. The result follows immediately from Theorem 3.2 using the T-coercivity operators of Proposition 3.3 and Proposition 3.5, respectively, for large or small contrasts. □

3.3. **Well-posedness of the problem in Ω^ε.** For every function $u \in H^1(\Omega^\varepsilon)$, we set

$$u_1 := u|_{\Omega_1^\varepsilon}, \quad u_2 := u|_{\Omega_2^\varepsilon}.$$

Let $H^1_{0,\Sigma}(\Omega_1^\varepsilon)$ be the space of functions u in $H^1(\Omega_1^\varepsilon)$ and such that $u = 0$ on the lateral boundary $\Sigma^{\varepsilon,D}$ of Ω^ε.

Like for the cell problem considered in the previous section, we need to distinguish the cases of large and small contrasts to study the well-posedness of (2.4) (or equivalently of its variational formulation (2.7)). The case of large contrasts can be easily adapted from [17] (even though the domain considered here is thin) and therefore we only state the results for the sake of completeness. For the case of small contrasts, we propose a new proof which is more elementary than the ones given for the case of a fixed domain in [16] (using the T-coercivity) and in [11] (via the analysis of the spectrum of the Neumann-Poincaré operator).

Large contrasts. As in [17, Proposition 2.4], one can prove the following result.

Proposition 3.7. *There is a family of linear bounded extension operators $(\mathbf{P}^\varepsilon)_{\varepsilon>0}$ from Ω_1^ε to Ω^ε such that*

(1) $\mathbf{P}^\varepsilon \in \mathcal{L}(H^1_{0,\Sigma}(\Omega_1^\varepsilon), V^\varepsilon)$.
(2) *For all $u \in H^1_{0,\Sigma}(\Omega_1^\varepsilon)$, $(\mathbf{P}^\varepsilon u)(x) = u(x)$ for all $x \in \Omega_1^\varepsilon$.*
(3) *If κ_Y is the constant defined in* (3.10), *one has for all $u \in H^1_{0,\Sigma}(\Omega_1^\varepsilon)$,*

$$\|\nabla(\mathbf{P}^\varepsilon u)\|^2_{L^2(\Omega^\varepsilon)} \leqslant \kappa_Y \|\nabla u\|^2_{L^2(\Omega_1^\varepsilon)}. \tag{3.19}$$

According to [17, Theorem 2.6], we also have the following result.

Proposition 3.8. *For $u \in H^1(\Omega^\varepsilon)$, set*

$$\mathbf{T}^\varepsilon u = \begin{cases} u_1 & \text{in } \Omega_1^\varepsilon, \\ -u_2 + 2\mathbf{P}^\varepsilon u_1 & \text{in } \Omega_2^\varepsilon. \end{cases} \tag{3.20}$$

Then, the following assertions hold:

(1) *$(\mathbf{T}^\varepsilon)_{\varepsilon>0}$ is a family of uniformly bounded linear operators of $\mathcal{L}(V^\varepsilon)$.*
(2) *For $\kappa > \kappa_Y$, where κ_Y is defined in* (3.10), *there exists $\gamma > 0$ such that for all $\varepsilon > 0$ and all $u \in V^\varepsilon$,*

$$\mathcal{A}^\varepsilon(u, \mathbf{T}^\varepsilon u) = \int_{\Omega^\varepsilon} a^\varepsilon(x)\nabla u(x) \cdot \nabla(\mathbf{T}^\varepsilon u)(x)\, \mathrm{d}x \geqslant \gamma \|\nabla u\|^2_{L^2(\Omega^\varepsilon)}.$$

Small contrasts. Given $u \in H^1(\Omega_2^\varepsilon)$, let $\mathcal{M}_2^\varepsilon(u) \in H^1(\Omega_2^\varepsilon)$ denote the piecewise constant function defined on each inclusion $\varepsilon Y_2^{\mathbf{k}} \subset \Omega^\varepsilon$, $\mathbf{k} \in \mathbb{Z}_\varepsilon$, by

$$\mathcal{M}_2^\varepsilon(u)(x) = \frac{1}{|\varepsilon Y_2^{\mathbf{k}}|} \int_{\varepsilon Y_2^{\mathbf{k}}} u \, \mathrm{d}x, \quad \forall x \in \varepsilon Y_2^{\mathbf{k}}.$$

We set

$$H^1_{\text{mean}}(\Omega_2^\varepsilon) = \left\{ u \in H^1(\Omega_2^\varepsilon) \mid \mathcal{M}_2^\varepsilon(u) = 0 \right\}.$$

As in [23, Lemma 2.3], one has the following result.

Proposition 3.9. *There is a family of linear bounded extension operators $(\mathbf{Q}^\varepsilon)_{\varepsilon>0}$ from Ω_2^ε to Ω^ε such that*

(1) $\mathbf{Q}^\varepsilon \in \mathcal{L}(H^1_{\text{mean}}(\Omega_2^\varepsilon), V^\varepsilon)$.
(2) *For all $u \in H^1_{\text{mean}}(\Omega_2^\varepsilon)$, $(\mathbf{Q}^\varepsilon u)(x) = u(x)$ for all $x \in \Omega_2^\varepsilon$.*
(3) *If κ'_Y is the constant defined in* (3.13)*, one has for all $u \in H^1_{\text{mean}}(\Omega_2^\varepsilon)$:*

$$\|\nabla(\mathbf{Q}^\varepsilon u)\|^2_{L^2(\Omega^\varepsilon)} \leqslant \kappa'_Y \|\nabla u\|^2_{L^2(\Omega_2^\varepsilon)}. \tag{3.21}$$

Proposition 3.10. *For $u \in H^1(\Omega^\varepsilon)$, let*

$$\mathbf{T}^\varepsilon u = \begin{cases} u_1 - 2\mathbf{Q}^\varepsilon(u_2 - \mathcal{M}_2^\varepsilon(u_2)) & \text{in } \Omega_1^\varepsilon, \\ -u_2 + 2\mathcal{M}_2^\varepsilon(u_2) & \text{in } \Omega_2^\varepsilon. \end{cases} \tag{3.22}$$

Then, the following assertions hold:

(1) *$(\mathbf{T}^\varepsilon)_{\varepsilon>0}$ is a family of uniformly bounded linear operators of $\mathcal{L}(V^\varepsilon)$.*
(2) *For $0 < \kappa < 1/\kappa'_Y$, where κ'_Y is defined in* (3.13)*, there exists $\gamma' > 0$ such that for all $\varepsilon > 0$ and all $u \in V^\varepsilon$,*

$$\mathcal{A}^\varepsilon(u, \mathbf{T}^\varepsilon u) = \int_{\Omega^\varepsilon} a^\varepsilon(x) \nabla u(x) \cdot \nabla(\mathbf{T}^\varepsilon u)(x) \, \mathrm{d}x \geqslant \gamma' \|\nabla u\|^2_{L^2(\Omega^\varepsilon)}.$$

Proof. 1. Let u be given in V^ε. Since $u_2 - \mathcal{M}_2^\varepsilon(u_2) \in H^1_{\text{mean}}(\Omega_2^\varepsilon)$, $\mathbf{Q}^\varepsilon(u_2 - \mathcal{M}_2^\varepsilon(u_2))$ is well-defined and so is $\mathbf{T}^\varepsilon u$. Moreover, $\mathbf{T}^\varepsilon u$ is continuous across the interface $\partial\Omega_1^\varepsilon \cap \partial\Omega_2^\varepsilon$ and satisfies $\mathbf{T}^\varepsilon u_{|\Sigma^{\varepsilon,D}} = 0$. Hence, $\mathbf{T}^\varepsilon u \in V^\varepsilon$ and one clearly has $\mathbf{T}^\varepsilon \in \mathcal{L}(V^\varepsilon)$ and is uniformly bounded.

2. For all $u \in H^1(\Omega^\varepsilon)$ and for all $\eta > 0$, we have

$$\begin{aligned}
\mathcal{A}^\varepsilon(u, \mathbf{T}^\varepsilon u) &= \int_{\Omega^\varepsilon} a^\varepsilon(x) \nabla u(x) \cdot \nabla(\mathbf{T}^\varepsilon u)(x) \, \mathrm{d}x \\
&= a_1 \int_{\Omega_1^\varepsilon} |\nabla u_1|^2 \, \mathrm{d}x + |a_2| \int_{\Omega_2^\varepsilon} |\nabla u_2|^2 \, \mathrm{d}x \\
&\quad - 2a_1 \int_{\Omega_1^\varepsilon} \nabla u_1 \cdot \nabla \left(\mathbf{Q}^\varepsilon(u_2 - \mathcal{M}_2^\varepsilon(u_2) \right) \mathrm{d}x \\
&\geqslant a_1 \|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)} + |a_2| \|\nabla u_2\|^2_{L^2(\Omega_2^\varepsilon)} - a_1 \eta \|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)} \\
&\quad - \frac{a_1}{\eta} \|\nabla \left(\mathbf{Q}^\varepsilon(u - \mathcal{M}_2^\varepsilon(u) \right) \|^2_{L^2(\Omega_1^\varepsilon)},
\end{aligned}$$

where we have used Young's inequality for the last estimate. Consequently, using (3.21), we obtain that

$$|\mathcal{A}^\varepsilon(u, \mathbf{T}^\varepsilon u)| \geqslant a_1 \Big\{ (1 - \eta) \|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)} + \Big(\frac{1}{\kappa} - \frac{\kappa'_Y}{\eta} \Big) \|\nabla u_2\|^2_{L^2(\Omega_2^\varepsilon)} \Big\}.$$

Thus, if the contrast satisfies $0 < \kappa < 1/\kappa'_Y$, we can choose $\eta \in]\kappa\kappa'_Y, 1[$ and get the existence of a constant $\gamma > 0$ such that

$$|\mathcal{A}^\varepsilon(u, \mathbf{T}^\varepsilon u)| \geqslant \gamma' \|\nabla u\|^2_{L^2(\Omega^\varepsilon)}, \quad \forall u \in H^1(\Omega^\varepsilon). \qquad \square$$

We can now state a well-posedness result for problem (2.4) for extreme contrasts.

Theorem 3.11. *Let κ_Y and κ'_Y denote the constants in* (3.19) *and* (3.21)*, respectively. If the contrast $\kappa := |\frac{a_1}{a_2}|$ satisfies*

$$\kappa > \kappa_Y \quad or \quad \kappa < 1/\kappa'_Y, \tag{3.23}$$

then, for every given $f \in L^2(\omega)$, problem (2.4) *(or, equivalently, its variational formulation* (2.7)*) admits a unique solution $u^\varepsilon \in V^\varepsilon$ and there exists a constant $C > 0$ independent of ε such that for all $\varepsilon > 0$,*

$$\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leqslant C\sqrt{\varepsilon}\|f\|_{L^2(\omega)}. \tag{3.24}$$

Proof. The claimed result follows immediately from Theorem 3.2, Proposition 3.8 and Proposition 3.10 and from the hypothesis on f (f does not depend on x_3), which yields $\|f\|_{L^2(\Omega^\varepsilon)} = \sqrt{\varepsilon}\|f\|_{L^2(\omega)}$. $\square$

4. Homogenization results in the case of a thin periodic layer

Our goal in this section is to pass to the limit, with $\varepsilon \to 0$, in the variational formulation (2.7) of problem (2.4). To this end, we use the two-scale convergence method [43, 1], adapted to thin periodic domains (see, for instance, [42]).

Let $\mathcal{C}^\infty_{\mathrm{per}}(\overline{Y})$ be the space of infinitely differentiable functions in $\mathbb{R}^3$ that are 1-periodic in the first two variables y_1 and y_2. We recall the definition of the weak two-scale convergence from [42].

Definition 4.1. A sequence (v^ε) in $L^2(\Omega^\varepsilon)$ weakly two-scale converges to $v \in L^2(\omega \times Y)$ if one has

$$\frac{1}{\varepsilon}\int_{\Omega^\varepsilon} v^\varepsilon(x)\psi\big(\bar{x}, \frac{x}{\varepsilon}\big)\,\mathrm{d}x \to \int_{\omega\times Y} v(\bar{x}, y)\psi(\bar{x}, y)\,\mathrm{d}\bar{x}\,\mathrm{d}y,$$

for all $\psi \in \mathcal{C}^\infty_0(\omega, \mathcal{C}^\infty_{\mathrm{per}}(\overline{Y}))$. Then, we denote

$$v^\varepsilon \xrightharpoonup{2-s} v.$$

Using the *a priori* estimate (3.24), Poincaré inequality in Ω^ε (see, for instance, [40], Section 2.1) and [42, Prop. 4.4(i)], it follows that there exist $u \in H^1_0(\omega)$ and $\widehat{u} \in L^2(\omega, W_{\overline{\mathrm{per}}}(Y))$ (recall that $W_{\overline{\mathrm{per}}}(Y)$ is defined in (3.6)), such that, up to a subsequence, for $\varepsilon \to 0$, we obtain

$$\begin{gathered} u^\varepsilon \xrightharpoonup{2-s} u, \\ \nabla_{\bar{x}} u^\varepsilon \xrightharpoonup{2-s} \nabla_{\bar{x}}\, u + \nabla_{\bar{y}}\widehat{u}, \\ \partial_{x_3} u^\varepsilon \xrightharpoonup{2-s} \partial_{y_3}\widehat{u}. \end{gathered} \tag{4.1}$$

The special form of the limits in convergences $(4.1)_2$ and $(4.1)_3$ suggests to introduce the following notation: to every $w \in H^1(\omega)$, whose gradient $\nabla_{\bar{x}} w(\bar{x})$ has two components, we associate the tridimensional vector $\overline{\nabla} w(\bar{x})$ defined by

$$\overline{\nabla} w(\bar{x}) = (\nabla_{\bar{x}} w(\bar{x}), 0).$$

We introduce the space

$$\mathcal{H} = H_0^1(\omega) \times L^2\left(\omega, W_{\overline{\mathrm{per}}}(Y)\right)$$

and for all $\mathcal{V} = (v, \widehat{v}) \in \mathcal{H}$ we define the norm

$$\|\mathcal{V}\|_{\mathcal{H}}^2 = \|\overline{\nabla} v + \nabla_y \widehat{v}\|_{L^2(\omega\times Y)}^2,$$

which is equivalent to the norm $\|\overline{\nabla} v\|_{L^2(\omega)}^2 + \|\nabla_y \widehat{v}\|_{L^2(\omega\times Y)}^2$ (see, for instance, [26, Lemma 5.4]).

Theorem 4.2. *Assume that the contrast $\kappa := \left|\frac{a_1}{a_2}\right|$ satisfies*

$$\kappa > \kappa_Y \quad \text{or} \quad \kappa < 1/\kappa_Y',$$

where κ_Y and κ_Y' denote the constants in (3.19) *and* (3.21)*, respectively.*

The unique solution u^ε of the variational problem (2.7) *converges, in the sense of* (4.1)*, to the unique solution $(u, \widehat{u}) \in \mathcal{H}$ of the well-posed two-scale limit problem*

$$\int_{\omega\times Y} a(y)(\overline{\nabla} u + \nabla_y \widehat{u}) \cdot (\overline{\nabla}\varphi + \nabla_y \Phi)\, \mathrm{d}\bar{x}\, \mathrm{d}y = \int_\omega f(\bar{x})\varphi(\bar{x})\, \mathrm{d}\bar{x}, \tag{4.2}$$

for all $\varphi \in H_0^1(\omega)$ and $\Phi \in L^2(\omega, H^1_{\overline{\mathrm{per}}}(Y))$.

Proof. According to Theorem 3.11, the variational problem (2.7) admits a unique solution u^ε which satisfies the energy estimate (3.24). Dividing (2.7) by ε and then choosing the test function (recalling that $x = (\bar{x}, x_3)$)

$$v(\bar{x}, x_3) = \varphi(\bar{x}) + \varepsilon\psi\Big(\bar{x}, \frac{x}{\varepsilon}\Big), \tag{4.3}$$

with $\varphi \in \mathcal{D}(\omega)$, $\psi^\varepsilon \in \mathcal{C}_0^\infty(\omega, \mathcal{C}^\infty_{\overline{per}}(\overline{Y}))$, we can pass to the limit as $\varepsilon \to 0$. Using the convergences (4.1), we obtain by standard density arguments the two-scale limit problem (4.2).

We prove now the well-posedness of the limit problem (4.2). Define on $\mathcal{H}$ the bilinear form

$$\mathcal{B}(\mathcal{U}, \mathcal{V}) := \int_{\omega\times Y} a(y)\left(\overline{\nabla} u(\bar{x}) + \nabla_y \hat{u}(\bar{x}, y)\right) \cdot \left(\overline{\nabla} v(\bar{x}) + \nabla_y \hat{v}(\bar{x}, y)\right)\, \mathrm{d}\bar{x}\, \mathrm{d}y,$$

for all $\mathcal{U} = (u, \hat{u}) \in \mathcal{H}$ and all $\mathcal{V} = (v, \hat{v}) \in \mathcal{H}$. Given $\mathcal{F} \in \mathcal{H}'$ ($\mathcal{H}'$ denotes here the dual space of $\mathcal{H}$), consider the variational problem

$$\text{Find } \mathcal{U} = (u, \hat{u}) \in \mathcal{H} \text{ such that } \mathcal{B}(\mathcal{U}, \mathcal{V}) = \langle \mathcal{F}, \mathcal{V}\rangle_{\mathcal{H}',\mathcal{H}}, \quad \forall \mathcal{V} = (v, \hat{v}) \in \mathcal{H}. \tag{4.4}$$

We remark that the two-scale limit problem (4.2) fits in the frame of problem (4.4), for which we shall prove the well-posedness successively for large and small contrasts.

(i) Large contrasts. Let $\mathcal{T}$ be the operator defined on $\mathcal{H}$ by

$$\mathcal{T}\mathcal{U} = (u, \mathbf{T}_Y \hat{u}), \quad \forall \mathcal{U} = (u, \hat{u}) \in \mathcal{H}, \tag{4.5}$$

where $\mathbf{T}_Y \in \mathcal{L}(W_{\overline{\mathrm{per}}}(Y))$ is defined in (3.11). Since $\mathbf{T}_Y \in \mathcal{L}(W_{\overline{\mathrm{per}}}(Y))$, we clearly have $\mathcal{T} \in \mathcal{L}(\mathcal{H})$. For every $\mathcal{U} = (u, \hat{u}) \in \mathcal{H}$, we note that

$$\overline{\nabla} u(\bar{x}) + \nabla_y \hat{u}(\bar{x}, y) = \nabla_y\left(\overline{\nabla} u(\bar{x}) \cdot y^c + \hat{u}(\bar{x}, y)\right),$$

where we have set

$$y^c := y - \mathcal{M}(y).$$

Therefore,

$\mathcal{B}(\mathcal{U}, \mathcal{T}\mathcal{U})$

$$= \int_{\omega\times Y} a(y)\nabla_y \left(\overline{\nabla} u(\bar{x})\cdot y^c + \hat{u}(\bar{x},y)\right)\cdot \nabla_y \left(\overline{\nabla} u(\bar{x})\cdot y^c + \mathbf{T}_Y\hat{u}(\bar{x},y)\right) \,\mathrm{d}\bar{x}\,\mathrm{d}y.$$

According to Proposition 3.3, we have $\mathbf{T}_Y y^c = y^c$ and, thus,

$$\begin{aligned}\nabla_y \left(\overline{\nabla} u(\bar{x})\cdot y^c + \mathbf{T}_Y\hat{u}(\bar{x},y)\right) &= \nabla_y \left(\overline{\nabla} u(\bar{x})\cdot \mathbf{T}_Y y^c + \mathbf{T}_Y\hat{u}(\bar{x},y)\right)\\ &= \nabla_y \mathbf{T}_Y \left(\overline{\nabla} u(\bar{x})\cdot y^c + \hat{u}(\bar{x},y)\right).\end{aligned}$$

Consequently, setting $U_{\bar{x}}(y) := \overline{\nabla} u(\bar{x})\cdot y^c + \hat{u}(\bar{x},y)$, we have

$$\mathcal{B}(\mathcal{U},\mathcal{T}\mathcal{U}) = \int_{\omega\times Y} a(y)\overline{\nabla}_y U_{\bar{x}}(y)\cdot\nabla_y \mathbf{T}_Y U_{\bar{x}}(y)\,\mathrm{d}\bar{x}\,\mathrm{d}y = \int_\omega \mathcal{A}(U_{\bar{x}},\mathbf{T}_Y U_{\bar{x}})\,\mathrm{d}\bar{x},$$

where $\mathcal{A}(\cdot,\cdot)$ is defined in (3.9). Using the T-coercivity inequality (3.12) in the above relation, we obtain that, for $\kappa > \kappa_Y$, there exists $\gamma > 0$ such that

$$\mathcal{B}(\mathcal{U},\mathcal{T}\mathcal{U}) \geqslant \gamma \int_\omega \|\nabla_y U_{\bar{x}}\|^2_{L^2(Y)}\,\mathrm{d}\bar{x} = \gamma\|\overline{\nabla} u(\bar{x}) + \nabla_y\hat{u}(\bar{x},y)\|^2_{L^2(\omega\times Y)} = \gamma\|\mathcal{U}\|^2_{\mathcal{H}}.$$

(ii) Small contrasts. As above, the idea is to use the T-coercivity of the reference cell problem given in (3.14) to derive a suitable two-scale T-coercivity operator $\mathcal{T}$. However, since affine functions are not anymore stable by the extension operator $\mathbf{Q}$ used to define $\mathbf{T}_Y$, the proof needs to be adapted. Let $\mathcal{T}$ be the operator defined on $\mathcal{H}$ by

$$\mathcal{T}\mathcal{U} = (u, \mathbf{T}_Y\hat{u} + \mathbf{T}_Y\psi - \psi), \quad \forall \mathcal{U} = (u,\hat{u}) \in \mathcal{H}, \tag{4.6}$$

where $\mathbf{T}_Y \in \mathcal{L}(W_{\overline{\mathrm{per}}}(Y))$ is defined in (3.14) and

$$\psi := \overline{\nabla} u\cdot y^c.$$

Let us first check that $\mathcal{T} \in \mathcal{L}(\mathcal{H})$. Obviously, we only need to prove that $\mathbf{T}_Y\hat{u} + \mathbf{T}_Y\psi - \psi \in W_{\overline{\mathrm{per}}}(Y)$ for all $\mathcal{U} = (u,\hat{u}) \in \mathcal{H}$. First of all, we have

$$\mathcal{M}\left(\mathbf{T}_Y\hat{u} + \mathbf{T}_Y\psi - \psi\right) = -\mathcal{M}(\psi) = -\overline{\nabla} u\cdot\mathcal{M}(y^c) = 0.$$

Next, by the definition of $\mathbf{T}_Y$ (see (3.14)), $\mathbf{T}_Y\hat{u}$ is 1-periodic in the variables y_1 and y_2. Furthermore, it follows from the definition of $\widetilde{\mathbf{T}}_Y$ (see (3.15)) that $\widetilde{\mathbf{T}}_Y\psi = \psi$ on the lateral boundary Σ^D of ∂Y. Hence,

$$\mathbf{T}_Y\psi - \psi = \widetilde{\mathbf{T}}_Y\psi - \mathcal{M}(\widetilde{\mathbf{T}}_Y\psi) - \psi = -\mathcal{M}(\widetilde{\mathbf{T}}_Y\psi), \quad \text{on } \Sigma^D.$$

From the above relation we deduce that $\mathbf{T}_Y\psi - \psi$ is constant on Σ^D, and thus, $\mathbf{T}_Y\hat{u} + (\mathbf{T}_Y\psi - \psi)$ is 1-periodic in y_1 and y_2. Consequently, $\mathbf{T}_Y\hat{u} + \mathbf{T}_Y\psi - \psi \in W_{\overline{\mathrm{per}}}(Y)$, which shows that $\mathcal{T} \in \mathcal{L}(\mathcal{H})$.

Moreover, for every $\mathcal{U} = (u,\hat{u}) \in \mathcal{H}$, one can easily check that

$$\mathcal{B}(\mathcal{U},\mathcal{T}\mathcal{U}) = \int_{\omega\times Y} a(y)\nabla_y\Big(\psi(\bar{x},y) + \hat{u}(\bar{x},y)\Big)\cdot\nabla_y\Big(\mathbf{T}_Y\psi(\bar{x},y) + \mathbf{T}_Y\hat{u}(\bar{x},y)\Big)\,\mathrm{d}\bar{x}\,\mathrm{d}y.$$

Hence, setting $U_{\bar{x}}(y) := \psi(\bar{x},y) + \hat{u}(\bar{x},y)$, we have

$$\mathcal{B}(\mathcal{U},\mathcal{T}\mathcal{U}) = \int_{\omega\times Y} a(y)\nabla_y U_{\bar{x}}(y)\cdot\nabla_y \mathbf{T}_Y U_{\bar{x}}(y)\,\mathrm{d}\bar{x}\,\mathrm{d}y = \int_\omega \mathcal{A}(U_{\bar{x}},\mathbf{T}_Y U_{\bar{x}})\,\mathrm{d}\bar{x}.$$

Using the T-coercivity inequality (3.16) in the above relation, we obtain that, for $\kappa < 1/\kappa'_Y$, there exists $\gamma' > 0$ such that

$$\mathcal{B}(\mathcal{U},\mathcal{T}\mathcal{U}) \geqslant \gamma' \int_\omega \|\nabla_y U_{\bar{x}}\|^2_{L^2(Y)}\,\mathrm{d}\bar{x} = \gamma'\|\overline{\nabla} u(\bar{x}) + \nabla_y\hat{u}(\bar{x},y)\|^2_{L^2(\omega\times Y)} = \gamma'\|\mathcal{U}\|^2_{\mathcal{H}}.$$

From the uniqueness of $(u, \widehat{u}) \in \mathcal{H}$, all the above convergences hold for the whole sequence. $\square$

Theorem 4.3. *Under the hypotheses of Theorem 4.2, the unique solution u^ε of the variational problem (2.7) converges, in the sense of (4.1), to $(u, \widehat{u}) \in \mathcal{H}$, where u is the unique solution of the homogenized problem*

$$\begin{gathered} -\operatorname{div}_{\bar{x}}\left(a^{\mathrm{hom}} \nabla_{\bar{x}} u(\bar{x})\right) = f(\bar{x}) \quad \text{in } \omega, \\ u = 0 \quad \text{on } \partial\omega \end{gathered} \tag{4.7}$$

and

$$\widehat{u}(\bar{x}, y) = -\sum_{j=1}^{2} \frac{\partial u}{\partial x_j}(\bar{x}) \chi^j(y) \quad \forall (\bar{x}, y) \in \omega \times Y. \tag{4.8}$$

Here, a^{hom} is the constant homogenized 2×2 matrix whose entries are defined, for $i, j \in \{1, 2\}$, by

$$a_{ij}^{\mathrm{hom}} = \int_Y \Big(a_{ij} - \sum_{k=1}^{3} a_{ik} \frac{\partial \chi^j}{\partial y_k} \Big) \,\mathrm{d}y, \tag{4.9}$$

where the coefficients $(a_{ij})_{1 \leqslant i,j \leqslant 3}$ are those of the (diagonal) matrix $\mathbf{a}$ defined in (2.5).

The function $\chi = (\chi^1, \chi^2) \in (H^1_{\mathrm{per}}(Y))^2$ is the weak solution of the cell problem ($j = 1, 2$),

$$\begin{gathered} -\operatorname{div}_y \left(a(y)(\nabla_y \chi^j - e_j) \right) = 0, \quad \text{in } Y, \\ (\nabla_y \chi^j - e_j) \cdot \nu_\pm = 0, \quad \text{on } \Sigma^N_\pm, \\ \mathcal{M}_Y(\chi^j) = 0, \end{gathered} \tag{4.10}$$

where $\Sigma^N_+ = (0,1)^2 \times \{1\}$, $\Sigma^N_- = (0,1)^2 \times \{0\}$, and $\nu_\pm = (0, 0, \pm 1)$.

Proof. By choosing $\varphi = 0$ in the two-scale limit problem (4.2), we obtain

$$\int_{\omega \times Y} a(y)(\overline{\nabla} u + \nabla_y \widehat{u}) \cdot \nabla_y \Phi \,\mathrm{d}\bar{x}\,\mathrm{d}y = 0, \tag{4.11}$$

for all $\Phi \in L^2(\omega, H^1_{\mathrm{per}}(Y))$. Choosing $\Phi \in L^2(\omega, H^1_{\mathrm{per}}(Y))$ such that $\Phi = 0$ on $\omega \times \partial Y$ and integrating by parts with respect to y, we are formally led to

$$-\operatorname{div}_y \left(a(y) \nabla_y \widehat{u} \right) = \operatorname{div}_y (a(y) \overline{\nabla} u) \quad \text{in } \omega \times Y.$$

Taking in (4.11) a test function $\Phi \in L^2(\omega, H^1_{\mathrm{per}}(Y))$ which is zero on Σ^N_- (and, respectively, which is zero on Σ^N_+), we formally obtain

$$\begin{aligned} \nabla_y \widehat{u} \cdot \nu_+ &= -\overline{\nabla} u \cdot \nu_+ \quad \text{on } \omega \times \Sigma^N_+, \\ \nabla_y \widehat{u} \cdot \nu_- &= -\overline{\nabla} u \cdot \nu_- \quad \text{on } \omega \times \Sigma^N_-. \end{aligned}$$

The linearity of the problem suggests us to search $\widehat{u}(\bar{x}, y) = -\overline{\nabla} u(\bar{x}) \cdot \chi(y)$, where the vector $\chi(y) = (\chi^1(y), \chi^2(y), \chi^3(y))$ belonging to $(H^1_{\mathrm{per}}(Y))^3$ has to be determined. Recalling that $\overline{\nabla} u(\bar{x}) = (\nabla_{\bar{x}} u(\bar{x}), 0)$, we notice that only the first two components of χ will play a role in our analysis. Inserting this factorization into the equation, we therefore obtain the two local problems (4.10).

By choosing now $\Phi = 0$ in (4.2), we obtain

$$\int_{\omega \times Y} a(y) \left(\overline{\nabla} u + \nabla_y \widehat{u} \right) \cdot \overline{\nabla} \varphi \,\mathrm{d}\bar{x}\,\mathrm{d}y = \int_\omega f(\bar{x}) \varphi(\bar{x}) \,\mathrm{d}\bar{x}. \tag{4.12}$$

We integrate this last equality by parts with respect to $\bar{x}$ and, by using (4.8) and (4.10), we are led to the homogenized problem (4.7) with the homogenized coefficients given by (4.9). □

Remark 4.4. The expected dimension reduction effect can be seen in the fact that the 2×2 homogenized matrix a^{hom} is lower dimensional with respect to the initial matrix. However, information coming from the vertical direction of the initial problem is preserved. Indeed, the value of the coefficients (4.9) is influenced by the vertical local variable y_3, through the solution of the cell problem (4.10). Consequently, despite the fact that the derivative in the third direction does not explicitly appear in the homogenized problem (4.7), its solution u is implicitly influenced by the third local variable y_3.

Due to the diagonal form of the matrix $\mathbf{a} = (a_{ij})_{1\leqslant i,j\leqslant 3}$ (see (2.5)), the homogenized coefficients (4.9) can be written as

$$\begin{aligned} a_{11}^{\text{hom}} &= \int_Y a(y)\Big(1 - \frac{\partial \chi^1}{\partial y_1}\Big)\,\mathrm{d}y,\\ a_{12}^{\text{hom}} = a_{21}^{\text{hom}} &= -\int_Y a(y)\frac{\partial \chi^2}{\partial y_1}\,\mathrm{d}y,\\ a_{22}^{\text{hom}} &= \int_Y a(y)\Big(1 - \frac{\partial \chi^2}{\partial y_2}\Big)\,\mathrm{d}y, \end{aligned}$$

The above coefficients can be further simplified in the particular, but relevant, situation in which the reference cell Y is invariant under rotation of angle $\pi/2$ (see [17, Proposition 4.2]).

5. The case of a thin periodic rod

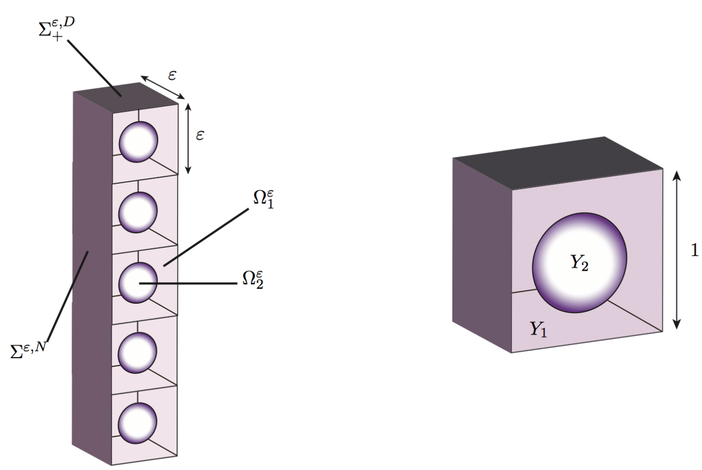

Figure 2. Example of a thin periodic rod and the corresponding reference cell Y.

In this section, we study the case of a thin rod. We start by describing the geometry of the domain Ω^ε, which represents now a two-phase thin rod (see Figure

2). We denote the independent variable $x \in \mathbb{R}^3$ by $x = (x_1, x_2, x_3)$ and we define

$$\Omega^\varepsilon = \varepsilon Y' \times (0, L), \tag{5.1}$$

where $Y' = (0,1)^2$, $L > 0$ and $\varepsilon \in (0,1)$ is a sequence of strictly positive numbers such that $\frac{L}{\varepsilon} \in \mathbb{N}^*$. This small parameter is related to the characteristic dimension of our domain. Thus, Ω^ε is a thin heterogeneous rod, its thickness, the periodicity of its heterogeneities and their size being of order ε. More precisely, the microscopic structure of Ω^ε consists of an exact number of replicated unit cells $Y = Y' \times (0,1) = (0,1)^3$, rescaled with ε. One has $Y = Y_1 \cup \overline{Y}_2$, where Y_1 and Y_2 are two non-empty disjoint connected open subsets of Y such that $\overline{Y}_2 \subset Y$. We assume that ∂Y_2 is Lipschitz continuous. For each $\mathbf{k} \in \mathbb{Z}^3$, we denote $Y_\alpha^{\mathbf{k}} = \mathbf{k} + Y_\alpha$, for $\alpha \in \{1, 2\}$. We also define, for each ε, $\mathbb{Z}_\varepsilon = \left\{\mathbf{k} \in \mathbb{Z}^3 : \varepsilon \overline{Y}_2^{\mathbf{k}} \subset \Omega^\varepsilon\right\}$; we set $\Omega_2^\varepsilon = \cup_{\mathbf{k} \in \mathbb{Z}_\varepsilon} \left(\varepsilon Y_2^{\mathbf{k}}\right)$ and $\Omega_1^\varepsilon = \Omega \setminus \overline{\Omega}_2^\varepsilon$. The boundary of the domain Ω^ε is split into three parts: Σ_N^ε, the lateral boundary of the domain Ω^ε, $\Sigma^{\varepsilon,N} = \{x \in \mathbb{R}^3 : (x_1, x_2) \in \partial \varepsilon Y',\ 0 < x_3 < L\}$, and $\Sigma_+^{\varepsilon,D}$, $\Sigma_-^{\varepsilon,D}$, the top and the bottom boundaries, respectively, $\Sigma_+^{\varepsilon,D} = \{x \in \mathbb{R}^3 : (x_1, x_2) \in \varepsilon Y',\ x_3 = L\}$, $\Sigma_-^{\varepsilon,D} = \{x \in \mathbb{R}^3 : (x_1, x_2) \in \varepsilon Y',\ x_3 = 0\}$.

The goal in this section is to analyze the asymptotic behavior, as $\varepsilon \to 0$, of the solution u^ε of the problem

$$\begin{gathered} -\operatorname{div}(a^\varepsilon \nabla u^\varepsilon) = f \quad \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \Sigma_\pm^{\varepsilon,D}, \\ \frac{\partial u^\varepsilon}{\partial \nu^\varepsilon} = 0 \quad \text{on } \Sigma^{\varepsilon,N}, \end{gathered} \tag{5.2}$$

where the function $f \in L^2((0,L))$ is given and ν^ε is the unit outward normal to the lateral boundary $\Sigma^{\varepsilon,N}$. Here, we have set as before

$$a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right), \quad \forall x \in \Omega^\varepsilon,$$

where $a(y)$ denotes the 1-periodic in the variable y_3 function defined on Y by

$$a(y) = a_1 \mathbf{1}_{Y_1}(y) + a_2 \mathbf{1}_{Y_2}(y),$$

with $a_1, a_2 \in \mathbb{R}$ and $a_1 a_2 < 0$.

To write the variational formulation of problem (5.2), we introduce, for every positive $\varepsilon < 1$, the Hilbert space

$$V^\varepsilon = \left\{v \in H^1(\Omega^\varepsilon) : v = 0 \text{ on } \Sigma_\pm^{\varepsilon,D}\right\},$$

endowed with the norm $\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega^\varepsilon)}$, for any $v \in V^\varepsilon$.

The variational formulation of problem (5.2) is the following: find $u^\varepsilon \in V^\varepsilon$ such that

$$\mathcal{A}^\varepsilon(u^\varepsilon, v) = \ell^\varepsilon(v), \quad \forall v \in V^\varepsilon, \tag{5.3}$$

where the bilinear form $\mathcal{A}^\varepsilon : V^\varepsilon \times V^\varepsilon \to \mathbb{R}$ and the linear form $\ell^\varepsilon : V^\varepsilon \to \mathbb{R}$ are given by

$$\begin{gathered} \mathcal{A}^\varepsilon(u, v) = \int_{\Omega^\varepsilon} a^\varepsilon \nabla u \cdot \nabla v \, \mathrm{d}x, \\ \ell^\varepsilon(v) = \int_{\Omega^\varepsilon} f v \, \mathrm{d}x \,. \end{gathered}$$

As in the case of the thin periodic layer, the aim is to prove the well-posedness of the variational problem (5.3) and then to pass to the limit with $\varepsilon \to 0$ in this problem. The proofs follow the same steps as in the previous case, with the use of an appropriate weakly two-scale convergence (see Definition 5.2 below) and with a slight modification in the definition of the extension operator $\mathbf{Q}$, as follows. In this case, we denote by $H^1_{0,\Sigma}(Y)$ the space of functions in $H^1(Y)$ which are zero on the upper and lower boundaries $\Sigma^D_\pm$ of the domain Y (the lateral boundary is denoted Σ^N). Let $\mathbf{Q}$ denote the harmonic extension operator from $H^1(Y_2)$ onto $H^1_{0,\Sigma}(Y)$. In other words, given $u \in H^1(Y_2)$, $\mathbf{Q}u \in H^1_{0,\Sigma}(Y)$ is defined by setting $\mathbf{Q}u = u$ in Y_2, while, in Y_1, $\mathbf{Q}u$ is the unique solution of the boundary value problem

$$\begin{gathered} -\Delta(\mathbf{Q}u) = 0 \quad \text{in } Y_1, \\ \mathbf{Q}u = u \quad \text{on } \partial Y_2, \\ \mathbf{Q}u = 0 \quad \text{on } \Sigma^D_\pm, \\ \frac{\partial(\mathbf{Q}u)}{\partial \nu} = 0 \quad \text{on } \Sigma^N. \end{gathered}$$

With the notation

$$\mathcal{M}_2(u) = \frac{1}{|Y_2|}\int_{Y_2} u \,\mathrm{d}y$$

and setting

$$H^1_{\rm mean}(Y_2) = \left\{u \in H^1(Y_2) \mid \mathcal{M}_2(u) = 0\right\},$$

it can be easily seen that there exists $\kappa'_Y > 0$ such that

$$\|\nabla(\mathbf{Q}u)\|^2_{L^2(Y)} \leqslant \kappa'_Y \|\nabla u\|^2_{L^2(Y_2)}, \quad \forall u \in H^1_{\rm mean}(Y_2). \tag{5.4}$$

Next we have a well-posedness result.

Theorem 5.1. *Assume that the contrast $\kappa := \left|\frac{a_1}{a_2}\right|$ satisfies*

$$\kappa > \kappa_Y \quad \text{or} \quad \kappa < 1/\kappa'_Y,$$

where κ_Y and κ'_Y denote the constants in (3.19) and (5.4) respectively.

Then, for any $\varepsilon \in (0,1)$, the variational problem (5.3) has a unique solution $u^\varepsilon \in V^\varepsilon$. Moreover, there exists a constant $C > 0$, independent of ε, such that

$$\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon \|f\|_{L^2(\omega)}. \tag{5.5}$$

To pass to the limit, with $\varepsilon \to 0$, in the variational formulation (5.3) of problem (5.2), we use again the two-scale convergence method [43, 1], adapted to the thin periodic rod case. Let $\mathcal{C}^\infty_{\rm per}(\overline{Y})$ be the space of infinitely differentiable functions in $\mathbb{R}^3$ that are 1-periodic in the variable y_3. We recall the definition of the weakly two-scale convergence (see, for instance, [39]).

Definition 5.2. A sequence (v^ε) in $L^2(\Omega^\varepsilon)$ weakly two-scale converges to $v \in L^2((0,L)\times Y)$ if one has

$$\frac{1}{\varepsilon^2}\int_{\Omega^\varepsilon} v^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right)\mathrm{d}x \to \int_{(0,L)\times Y} v(x_3,y)\psi(x_3,y)\,\mathrm{d}x_3\,\mathrm{d}y,$$

for all $\psi \in \mathcal{C}^\infty_0((0,L), \mathcal{C}^\infty_{\rm per}(\overline{Y}))$. Then, we denote

$$v^\varepsilon \xrightarrow{2-s} v.$$

Using the *a priori* estimate (5.5) and well-adapted compactness results (see, for instance, [39]), it follows that there exist $u \in H_0^1(0,L)$ and $\widehat{u} \in L^2((0,L), H^1_{\overline{\mathrm{per}}}(Y))$, with $\mathcal{M}_Y(\widehat{u}) = 0$, such that, up to a subsequence, for $\varepsilon \to 0$, we obtain

$$\begin{aligned} u^\varepsilon &\xrightharpoonup{2-s} u, \\ \nabla_{x'} u^\varepsilon &\xrightharpoonup{2-s} \nabla_{y'} \widehat{u}, \\ \partial_{x_3} u^\varepsilon &\xrightharpoonup{2-s} \frac{\mathrm{d}u}{\mathrm{d}x_3} + \partial_{y_3}\widehat{u}, \end{aligned} \tag{5.6}$$

where $x' = (x_1, x_2)$ and $y' = (y_1, y_2)$. Here, the space $H^1_{\overline{\mathrm{per}}}(Y)$ is defined by

$$H^1_{\overline{\mathrm{per}}}(Y) = \{v \in H^1(Y) : v \text{ is 1-periodic in } y_3\}.$$

The special form of the limits in convergences $(5.6)_2$ and $(5.6)_3$ suggests the introduction of the following notation: to every $w \in H^1(0,L)$, we associate the three-dimensional vector $\overline{\nabla} w(x_3)$ defined by

$$\overline{\nabla} w(x_3) = \Big(0, 0, \frac{\mathrm{d}w}{\mathrm{d}x_3}\Big).$$

Let $W_{\overline{\mathrm{per}}}(Y) = \{v \in H^1_{\overline{\mathrm{per}}}(Y) : \mathcal{M}_Y(v) = 0\}$. We introduce the space

$$\mathcal{H} = H_0^1(0,L) \times L^2\left((0,L), W_{\overline{\mathrm{per}}}(Y)\right)$$

and for all $\mathcal{V} = (v, \widehat{v}) \in \mathcal{H}$ we define the norm

$$\|\mathcal{V}\|^2_{\mathcal{H}} = \left\|\overline{\nabla} v + \nabla_y \widehat{v}\right\|^2_{L^2((0,L)\times Y)}.$$

Theorem 5.3. *Assume that the contrast* $\kappa := \left|\frac{a_1}{a_2}\right|$ *satisfies*

$$\kappa > \kappa_Y \quad \text{or} \quad \kappa < 1/\kappa'_Y,$$

where κ_Y and κ'_Y denote the constants in (3.19) *and* (5.4)*, respectively. Then, the unique solution u^ε of the variational problem* (5.3) *converges, in the sense of* (5.6)*, to $(u, \widehat{u}) \in \mathcal{H}$, where u is the unique solution of the homogenized problem*

$$\begin{gathered} -a^{\mathrm{hom}} \frac{\mathrm{d}^2 u}{\mathrm{d}x_3^2} = f \quad \text{on } (0,L), \\ u(0) = u(L) = 0 \end{gathered} \tag{5.7}$$

and

$$\widehat{u}(x_3, y) = -\frac{\mathrm{d}u}{\mathrm{d}x_3}(x_3)\chi^3(y), \quad \forall (x_3, y) \in (0,L) \times Y. \tag{5.8}$$

Here, a^{hom} is the scalar defined by

$$a^{\mathrm{hom}} = \int_Y a(y)\Big(1 - \frac{\partial \chi^3}{\partial y_3}\Big)\,\mathrm{d}y. \tag{5.9}$$

The function $\chi^3 \in H^1_{\overline{\mathrm{per}}}(Y)$ is the weak solution of the cell problem

$$\begin{gathered} -\operatorname{div}_y(a(y)(\nabla_y \chi^3 - e_3)) = 0 \quad \text{in } Y, \\ (\nabla_y \chi^3 - e_3)\cdot \nu = 0 \quad \text{on } \Sigma^N, \\ \mathcal{M}(\chi^3) = 0, \end{gathered} \tag{5.10}$$

where Σ^N denotes the lateral boundary of the reference cell Y and ν is the unit outward normal to Σ^N.

Remark 5.4. Because of the dimension-reduction effect in the first two directions, the homogenized matrix reduces in this case to a scalar. However, the value of this scalar coefficient a^{hom} is influenced by the local variables y_1 and y_2 through the solution χ^3 of the cell problem (5.10).

Acknowledgments. The authors would like to thank Lucas Chesnel and Mahran Rihani for fruitful discussions on T-coercivity technics and for suggesting the idea used in (4.6) to cope with the fact that affine functions are not preserved by the extension operator. This paper was completed during the visit of the third author at Institut Elie Cartan de Lorraine, whose warm hospitality is gratefully acknowledged.

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