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PERIODIC ORBITS OF THE SPATIAL ANISOTROPIC KEPLER PROBLEM WITH ANISOTROPIC PERTURBATIONS

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Abstract. In this article, we study the periodic orbits of the spatial anisotropic Kepler problem with anisotropic perturbations on each negative energy surface, where the perturbations are homogeneous functions of arbitrary integer degree p. By choosing the different ranges of a parameter β, we show that there exist at least 6 periodic solutions for $p > 1$, while there exist at least 2 periodic solutions for $p \leq 1$ on each negative energy surface. The proofs of main results are based on symplectic Delaunay coordinates, residue theorem, and averaging theory.

1. Introduction

Celestial mechanics has stimulated the development of many branches of Mathematics [12, 14]. The two-body problem or Kepler problem is the basic model of celestial mechanics, which attracts the vivid interest of many mathematicians who have studied its classical form with different types of perturbations. The anisotropic problem arises in many areas, and it studies the motion of particles, where the interaction law is different in each direction of the space. The study of the dynamics of the planar anisotropic celestial problem mainly includes the anisotropic Manev problem and the anisotropic Kepler problem with perturbations, see [24, 31].

The plane anisotropic Kepler problem with perturbations is described by the Hamiltonian

$$H = \frac{1}{2}\big(X^2 + Y^2\big) - \frac{1}{\sqrt{\mu x^2 + y^2}} - \frac{\epsilon\beta}{(\mu x^2 + y^2)^{p/2}}, \tag{1.1}$$

where β is a constant, p is a integer and ϵ is a small parameter.

The Hamiltonian (1.1) for $\beta = 0$ and $\mu = 1$ corresponds to the classical Kepler problem. In this respect, quite mature work has been done. See for example the books [5, 38] for a detailed introduction. If $\beta \neq 0$ and $\mu = 1$, we have the Kepler problem with perturbations which are symmetric with respect to the origin. Vidal [46] proved that each circular solution of the unperturbed problem gives rise to a periodic solution of the perturbed system.

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For $\beta = 0$ and $\mu \neq 1$, the Hamiltonian (1.1) becomes

$$H = \frac{1}{2}\big(X^2 + Y^2\big) - \frac{1}{\sqrt{\mu x^2 + y^2}}, \tag{1.2}$$

and we have the anisotropic Kepler problem which comes originally from quantum mechanics. It has been studied by Gutzwiller to establish a link between quantum mechanics and classical mechanics, see [21, 20, 22, 24], and these results were later extensively extended by Devaney [8, 9, 10], Casasayas and Lliber [3]. Devaney [8] studied the collision manifold of the anisotropic Kepler problem and obtained that, for all $\mu > 1$, the primary bi-collision orbits along the x-axis are transversal heteroclinic solutions of the system. Moreover, based on the invariant boundary (McGehee's method) to "blown up" the singularity, he obtained a qualitative picture of the behavior of a mechanical system near a singularity, see [9, 10]. Casasayas and Llibre exhibited phenomena such as non-integrability and chaotic behaviour, and also surveyed the recent techniques and results from the anisotropic Kepler problem, see [3]. By using the Lyapunov characteristic number (LCN), Contopoulos and Harsoula studied the degree of order and chaos for the anisotropic Kepler problem, calculate the asymptotic curves from unstable periodic orbits, and conjecture that there are stable orbits of high multiplicities for larger values of μ. These curves go through the collision orbits and continue beyond these points, see [4].

The existence of periodic orbits of the anisotropic Kepler problem is of special interest for many mathematicians. Firstly, Gutzwiller studied numerical periodic solutions of the plane anisotropic Kepler problem, and show that periodic orbits in the plane anisotropic Kepler problem with mass-ratio five are found numerically up to length five; see [23] for details. By using the McGehee coordinates and averaging theory, Abouelmagd, Llibre and Guirao [1] proved that at every energy level the anisotropic Kepler problem with small anisotropy has at least two periodic orbits. These result has been generalized by Lliber and Valls in [31].

When $\beta \neq 0$, $\mu = 1$ and $p = 2$, we have

$$H = \frac{1}{2}\big(X^2 + Y^2\big) - \frac{1}{\sqrt{x^2 + y^2}} - \frac{\epsilon\beta}{x^2 + y^2} \tag{1.3}$$

which corresponds to the Manev problem. One of the advantages of the Manev problem over the Keplerian is that it explains the perihelion advance of the inner planets with the same accuracy as relativity [44]. Firstly, Delgado et al. founded analytic solutions of Manev systems corresponding to (1.3) and described completely the global flows by using the McGehee coordinates and topological methods, see [7]. They showed that if the energy constant is negative, then the orbits are generically precessional ellipses; for zero energy, the orbits are precessional parabolas, and for positive energy, they are precessional hyperbolas. Lliber, Teruel and Fuente [30] characterized the global flows of Manev systems, and gave the phase portraits. Szenkovits, Mioc and Stoica [42] tackled the Manev problem from the standpoint of topology and pointed out the first integrals of energy and angular momentum. We can refer to [17, 34, 35, 36, 37, 39, 40, 43] for related developments.

When $\beta \neq 0$, $\mu \neq 1$ and $p = 2$, the corresponding problem is called the anisotropic Manev problem (AMP) which is inspired by the anisotropic Kepler problem. One of the main purpose for studying the anisotropic Manev problem, is to further analyze the similarities between classical mechanics and quantum theory. This was introduced by Craig, Diacu, Lacomba and Pérez in [6]. With the

McGehee coordinates, they found a positive-measure set of collision orbits and tackled capture and escape solutions in the zero-energy case. Santoprete [41] studied the existence of periodic solutions for weak anisotropy and found that the symmetric periodic orbits of the Manev system are perturbed to periodic orbits in the anisotropic problem. Moreover, Diacu and Santoprete proved that for weak anisotropy, chaos shows up on the zero-energy manifold by using an extension of the Poincaré-Melnikov method, see [15]. Recently, Based on the method of averaging, Llibre and Yuan showed that the Kepler problem has a unique elliptic periodic solution can be continued to the plane anisotropic Manev problem under a certain condition, see [32] for details. We refer to the related papers [11, 16].

If $\beta \neq 0$, $\mu \neq 1$, two kinds of models are considered. On the one hand, the plane Kepler problem with anisotropic perturbations is given by the Hamiltonian

$$H = \frac{1}{2}\big(X^2 + Y^2\big) - \frac{1}{\sqrt{x^2 + y^2}} - \frac{\epsilon\beta}{(\mu x^2 + y^2)^{p/2}}, \tag{1.4}$$

has been first studied by Diacu, Pérez-Chavela and Santoprete in [13], where μ is near 1 and β is a constant, p is a integer and ϵ is a small parameter. They obtained that for $p > 2$, the sets of initial conditions leading to collisions/ejections and the one leading to escapes/captures have positive measure; for $p > 2$ and $p \neq 3$, the flows on the zero-energy manifold are chaotic and for $p = 2$, the infinity manifold of the zero-energy level has heteroclinic connections with the collision manifold. The results are extended by Escalona-Buendia and Pérez-Chavela in [18]. On the other hand, the plane anisotropic Kepler problem with anisotropic perturbation has been considered by Miguel, Raquel and Juan in [33]. Based on averaging theory, by using symplectic Delaunay coordinates they analyzed the sufficient conditions for existence and kind of stability of periodic orbits.

The planar models in celestial mechanics are naturally extended to the spatial cases and there also are a few results about the dynamics of the spatial anisotropic problem. See for example[19, 29]. The spatial anisotropic Kepler problem with anisotropic perturbations is given by the Hamiltonian

$$H = \frac{1}{2}\big(X^2 + Y^2 + Z^2\big) - \frac{1}{\sqrt{\mu\big(x^2 + y^2\big) + z^2}} - \frac{\epsilon\beta}{\big(\mu\big(x^2 + y^2\big) + z^2\big)^{p/2}}, \tag{1.5}$$

where β is a constant, p is a integer and ϵ is a small parameter.

The spatial Kepler problem and the spatial Manev problem are corresponding to the Hamiltonian (1.5) with the parameters $\beta = 0$, $\mu = 1$ and $\beta \neq 0$, $\mu = 1$, $p = 2$, respectively. One of the advantages of the Manev problem over the Keplerian is that it explains the perihelion advance of the inner planets with the same accuracy as relativity [45].

If $\beta = 0$ and $\mu \neq 1$, we have spatial anisotropic Kepler problem. The existence of periodic orbits is firstly studied by Guriao, Llibre and Vera in [19], where periodic orbits of the perturbed spatial Keplerian Hamiltonians with axial symmetry, such as Matese-Whitman Hamiltonian and spatial generalized van der Waals Hamiltonian were considered.

If $\beta \neq 0$, $\mu \neq 1$ and $p = 2$, we have spatial anisotropic Manev problem (SAMP). Llibre and Makhlouf have proved the existence of periodic orbits on the every level $H = h < 0$ by using the method of averaging in [29]. Later, the results were generalized by Alberti and Vidal in [2] by using symmetry conditions. They found the existence of new families of periodic orbits.

In this article, inspired by the study of plane anisotropy Kepler problem with anisotropy perturbations, we study the existence of periodic orbits on negative energy level $H = h$ ($h < 0$) for the spatial anisotropic problem with the Hamiltonian

$$H = \frac{1}{2}\big(X^2 + Y^2 + Z^2\big) - \frac{1}{\sqrt{\mu\big(x^2 + y^2\big) + z^2}} - \frac{\epsilon\beta}{\big(\mu\big(x^2 + y^2\big) + z^2\big)^{p/2}}, \tag{1.6}$$

where μ is near 1, β is a constant, p is an arbitrary integer and ϵ is a small parameter. Originally, in quantum mechanics, the parameter μ of (1.6) is taken as $\mu \neq 1$ and the problem is called a spatial anisotropic problem.

In this paper, we always assume that $\mu = 1 - \epsilon$. As usual, firstly we use a series of canonical transformations to transform (1.6) into the spatial Delaunay variables (see [5, 38])

$$H = -\frac{1}{2L^2} + \epsilon H_1(l, g, L, G, K) + O(\epsilon^2), \tag{1.7}$$

where $H_1(l, g, L, G, K)$ is equal to

$$\begin{aligned} &\frac{2}{4L\left(L - \sqrt{L^2 - G^2}\cos E\right)} - \beta\left(L\left(L - \sqrt{L^2 - G^2}\cos E\right)\right)^{-p} \\ &- \frac{2\left(G^2 - K^2\right)\left(\left(L\cos E - \sqrt{L^2 - G^2}\right)\sin g + G\sin E\cos g\right)^2}{4G^2L\left(L - \sqrt{L^2 - G^2}\cos E\right)^3} \end{aligned}$$

and $E = E(l, L, G)$ is defined by $l = E - e\sin E$, l is the mean anomaly, g is the argument of the perigee of the unperturbed elliptic orbit measured in the invariant plane, L is the square root of the semimajor axis of the unperturbed elliptic orbit, G is the modulus of the total angular momentum, and K is the third component of the angular momentum.

To state our main results, for any fixed constant h, p, we define two functions as follows

$$\begin{aligned} D_1(G; h, p) = &-\beta\sqrt{-2h}^{\,p-3}G^{-p-1}\sum_{m=0}^{p-2} C_{p-2}^m 2^{-m}\frac{(m+p-2)!}{m!(p-2)!} \\ &\times\Big((1-p)(1-G\sqrt{-2h}) - m\Big)\Big(\frac{1 - G\sqrt{-2h}}{G\sqrt{-2h}}\Big)^{m-1} \end{aligned}$$

and

$$\begin{aligned} D_2(G; h, p) = &-\beta 2^{p-1}\sqrt{-2h}^{\,2p-2}\Big(G\sqrt{-2h} + 1\Big)^{-p-1}\sum_{m=0}^{1-p} C_{1-p}^m\frac{(1-p)!}{m!(-m-p+1)!} \\ &\times\Big((1-p)(1-G\sqrt{-2h}) - 2m\Big)\Big(\frac{1 - G\sqrt{-2h}}{1 + G\sqrt{-2h}}\Big)^{m-1}. \end{aligned}$$

The existence of periodic orbits of Hamiltonian system corresponding to (1.6) is determined by the functions $D_1(G; h, p)$ and $D_2(G; h, p)$. Considering the calculation of abnormal integrals for averaging systems, we divided into two cases, $p > 1$ and $p \leq 1$. Now we state the first result as follows.

Theorem 1.1. *Assume that $p > 1$. For every $k_0 \in [0, 2\pi)$, there exists a sufficiently small positive constant ϵ_0 such that, for each $\epsilon \in (-\epsilon_0, \epsilon_0)$, on every energy level $H = h < 0$, the spatial anisotropic Kepler problem with anisotropic perturbations given by the Hamiltonian (1.7) has*

(i) *Two 2π-periodic solutions $\gamma_\epsilon{}^{\pm}(l) = (g(l,\epsilon), k(l,\epsilon), L(l,\varepsilon), G(l,\epsilon), K(l,\epsilon))$ if*
$$|\beta| < \frac{1}{2^{p+1}p(p-1)|h|^{p-1}},$$
such that
$$\lim_{\epsilon\to 0}\gamma_\epsilon{}^{\pm}(0) = \Big(\pm\frac{1}{2}\arccos\big(-\beta 2^{p+1}p(p-1)(-h)^{p-1}\big), k_0, \frac{1}{\sqrt{-2h}}, \frac{1}{\sqrt{-2h}}, 0\Big).$$

(ii) *Four 2π-periodic solutions $\gamma_\epsilon{}^{(i)}(l) = (g(l,\epsilon), k(l,\epsilon), L(l,\epsilon), G(l,\epsilon), K(l,\epsilon))$ if*
$$\beta \in \Big(-\frac{1}{2^{p+1}(-h)^{p-1}p(p-1)}, 0\Big),$$
and $\widetilde{G}_i$ are two nondegenerate zeros of $D_1(G;h,p) + \big(2G\sqrt{-h}+\sqrt{2}\big)^{-2}$, such that
$$\lim_{\epsilon\to 0}\gamma_\epsilon{}^{(i)}(0) = \Big(g_0, k_0, \frac{1}{\sqrt{-2h}}, \widetilde{G}_i, 0\Big), \quad g_0 = 0 \text{ or } \pi, \ i = 1,2.$$

(iii) *Four 2π-periodic solutions $\gamma_\epsilon{}^{(i)}(l) = (g(l,\epsilon), k(l,\epsilon), L(l,\epsilon), G(l,\epsilon), K(l,\epsilon))$ if*
$$\beta \in \Big(0, \frac{1}{2^{p+1}(-h)^{p-1}p(p-1)}\Big),$$
and $\overline{G}_i$ are two nondegenerate zeros of $D_1(G;h,p) - \big(2G\sqrt{-h}+\sqrt{2}\big)^{-2}$, such that
$$\lim_{\epsilon\to 0}\gamma_\epsilon{}^{(i)}(0) = \Big(\pm\frac{\pi}{2}, k_0, \frac{1}{\sqrt{-2h}}, \overline{G}_i, 0\Big), \quad i = 1,\dots,4.$$

Theorem 1.1 establishes a generic result on the existence of periodic orbits of the spatial anisotropic Kepler problem with anisotropic perturbations, which generalizes the results of Lliber and Makhlouf in [29], where the only case $p = 2$ has been considered. The proof of Theorem 1.1 combines the Hamiltonian perturbation theory, Residue theory in complex analysis and the average theory. In the proof of Theorem 1.1, we have proved the functions $D_1(G;h,p) \pm \big(2G\sqrt{-h}+\sqrt{2}\big)^{-2}$ have at least two different zeros. Owing to lots of calculations, the proof is so long that we arrange the proof of Theorem 1.1 in Subsection 6.

Similarly, we can obtain the following result for the case $p \le 1$.

Theorem 1.2. *Assume that $p \le 1$. For every $k_0 \in [0, 2\pi)$, there exists a sufficiently small positive constant ϵ_0 such that, for each $\epsilon \in (-\epsilon_0, \epsilon_0)$, on every energy level $H = h < 0$, the spatial anisotropic Kepler problem with anisotropic perturbations given by the Hamiltonian (1.7) has*

(i) *Two 2π-periodic solutions $\gamma_\epsilon{}^{\pm}(l) = (g(l,\epsilon), k(l,\epsilon), L(l,\epsilon), G(l,\epsilon), K(l,\epsilon))$ if*
$$|\beta| < \frac{1}{2^{p+1}(p-1)p|h|^{p-1}},$$
such that
$$\lim_{\epsilon\to 0}\gamma_\epsilon{}^{\pm}(0) = \Big(\pm\frac{1}{2}\arccos\big(\beta 2^{p+1}(1-p)p(-h)^{p-1}\big), k_0, \frac{1}{\sqrt{-2h}}, \frac{1}{\sqrt{-2h}}, 0\Big).$$

(ii) *Two 2π-periodic solutions* $\gamma_\epsilon(l) = (g(l,\epsilon), k(l,\epsilon), L(l,\epsilon), G(l,\epsilon), K(l,\epsilon))$ *if* $p < 0$ *,*
$$\beta \in \Big(-\infty, -\frac{1}{2^{p+1}(p-1)p(-h)^{p-1}}\Big),$$
and $\widetilde{G}$ *is a nondegenerate zero of the* $D_2(G;h,p) + \big(2G\sqrt{-h}+\sqrt{2}\big)^{-2}$*, such that*
$$\lim_{\epsilon\to 0} {\gamma_\epsilon}^{(i)}(0) = \Big(g_0, k_0, \frac{1}{\sqrt{-2h}}, \widetilde{G}, 0\Big), \quad g_0 = 0 \ \text{or}\ \pi,\ i = 1, 2.$$

(iii) *Two 2π-periodic solutions* ${\gamma_\epsilon}^{\pm}(l) = (g(l,\epsilon), k(l,\epsilon), L(l,\epsilon), G(l,\epsilon), K(l,\epsilon))$ *if* $p < 0$*,*
$$\beta \in \Big(\frac{1}{2^{p+1}(p-1)p(-h)^{p-1}}, +\infty\Big),$$
and $\overline{G}$ *is a nondegenerate zero of the* $D_2(G;h,p) - \big(2G\sqrt{-h}+\sqrt{2}\big)^{-2}$*, such that*
$$\lim_{\epsilon\to 0} {\gamma_\epsilon}^{\pm}(0) = \Big(\pm\frac{\pi}{2}, k_0, \frac{1}{\sqrt{-2h}}, \overline{G}, 0\Big).$$

The proof of Theorem 1.2 shows that $D_2(G;h,p) \pm \big(2G\sqrt{-h}+\sqrt{2}\big)^{-2}$ have no zero points for the case of $p = 0, 1$, while for $p < 0$, these functions have at least one zero point. The case $p \le 0$ is corresponding to the anisotropic perturbations without singularities. We arrange the proof of Theorem 1.2 in Subsection 6.1.

The rest paper is organized as follows. In Section 2, we give some applications related to spatial anisotropic problem. To demonstrate the applications of Theorems 1.1 and 1.2, we consider two cases $p = 2$ and $p = -1$. The numerical simulations of periodic orbits are shown in configuration space $xyOz$. In Section 3, we introduce two canonical transformations. One is the spherical transformation $\Psi_0 : (x, y, z, X, Y, Z) \to (\rho, \theta, \phi, P, \Theta, \Phi)$, and the other one is the spatial Delaunay transformation $\Psi_1 : (\rho, \theta, \phi, P, \Theta, \Phi) \to (l, g, k, L, G, K)$, which transforms Hamiltonian (1.6) into Hamiltonian (1.7) with the Delaunay coordinates. In Section 4, we obtain the Hamiltonian system corresponding to the Hamiltonian (1.7), and reduce this system to a two dimensional system on the energy surface Ω, see (4.4). We emphasize on that we are different from [29] in methodology in this content. In Section 5, we use the Residue theorem to obtain the average system of the Hamiltonian system corresponding to the Hamiltonian (1.7) for $p > 1$ and $p \le 1$, respectively. This is the preparation for the next section to use the averaging method to find the equilibrium points of system. In Section 6, we use averaging theory to prove Theorem 1.1 and Theorem 1.2, and calculate the equilibrium points of the average system (5.7) in three cases.

2. Applications related to spatial anisotropic problems

In this section, we give some applications related to spatial anisotropic problem. The main purpose is to demonstrate the applications of Theorems 1.1 and 1.2. We shall consider two cases $p = 2$ and $p = -1$. The first case is corresponding to the spatial anisotropic Manev problem. Although this situation has been studied in [29], only the transformed system is considered. We will return the result of Theorem 1.1 (see Theorem 2.1 below) to the original system with the Cartesian coordinates. Moreover, some numerical examples will be exhibited. The second case is corresponding to the spatial anisotropic Kepler problem with anisotropic perturbations.

2.1. **The spatial anisotropic Manev problem.** When $p=2$, we have the spatial anisotropic Manev problem, which is given by the Hamiltonian

$$H=\frac{1}{2}\big(X^2+Y^2+Z^2\big)-\frac{1}{\sqrt{\mu\big(x^2+y^2\big)+z^2}}-\frac{\epsilon\beta}{\mu\big(x^2+y^2\big)+z^2}. \tag{2.1}$$

Without loss of generality, we consider the periodic orbits of the spatial anisotropic Manev problem on the energy level $H=h=-1$. By using Theorem 1.1, we have the following result.

Theorem 2.1. *For every $k_0\in[0,2\pi)$, there exists a sufficiently small positive constant ϵ_0 such that, for each $\epsilon\in(-\epsilon_0,\epsilon_0)$, on energy level $H=h=-1$, the spatial anisotropic Manev problem given by the Hamiltonian (2.1) has:*

(i) *Two 2π-periodic solutions ${\zeta_\epsilon}^{\pm}(t)=(x(t,\epsilon),y(t,\epsilon),z(t,\varepsilon),X(t,\epsilon),Y(t,\epsilon),Z(t,\epsilon))$ if $|\beta|<1/16$, such that ${\zeta_\epsilon}^{\pm}(0)$ tends to*

$$\Big(\frac{1}{2},\frac{1}{2}\cos g_0\sin k_0,\frac{1}{2}\sin g_0,\sqrt{2}\sin g_0\cos k_0,\sqrt{2}\sin g_0\sin k_0,-\sqrt{2}\cos g_0\Big),$$

as $\epsilon\to 0$ where

$$g_0=\pm\frac{1}{2}\arccos\left(-16\beta\right).$$

(ii) *Four 2π-periodic solutions*

$${\zeta_\epsilon}^{i}(t)=\big(x(t,\epsilon),y(t,\epsilon),z(t,\varepsilon),X(t,\epsilon),Y(t,\epsilon),Z(t,\epsilon)\big)$$

if $\beta\in\big(-1/16,0\big)$, such that ${\zeta_\epsilon}^{i}(0)$ tends to

$$\Big(\pm\frac{1}{2}\Big(1-\sqrt{1-2G_0^2}\Big)\cos k_0,\pm\frac{1}{2}\Big(1-\sqrt{1-2G_0^2}\Big)\sin k_0,0,0,0,\frac{\mp 2|G_0|}{1-\sqrt{1-2G_0^2}}\Big),$$

as $\epsilon\to 0$ where

$$G_0=\frac{2\sqrt{2}\beta\pm\sqrt{-2\beta}}{-1-4\beta}.$$

(iii) *Four 2π-periodic solutions*

$${\zeta_\epsilon}^{i}(t)=\big(x(t,\epsilon),y(t,\epsilon),z(t,\varepsilon),X(t,\epsilon),Y(t,\epsilon),Z(t,\epsilon)\big)$$

if $\beta\in(0,1/16)$, such that ${\zeta_\epsilon}^{i}(0)$ tends to

$$\Big(0,0,\pm\frac{1}{2}\Big(1-\sqrt{1-2G_0^2}\Big),\pm\frac{2|G_0|\cos k_0}{1-\sqrt{1-2G_0^2}},\pm\frac{2|G_0|\sin k_0}{1-\sqrt{1-2G_0^2}},0\Big),$$

as $\epsilon\to 0$ where

$$G_0=\frac{2\sqrt{2}\beta\pm\sqrt{2\beta}}{1-4\beta}.$$

Proof. When $|\beta|<1/16$, from the statement (i) of Theorem 1.1, we obtain

$$\lim_{\epsilon\to 0}{\gamma_\epsilon}^{\pm}(0)=\Big(\pm\frac{1}{2}\arccos\left(-16\beta\right),k_0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\Big),$$

or equivalently,

$$\gamma_0^{\pm}=(l_0,g_0,k_0,L_0,G_0,K_0)=\Big(0,\pm\frac{1}{2}\arccos\left(-16\beta\right),k_0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\Big).$$

We insert $p=2$ and $h=-1$ back into the function $D_1(G;h,p)+\big(2G\sqrt{-h}+\sqrt{2}\big)^{-2}$ when $\beta\in\big(-\frac{1}{16},0\big)$, then obtain

$$D_1(G;h,p)+\Big(2G\sqrt{-h}+\sqrt{2}\Big)^{-2}=\frac{G^2+2\beta\left(G\sqrt{2}+1\right)^2}{2G^2\left(G\sqrt{2}+1\right)^2}. \tag{2.2}$$

By direct computation, equation (2.2) has two zero points as

$$G_0=\frac{2\sqrt{2}\beta\pm\sqrt{-2\beta}}{-1-4\beta},$$

and these zero points are nondegenerate, i.e.,

$$\frac{d}{dG}\Big(D_1(G_0)+\Big(2G_0+\sqrt{2}\Big)^{-2}\Big)\neq 0.$$

From statement (ii) of Theorem 1.1, we obtain

$$\lim_{\epsilon\to 0}{\gamma_\epsilon}^i(0)=\Big(g_0,k_0,\frac{1}{\sqrt{2}},\frac{2\sqrt{2}\beta\pm\sqrt{-2\beta}}{-1-4\beta},0\Big),\quad g_0=0 \text{ or } \pi,\ i=1,\dots,4,$$

or equivalent to

$$\gamma_0^i=(l_0,g_0,k_0,L_0,G_0,K_0)=\Big(0,g_0,k_0,\frac{1}{\sqrt{2}},\frac{2\sqrt{2}\beta\pm\sqrt{-2\beta}}{-1-4\beta},0\Big),$$
$$g_0=0 \text{ or } \pi, i=1,\dots,4.$$

We insert $p=2$ and $h=-1$ back into $D_1(G;h,p)-\big(2G\sqrt{-h}+\sqrt{2}\big)^{-2}$ when $\beta\in(0,1/16)$, then obtain

$$D_1(G;h,p)-\Big(2G\sqrt{-h}+\sqrt{2}\Big)^{-2}=\frac{-G^2+2\beta\left(G\sqrt{2}+1\right)^2}{2G^2\left(G\sqrt{2}+1\right)^2}. \tag{2.3}$$

The zero points of (2.3) are

$$G_0=\frac{2\sqrt{2}\beta\pm\sqrt{2\beta}}{1-4\beta}.$$

We derive $D_1(G)-\big(2G+\sqrt{2}\big)^{-2}$ and replace G in the derived equation with G_0, obtain

$$\frac{d}{dG}\Big(D_1(G_0)-\Big(2G_0+\sqrt{2}\Big)^{-2}\Big)\neq 0.$$

Therefore, G_0 are the nondegenerate zero of the $D_1(G)-\big(2G+\sqrt{2}\big)^{-2}$. From the statement (iii) of Theorem 1.1, we obtain

$$\lim_{\epsilon\to 0}{\gamma_\epsilon}^i(0)=\Big(\pm\frac{\pi}{2},k_0,\frac{1}{\sqrt{2}},\frac{2\sqrt{2}\beta\pm\sqrt{2\beta}}{1-4\beta},0\Big),\quad i=1,\dots,4,$$

or equivalent to

$$\gamma_0^i=(l_0,g_0,k_0,L_0,G_0,K_0)=\Big(0,\pm\frac{\pi}{2},k_0,\frac{1}{\sqrt{2}},\frac{2\sqrt{2}\beta\pm\sqrt{2\beta}}{1-4\beta},0\Big),\quad i=1,\dots,4.$$

To study periodic solutions of the system corresponding to Hamiltonian (2.1), we need two to transform the Delaunay coordinates (l, g, k, L, G, K) into the Cartesian coordinates (x, y, z, X, Y, Z). Firstly, inserting

$$(l_0, k_0, L_0, K_0) = \Big(0, k_0, \frac{1}{\sqrt{2}}, 0\Big), \quad k_0 \in [0, 2\pi)$$

into the Delaunay transformation Ψ_1 (see (3.3)), and transforming the Delaunay coordinates (l, g, k, L, G, K) into the spatial spherical coordinates $(\rho, \theta, \phi, P, \Theta, \Phi)$, we have

$$\rho = \frac{1}{2}\left(1 - \sqrt{1 - 2G^2}\right), \theta = k_0, \phi = \frac{\pi}{2} - g, P = 0, \Theta = 0, \Phi = |G|. \tag{2.4}$$

Then we put (2.4) into the spherical transformation Ψ_0 (see (3.1)) and obtain

$$\begin{gathered}
x = \frac{1}{2}\left(1 - \sqrt{1 - 2G^2}\right)\cos g \cos k_0, \\
y = \frac{1}{2}\left(1 - \sqrt{1 - 2G^2}\right)\cos g \sin k_0, \\
z = \frac{1}{2}\left(1 - \sqrt{1 - 2G^2}\right)\sin g, \\
\cos g\,(X\cos k_0 + Y\sin k_0) + Z\sin g = 0, \\
\frac{1}{2}\left(1 - \sqrt{1 - 2G^2}\right)\cos g\,(-X\sin k_0 + Y\cos k_0) = 0, \\
\frac{1}{2}\left(1 - \sqrt{1 - 2G^2}\right)[\sin g\,(X\cos k_0 + Y\sin k_0) - Z\cos g] = |G|.
\end{gathered} \tag{2.5}$$

Considering the statement (i) of Theorem 1.1, we insert

$$(g_0, G_0) = \Big(\pm\frac{1}{2}\arccos(-16\beta), \frac{1}{\sqrt{2}}\Big)$$

into (2.5) to obtain

$$\begin{aligned}
\zeta_0^{\pm} &= (x_0, y_0, z_0, X_0, Y_0, Z_0) \\
&= \Big(\frac{1}{2}\cos g_0 \cos k_0, \frac{1}{2}\cos g_0 \sin k_0, \frac{1}{2}\sin g_0, \\
&\quad \sqrt{2}\sin g_0 \cos k_0, \sqrt{2}\sin g_0 \sin k_0, -\sqrt{2}\cos g_0\Big).
\end{aligned}$$

Similarly, considering the statement (ii) of Theorem 1.1, we take

$$(g_0, G_0) = \Big(0 \text{ or } \pi, \frac{2\sqrt{2}\beta \pm \sqrt{-2\beta}}{-1 - 4\beta}\Big)$$

into (2.5) to obtain

$$\begin{aligned}
\zeta_0^{i} =& \Big(\pm\frac{1}{2}\Big(1 - \sqrt{1 - 2G_0^2}\Big)\cos k_0, \pm\frac{1}{2}\Big(1 - \sqrt{1 - 2G_0^2}\Big)\sin k_0, \\
&0, 0, 0, \mp\frac{2|G_0|}{1 - \sqrt{1 - 2G_0^2}}\Big), \quad i = 1, \dots, 4.
\end{aligned}$$

Considering the statement (iii) of Theorem 1.1, we take

$$(g_0, G_0) = \Big(\pm\frac{\pi}{2}, \frac{2\sqrt{2}\beta \pm \sqrt{2\beta}}{1 - 4\beta}\Big)$$

into (2.5) to obtain

$$(x_0, y_0, z_0, Z_0) = \Big(0, 0, \pm\frac{1}{2}\Big(1 - \sqrt{1 - 2G_0^2}\Big), 0\Big), \tag{2.6}$$

$$\pm\frac{1}{2}\Big(1 - \sqrt{1 - 2G_0^2}\Big)\,(X_0 \cos k_0 + Y_0 \sin k_0) = |G_0|. \tag{2.7}$$

Inserting (2.6) and $H = h = -1$ back into Hamiltonian (2.1), we obtain

$$X_0^2 + Y_0^2 = -2 + \frac{4(1 - \sqrt{1 - 2G_0^2}) + 8\epsilon\beta}{(1 - \sqrt{1 - 2G_0^2})^2}. \tag{2.8}$$

When $\epsilon \to 0$, combining (2.7) with (2.8), we obtain

$$(X_0, Y_0) = \Big(\pm\frac{2|G_0|\cos k_0}{1 - \sqrt{1 - 2G_0^2}}, \pm\frac{2|G_0|\sin k_0}{1 - \sqrt{1 - 2G_0^2}}\Big).$$

Therefore,

$$\zeta_0^i = \Big(0, 0, \pm\frac{1}{2}\Big(1 - \sqrt{1 - 2G_0^2}\Big), \pm\frac{2|G_0|\cos k_0}{1 - \sqrt{1 - 2G^2}}, \pm\frac{2|G_0|\sin k_0}{1 - \sqrt{1 - 2G_0^2}}, 0\Big),$$

for $i = 1, \dots, 4$. Thus the proof is complete. □

From Theorem 2.1, when $\beta \in (0, 1/16)$ or $\beta \in (-1/16, 0)$, for sufficiently small $|\epsilon| \ll 1$, the spatial anisotropic Manev problem corresponding to Hamiltonian (2.1) have at least six kinds of periodic solutions, which constitute a continuum of periodic solutions by continuously changing k_0 on the interval $[0, 2\pi)$.

To demonstrate Theorem 2.1, we take the parameter $\beta = 1/32$ and $k_0 = \pi/3$. According to Theorem 2.1, there exists six periodic orbits $\zeta_\epsilon^i(t)$ such that

$$\begin{aligned}
&\lim_{\epsilon\to 0} {\zeta_\epsilon}^i(0) = \zeta^i(0)\\
&= \begin{cases}
\Big(\frac{1}{8}, \frac{\sqrt{3}}{8}, \frac{\sqrt{3}}{4}, \frac{\sqrt{6}}{4}, \frac{3\sqrt{2}}{4}, -\frac{1}{\sqrt{2}}\Big) \quad \text{for } i = 1,\\
\Big(\frac{1}{8}, \frac{\sqrt{3}}{8}, -\frac{\sqrt{3}}{4}, -\frac{\sqrt{6}}{4}, -\frac{3\sqrt{2}}{4}, -\frac{1}{\sqrt{2}}\Big) \quad \text{for } i = 2,\\
\Big(0, 0, \frac{1}{2} - \frac{\sqrt{10-\sqrt{2}}}{7}, -\frac{\sqrt{2}}{2} + \sqrt{4 - 2\sqrt{2}} + 2, -\frac{\sqrt{6}}{2} + \sqrt{12 - 6\sqrt{2}} + 2\sqrt{3}, 0\Big),\\
\Big(0, 0, \frac{1}{2} - \frac{\sqrt{10+\sqrt{2}}}{7}, \frac{\sqrt{2}}{2} + \sqrt{4 + 2\sqrt{2}} + 2, \frac{\sqrt{6}}{2} + \sqrt{12 + 6\sqrt{2}} + 2\sqrt{3}, 0\Big),\\
\Big(0, 0, \frac{\sqrt{10-\sqrt{2}}}{7} - \frac{1}{2}, \frac{\sqrt{2}}{2} - \sqrt{4 - 2\sqrt{2}} - 2, \frac{\sqrt{6}}{2} - \sqrt{12 - 6\sqrt{2}} - 2\sqrt{3}, 0\Big),\\
\Big(0, 0, \frac{\sqrt{10+\sqrt{2}}}{7} - \frac{1}{2}, -\frac{\sqrt{2}}{2} - \sqrt{4 + 2\sqrt{2}} - 2, -\frac{\sqrt{6}}{2} - \sqrt{12 + 6\sqrt{2}} - 2\sqrt{3}, 0\Big).
\end{cases}
\end{aligned}$$

For numerical demonstration of periodic orbits, we take $\epsilon = 0.002$ and $\mu = 0.998$. Since for sufficiently small $|\epsilon| \ll 1$, the initial values of periodic solutions $\zeta_\epsilon^i(t), i = 1, 2, \dots, 6$ are near $\zeta^i(0)$, we plot the curves of the solutions starting at $\zeta^i(0)$ on the time interval $t \in (990, 1000)$, see Figure 1. As is shown in Figure 1, the solutions starting at $\zeta^i(0)$ are periodic. Since the flows of autonomous Hamiltonian system have the semigroup property, two different periodic solutions may be have the same orbits. In order to test different periodic orbits, we plot the periodic orbits $\zeta_\epsilon^i(t)$ $(i = 1, 2, \dots, 6)$ on the projective configuration space $xOyz$, see Figure 2(a). As can be seen from Figure 2(a), there are five differential periodic orbits, since periodic

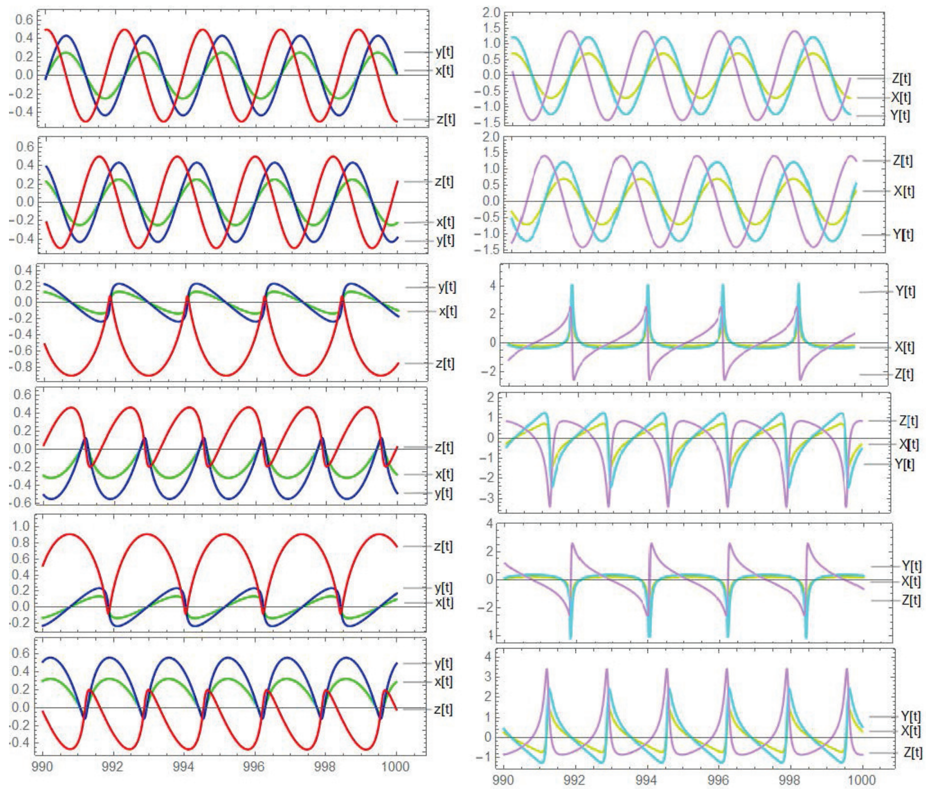


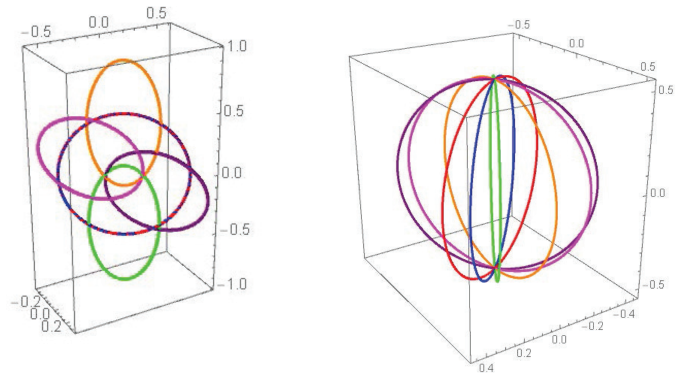
Figure 1. Periodic orbits of Hamiltonian (2.1) with the parameters $\epsilon = 0.002$, $\beta = 1/32$ and $k_0 = \pi/3$. In the i-th ($i = 1, 2, \dots, 6$) line, the curves represent periodic orbit $\zeta_\epsilon^i(t)$ corresponding to the initial value $\zeta^i(0)$, respectively. On the left, the pictures denote by the curves $x(t), y(t), z(t)$, while on the right, the pictures denote by the curves $X(t), Y(t), Z(t)$.

solutions $\zeta_\epsilon^1(t)$ and $\zeta_\epsilon^2(t)$ have the same periodic orbit. Moreover, we plot the orbits on the projective configuration space $xOyz$ starting from the initial values

$$\zeta^1(0) = \Big(\frac{1}{4}\cos k_0, \frac{1}{4}\sin k_0, \frac{\sqrt{3}}{4}, \sqrt{\frac{3}{2}}\cos k_0, \sqrt{\frac{3}{2}}\sin k_0, -\frac{1}{\sqrt{2}}\Big)$$

by varying $k_0 = i\pi/6, i = 1, 2, \dots, 6$, see Figure 2(b). As is shown from Figure 2(b), if we continuously change k_0 on the interval $(0, 2\pi)$, all of periodic orbits are full of an ellipsoid.

2.2. **The spatial anisotropic Kepler problem with perturbations.** In this subsection, we consider the spatial anisotropic Kepler problem with anisotropic perturbations. We only consider $p = -1$ as an example for the case $p \le 1$. In this case, the integral computation of the perturbations does not need the Residue



$k_0 = \pi/3$ and $(x_\epsilon^i(t), y_\epsilon^i(t), z_\epsilon^i(t))$ $\qquad$ $k_0 = i\pi/6$ $(i = 1, 2, \dots, 6)$

Figure 2. Periodic orbits on the configuration space $xOyz$, where we choose the parameters as $\epsilon = 0.002$ and $\beta = 1/32$. We plot the periodic orbits $\zeta_\epsilon^i(t)$ $(i = 1, 2, \dots, 6)$ on the projective configuration space $xOyz$ on the left, while on the right we plot the periodic orbits starting from the initial values $\zeta^1(0) = (\cos k_0/4, \sin k_0/4, \sqrt{3}/4, \sqrt{3/2}\cos k_0, \sqrt{3/2}\sin k_0, -1/\sqrt{2})$ by varying $k_0 = i\pi/6, i = 1, 2, \dots, 6$.

theorem. The Hamiltonian is

$$H = \frac{1}{2}\big(X^2 + Y^2 + Z^2\big) - \frac{1}{\sqrt{\mu\big(x^2 + y^2\big) + z^2}} - \epsilon\beta\sqrt{\mu\big(x^2 + y^2\big) + z^2}. \tag{2.9}$$

Without loss of generality, we consider the periodic orbits of the spatial anisotropic Kepler problem with anisotropic perturbations on the energy level $H = h = -1$. By using Theorem 1.2, we have the following result.

Theorem 2.2. *For every $k_0 \in [0, 2\pi)$, there exists a sufficiently small positive constant ϵ_0 such that, for each $\epsilon \in (-\epsilon_0, \epsilon_0)$, on energy level $H = h = -1$, the spatial anisotropic Kepler problem with anisotropic perturbations defined by the Hamiltonian* (2.9) *has:*

(i) *Two 2π-periodic solutions*

$$\zeta_\epsilon{}^{\pm}(t) = (x(t,\epsilon), y(t,\epsilon), z(t,\varepsilon), X(t,\epsilon), Y(t,\epsilon), Z(t,\epsilon))$$

such that if $|\beta| < 1/2$, then $\zeta_\epsilon{}^{\pm}(0)$ tends to

$$\Big(\frac{1}{2}\cos g_0 \cos k_0, \frac{1}{2}\cos g_0 \sin k_0, \frac{1}{2}\sin g_0, \sqrt{2}\sin g_0 \cos k_0, \sqrt{2}\sin g_0 \sin k_0, -\sqrt{2}\cos g_0\Big),$$

as $\epsilon \to 0$, where

$$g_0 = \pm\frac{1}{2}\arccos\left(-2\beta\right).$$

(ii) *Two 2π-periodic solutions*

$$\zeta_\epsilon{}^{\pm}(t) = (x(t,\epsilon), y(t,\epsilon), z(t,\varepsilon), X(t,\epsilon), Y(t,\epsilon), Z(t,\epsilon))$$

such that if $\beta \in (-\infty, -1/2)$, then ${\zeta_\epsilon}^{\pm}(0)$ tends to

$$\Big(\pm\frac{1}{2}\Big(1-\sqrt{1-2G_0^2}\Big)\cos k_0, \pm\frac{1}{2}\Big(1-\sqrt{1-2G_0^2}\Big)\sin k_0, 0, 0, 0, \mp\frac{2|G_0|}{1-\sqrt{1-2G_0^2}}\Big),$$

as $\epsilon \to 0$, where

$$G_0 = -\frac{\sqrt[3]{-\beta^2\left(\beta+3\left(\sqrt{81-6\beta}-9\right)\right)}}{3\sqrt{2}\beta} + \frac{1}{3\sqrt{2}}\sqrt[3]{\frac{\beta}{\beta+3\left(\sqrt{81-6\beta}-9\right)}} - \frac{\sqrt{2}}{3}.$$

(iii) *Two 2π-periodic solutions*

$${\zeta_\epsilon}^{\pm}(t) = (x(t,\epsilon), y(t,\epsilon), z(t,\varepsilon), X(t,\epsilon), Y(t,\epsilon), Z(t,\epsilon))$$

such that if $\beta \in (1/2, +\infty)$, then ${\zeta_\epsilon}^{\pm}(0)$ tends to

$$\Big(0, 0, \pm\frac{1}{2}\Big(1-\sqrt{1-2G_0^2}\Big), \pm\frac{2|G_0|\cos k_0}{1-\sqrt{1-2G_0^2}}, \pm\frac{2|G_0|\sin k_0}{1-\sqrt{1-2G_0^2}}, 0\Big),$$

as $\epsilon \to 0$, where

$$G_0 = \frac{\big(\beta - \big(3\sqrt{3}\sqrt{\beta^4(2\beta+27)} + (\beta+27)\beta^2\big)^{1/3}\big)^2}{3\sqrt{2}\beta\big(3\sqrt{3}\sqrt{\beta^4(2\beta+27)} + (\beta+27)\beta^2\big)^{1/3}}.$$

Proof. By using Theorem 2.2, in each case we can obtain two periodic solutions of Hamiltonian (2.9) such that $\lim_{\epsilon\to 0} {\gamma_\epsilon}^{\pm}(0) = (g_0, k_0, L_0, G_0, K_0)$. Then we use the Delaunay transformation Ψ_1 (see (3.3)) to change the Delaunay coordinates $(0, g_0, k_0, L_0, G_0, K_0)$ into the spatial spherical coordinates $(\rho_0, \theta_0, \phi_0, P_0, \Theta_0, \Phi_0)$. Subsequently, we use the spherical transformation Ψ_0 (see (3.1)) to transform the spatial spherical coordinates into the Cartesian coordinates $(x_0, y_0, z_0, X_0, Y_0, Z_0)$. The rest proof of Theorem 2.2 is similar to the one of Theorem 2.1. To avoid repetition, we do not do it. $\square$

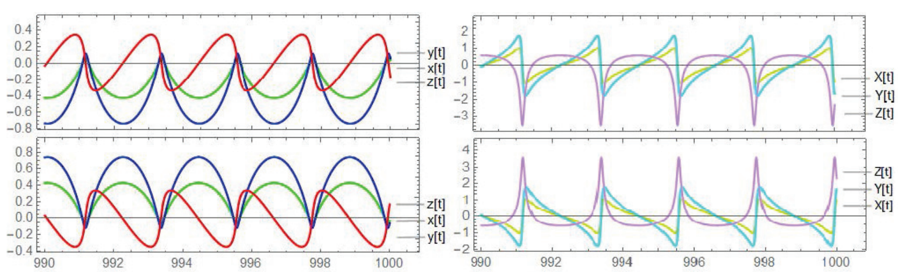
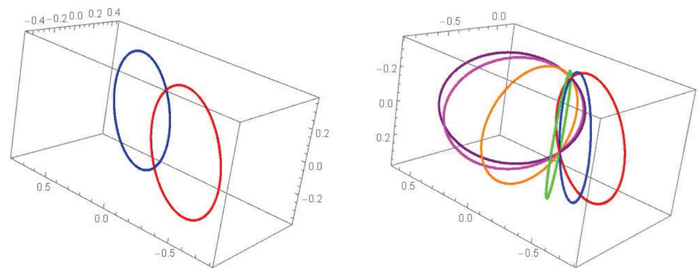


FIGURE 3. Periodic orbits of Hamiltonian (2.9) with the parameters $\epsilon = 0.002$, $\beta = -1$ and $k_0 = \pi/3$. In the i-th ($i = 1, 2$) line, the curves represent periodic orbit $\zeta_\epsilon^i(t)$ corresponding to the initial value $\zeta^i(0)$, respectively. On the left, the pictures denote by the curves $x(t), y(t), z(t)$, while on the right, the pictures denote by the curves $X(t), Y(t), Z(t)$.

From Theorem 2.2, when $\beta \in (-\infty, -1/2)$, $\beta \in (-1/2, 0)$ or $\beta \in (0, 1/2)$, $\beta \in (1/2, +\infty)$, for sufficiently small $|\epsilon| \ll 1$, the spatial anisotropic Kepler problem with perturbations corresponding to Hamiltonian (2.9) have at least two kinds of



$k_0 = \pi/3$ and $(x_\epsilon^i(t), y_\epsilon^i(t), z_\epsilon^i(t))$ $\qquad$ $k_0 = i\pi/6$ $(i = 1, 2, \dots, 6)$

FIGURE 4. Periodic orbits on the configuration space $xOyz$, where we choose the parameters as $\epsilon = 0.002$ and $\beta = -1$. We plot the periodic orbits $\zeta_\epsilon^i(t)$ $(i = 1, 2)$ on the projective configuration space $xOyz$ on the left, while on the right we plot the periodic orbits starting from the initial values $\zeta^1(0) = (0.140795 \cos k_0, 0.140795 \sin k_0, 0, 0, -3.49357)$ by varying $k_0 = i\pi/6, i = 1, 2, \dots, 6$.

periodic solutions, which constitute a continuum of periodic solutions by continuously changing k_0 on the interval $[0, 2\pi)$.

To demonstrate Theorem 2.2, we take the parameter $\beta = -1$ and $k_0 = \pi/3$. According to Theorem 2.2, there exists two periodic orbits $\zeta_\epsilon^i(t), i = 1, 2$ such that

$$\lim_{\epsilon \to 0} {\zeta_\epsilon}^i(0) = \zeta^i(0) = \begin{cases} (0.070398, 0.121932, 0, 0, 0, -3.493567) \\ (-0.070398, -0.121932, 0, 0, 0, 3.493567) \,. \end{cases}$$

For numerical demonstration of periodic orbits, we take $\epsilon = 0.002$ and $\mu = 0.998$. Since for sufficiently small $|\epsilon| \ll 1$, the initial values of periodic solutions $\zeta_\epsilon^i(t), i = 1, 2$ are near $\zeta^i(0)$, we plot the curves of the solutions starting at $\zeta^i(0)$ on the time interval $t \in (990, 1000)$, see Figure 3. As is shown in Figure 3, the solutions starting at $\zeta^i(0)$ are periodic. In order to test different periodic orbits, we plot the periodic orbits $\zeta_\epsilon^i(t)$ $(i = 1, 2)$ on the projective configuration space $xOyz$, see Figure 4(a). As can be seen from Figure 4(a), there are two differential periodic orbits. Moreover, we plot the orbits on the projective configuration space $xOyz$ starting from the initial values

$$\zeta^1(0) = (0.140795 \cos k_0, 0.140795 \sin k_0, 0, 0, 0, -3.493567)$$

by varying $k_0 = i\pi/6, i = 1, 2, \dots, 6$, see Figure 4(b). As shown IN Figure 4(b), if we continuously change k_0 on the interval $[0, 2\pi)$, all of periodic orbits are full of an ellipsoid.

Similarly, to demonstrate Theorem 2.2, we take the parameter $\beta = 1/32$ and $k_0 = \pi/3$. According to Theorem 2.2, there exists two periodic orbits $\zeta_\epsilon^i(t)$, $i = 1, 2$ such that

$$\lim_{\epsilon \to 0} {\zeta_\epsilon}^i(0) = \zeta^i(0) = \begin{cases} \Big(\frac{\sqrt{15}}{16\sqrt{2}}, \frac{3\sqrt{5}}{16\sqrt{2}}, \frac{\sqrt{17}}{8\sqrt{2}}, \frac{\sqrt{17}}{8}, \frac{\sqrt{51}}{8}, -\frac{\sqrt{15}}{4}\Big) & \text{for } i = 1, \\ \Big(\frac{\sqrt{15}}{16\sqrt{2}}, \frac{3\sqrt{5}}{16\sqrt{2}}, -\frac{\sqrt{17}}{8\sqrt{2}}, -\frac{\sqrt{17}}{8}, -\frac{\sqrt{51}}{8}, -\frac{\sqrt{15}}{4}\Big) & \text{for } i = 2. \end{cases}$$

For numerical examples of periodic orbits, we take $\epsilon = 0.002$ and $\mu = 0.998$. Since for sufficiently small $|\epsilon| \ll 1$, the initial values of periodic solutions $\zeta_\epsilon^i(t), i =$

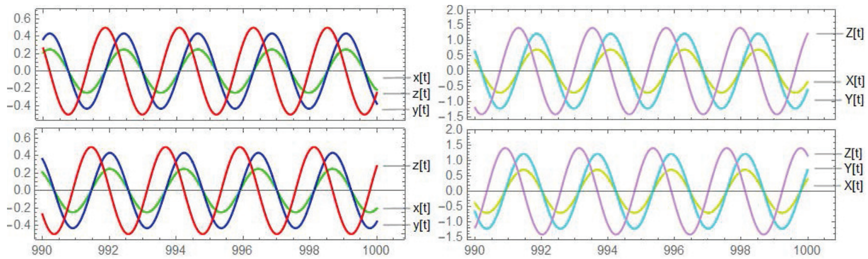
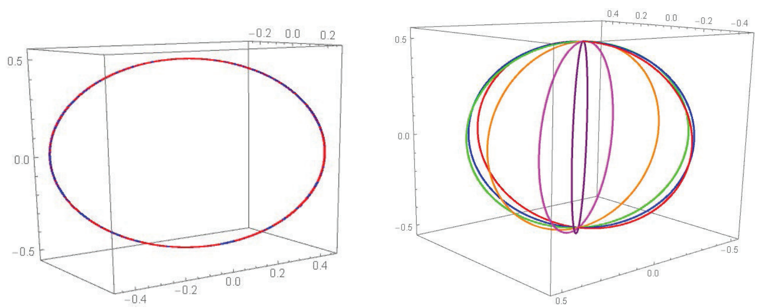


FIGURE 5. Periodic orbits of Hamiltonian (2.9) with the parameters $\epsilon = 0.002$, $\beta = 1/32$ and $k_0 = \pi/3$. In the i-th ($i = 1, 2$) line, the curves represent periodic orbit $\zeta_\epsilon^i(t)$ corresponding to the initial value $\zeta^i(0)$, respectively. On the left, the pictures denote by the curves $x(t), y(t), z(t)$, while on the right, the pictures denote by the curves $X(t), Y(t), Z(t)$.



$k_0 = \pi/3$ and $(x_\epsilon^i(t), y_\epsilon^i(t), z_\epsilon^i(t))$ $k_0 = ipi/6$ $(i = 1, 2, \dots, 6)$

FIGURE 6. Periodic orbits on the configuration space $xOyz$, where we choose the parameters as $\epsilon = 0.002$ and $\beta = 1/32$. We plot the periodic orbits $\zeta_\epsilon^i(t)$ $(i = 1, 2)$ on the projective configuration space $xOyz$ on the left, while on the right we plot the periodic orbits starting from the initial values $\zeta^1(0) = (\sqrt{15}\cos k_0/8\sqrt{2}, \sqrt{15}\sin k_0/8\sqrt{2}, \sqrt{17}/8\sqrt{2}, \sqrt{17}\cos k_0/4, \sqrt{17}\sin k_0/4, -\sqrt{15}/4)$ by varying $k_0 = i\pi/6, i = 1, 2, \dots, 6$.

1, 2 are near $\zeta^i(0)$, we plot the curves of the solutions starting at $\zeta^i(0)$ on the time interval $t \in (990, 1000)$, see Figure 5. As is shown in Figure 5, the solutions starting at $\zeta^i(0)$ are periodic. Since the flows of autonomous Hamiltonian system have the semigroup property, two different periodic solutions may be have the same orbits. In order to test different periodic orbits, we plot the periodic orbits $\zeta_\epsilon^i(t)$ $(i = 1, 2)$ on the projective configuration space $xOyz$, see Figure 6(a). As can be seen from Figure 6(a), there is only one periodic orbit, since periodic solutions $\zeta_\epsilon^1(t)$ and $\zeta_\epsilon^2(t)$ have the same periodic orbit. Moreover, we plot the orbits on the projective configuration space $xOyz$ starting from the initial values

$$\zeta^1(0) = \Big(\frac{\sqrt{15}}{8\sqrt{2}}\cos k_0, \frac{\sqrt{15}}{8\sqrt{2}}\sin k_0, \frac{\sqrt{17}}{8\sqrt{2}}, \frac{\sqrt{17}}{4}\cos k_0, \frac{\sqrt{17}}{4}\sin k_0, -\frac{\sqrt{15}}{4}\Big)$$

by varying $k_0 = i\pi/6, i = 1, 2, \dots, 6$, see Figure 6(b). As is shown from Figure 6(b), if we continuously change k_0 on the interval $(0, 2\pi)$, all of periodic orbits are full of an ellipsoid.

3. Canonical transformations

In this section, we introduce two canonical transformations, which will be used to transform Hamiltonian (1.6) with the Cartesian coordinates (x, y, z, X, Y, Z) into Hamiltonian (1.7) with the Delaunay coordinates (l, g, k, L, G, K). The content of this part is known and we can refer to some classical books (for example, see [5, 38]) for details.

3.1. **Spherical coordinates.**

Firstly, we shall introduce a symplectic transformations

$$\Psi_0 : (x, y, z, X, Y, Z) \to (\rho, \theta, \phi, P, \Theta, \Phi)$$

by Ψ_0:

$$\begin{gathered} x = \rho\cos\theta\sin\phi,\ \ y = \rho\sin\theta\sin\phi,\ \ z = \rho\cos\phi, \\ X = \frac{\cos\theta(P\rho\sin\phi + \Phi\cos\phi) - \Theta\sin\theta\csc\phi}{\rho}, \\ Y = \frac{\Theta\cos\theta\csc\phi + \sin\theta(P\rho\sin\phi + \Phi\cos\phi)}{\rho}, \\ Z = P\cos\phi - \frac{\Phi\sin\phi}{\rho}, \end{gathered} \tag{3.1}$$

which is generated by the Mathieu generating function

$$W_0(\rho, \theta, X, Y, Z) = \rho X\cos\theta\sin\phi + \rho Y\sin\theta\sin\phi + \rho Z\cos\phi.$$

Under the symplectic transformations Ψ_0, the Hamiltonian (1.6) becomes

$$\begin{aligned} H = \frac{1}{2}\Big(&\frac{\Theta^2\csc^2(\phi)}{\rho^2} - \frac{2\sqrt{2}}{\sqrt{\rho^2(1+\mu+(1-\mu)\cos(2\phi))}} + P^2 + \frac{\Phi^2}{\rho^2} \\ &- \epsilon 2^{\frac{p}{2}+1}\beta\left(\rho^2\Big(1+\mu+(1-\mu)\cos(2\phi)\Big)\right)^{-p/2}\Big). \end{aligned} \tag{3.2}$$

3.2. **Spatial Delaunay elements.**

To change from the spherical coordinates $(\rho, \theta, \phi, P, \Theta, \Phi)$ to the Delaunay elements (l, g, k, L, G, K) with the first three variables defined mod 2π, we consider the generating function

$$W(\rho, \theta, \phi, L, G, K) = \theta K + \int_{\pi/2}^{\phi}\Big(G^2 - \frac{K^2}{\sin^2\xi}\Big)^{1/2} \mathrm{d}\xi + \int_{\rho_0}^{\rho}\Big(-\frac{G^2}{\xi^2} - \frac{1}{L^2} + \frac{2}{\xi}\Big)^{1/2}\mathrm{d}\xi,$$

where $\rho_0 = L^2(1 - \sqrt{1 - G^2/L^2})$.

Then the change of coordinates is Ψ_1:

$$
\begin{gathered}
P=\Big(-\frac{G^2}{\rho^2}-\frac{1}{L^2}+\frac{2}{\rho}\Big)^{1/2},\\
\Theta=K,\\
\Phi=\Big(G^2-\frac{K^2}{\sin^2\phi}\Big)^{1/2},\\
l=\frac{1}{L^3}\int_{\rho_0}^{\rho}\Big(-\frac{G^2}{\xi^2}-\frac{1}{L^2}+\frac{2}{\xi}\Big)^{-1/2}\mathrm{d}\xi,\\
g=-\int_{\pi/2}^{\phi}\Big(G^2-\frac{K^2}{\sin^2\xi}\Big)^{-1/2}G\mathrm{d}\xi-\int_{\rho_0}^{\rho}\Big(-\frac{G^2}{\xi^2}-\frac{1}{L^2}+\frac{2}{\xi}\Big)^{-1/2}\Big(\frac{G}{\xi^2}\Big)\mathrm{d}\xi,\\
k=\theta-\int_{\pi/2}^{\phi}\Big(G^2-\frac{K^2}{\sin^2\xi}\Big)^{-1/2}\Big(\frac{K^2}{\sin^2\xi}\Big)\mathrm{d}\xi.
\end{gathered}
\tag{3.3}
$$

The first integral in (3.3) can be obtained explicitly by

$$
l=\frac{1}{L^3}\int_{\rho_0}^{\rho}\Big(-\frac{G^2}{\xi^2}-\frac{1}{L^2}+\frac{2}{\xi}\Big)^{-1/2}\mathrm{d}\xi=E-e\sin E,
\tag{3.4}
$$

where E is determined by $\rho=a(1-e\cos E)$ with $a=L^2$ and $e=\sqrt{1-G^2/L^2}$. Thus we can see that the variable $\rho=\rho(l,L,G)$ depends on l,L,G. We use $-\sigma,f$ to denote the second integral and the third integral, respectively, where $\cos\phi=\sin\sigma\sin i=\sin(f+g)\sin i$ with

$$
\cos i=K/G,\quad \rho\cos f=a(\cos E-e),\quad \rho\sin f=\frac{a}{L}G\sin E.
$$

From the equalities above, we know that the variable $\phi=\phi(l,g,L,G,K)$ depends on l,g,L,G,K. The last equation of (3.3) implies that $\sin(k-\theta)=(\cot\phi)/\gamma,\ \gamma^2=(G^2-K^2)/K^2$.

Under the symplectic transformations Ψ_1, the Hamiltonian (3.2) becomes

$$
\begin{aligned}
H=&-\frac{1}{2L^2}+\frac{1}{\rho}-\frac{\sqrt{2}}{\rho\sqrt{(1+\mu+(1-\mu)\cos(2\phi))}}\\
&-\epsilon 2^{\frac{p}{2}}\beta\Big[\rho^2\Big(1+\mu+(1-\mu)\cos(2\phi)\Big)\Big]^{-p/2},
\end{aligned}
\tag{3.5}
$$

where $\rho=\rho(l,L,G),\phi=\phi(l,g,L,G,K)$. We emphasize that the Hamiltonian (3.5) is independent of the variable k, which implies that $K(t)$ is always constant.

Let $\mu=1-\varepsilon$. We expand the Hamiltonian function (3.5) at $\epsilon=0$ into

$$
\begin{aligned}
H=&-\frac{1}{2L^2}-\epsilon\Big(\beta\rho^{-p}+\frac{1-\cos(2\phi)}{4\rho}\Big)\\
&-\frac{1}{8}\epsilon^2\Big(2\beta p\rho^{-p}\big(1-\cos(2\phi)\big)+\frac{3\big(3-4\cos(2\phi)+\cos(4\phi)\big)}{8\rho}\Big)+O(\epsilon^3).
\end{aligned}
\tag{3.6}
$$

Since

$$
\rho^2\cos^2\phi=\rho^2\sin^2(f+g)\sin^2 i=(\rho\sin f\cos g+\rho\cos f\sin g)^2\,(1-\cos^2 i),
$$

by using the relationships between various variables, we know that

$$
\begin{aligned}
\rho^2\cos(2\phi) =& 2\left(\rho\sin f\cos g+\rho\cos f\sin g\right)^2\left(1-\cos^2 i\right)-\rho^2 \\
=& 2\left(\frac{a}{L}G\sin E\cos g+a(\cos E-e)\sin g\right)^2\left(1-\frac{K^2}{G^2}\right)-\rho^2.
\end{aligned}
\tag{3.7}
$$

Substituting (3.7) and $\rho=a(1-e\cos E)$ into (3.6), we obtain

$$
\begin{aligned}
H=&-\frac{1}{2L^2}+\epsilon\biggl(\frac{1}{2L\left(L-\sqrt{L^2-G^2}\cos E\right)}-\beta\left(L\left(L-\sqrt{L^2-G^2}\cos E\right)\right)^{-p} \\
&-\frac{(G^2-K^2)\big((L\cos E-\sqrt{L^2-G^2})\sin g+G\sin E\cos g\big)^2}{2G^2L(L-\sqrt{L^2-G^2}\cos E)^3}\biggr)+O(\epsilon^2) \\
:=& H_0(L)+\epsilon H_1(l,g,L,G,K)+O(\epsilon^2),
\end{aligned}
\tag{3.8}
$$

where $E=E(l,L,G)$ is defined by $l=E-e\sin E$.

4. Reduction of system on energy surface

The Hamiltonian system corresponding to the Hamiltonian (3.8) is given by

$$
\begin{gathered}
\frac{\mathrm{d}l}{\mathrm{d}t}=\frac{\partial H}{\partial L}=\frac{1}{L^3}+O(\epsilon), \\
\frac{\mathrm{d}g}{\mathrm{d}t}=\frac{\partial H}{\partial G}=\epsilon\frac{\partial H_1}{\partial G}(l,g,L,G,K)+O(\epsilon^2), \\
\frac{\mathrm{d}k}{\mathrm{d}t}=\frac{\partial H}{\partial K}=\epsilon\frac{\partial H_1}{\partial K}(l,g,L,G,K)+O(\epsilon^2), \\
\frac{\mathrm{d}L}{\mathrm{d}t}=-\frac{\partial H}{\partial l}=-\epsilon\frac{\partial H_1}{\partial l}(l,g,L,G,K)+O(\epsilon^2), \\
\frac{\mathrm{d}G}{\mathrm{d}t}=-\frac{\partial H}{\partial g}=-\epsilon\frac{\partial H_1}{\partial g}(l,g,L,G,K)+O(\epsilon^2), \\
\frac{\mathrm{d}K}{\mathrm{d}t}=-\frac{\partial H}{\partial k}=0,
\end{gathered}
\tag{4.1}
$$

where the last equation is owing to that H is independent of k, see (3.5).

For sufficiently small $|\epsilon|\ll 1$ and $L>0$, we know that $\mathrm{d}l/\mathrm{d}t=1/L^3+O(\epsilon)>0$. Then we regard the angular variable l as a new time variable. From (4.1), we can obtain a new system of differential equations

$$
\begin{gathered}
\frac{\mathrm{d}g}{\mathrm{d}l}=\epsilon L^3\frac{\partial H_1}{\partial G}(l,g,L,G,K)+O(\epsilon^2), \\
\frac{\mathrm{d}k}{\mathrm{d}l}=\epsilon L^3\frac{\partial H_1}{\partial K}(l,g,L,G,K)+O(\epsilon^2), \\
\frac{\mathrm{d}L}{\mathrm{d}l}=-\epsilon L^3\frac{\partial H_1}{\partial l}(l,g,L,G,K)+O(\epsilon^2), \\
\frac{\mathrm{d}G}{\mathrm{d}l}=-\epsilon L^3\frac{\partial H_1}{\partial g}(l,g,L,G,K)+O(\epsilon^2), \\
\frac{\mathrm{d}K}{\mathrm{d}l}=-\frac{\partial H}{\partial k}=0.
\end{gathered}
\tag{4.2}
$$

Since l is an angular variable defined mod 2π, H,H_1 and system (4.2) are also 2π-periodic in l. For any $h<0$, we define the energy surface by

$$
\Omega=\big\{(l,g,k,L,G,K)\in\mathbb{R}^6: H(l,g,k,L,G,K)=h\big\}. \tag{4.3}
$$

For sufficiently small $|\epsilon| \ll 1$, we obtain that $L = 1/\sqrt{-2h} + O(\epsilon)$. By using the first order expansion of L, we can discard the third equation in system (4.2). Moreover, from the last equation of (4.2) we have $K(l) \equiv K_0$, where K_0 is a constant to be determined later to guarantee that $k(l)$ is 2π-periodic, since the right side of the second equation of (4.2) dose not depend on k. Therefore, to prove the existence of periodic solutions of (4.2), we only need to consider

$$\begin{aligned} \frac{\mathrm{d}g}{\mathrm{d}l} &= \epsilon L^3 \frac{\partial H_1}{\partial G}(l, g, L, G, K) + O(\epsilon^2), \\ \frac{\mathrm{d}G}{\mathrm{d}l} &= -\epsilon L^3 \frac{\partial H_1}{\partial g}(l, g, L, G, K) + O(\epsilon^2), \end{aligned} \tag{4.4}$$

where $K = K_0$ and $L = 1/\sqrt{-2h}$ are considered as constants in system (4.4) at this stage, and the high order part $O(\epsilon^2)$ depends on l, g, h, G, K.

5. Standard form of averaging

To investigate the existence of 2π-periodic solutions of system (4.4), we perform the method of averaging. The averaged system of (4.4) is

$$\begin{aligned} \frac{\mathrm{d}g}{\mathrm{d}l} &= \epsilon \frac{L^3}{2\pi} \int_0^{2\pi} \frac{\partial H_1}{\partial G}(l, g, L, G, K)\mathrm{d}l, \\ \frac{\mathrm{d}G}{\mathrm{d}l} &= -\epsilon \frac{L^3}{2\pi} \int_0^{2\pi} \frac{\partial H_1}{\partial g}(l, g, L, G, K)\mathrm{d}l. \end{aligned} \tag{5.1}$$

Since $H_1, \partial H_1/\partial G$ and $\partial H_1/\partial g$ are continuous, system (5.1) is equivalent to

$$\begin{aligned} \frac{\mathrm{d}g}{\mathrm{d}l} &= \epsilon \frac{L^3}{2\pi} \frac{\partial}{\partial G}\Big(\int_0^{2\pi} H_1(l, g, L, G, K)\mathrm{d}l \Big), \\ \frac{\mathrm{d}G}{\mathrm{d}l} &= -\epsilon \frac{L^3}{2\pi} \frac{\partial}{\partial g}\Big(\int_0^{2\pi} H_1(l, g, L, G, K)\mathrm{d}l \Big). \end{aligned} \tag{5.2}$$

To obtain an explicit expression of system (5.2), we have to calculate the definite integral above. It can be seen from (3.4) that, E varies form 0 to 2π when the integral variable l varies form 0 to 2π. By the integral method of substitution, we have

$$\frac{1}{2\pi} \int_0^{2\pi} H_1(l, g, L, G, K)\mathrm{d}l = \frac{1}{2\pi} \int_0^{2\pi} \Big(H_1(l, g, L, G, K) \frac{\mathrm{d}l}{\mathrm{d}E} \Big) \mathrm{d}E. \tag{5.3}$$

Recall that $\rho = L^2(1 - e\cos E)$, and by (3.4) we have

$$\frac{\mathrm{d}l}{\mathrm{d}E} = 1 - e\cos E = \frac{\rho}{L^2}.$$

For convenience of calculations, we divide $H_1 \cdot \mathrm{d}l/\mathrm{d}E$ into two parts:

$$\begin{aligned} H_1(l, g, L, G, K)\frac{\mathrm{d}l}{\mathrm{d}E} &= \frac{\rho}{L^2}\Big(-\beta\rho^{-p} - \frac{\sin^2(\phi)}{2\rho} \Big) \\ &= \Big(-\frac{\beta\rho^{1-p}}{L^2} \Big) + \Big(-\frac{\sin^2(\phi)}{2L^2} \Big) \\ &:= H_{10}(E, g, L, G, K) + H_{11}(E, g, L, G, K). \end{aligned}$$

Since

$$\rho^2\cos^2\phi = \rho^2\sin^2(f+g)\sin^2 i = (\rho\sin f\cos g + \rho\cos f\sin g)^2\,(1 - \cos^2 i),$$

by using the relationships between various variables, we know that

$$\begin{aligned} \rho^2\sin^2\phi =&\rho^2 - (\rho\sin f\cos g + \rho\cos f\sin g)^2(1-\cos^2 i)\\ =&\rho^2 - \Big(\frac{a}{L}G\sin E\cos g + a(\cos E - e)\sin g\Big)^2\Big(1-\frac{K^2}{G^2}\Big). \end{aligned} \tag{5.4}$$

Substituting (5.4), $a = L^2$ and $\rho = a(1-e\cos E)$ into H_{10} and H_{11}, we obtain

$$\begin{aligned} H_{10}(E,g,L,G,K) &= -\frac{\beta\rho^{1-p}}{L^2} = -\frac{\beta\left(L^2(1-e\cos E)\right)^{1-p}}{L^2}\\ &= -\frac{\beta L^{-2p}}{(1-e\cos E)^{p-1}} \end{aligned}$$

and

$$\begin{aligned} &H_{11}(E,g,L,G,K)\\ &= -\frac{\sin^2\phi}{2L^2} = \frac{(\sin f\cos g + \cos f\sin g)^2\frac{G^2-K^2}{G^2} - 1}{2L^2}\\ &= \frac{1}{2L^2}\Big(\frac{(G-K)(G+K)(L\sin g(\cos E - e) + G\sin E\cos g)^2}{G^2L^2(e\cos E-1)^2} - 1\Big)\\ &= -\frac{\sin^2 E\sin^2 g\left(G^2-K^2\right)}{4L^4(e\cos E-1)^2} + \frac{\sin^2 E\cos^2 g\left(G^2-K^2\right)}{4L^4(e\cos E-1)^2}\\ &\quad + \frac{\sin E\sin g\cos g\left(G^2-K^2\right)(\cos E - e)}{GL^3(e\cos E-1)^2}\\ &\quad - \frac{\cos^2 g\left(G^2-K^2\right)(\cos E-e)^2}{4G^2L^2(e\cos E-1)^2} + \frac{\sin^2 g\left(G^2-K^2\right)(\cos E-e)^2}{4G^2L^2(e\cos E-1)^2}\\ &\quad - \frac{G^2+K^2}{4G^2L^2}\\ &:= P_1+P_2+P_3+P_4+P_5+P_6. \end{aligned}$$

5.1. **Calculation of integral for H_{10}.** To calculate the definite integral (5.3), we calculate the integral for H_{10}

$$\overline{H}_{10} := \frac{1}{2\pi}\int_0^{2\pi} H_{10}(E,g,L,G,K)\,d\mathrm{E} = -\frac{1}{2\pi}\beta L^{-2p}\int_0^{2\pi}\frac{1}{(1-e\cos E)^{p-1}}\mathrm{d}E.$$

In the following, by using Residue theory of complex functions we calculate the definite integral

$$\int_0^{2\pi}\frac{1}{(1-e\cos E)^{p-1}}\mathrm{d}E$$

with two different cases for the range of the integer p.

Case I: $p > 1$. Let $z = \mathrm{e}^{iE}$, then $\cos E = (z+z^{-1})/2$. When E experiences the interval $[0,2\pi]$, z goes through the unit circle $|z| = 1$ one round in the positive direction. Then it follows that

$$\int_0^{2\pi}\frac{1}{(1-e\cos E)^{p-1}}\,\mathrm{d}E = \frac{1}{i}\int_{|z|=1}\frac{2^{p-1}z^{p-2}}{(-ez^2-e+2z)^{p-1}}\mathrm{d}z := \frac{1}{i}\int_{|z|=1} f(z)\mathrm{d}z.$$

The function

$$f(z) := \frac{2^{p-1}z^{p-2}}{(-ez^2-e+2z)^{p-1}} = \frac{2^{p-1}z^{p-2}}{(-e)^{p-1}\big(z-\frac{1-\sqrt{1-e^2}}{e}\big)^{p-1}\big(z-\frac{\sqrt{1-e^2}+1}{e}\big)^{p-1}}$$

has only a singular point $z_0 = (1-\sqrt{1-e^2})/e$ in the open unit disc, which is a pole of order $p-1$. At the same time, the function f is continuous on the unit circle $|z| = 1$. Let

$$g(z) = \frac{2^{p-1} z^{p-2}}{(-e)^{p-1}\big(z - \frac{\sqrt{1-e^2}+1}{e}\big)^{p-1}}.$$

Using the Leibniz expansive formula, we have

$$\begin{aligned}
&\frac{d^{p-2}}{dz^{p-2}} g(z)\Big|_{z=z_0}\\
&= \frac{2^{p-1}}{(-e)^{p-1}} \sum_{m=0}^{p-2} C_{p-2}^{m} \left(z^{p-2}\right)^{(-m+p-2)} \Big(\Big(z - \frac{\sqrt{1-e^2}+1}{e}\Big)^{1-p}\Big)^{(m)}\Big|_{z=z_0}\\
&= \sum_{m=0}^{p-2} C_{p-2}^{m} 2^{-m} \frac{(m+p-2)!}{m!} \Big(1-\sqrt{1-e^2}\Big)^{m} \Big(\sqrt{1-e^2}\Big)^{-m-p+1}\\
&= \sum_{m=0}^{p-2} C_{p-2}^{m} 2^{-m} \frac{(m+p-2)!}{m!} \Big(1-\frac{G}{L}\Big)^{m} \big(\frac{G}{L}\big)^{-m-p+1},
\end{aligned}$$

where $u^{(m)}(z)$ denotes the m-th order derivatives of u with respect to z, and the last equality is obtained by using $e^2 = 1 - G^2/L^2$. By the Residue theorem, we have

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{(1-e\cos E)^{p-1}}\,\mathrm{d}E &= \frac{1}{i}\int_{|z|=1} \frac{2^{p-1} z^{p-2}}{(-ez^2 - e + 2z)^{p-1}}\mathrm{d}z\\
&= 2\pi \operatorname*{Res}_{z=\frac{1-\sqrt{1-e^2}}{e}} f(z)\\
&= 2\,\pi \frac{1}{(p-2)!} \frac{d^{p-2}}{dz^{p-2}} g(z)\Big|_{z=\frac{1-\sqrt{1-e^2}}{e}}\\
&= 2\pi \sum_{m=0}^{p-2} C_{p-2}^{m} 2^{-m} \frac{(m+p-2)!}{m!(p-2)!} \Big(1-\frac{G}{L}\Big)^{m} \Big(\frac{G}{L}\Big)^{-m-p+1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{H}_{10} &= -\beta L^{-2p} \sum_{m=0}^{p-2} C_{p-2}^{m} 2^{-m} \frac{(m+p-2)!}{m!(p-2)!} \Big(1-\frac{G}{L}\Big)^{m} \Big(\frac{G}{L}\Big)^{-m-p+1}\\
&= -\beta L^{-p-1} \sum_{m=0}^{p-2} C_{p-2}^{m} 2^{-m} \frac{(m+p-2)!}{m!(p-2)!} \left(L-G\right)^{m} G^{-m-p+1}.
\end{aligned} \tag{5.5}$$

Case II: $p \le 1$. In this case,

$$\int_0^{2\pi} \frac{1}{(1-e\cos E)^{p-1}}\,\mathrm{d}E = \frac{1}{i}\int_{|z|=1} \frac{\left(-ez^2 - e + 2z\right)^{1-p}}{2^{1-p} z^{2-p}}\mathrm{d}z := \frac{1}{i}\int_{|z|=1} f(z)\mathrm{d}z.$$

We know the singular point $z = 0$ is a pole of order $2-p$. With the same argument above, we obtain

$$\frac{1}{i}\int_{|z|=1} \frac{\big(-ez^2 - e + 2z\big)^{1-p}}{2^{1-p} z^{2-p}}\mathrm{d}z$$

$$
\begin{aligned}
&= \frac{1}{i}\int_{|z|=1} \frac{(-e)^{1-p}\big(z-\frac{1-\sqrt{1-e^2}}{e}\big)^{1-p}\big(z-\frac{\sqrt{1-e^2}+1}{e}\big)^{1-p}}{2^{1-p}z^{2-p}}\mathrm{d}z \\
&= 2\pi\,\mathrm{Res}_{z=0}\, f(z) \\
&= 2\pi\frac{(-e)^{1-p}}{2^{1-p}}\sum_{m=0}^{1-p} C_{1-p}^{m}\Big(\Big(z-\frac{1-\sqrt{1-e^2}}{e}\Big)^{1-p}\Big)^{(1-m-p)} \\
&\quad \times \Big(\Big(z-\frac{\sqrt{1-e^2}+1}{e}\Big)^{1-p}\Big)^{(m)}\Big|_{z=0} \\
&= 2\pi 2^{p-1}\sum_{m=0}^{1-p} C_{1-p}^{m}\frac{(1-p)!}{m!(-m-p+1)!}\Big(1-\sqrt{1-e^2}\Big)^{m}\Big(\sqrt{1-e^2}+1\Big)^{-m-p+1} \\
&= 2\pi 2^{p-1}\sum_{m=0}^{1-p} C_{1-p}^{m}\frac{(1-p)!}{m!(-m-p+1)!}\Big(1-\frac{G}{L}\Big)^{m}\Big(\frac{G}{L}+1\Big)^{-m-p+1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\overline{H}_{10} \\
&= -\beta L^{-2p}2^{p-1}\sum_{m=0}^{1-p} C_{1-p}^{m}\frac{(1-p)!}{m!(-m-p+1)!}\Big(1-\frac{G}{L}\Big)^{m}\Big(\frac{G}{L}+1\Big)^{-m-p+1} \\
&= -\beta L^{-1-p}2^{p-1}\sum_{m=0}^{1-p} C_{1-p}^{m}\frac{(1-p)!}{m!(-m-p+1)!}\left(L-G\right)^{m}\left(L+G\right)^{-m-p+1}.
\end{aligned}
\tag{5.6}
$$

5.2. **Calculation of integral for H_{11}.** With the same argument for calculating the integral of H_{01}, we also use the Residue theorem to calculate the integral of H_{11}. To calculate the integrals of the first two terms P_1, P_2 of H_{11}, we consider the following definite integral

$$
\int_0^{2\pi}\frac{\sin^2 E}{(e\cos E-1)^2}\mathrm{d}E = \frac{1}{i}\int_{|z|=1} -\frac{\left(z^2-1\right)^2}{z\left(ez^2+e-2z\right)^2}\mathrm{d}z := \frac{1}{i}\int_{|z|=1} f_1(z)\mathrm{d}z,
$$

by setting $z=\mathrm{e}^{iE}$. The function $f_1(z)$ has two singular points $z_1=0$ and $z_2=(1-\sqrt{1-e^2})/e$ in the open unit disc. Moreover, z_1 is a pole of order one while z_2 is a pole of order two. Then by using the Residue theorem we have

$$
\begin{aligned}
\int_0^{2\pi}\frac{\sin^2 E}{(e\cos E-1)^2}\mathrm{d}E &= 2\pi\sum_{k=1}^{2}\operatorname*{Res}_{z=z_k} f(z) \\
&= 2\pi\Big(-\frac{1}{e^2}+\frac{-e^2-2\sqrt{1-e^2}+2}{e^2\sqrt{1-e^2}\left(\sqrt{1-e^2}-1\right)^2}\Big) \\
&= 2\pi\frac{1-\sqrt{1-e^2}}{e^2\sqrt{1-e^2}} \\
&= \frac{2\pi L^2}{G(G+L)}.
\end{aligned}
$$

Therefore,

$$
\overline{P}_1 := \frac{1}{2\pi}\int_0^{2\pi} -\frac{\sin^2 E\sin^2 g\left(G^2-K^2\right)}{4L^4(e\cos E-1)^2}\,\mathrm{d}E = -\frac{\sin^2 g\left(G^2-K^2\right)}{4GL^2(G+L)},
$$

$$\overline{P}_2 := \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin^2 E\cos^2 g\left(G^2-K^2\right)}{4L^4(e\cos E-1)^2}\,\mathrm{d}E = \frac{\cos^2 g\left(G^2-K^2\right)}{4GL^2(G+L)}.$$

With the same argument, to calculate the integrals of the first two terms P_4, P_5; we consider the definite integral

$$\int_0^{2\pi} \frac{(\cos E-e)^2}{(e\cos E-1)^2}\mathrm{d}E = \frac{1}{i}\int_{|z|=1} \frac{\left(-2ez+z^2+1\right)^2}{\left(ez^2+e-2z\right)^2}\mathrm{d}z := \frac{1}{i}\int_{|z|=1} f_2(z)\mathrm{d}z.$$

The function $f_2(z)$ has a pole of order one $z_1=0$ and a pole of order two $z_2=(1-\sqrt{1-e^2})/e$. Then by using the Residue theorem we have

$$\begin{aligned}
\int_0^{2\pi} \frac{(\cos E-e)^2}{(e\cos E-1)^2}\mathrm{d}E &= 2\pi\sum_{k=1}^{2}\operatorname*{Res}_{z=z_k} f(z)\\
&= 2\pi\Big(\frac{1}{e^2} - \frac{\left(e^2-1\right)\left(e^2+2\sqrt{1-e^2}-2\right)}{e^2\sqrt{1-e^2}\left(\sqrt{1-e^2}-1\right)^2}\Big)\\
&= 2\pi\frac{1}{\sqrt{1-e^2}+1}\\
&= \frac{2\pi L}{G+L}.
\end{aligned}$$

Therefore,

$$\begin{gathered}
\overline{P}_4 := \frac{1}{2\pi}\int_0^{2\pi} -\frac{\cos^2 g\left(G^2-K^2\right)(\cos E-e)^2}{4G^2L^2(e\cos E-1)^2}\,\mathrm{d}E = -\frac{\cos^2 g\left(G^2-K^2\right)}{4G^2L(G+L)},\\
\overline{P}_5 := \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin^2 g\left(G^2-K^2\right)(\cos E-e)^2}{4G^2L^2(e\cos E-1)^2} = \frac{\sin^2 g\left(G^2-K^2\right)}{4G^2L(G+L)}.
\end{gathered}$$

Since the term P_3 is an odd function with respect to E, we know the mean value of P_3 along a period 2π is zero, that is, $\overline{P}_3=0$. Moreover, the term P_6 does not depend on E, then we have the mean value of P_6 $\overline{P}_6=P_6$.

Finally, we conclude that

$$\begin{aligned}
\overline{H}_{11} :&= \frac{1}{2\pi}\int_0^{2\pi} H_{11}(E,g,L,G,K)\,d\mathrm{E}\\
&= \overline{P}_1+\overline{P}_2+\overline{P}_3+\overline{P}_4+\overline{P}_5+\overline{P}_6\\
&= -\frac{\sin^2 g\left(G^2-K^2\right)}{4GL^2(G+L)} + \frac{\cos^2 g\left(G^2-K^2\right)}{4GL^2(G+L)}\\
&\quad -\frac{\cos^2 g\left(G^2-K^2\right)}{4G^2L(G+L)} + \frac{\sin^2 g\left(G^2-K^2\right)}{4G^2L(G+L)} - \frac{\left(G^2+K^2\right)}{4G^2L^2}\\
&= \frac{\cos(2g)(G-K)(G+K)(G-L)}{4G^2L^2(G+L)} - \frac{\left(G^2+K^2\right)}{4G^2L^2}\\
&= \frac{\cos(2g)(G-K)(G+K)(G-L)-\left(G^2+K^2\right)(G+L)}{4G^2L^2(G+L)}.
\end{aligned}$$

5.3. **Averaged system.** Now we state the averaged system of (4.4) into an explicit expression. Since $\overline{H}_{10}$ is independent of g, the averaged system (5.2) becomes

$$
\begin{aligned}
\frac{\mathrm{d}g}{\mathrm{d}l} &= \epsilon L^3\Big(\frac{\partial \overline{H}_{10}}{\partial G}(g,G;L,K)+\frac{\partial \overline{H}_{11}}{\partial G}(g,G;L,K)\Big) := \epsilon F_1(g,G;h,K_0),\\
\frac{\mathrm{d}G}{\mathrm{d}l} &= -\epsilon L^3\frac{\partial \overline{H}_{11}}{\partial g}(g,G;L,K) := \epsilon F_2(g,G;h,K_0),
\end{aligned}
\tag{5.7}
$$

where $K=K_0$ and $L=1/\sqrt{-2h}$ are considered as constants. Taking the partial derivative of $\overline{H}_{11}$ with respect to the variable g and substituting $L=1/\sqrt{-2h}$, we obtain

$$
F_2(g,G;h,K_0) = -\frac{\sin(2g)\left(1-G\sqrt{-2h}\right)\left(G^2-K_0^2\right)}{2\sqrt{-2h}G^2\left(G\sqrt{-2h}+1\right)}.
$$

Moreover, taking the partial derivative of $\overline{H}_{10}$ with respect to the variable G and substituting $L=1/\sqrt{-2h}$, we obtain

$$
\begin{aligned}
F_1(g,G;h,K_0) = D(G;h) &- \frac{1}{\sqrt{-2h}G^3\left(2G\sqrt{-h}+\sqrt{2}\right)^2}\\
&\times\Big(\cos(2g)\Big(-G^3\sqrt{-2h}+2G^2hK_0^2+G\sqrt{-2h}K_0^2+K_0^2\Big)\\
&+K_0^2\Big(2G^2h-2G\sqrt{-2h}-1\Big)\Big),
\end{aligned}
$$

where for $p>1$,

$$
\begin{aligned}
D(G;h) =& L^3\frac{\partial \overline{H}_{10}}{\partial G}(g,G;L,K)\\
=& -\beta\sqrt{-2h}^{2p-3}\sum_{m=0}^{p-2}C_{p-2}^m 2^{-m}\frac{(m+p-2)!}{m!(p-2)!}\frac{1}{G}\\
&\times\Big(-m\Big(\frac{1}{G\sqrt{-2h}}-1\Big)^{m-1}\big(G\sqrt{-2h}\big)^{-p}\\
&+(1-p)\Big(\frac{1}{G\sqrt{-2h}}-1\Big)^{m}\Big(G\sqrt{-2h}\Big)^{1-p}\Big)\\
=& -\beta\sqrt{-2h}^{2p-3}\sum_{m=0}^{p-2}C_{p-2}^m 2^{-m}\frac{(m+p-2)!}{m!(p-2)!}\frac{1}{G}\\
&\times\Big((1-p)(1-G\sqrt{-2h})-m\Big)\Big(\frac{1}{G\sqrt{-2h}}-1\Big)^{m-1}\big(G\sqrt{-2h}\big)^{-p}\\
=& -\beta\sqrt{-2h}^{p-3}G^{-p-1}\sum_{m=0}^{p-2}C_{p-2}^m 2^{-m}\frac{(m+p-2)!}{m!(p-2)!}\\
&\times\Big((1-p)(1-G\sqrt{-2h})-m\Big)\Big(\frac{1-G\sqrt{-2h}}{G\sqrt{-2h}}\Big)^{m-1}\\
:=& D_1(G;h,p),
\end{aligned}
$$

and for $p\le 1$,

$$
D(G;h) = L^3\frac{\partial \overline{H}_{10}}{\partial G}(g,G;L,K)
$$

$$
\begin{aligned}
&= -\beta 2^{p-1}\sqrt{-2h}^{2p-2}\sum_{m=0}^{1-p} C_{1-p}^{m}\frac{(1-p)!}{m!(-m-p+1)!}\Big(\frac{1-G\sqrt{-2h}}{1+G\sqrt{-2h}}\Big)^{m-1}\\
&\quad\times\Big(G\sqrt{-2h}+1\Big)^{-p-1}\Big((-2m-p+1)-(1-p)G\sqrt{-2h}\Big)\\
&= -\beta 2^{p-1}\sqrt{-2h}^{2p-2}\Big(G\sqrt{-2h}+1\Big)^{-p-1}\sum_{m=0}^{1-p} C_{1-p}^{m}\frac{(1-p)!}{m!(-m-p+1)!}\\
&\quad\times\Big((-2m-p+1)-(1-p)G\sqrt{-2h}\Big)\Big(\frac{1-G\sqrt{-2h}}{1+G\sqrt{-2h}}\Big)^{m-1}\\
&= -\beta 2^{p-1}\sqrt{-2h}^{2p-2}\Big(G\sqrt{-2h}+1\Big)^{-p-1}\sum_{m=0}^{1-p} C_{1-p}^{m}\frac{(1-p)!}{m!(-m-p+1)!}\\
&\quad\times\Big((1-p)(1-G\sqrt{-2h})-2m\Big)\Big(\frac{1-G\sqrt{-2h}}{1+G\sqrt{-2h}}\Big)^{m-1}\\
&:= D_2(G;h,p).
\end{aligned}
$$

6. Proof of main results

In this section, we give the proofs of Theorems 1.1 and 1.2. The proofs are based on the method of averaging; For example, see [25, 28, 26, 27]. Generally, averaging method involves two steps: transforming to standard form; solving the averaging equation. Now we compute the equilibrium points of the averaged system (5.7). From the second equation of the averaged system (5.7), it is easy to have that $\sin(2g)\left(1-G\sqrt{-2h}\right)\left(G^2-K_0^2\right)=0$.

Proof of Theorem 1.1. In this subsection, we always assume that $p>1$. We separate the computation of equilibrium points in three cases.

Case I: $\left(1-G\sqrt{-2h}\right)=0$**.** This means that $G^*=\frac{1}{\sqrt{-2h}}$. The equilibrium points must satisfy the first equation of the averaged system (5.7). Inserting G^* back into $F_1(g,G;h,K_0)=0$, we obtain

$$
F_1\Big(g,\frac{1}{\sqrt{-2h}};h,K_0\Big)=\beta 2^{p-2}p(p-1)(-h)^{p-1}+\frac{1}{8}\cos(2g)\left(2hK_0^2+1\right)-hK_0^2=0.
$$

Therefore,

$$
\cos(2g)=\frac{8\big(-\beta 2^{p-2}p(p-1)(-h)^{p-1}+hK_0^2\big)}{2hK_0^2+1}
$$

which possesses the solutions

$$
g^*=\pm\frac{1}{2}\arccos\Big(\frac{8\left(-\beta 2^{p-2}p(p-1)(-h)^{p-1}+hK_0^2\right)}{2hK_0^2+1}\Big),
$$

which exist when

$$
\frac{8\left(-\beta 2^{p-2}p(p-1)(-h)^{p-1}+hK_0^2\right)}{2hK_0^2+1}\in[-1,1].
$$

Therefore, we have two solutions

$$
(g^*,G^*;h,K_0)=\Big(\pm\frac{1}{2}\arccos\Big(\frac{8\left(-\beta 2^{p-2}p(p-1)(-h)^{p-1}+hK_0^2\right)}{2hK_0^2+1}\Big),\frac{1}{\sqrt{-2h}}\Big).
$$

The Jacobian determinant of the averaged equations at these two solutions is

$$\Big|\frac{\partial(F_1,F_2)}{\partial(g,G)}\Big|_{(g^*,G^*)} = \frac{1}{16}\left(2hK_0^2+1\right)^2\Big(1-\frac{4\left(\beta 2^p p(p-1)(-h)^p+4h^2K_0^2\right)^2}{h^2\left(2hK_0^2+1\right)^2}\Big).$$

Only when

$$K_0 \neq \frac{1}{\sqrt{-2h}}, \quad |\beta| \neq \frac{8hK_0{}^2+\left|2hK_0{}^2+1\right|}{2^{p+1}p(p-1)|h|^{p-1}},$$

we can get the nonzero Jacobian determinant.

There exists two 2π-periodic solutions $(g(l,\epsilon),G(l,\epsilon))=(g^*+O(\epsilon),G^*+O(\epsilon))$ of the system (4.4). In order for $k(l)$ to be 2π-periodic, we plug $(g(l,\epsilon),G(l,\epsilon))$ back into the second equation of the system (4.2), and obtain

$$\frac{\mathrm{d}k}{\mathrm{d}l} = \epsilon L^3\frac{\partial H_1}{\partial K}(l,g(l,\epsilon),L,G(l,\epsilon),K)+O\left(\epsilon^2\right).$$

Next, we prove that there exists $K=K_0$ such that

$$F_3(K_0,\epsilon) := \int_0^{2\pi} L^3\frac{\partial H_1}{\partial K}(l,g(l,\epsilon),L,G(l,\epsilon),K_0)\mathrm{d}l+O(\epsilon)=0.$$

Since the H_1 and $\partial H_1/\partial K$ are continuous, and we obtain

$$\begin{aligned}F_3(K_0,\epsilon) &= \int_0^{2\pi} L^3\frac{\partial H_1}{\partial K}(l,g(l,\epsilon),L,G(l,\epsilon),K_0)\mathrm{d}l+O(\epsilon)\\ &=L^3\frac{\partial}{\partial K}\int_0^{2\pi} H_1(l,g(l,\epsilon),L,G(l,\epsilon),K_0)\mathrm{d}l+O(\epsilon).\end{aligned}$$

Substituting $L=1/\sqrt{-2h}$ and $(g(l,\epsilon),G(l,\epsilon))=(g^*+O(\epsilon),G^*+O(\epsilon))$, we obtain

$$\begin{aligned}F_3(K_0,\epsilon) &= -\frac{\pi K_0\left(-\cos(2g^*)\left(1-G^*\sqrt{-2h}\right)+G^*\sqrt{-2h}+1\right)}{\sqrt{-2h}(G^*)^2\left(G^*\sqrt{-2h}+1\right)}+O(\epsilon)\\ &= -\sqrt{-2h}\pi K_0+O(\epsilon).\end{aligned}$$

Hence there exists $(K_0,\epsilon)=(0,0)$ such that

$$F_3(0,0)=0, \quad \frac{\mathrm{d}}{\mathrm{d}K_0}F_3(K_0,\epsilon)|_{(0,0)}\neq 0.$$

Using implicit function theorem we obtain that there exists $K_0=K_0(\epsilon)$ such that $F_3(K_0(\epsilon),\epsilon)=0$. Therefore, there exists $K=K_0(\epsilon)$ such that $k(l)$ is 2π-periodic. We plug $K_0=0$ back into the two solutions $(g^*,G^*;h,K_0)$, and obtain

$$(g^*,G^*;h)=\Big(\pm\frac{1}{2}\arccos\left(-\beta 2^{p+1}p(p-1)(-h)^{p-1}\right),\frac{1}{\sqrt{-2h}}\Big)$$

with

$$|\beta| < \frac{1}{2^{p+1}p(p-1)|h|^{p-1}}.$$

Then, statement (i) of Theorem 1.1 is proved.

Case II: $\left(G^2-K_0^2\right)=0$. This equation have two solutions $G^*=\pm K_0$. Because the equilibrium points must satisfy the first equation of the averaged system (5.7). Inserting G^* back into $F_1(g,G;h,K_0)=0$, we obtain

$$F_1(g,\pm K_0;h,K_0)=D_1(\pm K_0;h,p)+\frac{1}{2\sqrt{-2h}K_0}\mp\frac{\cos(2g)\left(1-\sqrt{-2h}K_0\right)}{2\sqrt{-2h}K_0\left(1+\sqrt{-2h}K_0\right)}=0.$$

Because we consider the existence of periodic orbits of the system, when K_0 near to origin. $K_0 = 0$ is the singular point of function $F_1(g, \pm K_0; h, K_0)$, so this case can not be discussed.

Case III: $\sin(2g) = 0$**.** This equation has four solutions $g^* = 0, \pm\frac{\pi}{2}, \pi$. Next we consider four subcases.

Subcase A: $g^* = 0$**.** The equilibrium points must satisfy the first equation of the averaged system (5.7). Inserting g^* back into $F_1(g, G; h, K_0) = 0$, we obtain

$$\begin{aligned} F_1(0, G; h, K_0) =& D_1(G; h, p) - \frac{1}{\sqrt{-2h}G^3\left(2G\sqrt{-h}+\sqrt{2}\right)^2} \\ &\times \Big(\Big(- G^3\sqrt{-2h} + 2G^2hK_0^2 + G\sqrt{-2h}K_0^2 + K_0^2\Big) \\ &+ K_0^2\big(2G^2h - 2G\sqrt{-2h} - 1\big)\Big) = 0. \end{aligned} \tag{6.1}$$

We restrict K_0 to a sufficiently small neighborhood of the origin, and obtain

$$\begin{aligned} F_1(0, G; h, K_0) =& D_1(G; h, p) + \frac{1}{2\left(G\sqrt{-2h}+1\right)^2} + O(K_0) \\ =& -\beta\sqrt{-2h}^{p-3}G^{-p-1}\sum_{m=0}^{p-2} C_{p-2}^m 2^{-m}\frac{(m+p-2)!}{m!(p-2)!} \\ &\times \Big((1-p)(1-G\sqrt{-2h}) - m\Big)\Big(\frac{1-G\sqrt{-2h}}{G\sqrt{-2h}}\Big)^{m-1} \\ &+ \frac{1}{2\left(G\sqrt{-2h}+1\right)^2} + O(K_0) \\ =& -\beta\sqrt{-2h}^{p-2}G^{-p}\Big((1-p) + \frac{(p-1)(p-2)}{2} \\ &\times \Big(-p\frac{1-G\sqrt{-2h}}{G\sqrt{-2h}} - 1\Big)\sum_{m=2}^{p-2} C_{p-2}^m 2^{-m}\frac{(m+p-2)!}{m!(p-2)!} \\ &\times \Big((1-p)(1-G\sqrt{-2h}) - m\Big)\Big(\frac{1-G\sqrt{-2h}}{G\sqrt{-2h}}\Big)^{m-1}\Big) \\ &+ \frac{1}{2\left(G\sqrt{-2h}+1\right)^2} + O(K_0). \end{aligned} \tag{6.2}$$

From this equality, we obtain that $F_1(0, G; h, K_0)$ have two singularities $G = -\frac{1}{\sqrt{-2h}}, 0$. Therefore $G \in \big(-1/\sqrt{-2h}, 0\big) \cup \big(0, 1/\sqrt{-2h}\big)$.

When $G \to -1/\sqrt{-2h}$, we obtain $F_1(0, G; h, K_0) \to +\infty$. When $G \to 0$ and $\beta > 0$, we obtain $F_1(0, G; h, K_0) \to +\infty$, while if $\beta < 0$ we obtain $F_1(0, G; h, K_0) \to -\infty$. When $G \to 1/\sqrt{-2h}$ and

$$\beta > -\frac{1}{2^{p+1}(-h)^{p-1}p(p-1)},$$

we have $F_1(0, G; h, K_0) > 0$, while if

$$\beta < -\frac{1}{2^{p+1}(-h)^{p-1}p(p-1)}$$

we have $F_1(0, G; h, K_0) < 0$.

Using zero point theorem, we obtain that when

$$\beta \in \Big(-\frac{1}{2^{p+1}(-h)^{p-1}p(p-1)}, 0 \Big),$$

there exist two solutions $\widetilde{G}_1(K_0,h) \in (-1/\sqrt{-2h}, 0)$ and $\widetilde{G}_2(K_0,h) \in (0, 1/\sqrt{-2h})$ of (6.1). Therefore, when

$$\beta \in \Big(-\frac{1}{2^{p+1}(-h)^{p-1}p(p-1)}, 0 \Big),$$

we have two solutions $(g^*, G^{*(1)}; K_0, h) = (0, \widetilde{G}_1(K_0,h))$ and $(g^*, G^{*(2)}; K_0, h) = (0, \widetilde{G}_2(K_0,h))$.

Next, we calculate the Jacobian determinant of the averaged equation at these solutions. Firstly, we consider the derivative of $D_1(G;h,p)$ with respect to G,

$$\begin{aligned}
D_1'(G;h,p) =& \frac{\mathrm{d}}{\mathrm{d}G} D_1(G;h,p) \\
=& -\beta \sqrt{-2h}^{p-4} G^{-p-3} \sum_{m=0}^{p-2} 2^{-m} C_{p-2}^{m} \frac{(m+p-2)!}{m!(p-2)!} \Big(\frac{1-G\sqrt{-2h}}{G\sqrt{-2h}} \Big)^{m-2} \\
& \times \Big(\Big((1-p)\big(1-G\sqrt{-2h}\big) - m \Big) \Big((-1-p)\big(1-G\sqrt{-2h}\big) + 1 - m \Big) \\
& + G(p-1)\sqrt{-2h}\big(1-G\sqrt{-2h}\big) \Big).
\end{aligned}$$

Therefore we can obtain the derivative of $F_1(g,G;h,K_0)$ with respect to G. Then we plug $(g^*, G^{*(i)}; K_0, h) = (0, \widetilde{G}_i(K_0,h))$, $i=1,2$ back into the derivative, and obtain

$$\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}G} F_1(g,G;h,K_0)\big|_{(g^*,G^{*(i)})} \\
& = D_1'(\widetilde{G}_i;h,p) - \frac{\big(\sqrt{-2h}\widetilde{G}_i^3 - 6\widetilde{G}_i^2 h {K_0}^2 + 3\sqrt{-2h}\widetilde{G}_i {K_0}^2 + {K_0}^2\big)}{\widetilde{G}_i^3 \big(\sqrt{-2h}\widetilde{G}_i + 1\big)^3} \\
& = \Big(D_1'(\widetilde{G}_i;h,p) \widetilde{G}_i^3 \big(\sqrt{-2h}\widetilde{G}_i + 1\big)^3 \\
& \quad - \big(\sqrt{-2h}\widetilde{G}_i^3 - 6\widetilde{G}_i^2 h {K_0}^2 + 3\sqrt{-2h}\widetilde{G}_i {K_0}^2 + {K_0}^2\big) \Big) \Big/ \Big(\widetilde{G}_i^3 \big(\sqrt{-2h}\widetilde{G}_i + 1\big)^3 \Big).
\end{aligned}$$

The Jacobian determinant of the averaged equations with these solutions are

$$\begin{aligned}
& \Big| \frac{\partial(F_1,F_2)}{\partial(g,G)} \Big|_{(g^*,G^{*(i)})} \\
& = -\frac{\mathrm{d}}{\mathrm{d}G} F_1(g,G;h,K_0)\big|_{(g^*,G^{*(i)})} \cdot \frac{\mathrm{d}}{\mathrm{d}g} F_2(g,G;h,K_0)\big|_{(g^*,G^{*(i)})} \\
& = \frac{\big(\sqrt{-2h}\widetilde{G}_i - 1\big)\big(\widetilde{G}_i^2 - {K_0}^2\big)}{\sqrt{-2h}\widetilde{G}_i^2 \big(1 + \sqrt{-2h}\widetilde{G}_i\big)} \Big(\frac{\mathrm{d}}{\mathrm{d}G} F_1(g,G;h,K_0)\big|_{(g^*,G^{*(i)})} \Big).
\end{aligned}$$

We restrict K_0 to a sufficiently small neighborhood of the origin, and obtain

$$\begin{aligned}
&\Big|\frac{\partial(F_1,F_2)}{\partial(g,G)}\Big|_{(g^*,G^{*(i)})}\\
&= \frac{\big(\sqrt{-2h}\widetilde{G}_i-1\big)\widetilde{G}_i^2}{\sqrt{-2h}\widetilde{G}_i^2\big(1+\sqrt{-2h}\widetilde{G}_i\big)}\Big(\frac{\mathrm{d}}{\mathrm{d}G}F_1(g,G;h)|_{(g^*,G^{*(i)})}\Big)+O(K_0)\\
&= \frac{\big(\sqrt{-2h}\widetilde{G}_i-1\big)}{\sqrt{-2h}\big(1+\sqrt{-2h}\widetilde{G}_i\big)}\frac{\big(D_1'(\widetilde{G}_i;h,p)\big(\sqrt{-2h}\widetilde{G}_i+1\big)^3-\sqrt{-2h}\big)}{\big(\sqrt{-2h}\widetilde{G}_i+1\big)^3}+O(K_0)\\
&= \frac{\big(\sqrt{-2h}\widetilde{G}_i-1\big)\big(D_1'(\widetilde{G}_i;h,p)\big(\sqrt{-2h}\widetilde{G}_i+1\big)^3-\sqrt{-2h}\big)}{\sqrt{-2h}\big(\sqrt{-2h}\widetilde{G}_i+1\big)^4}+O(K_0).
\end{aligned} \tag{6.3}$$

Because $\widetilde{G}_i \neq \pm 1/\sqrt{-2h}$, to get a nonzero Jacobian determinant, expression (6.3) needs to satisfy

$$\Big(D_1'(\widetilde{G}_i;h,p)\big(\sqrt{-2h}\widetilde{G}_i+1\big)^3-\sqrt{-2h}\Big)\neq 0.$$

Because $\widetilde{G}_i$ is the nondegenerate zero of (6.2) and

$$\frac{\mathrm{d}}{\mathrm{d}G}\Big(D_1(G;h,p)+\frac{1}{2\big(G\sqrt{-2h}+1\big)^2}\Big)=D_1'(G;h,p)-\frac{\sqrt{-2h}}{\big(G\sqrt{-2h.}+1\big)^3},$$

we have

$$\Big(D_1'(\widetilde{G}_i;h,p)\big(\sqrt{-2h}\widetilde{G}_i+1\big)^3-\sqrt{-2h}\Big)=D_1'(\widetilde{G}_i;h,p)-\frac{\sqrt{-2h}}{\big(\widetilde{G}_i\sqrt{-2h}+1\big)^3}\neq 0.$$

Then there exists 2π-periodic solutions $(g(l,\epsilon),G(l,\epsilon))=\big(g^*+O(\epsilon),G^{*(i)}+O(\epsilon)\big)$, $i=1,2$ of the system (4.4). Similarly, in order for $k(l)$ to be 2π-periodic, we plug $(g(l,\epsilon),G(l,\epsilon))$ back into $F_3(K_0,\epsilon)$, and obtain

$$\begin{aligned}
F_3(K_0,\epsilon) &= -\frac{\pi K_0\left(-\cos(2g^*)\left(\sqrt{2}-2G^{*(i)}\sqrt{-h}\right)+2G^{*(i)}\sqrt{-h}+\sqrt{2}\right)}{\sqrt{-2h}(G^{*(i)})^2\left(2G^{*(i)}\sqrt{-h}+\sqrt{2}\right)}+O(\epsilon)\\
&= -\frac{2\pi K_0}{\widetilde{G}_i\big(\widetilde{G}_i\sqrt{-2h}+1\big)}+O(\epsilon).
\end{aligned}$$

There exists $(K_0,\epsilon)=(0,0)$, such that

$$F_3(0,0)=0,\quad \frac{\mathrm{d}}{\mathrm{d}K_0}F_3(K_0,\epsilon)|_{(0,0)}\neq 0.$$

Using implicit function theorem we obtain that there exists $K_0=K_0(\epsilon)$, such that $F_3(K_0(\epsilon),\epsilon)=0$. Therefore there exists $K=K_0(\epsilon)$ such that $k(l)$ is 2π-periodic.

Then, statement (ii) of Theorem 1.1 is proved when $g^*=0$.

Subcase B: $g^*=\pi$. The proof of statement (ii) of Theorem 1.1 when $g^*=\pi$ is completely similar to case $g^*=0$.

Subcase C: $g^*=\frac{\pi}{2}$. The equilibrium points must satisfy the first equation of the averaged system (5.7). Inserting g^* back into $F_1(g,G;h,K_0)=0$, we obtain

$$\begin{aligned}F_1\big(\frac{\pi}{2},G;h,K_0\big)=&D_1(G;h,p)+\frac{1}{\sqrt{-2h}G^3\left(2G\sqrt{-h}+\sqrt{2}\right)^2}\\&\times\Big(\Big(-G^3\sqrt{-2h}+2G^2hK_0^2+G\sqrt{-2h}K_0^2+K_0^2\Big)\\&-K_0^2\big(2G^2h-2G\sqrt{-2h}-1\big)\Big)=0.\end{aligned}\tag{6.4}$$

We restrict K_0 to a sufficiently small neighborhood of the origin, and obtain

$$\begin{aligned}&F_1\big(\frac{\pi}{2},G;h,K_0\big)\\&=D_1(G;h,p)-\frac{1}{2\left(G\sqrt{-2h}+1\right)^2}+O(K_0)\\&=-\beta\sqrt{-2h}^{p-3}G^{-p-1}\sum_{m=0}^{p-2}C_{p-2}^m2^{-m}\frac{(m+p-2)!}{m!(p-2)!}\\&\quad\times\Big((1-p)(1-G\sqrt{-2h})-m\Big)\Big(\frac{1-G\sqrt{-2h}}{G\sqrt{-2h}}\Big)^{m-1}\\&\quad-\frac{1}{2\big(G\sqrt{-2h}+1\big)^2}+O(K_0)\\&=-\beta\sqrt{-2h}^{p-2}G^{-p}\Big((1-p)+\frac{(p-1)(p-2)}{2}\Big(-p\frac{1-G\sqrt{-2h}}{G\sqrt{-2h}}-1\Big)\\&\quad+\sum_{m=2}^{p-2}C_{p-2}^m2^{-m}\frac{(m+p-2)!}{m!(p-2)!}\Big((1-p)(1-G\sqrt{-2h})-m\Big)\\&\quad\times\Big(\frac{1-G\sqrt{-2h}}{G\sqrt{-2h}}\Big)^{m-1}\Big)-\frac{1}{2\big(G\sqrt{-2h}+1\big)^2}+O(K_0).\end{aligned}\tag{6.5}$$

From this equality, we obtain that $F_1(\frac{\pi}{2},G;h,K_0)$ have two singularities $G=-1/\sqrt{-2h},0$. Therefore $G\in(-1/\sqrt{-2h},0)\cup(0,1/\sqrt{-2h})$.

When $G\to-1/\sqrt{-2h}$, we obtain $F_1(\pi/2,G;h,K_0)\to-\infty$. When $G\to0$ and $\beta>0$, we obtain $F_1(\pi/2,G;h,K_0)\to+\infty$. When $G\to0$ and $\beta<0$, we obtain $F_1(\pi/2,G;h,K_0)\to-\infty$. When $G\to1/\sqrt{-2h}$, and

$$\beta>\frac{1}{2^{p+1}(-h)^{p-1}p(p-1)},$$

we obtain $F_1(\pi/2,G;h,K_0)>0$.

Similarity, when $G\to1/\sqrt{-2h}$, and

$$\beta<\frac{1}{2^{p+1}(-h)^{p-1}p(p-1)}$$

we obtain $F_1(\pi/2,G;h,K_0)<0$.

Using the zero point theorem, we obtain that when

$$\beta\in\Big(0,\frac{1}{2^{p+1}(-h)^{p-1}p(p-1)}\Big),$$

there exist two solutions $\overline{G}_1(K_0,h)\in(-1/\sqrt{-2h},0)$ and $\overline{G}_2(K_0,h)\in(0,1/\sqrt{-2h})$ of (6.4). Therefore, when

$$\beta\in\Big(0,\frac{1}{2^{p+1}(-h)^{p-1}p(p-1)}\Big),$$

we have two solutions

$$\begin{aligned}
(g^*,G^{*(1)};K_0,h)&=\Big(\frac{\pi}{2},\overline{G}_1(K_0,h)\Big),\\
(g^*,G^{*(2)};K_0,h)&=\Big(\frac{\pi}{2},\overline{G}_2(K_0,h)\Big).
\end{aligned}$$

Next, we calculate the Jacobian determinant of the averaged equation with these solutions. Firstly, we consider the derivative of $F_1(g,G;h,K_0)$ with respect to G. Then we insert

$$(g^*,G^{*(i)};h,K_0)=\Big(\frac{\pi}{2},\overline{G}_i(h,K_0)\Big),\quad i=1,2$$

back into the derivative, and obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}G}F_1(g,G;h,K_0)\big|_{(g^*,G^{*(i)})}\\
&=D_1'(\overline{G}_i;h,p)+\frac{-2h\overline{G}_i^4+12h\overline{G}_i^2{K_0}^2-\overline{G}_i8\sqrt{-2h}{K_0}^2-3{K_0}^2}{\overline{G}_i^4\sqrt{-2h}\left(\overline{G}_i\sqrt{-2h}+1\right)^3}\\
&=\Big(D_1'(\overline{G}_i;h,p)\overline{G}_i^4\sqrt{-2h}\left(\overline{G}_i\sqrt{-2h}+1\right)^3\\
&\quad+\Big(-2h\overline{G}_i^4+12h\overline{G}_i^2{K_0}^2-\overline{G}_i8\sqrt{-2h}{K_0}^2-3{K_0}^2\Big)\Big)\\
&\quad\div\Big(\overline{G}_i^4\sqrt{-2h}\left(\overline{G}_i\sqrt{-2h}+1\right)^3\Big).
\end{aligned}$$

The Jacobian determinant of the averaged equations with these solutions are

$$\begin{aligned}
&\Big|\frac{\partial(F_1,F_2)}{\partial(g,G)}\Big|_{(g^*,G^{*(i)})}\\
&=-\frac{d}{dG}F_1(g,G;h,K_0)\big|_{(g^*,G^{*(i)})}\cdot\frac{d}{dg}F_2(g,G;h,K_0)\big|_{(g^*,G^{*(i)})}\\
&=\frac{\left(\overline{G}_i\sqrt{-2h}-1\right)\left(\overline{G}_i^2-{K_0}^2\right)}{-\overline{G}_i^2\sqrt{-2h}\left(1+\overline{G}_i\sqrt{-2h}\right)}\Big(\frac{d}{dG}F_1(g,G;h,K_0)\big|_{(g^*,G^{*(i)})}\Big)
\end{aligned}$$

We restrict K_0 to a sufficiently small neighborhood of the origin, and obtain

$$
\begin{aligned}
&\Big|\frac{\partial(F_1,F_2)}{\partial(g,G)}\Big|_{(g^*,G^{*(i)})}\\
&= \frac{\big(\overline{G}_i\sqrt{-2h}-1\big)\,\overline{G}_i^2}{-\overline{G}_i^2\sqrt{-2h}\big(1+\sqrt{-2h}\overline{G}_i\big)}\Big(\frac{d}{dG}F_1(g,G;h)\big|_{(g^*,G^{*(i)})}\Big)+O(K_0)\\
&= \frac{\big(\overline{G}_i\sqrt{-2h}-1\big)}{-\sqrt{-2h}\,\big(1+\overline{G}_i\sqrt{-2h}\big)}\frac{\big(D_1'(\overline{G}_i;h,p)\,\big(\overline{G}_i\sqrt{-2h}+1\big)^3+\sqrt{-2h}\big)}{\big(\overline{G}_i\sqrt{-2h}+1\big)^3}\\
&\quad +O(K_0)\\
&= \frac{\big(\overline{G}_i\sqrt{-2h}-1\big)\big(D_1'(\overline{G}_i;h,p)\,\big(\overline{G}_i\sqrt{-2h}+1\big)^3+\sqrt{-2h}\big)}{-\sqrt{-2h}\,\big(\overline{G}_i\sqrt{-2h}+1\big)^4}+O(K_0).
\end{aligned}
\tag{6.6}
$$

Because $\overline{G}_i \neq \pm 1/\sqrt{-2h}$, to obtain the nonzero Jacobian determinant, (6.6) needs to satisfy

$$
\Big(D_1'(\overline{G}_i;h)\big(\overline{G}_i\sqrt{-2h}+1\big)^3+\sqrt{-2h}\Big)\neq 0.
$$

Because $\overline{G}_i$ is the nondegenerate zero of (6.5) and

$$
\frac{d}{dG}\Big(D_1(G;h,p)-\frac{1}{2\,\big(G\sqrt{-2h}+1\big)^2}\Big)=D_1'(G;h,p)+\frac{\sqrt{-2h}}{\big(G\sqrt{-2h}+1\big)^3},
$$

we have

$$
\Big(D_1'(\overline{G}_i;h,p)\big(\sqrt{-2h}\overline{G}_i+1\big)^3+\sqrt{-2h}\Big)=D_1'(\overline{G}_i;h,p)+\frac{\sqrt{-2h}}{\big(\overline{G}_i\sqrt{-2h}+1\big)^3}\neq 0.
$$

There exists 2π-periodic solutions $(g(l,\epsilon),G(l,\epsilon))=\big(g^*+O(\epsilon),G^{*(i)}+O(\epsilon)\big)$, $i=1,2$ of the system (4.4). Similarly, in order for $k(l)$ to be 2π-periodic, we plug $(g(l,\epsilon),G(l,\epsilon))$ back into $F_3(K_0,\epsilon)$ to obtain

$$
\begin{aligned}
F_3(K_0,\epsilon)&=-\frac{\pi K_0\,\big(-\cos(2g^*)\,\big(\sqrt{2}-2G^{*(i)}\sqrt{-h}\big)+2G^{*(i)}\sqrt{-h}+\sqrt{2}\big)}{\sqrt{-2h}(G^{*(i)})^2\,\big(2G^{*(i)}\sqrt{-h}+\sqrt{2}\big)}+O(\epsilon)\\
&=\frac{2\pi K_0}{\overline{G}_i\,\big(\overline{G}_i\sqrt{-2h}+1\big)}+O(\epsilon).
\end{aligned}
$$

There exists $(K_0,\epsilon)=(0,0)$ such that

$$
F_3(0,0)=0,\quad \frac{\mathrm{d}}{\mathrm{d}K_0}F_3(K_0,\epsilon)|_{(0,0)}\neq 0.
$$

Using the implicit function theorem we obtain that there exists $K_0=K_0(\epsilon)$, such that $F_3(K_0(\epsilon),\epsilon)=0$. Therefore there exists $K=K_0(\epsilon)$ such that $k(l)$ is 2π-periodic. Then, statement (iii) of Theorem 1.1 is proved when $g^*=\pi/2$.

Subcase D: $g^*=-\frac{\pi}{2}$. The proof of statement (iii) of Theorem 1.1 with $g^*=-\pi/2$ is completely similar to that of case $g^*=\pi/2$.

6.1. **Proof of Theorem 1.2.** In this subsection, we always assume that $p \le 1$. We separate the computation of the equilibrium points in three cases.

Case I: $\left(1 - G\sqrt{-2h}\right) = 0$. This means that $G^* = 1/\sqrt{-2h}$. The equilibrium points must satisfy the first equation of the averaged system (5.7). Inserting G^* back into $F_1(g, G; h, K_0) = 0$, we obtain

$$F_1\Big(g, \frac{1}{\sqrt{-2h}}; h, K_0\Big) = -\beta 2^{p-2}(1-p)p(-h)^{p-1} + \frac{1}{8}\cos(2g)\left(2h{K_0}^2 + 1\right) - h{K_0}^2 = 0.$$

Therefore,

$$\cos(2g) = \frac{8\left(h{K_0}^2 + \beta 2^{p-2}(1-p)p(-h)^{p-1}\right)}{2h{K_0}^2 + 1}.$$

It possesses the solutions

$$g^* = \pm\frac{1}{2}\arccos\Big(\frac{8\left(h{K_0}^2 + \beta 2^{p-2}(1-p)p(-h)^{p-1}\right)}{2h{K_0}^2 + 1}\Big),$$

which exist when

$$\frac{8\left(h{K_0}^2 + \beta 2^{p-2}(1-p)p(-h)^{p-1}\right)}{2h{K_0}^2 + 1} \in [-1, 1].$$

Therefore, we have two solutions:

$$(g^*, G^*; h, K_0) = \Big(\pm\frac{1}{2}\arccos\Big(\frac{8\left(hK^2 + \beta 2^{p-2}(1-p)p(-h)^{p-1}\right)}{2hK^2 + 1}\Big), \frac{1}{\sqrt{-2h}}\Big).$$

The Jacobian determinant of the averaged equations at these two solutions is

$$\Big|\frac{\partial(F_1, F_2)}{\partial(g, G)}\Big|_{(g^*, G^*)} = \frac{1}{16}\left(2hK_0^2 + 1\right)^2\Big(1 - \frac{4\left(4h^2K_0^2 + \beta 2^p(p-1)p(-h)^p\right)^2}{h^2\left(2hK_0^2 + 1\right)^2}\Big).$$

Only when

$$K_0 \ne \frac{1}{\sqrt{-2h}}, \quad |\beta| \ne \frac{8h{K_0}^2 + \left|2h{K_0}^2 + 1\right|}{2^{p+1}(p-1)p|h|^{p-1}}$$

we can get the nonzero Jacobian determinant.

There exist two 2π-periodic solutions $(g(l,\epsilon), G(l,\epsilon)) = (g^* + O(\epsilon), G^* + O(\epsilon))$ of system (4.4). Similarly, in order for $k(l)$ to be 2π-periodic, we plug $(g(l,\epsilon), G(l,\epsilon))$ back into $F_3(K,\epsilon)$, and obtain

$$\begin{aligned} F_3(K_0, \epsilon) &= -\frac{\pi K_0\left(-\cos(2g^*)\left(\sqrt{2} - 2G^*\sqrt{-h}\right) + 2G^*\sqrt{-h} + \sqrt{2}\right)}{\sqrt{-2h}(G^*)^2\left(2G^*\sqrt{-h} + \sqrt{2}\right)} + O(\epsilon) \\ &= -\sqrt{-2h}\pi K_0 + O(\epsilon). \end{aligned}$$

There exists $(K_0, \epsilon) = (0, 0)$ such that

$$F_3(0,0) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}K_0}F_3(K_0,\epsilon)|_{(0,0)} \ne 0.$$

Using implicit function theorem we obtain that there exists $K_0 = K_0(\epsilon)$ such that $F_3(K_0(\epsilon), \epsilon) = 0$. Therefore there exists $K_0 = K_0(\epsilon)$ such that $k(l)$ is 2π-periodic. We plug $K_0 = 0$ back into the two solutions $(g^*, G^*; h, K_0)$ and obtain

$$(g^*, G^*; h) = \Big(\pm\frac{1}{2}\arccos\left(\beta 2^{p+1}(1-p)p(-h)^{p-1}\right), \frac{1}{\sqrt{-2h}}\Big),$$

with

$$|\beta| < \frac{1}{2^{p+1}(p-1)p|h|^{p-1}}.$$

Then, statement (i) of Theorem 1.2 is proved.

Case II: $\left(G^2 - K_0^2\right) = 0$**.** This equation have two solutions $G^* = \pm K_0$. Due to the equilibrium points must satisfy the first equation of the averaged system (5.7). Inserting G^* back into $F_1(g, G; h, K_0) = 0$, we obtain

$$F_1(g, \pm K_0; h, K_0) = D_2(\pm K_0; h, p) + \frac{1}{2\sqrt{-2h}K_0} \mp \frac{\cos(2g)\left(1 - \sqrt{-2h}K_0\right)}{2\sqrt{-2h}K_0\left(1 + \sqrt{-2h}K_0\right)} = 0.$$

We consider the existence of periodic orbits of the system for the case that K_0 is near the origin. Since $K_0 = 0$ is the singular point of function $F_1(g, \pm K_0; h, K_0)$, this case is not feasible.

Case III: $\sin(2g) = 0$**.** We have four solutions $g^* = 0, \pm\frac{\pi}{2}, \pi$. Next consider four subcases.

Subcase A: $g^* = 0$**.** The equilibrium points must satisfy the first equation of the averaged system (5.7). Inserting g^* back into $F_1(g, G; h, K_0) = 0$, we obtain

$$\begin{aligned} F_1(0, G; h, K_0) =& D_2(G; h, p) - \frac{1}{\sqrt{-2h}G^3\left(2G\sqrt{-h} + \sqrt{2}\right)^2} \\ &\times \Big(\Big(- G^3\sqrt{-2h} + 2G^2hK_0^2 + G\sqrt{-2h}K_0^2 + K_0^2\Big) \\ &+ K_0^2\Big(2G^2h - 2G\sqrt{-2h} - 1\Big)\Big) = 0. \end{aligned} \tag{6.7}$$

We restrict K_0 to a sufficiently small neighborhood of the origin, and obtain

$$\begin{aligned} &F_1(0, G; h, K_0) \\ &= D_2(G; h, p) + \frac{1}{2\left(G\sqrt{-2h} + 1\right)^2} + O(K_0) \\ &= -\beta 2^{p-1}\sqrt{-2h}^{2p-2}\Big(G\sqrt{-2h} + 1\Big)^{-p-1}\sum_{m=0}^{1-p} C_{1-p}^m \frac{(1-p)!}{m!(-m-p+1)!} \\ &\times \Big((1-p)(1 - G\sqrt{-2h}) - 2m\Big)\Big(\frac{1 - G\sqrt{-2h}}{1 + G\sqrt{-2h}}\Big)^{m-1} \\ &\quad + \frac{1}{2\left(G\sqrt{-2h} + 1\right)^2} + O(K_0) \\ &= -\beta 2^{p-1}\sqrt{-2h}^{2p-2}\Big(G\sqrt{-2h} + 1\Big)^{-p-1}\Big((1-p)(1 + G\sqrt{-2h}) \\ &\quad + (1-p)^2\Big((1-p)(1 - G\sqrt{-2h}) - 2\Big) + \sum_{m=2}^{1-p} C_{1-p}^m \frac{(1-p)!}{m!(-m-p+1)!} \\ &\times \Big((1-p)(1 - G\sqrt{-2h}) - 2m\Big)\Big(\frac{1 - G\sqrt{-2h}}{1 + G\sqrt{-2h}}\Big)^{m-1}\Big) \\ &\quad + \frac{1}{2\left(G\sqrt{-2h} + 1\right)^2} + O(K_0). \end{aligned} \tag{6.8}$$

When $p = 1$ or $p = 0$, we obtain

$$F_1(0, G; h, K_0) = \frac{1}{2\left(G\sqrt{-2h} + 1\right)^2} + O(K_0),$$

there is no solution $G(K_0, h)$. When $p < 0$, from (6.8), we obtain that $F_1(0, G; h, K_0)$ has one singularity $G = -\frac{1}{\sqrt{-2h}}$. Therefore $G \in \big(-1/\sqrt{-2h}, 1/\sqrt{-2h}\big)$.

When $G \to 0$, we obtain $F_1(0, G; h, K_0) \to 1/2$. When $G \to 1/\sqrt{-2h}$ and

$$\beta > -\frac{1}{2^{p+1}(p-1)p(-h)^{p-1}},$$

we obtain $F_1(0, G; h, K_0) > 0$, while if

$$\beta < -\frac{1}{2^{p+1}(p-1)p(-h)^{p-1}},$$

we have $F_1(0, G; h, K_0) < 0$.

Using the zero point theorem, we obtain that when

$$\beta \in \Big(-\infty, -\frac{1}{2^{p+1}(p-1)p(-h)^{p-1}}\Big),$$

there exists a solution $\widetilde{G}(K_0, h) \in \Big(0, \frac{1}{\sqrt{-2h}}\Big)$ of (6.7).

Therefore, when $p < 0$ and

$$\beta \in \Big(-\infty, -\frac{1}{2^{p+1}(p-1)p(-h)^{p-1}}\Big)$$

we have a solution $(g^*, G^*; K_0, h) = (0, \widetilde{G}(K_0, h))$.

Next, we calculate the Jacobian determinant of the averaged equation at this solution. Firstly, we consider the derivative of $D_2(G; h, p)$ with respect to G,

$$\begin{aligned}
&D_2'(G; h, p)\\
&= \frac{\mathrm{d}}{\mathrm{d}G} D_2(G; h, p)\\
&= -\beta 2^{p-1}\sqrt{-2h}^{\,2p-2}\Big(G\sqrt{-2h} + 1\Big)^{-p-3} \sum_{m=0}^{1-p} C_{1-p}^m \frac{(1-p)!}{m!(-m-p+1)!}\\
&\quad \times \Big(\frac{1 - G\sqrt{-2h}}{1 + G\sqrt{-2h}}\Big)^{m-2} \Big(\Big(\sqrt{-2h}(-p-1)\big(1 - G\sqrt{-2h}\big) + (1-m)(1 + \sqrt{-2h})\Big)\\
&\quad \times \Big((1-p)\big(1 - G\sqrt{-2h}\big) - 2m\Big) - G\sqrt{-2h}(1-p)\big(1 - G\sqrt{-2h}\big)\Big).
\end{aligned}$$

Therefore we can obtain the derivative of $F_1(g, G; h, K_0)$ with respect to G. Then we plug $(g^*, G^*; K_0, h) = (0, \widetilde{G}(K_0, h))$ back into the derivative, to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}G} F_1(g, G; h, K_0)|_{(g^*, G^*)}\\
&= D_2'(\widetilde{G}; h, p) - \frac{\big(\sqrt{-2h}\widetilde{G}^3 - 6\widetilde{G}^2 h {K_0}^2 + 3\sqrt{-2h}\widetilde{G}{K_0}^2 + {K_0}^2\big)}{\widetilde{G}^3\big(\sqrt{-2h}\widetilde{G} + 1\big)^3}\\
&= \frac{D_2'(\widetilde{G}; h, p)\widetilde{G}^3\Big(\sqrt{-2h}\widetilde{G} + 1\Big)^3 - \Big(\sqrt{-2h}\widetilde{G}^3 - 6\widetilde{G}^2 h {K_0}^2 + 3\sqrt{-2h}\widetilde{G}{K_0}^2 + {K_0}^2\Big)}{\widetilde{G}^3\Big(\sqrt{-2h}\widetilde{G} + 1\Big)^3}.
\end{aligned}$$

The Jacobian determinant of the averaged equations with this solution is

$$\begin{aligned}\Big|\frac{\partial(F_1,F_2)}{\partial(g,G)}\Big|_{(g^*,G^*)} &= -\frac{\mathrm{d}}{\mathrm{d}G}F_1(g,G;h,K_0)|_{(g^*,G^*)}\cdot\frac{\mathrm{d}}{\mathrm{d}G}F_1(g,G;h,K_0)|_{(g^*,G^*)}\\ &= \frac{\big(\sqrt{-2h}\widetilde{G}-1\big)\big(\widetilde{G}^2-{K_0}^2\big)}{\sqrt{-2h}\widetilde{G}^2\big(1+\sqrt{-2h}\widetilde{G}\big)}\Big(\frac{\mathrm{d}}{\mathrm{d}G}F_1(g,G;h,K_0)|_{(g^*,G^*)}\Big).\end{aligned}$$

We restrict K_0 to a sufficiently small neighborhood of the origin, and obtain

$$\begin{aligned}&\Big|\frac{\partial(F_1,F_2)}{\partial(g,G)}\Big|_{(g^*,G^*)}\\ &= \frac{\big(\sqrt{-2h}\widetilde{G}-1\big)\widetilde{G}^2}{\sqrt{-2h}\widetilde{G}^2\big(1+\sqrt{-2h}\widetilde{G}\big)}\Big(\frac{\mathrm{d}}{\mathrm{d}G}F_1(g,G;h)|_{(g^*,G^*)}\Big)+O(K_0)\\ &= \frac{\big(\sqrt{-2h}\widetilde{G}-1\big)}{\sqrt{-2h}\big(1+\sqrt{-2h}\widetilde{G}\big)}\frac{\Big(D_2'(\widetilde{G};h,p)\big(\sqrt{-2h}\widetilde{G}+1\big)^3-\sqrt{-2h}\Big)}{\big(\sqrt{-2h}\widetilde{G}+1\big)^3}+O(K_0)\\ &= \frac{\big(\sqrt{-2h}\widetilde{G}-1\big)\Big(D_2'(\widetilde{G};h,p)\big(\sqrt{-2h}\widetilde{G}+1\big)^3-\sqrt{-2h}\Big)}{\sqrt{-2h}\big(\sqrt{-2h}\widetilde{G}+1\big)^4}+O(K_0).\end{aligned} \tag{6.9}$$

Because $\widetilde{G}\neq\pm 1/\sqrt{-2h}$, to obtain the nonzero Jacobian determinant, (6.9) needs to satisfy

$$\Big(D_2'(\widetilde{G};h,p)\big(\sqrt{-2h}\widetilde{G}+1\big)^3-\sqrt{-2h}\Big)\neq 0.$$

Because $\widetilde{G}$ is the nondegenerate zero of (6.8) and

$$\frac{\mathrm{d}}{\mathrm{d}G}\Big(D_2(G;h,p)+\frac{1}{2\left(G\sqrt{-2h}+1\right)^2}\Big)=D_2'(G;h,p)-\frac{\sqrt{-2h}}{\left(G\sqrt{-2h.}+1\right)^3}.$$

Therefore,

$$\Big(D_2'(\widetilde{G};h,p)\big(\sqrt{-2h}\widetilde{G}+1\big)^3-\sqrt{-2h}\Big)=D_2'(\widetilde{G};h,p)-\frac{\sqrt{-2h}}{\left(\widetilde{G}\sqrt{-2h}+1\right)^3}\neq 0.$$

There exists 2π-periodic solutions $(g(l,\epsilon),G(l,\epsilon))=(g^*+O(\epsilon),G^*+O(\epsilon))$ of the system (4.4). Similarly, in order for $k(l)$ to be 2π-periodic, we plug $(g(l,\epsilon),G(l,\epsilon))$ back into $F_3(K_0,\epsilon)$, and obtain

$$\begin{aligned}F_3(K_0,\epsilon) &= -\frac{\pi K_0\left(-\cos(2g^*)\left(\sqrt{2}-2G^*\sqrt{-h}\right)+2G^*\sqrt{-h}+\sqrt{2}\right)}{\sqrt{-2h}(G^*)^2\left(2G^*\sqrt{-h}+\sqrt{2}\right)}+O(\epsilon)\\ &= -\frac{2\pi K_0}{\widetilde{G}\left(\widetilde{G}\sqrt{-2h}+1\right)}+O(\epsilon).\end{aligned}$$

There exists $(K_0,\epsilon)=(0,0)$ such that

$$F_3(0,0)=0,\quad \frac{\mathrm{d}}{\mathrm{d}K_0}F_3(K_0,\epsilon)|_{(0,0)}\neq 0.$$

Using implicit function theorem we obtain that there exists $K_0=K_0(\epsilon)$, such that $F_3(K_0(\epsilon),\epsilon)=0$. Therefore there exists $K=K_0(\epsilon)$ such that $k(l)$ is 2π-periodic.

When $p=1$ or $p=0$, we obtain $D_2(G;h,p)=0$. Solving $F_1(0,G;h,K_0)=0$ with respect to G, we obtain

$$G_{\pm}^{*}=-\sqrt{-2h}K_0^2\pm\sqrt{-K_0^2-2hK_0^4}.$$

Then, we plug g^* and $G_{\pm}^*$ back into $F_3(K_0,\epsilon)$ to obtain

$$\begin{aligned}F_3^{\pm}(K_0,\epsilon)&=-\frac{\pi K_0\left(-\cos(2g^*)\left(\sqrt{2}-2G_+^*\sqrt{-h}\right)+2G_+^*\sqrt{-h}+\sqrt{2}\right)}{\sqrt{-2h}(G_+^*)^2\left(2G_+^*\sqrt{-h}+\sqrt{2}\right)}+O(\epsilon)\\&=-\frac{2\sqrt{-2h}\pi K_0\left(\sqrt{hK_0^2(1+2hK_0^2)}\mp\sqrt{2}hK_0^2\right)}{\sqrt{2}\sqrt{hK_0^2(1+2hK_0^2)}\left(\sqrt{hK_0^2(1+2hK_0^2)}\pm\sqrt{2}hK_0^2\right)}+O(\epsilon).\end{aligned}$$

There not exists $K_0(\epsilon)$ such that $F_3(K_0(\epsilon),\epsilon)=0$. Therefore there not exists $K=K_0(\epsilon)$ such that $k(l)$ is 2π-periodic. Then, statement (ii) of Theorem 1.2 is proved.

Subcase B: $g^*=\pi$. The proof of statement (ii) of Theorem 1.2 when $g^*=\pi$ is completely similar to case $g^*=0$.

Subcase C: $g^*=\frac{\pi}{2}$. The equilibrium points must satisfy the first equation of the averaged system (5.7). Inserting g^* back into $F_1(g,G;h,K_0)=0$, we obtain

$$\begin{aligned}F_1\Big(\frac{\pi}{2},G;h,K_0\Big)=&D_2(G;h,p)+\frac{1}{\sqrt{-2h}G^3\left(2G\sqrt{-h}+\sqrt{2}\right)^2}\\&\times\Big(\Big(-G^3\sqrt{-2h}+2G^2\mathrm{h}\ \mathrm{K}_0^2+G\sqrt{-2h}K_0^2-K_0^2\Big)\\&-K_0^2\Big(2G^2h-2G\sqrt{-2h}-1\Big)\Big)=0.\end{aligned}\tag{6.10}$$

We restrict K_0 to a sufficiently small neighborhood of the origin, and obtain

$$\begin{aligned}&F_1\big(\frac{\pi}{2},G;h,K_0\big)\\&=D_2(G;h,p)-\frac{1}{\left(2G\sqrt{-h}+\sqrt{2}\right)^2}+O(K_0)\\&=-\beta 2^{p-1}\sqrt{-2h}^{2p-2}\Big(G\sqrt{-2h}+1\Big)^{-p-1}\sum_{m=0}^{1-p}C_{1-p}^{m}\frac{(1-p)!}{m!(-m-p+1)!}\\&\quad\times\Big((1-p)(1-G\sqrt{-2h})-2m\Big)\Big(\frac{1-G\sqrt{-2h}}{1+G\sqrt{-2h}}\Big)^{m-1}\\&\quad-\frac{1}{2\left(G\sqrt{-2h}+1\right)^2}+O(K_0)\\&=-\beta 2^{p-1}\sqrt{-2h}^{2p-2}\Big(G\sqrt{-2h}+1\Big)^{-p-1}\Big((1-p)(1+G\sqrt{-2h})\\&\quad+(1-p)^2\Big((1-p)(1-G\sqrt{-2h})-2\Big)+\sum_{m=2}^{1-p}C_{1-p}^{m}\frac{(1-p)!}{m!(-m-p+1)!}\\&\quad\Big((1-p)(1-G\sqrt{-2h})-2m\Big)\Big(\frac{1-G\sqrt{-2h}}{1+G\sqrt{-2h}}\Big)^{m-1}\Big)\end{aligned}$$

$$
-\frac{1}{2\left(G\sqrt{-2h}+1\right)^2}+O(K_0). \tag{6.11}
$$

When $p=1$ or $p=0$, we obtain

$$
F_1\big(\frac{\pi}{2},G;h,K_0\big)=-\frac{1}{2\left(G\sqrt{-2h}+1\right)^2}+O(K_0),
$$

there is no solution $G(K_0,h)$. When $p<0$, from (6.11), we obtain that $F_1(\frac{\pi}{2},G;h,K_0)$ has one singularity $G=-\frac{1}{\sqrt{-2h}}$. Therefore $G\in\big(-1/\sqrt{-2h},1/\sqrt{-2h}\big)$.

When $G\to 0$, we obtain $F_1(\pi/2,G;h,K_0)\to -\frac{1}{2}$. When $G\to\frac{1}{\sqrt{-2h}}$ and

$$
\beta>\frac{1}{2^{p+1}(p-1)p(-h)^{p-1}},
$$

we obtain $F_1\left(\pi/2,G;h,K_0\right)>0$, while if

$$
\beta<\frac{1}{2^{p+1}(p-1)p(-h)^{p-1}},
$$

we have $F_1\left(\pi/2,G;h,K_0\right)<0$.

Using the zero point theorem, we obtain that when

$$
\beta\in\Big(\frac{1}{2^{p+1}(p-1)p(-h)^{p-1}},+\infty\Big),
$$

there exists a solution $\overline{G}(K_0,h)\in(0,1/\sqrt{-2h})$ of (6.10). Therefore, when $p<0$ and

$$
\beta\in\Big(0,\frac{1}{2^{p+1}(p-1)p(-h)^{p-1}}\Big),
$$

we have a solution

$$
(g^*,G^*;K_0,h)=\Big(\frac{\pi}{2},\overline{G}(K_0,h)\Big).
$$

Next, we calculate the Jacobian determinant of the averaged equation at this solution. Firstly, we consider the derivative of $F_1(g,G;h,K_0)$ with respect to G. Then we put $(g^*,G^*;h,K_0)=(\frac{\pi}{2},\overline{G}(h,K_0))$ back into the derivative, and obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}G}F_1(g,G;h,K_0)|_{(g^*,G^*)}\\
&=D_2'(\overline{G};h,p)+\frac{-2h\overline{G}^4+12h\overline{G}^2{K_0}^2-8\overline{G}\sqrt{-2h}{K_0}^2-3{K_0}^2}{G^4\sqrt{-2h}\left(\overline{G}\sqrt{-2h}+1\right)^3}\\
&=\Big(D_2'(\overline{G};h,p)\overline{G}^4\sqrt{-2h}\big(\overline{G}\sqrt{-2h}+1\big)^3\\
&\quad+\Big(-2h\overline{G}^4+12h\overline{G}^2{K_0}^2-8\overline{G}\sqrt{-2h}{K_0}^2-3{K_0}^2\Big)\Big)\\
&\quad\div\big(\overline{G}^4\sqrt{-2h}\big(\overline{G}\sqrt{-2h}+1\big)^3\big)
\end{aligned}
$$

The Jacobian determinant of the averaged equation with this solution is

$$
\begin{aligned}
\Big|\frac{\partial(F_1,F_2)}{\partial(g,G)}\Big|_{(g^*,G^*)}&=-\frac{\mathrm{d}}{\mathrm{d}G}F_1(g,G;h,K_0)|_{(g^*,G^*)}\cdot\frac{\mathrm{d}}{\mathrm{d}g}F_2(g,G;h,K_0)|_{(g^*,G^*)}\\
&=\frac{\big(\overline{G}\sqrt{-2h}-1\big)\left(\overline{G}^2-{K_0}^2\right)}{-\overline{G}^2\sqrt{-2h}\left(1+\overline{G}\sqrt{-2h}\right)}\Big(\frac{d}{dG}F_1(g,G;h,K_0)|_{(g^*,G^*)}\Big).
\end{aligned}
$$

We restrict K_0 to a sufficiently small neighborhood of the origin, and obtain

$$\begin{aligned}
&\Big|\frac{\partial(F_1,F_2)}{\partial(g,G)}\Big|_{(g^*,G^*)}\\
&= \frac{\left(\overline{G}\sqrt{-2h}-1\right)\overline{G}^2}{-\overline{G}^2\sqrt{-2h}\left(1+\overline{G}\sqrt{-2h}\right)}\Big(\frac{\mathrm{d}}{\mathrm{d}G}F_1(g,G;h)|_{(g^*,G^*)}\Big)+O(K_0)\\
&= \frac{\left(\overline{G}\sqrt{-2h}-1\right)}{-\sqrt{-2h}\left(1+\overline{G}\sqrt{-2h}\right)}\frac{\left(D_2'(\overline{G};h,p)\big(\overline{G}\sqrt{-2h}+1\big)^3+\sqrt{-2h}\right)}{\left(\overline{G}\sqrt{-2h}+1\right)^3}\\
&\quad +O(K_0)\\
&= \frac{\left(\overline{G}\sqrt{-2h}-1\right)\left(D_2'(\overline{G};h,p)\big(\overline{G}\sqrt{-2h}+1\big)^3+\sqrt{-2h}\right)}{-\sqrt{-2h}\left(\overline{G}\sqrt{-2h}+1\right)^4}+O(K_0).
\end{aligned}\tag{6.12}$$

Because $\overline{G}\neq \pm 1/\sqrt{-2h}$, to obtain the nonzero Jacobian determinant, (6.12) needs to satisfy

$$\Big(D_2'(\overline{G};h,p)\big(\overline{G}\sqrt{-2h}+1\big)^3+\sqrt{-2h}\Big)\neq 0.$$

Because $\overline{G}$ is the nondegenerate zero of (6.11) and

$$\frac{\mathrm{d}}{\mathrm{d}G}\Big(D_2(G;h,p)-\frac{1}{2\left(G\sqrt{-2h}+1\right)^2}\Big)=D_2'(G;h,p)+\frac{\sqrt{-2h}}{\left(G\sqrt{-2h}+1\right)^3}.$$

Therefore,

$$\Big(D_2'(\overline{G};h,p)\big(\sqrt{-2h}\overline{G}+1\big)^3+\sqrt{-2h}\Big)=D_2'(\overline{G};h,p)+\frac{\sqrt{-2h}}{\left(\overline{G}\sqrt{-2h}+1\right)^3}\neq 0.$$

There exists a 2π-periodic solution $(g(l,\epsilon),G(l,\epsilon))=(g^*+O(\epsilon),G^*+O(\epsilon))$ of system (4.4). Similarly, in order for $k(l)$ to be 2π-periodic, we plug $(g(l,\epsilon),G(l,\epsilon))$ back into $F_3(K_0,\epsilon)$, obtain

$$\begin{aligned}
F_3(K_0,\epsilon)&=-\frac{\pi K_0\left(-\cos(2g^*)\left(\sqrt{2}-2G^*\sqrt{-h}\right)+2G^*\sqrt{-h}+\sqrt{2}\right)}{\sqrt{-2h}(G^*)^2\left(2G^*\sqrt{-h}+\sqrt{2}\right)}+O(\epsilon)\\
&=\frac{2\pi K_0}{\overline{G}\left(\overline{G}\sqrt{-2h}+1\right)}+O(\epsilon).
\end{aligned}$$

There exists $(K_0,\epsilon)=(0,0)$ such that

$$F_3(0,0)=0,\quad \frac{\mathrm{d}}{\mathrm{d}K_0}F_3(K_0,\epsilon)|_{(0,0)}\neq 0.$$

Using the implicit function theorem we obtain that there exists $K_0=K_0(\epsilon)$, such that $F_3(K_0(\epsilon),\epsilon)=0$. Therefore there exists $K=K_0(\epsilon)$ such that $k(l)$ is 2π-periodic.

When $p=1$ or $p=0$, we obtain $D_2(G;h,p)=0$. Solving $F_1(\pi/2,G;h,K_0)=0$ with respect to G, we obtain

$$G^*=\frac{\left(-h\left(\sqrt{2hK^6+K^4}+K^2\right)\right)^{2/3}-2^{1/6}hK^2}{2^{1/6}(-h)^{5/6}\left(\sqrt{2hK^6+K^4}+K^2\right)^{1/3}}.$$

Then, we plug g^* and G^* back into $F_3(K_0,\epsilon)$, and obtain

$$
\begin{aligned}
F_3(K_0,\epsilon) &= -\frac{\pi K_0\left(-\cos(2g^*)\left(\sqrt{2}-2G^*\sqrt{-h}\right)+2G^*\sqrt{-h}+\sqrt{2}\right)}{\sqrt{-2h}(G^*)^2\left(2G^*\sqrt{-h}+\sqrt{2}\right)}+O(\epsilon)\\
&= \frac{2^{5/6}\sqrt{-h}hK\left(\sqrt{2hK^6+K^4}+K^2\right)}{\left(\left(-h\left(\sqrt{2hK^6+K^4}+K^2\right)\right)^{2/3}-2^{1/6}hK^2\right)^2}\\
&\quad\times\Big(-\sqrt{2}hK^2+\left(-h\left(\sqrt{2hK^6+K^4}+K^2\right)\right)^{1/3}\\
&\quad+2^{1/3}\left(-h\left(\sqrt{2hK^6+K^4}+K^2\right)\right)^{2/3}\Big)^{-1}.
\end{aligned}
$$

There is no $K_0(\epsilon)$ such that $F_3(K_0(\epsilon),\epsilon)=0$. Therefore there is not $K=K_0(\epsilon)$ such that $k(l)$ is 2π-periodic. Then, statement (iii) of Theorem 1.2 is proved when $g^*=\pi/2$.

Subcase D: $g^*=-\pi/2$. The proof of statement (iii) of Theorem 1.2 with $g^*=-\pi/2$ is completely similar to case $g^*=\pi/2$, and we do not repeat again. Therefore, we conclude the proof of Theorem 1.2.

7. Conclusion

In this article, the periodic orbits of the spatial anisotropic Kepler problem with anisotropic perturbations on the energy surface are discussed by using canonical transformations, Residue theory in complex analysis and the averaging method, see Theorem 1.1 and Theorem 1.2. Moreover, we also give some applications related to spatial anisotropic problem. Comparing the previous work, we are not limited to the study of the spatial anisotropic Manev problem when $p=2$, but we extend p to the integer domain. This is a new result in this context. From the view point of a physical application, it might be reasonable to use the averaging principle to replace a mathematical model by the corresponding averaged system, to use the averaged system to make a prediction. The study of this paper exactly gives a frame for application of the averaging method to spatial anisotropic Kepler problem with anisotropic perturbations.

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