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EXISTENCE OF BOUNDED GLOBAL SOLUTIONS FOR FULLY PARABOLIC ATTRACTION-REPULSION CHEMOTAXIS SYSTEMS WITH SIGNAL-DEPENDENT SENSITIVITIES AND WITHOUT LOGISTIC SOURCE

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Abstract. This article concerns the parabolic attraction-repulsion chemotaxis system with signal-dependent sensitivities

$$\begin{aligned} u_t &= \Delta u - \nabla \cdot (u\chi(v)\nabla v) + \nabla \cdot (u\xi(w)\nabla w), \quad x \in \Omega,\ t > 0, \\ v_t &= \Delta v - v + u, \quad x \in \Omega,\ t > 0, \\ w_t &= \Delta w - w + u, \quad x \in \Omega,\ t > 0 \end{aligned}$$

under homogeneous Neumann boundary conditions and initial conditions, where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary, χ, ξ are functions satisfying certain conditions. Existence of bounded global classical solutions to the system with logistic source and logistic damping have been obtained in [1]. This article establishes the existence of global bounded classical solutions with logistic damping.

1. Introduction

Recently, in [1], we studied the fully parabolic attraction-repulsion chemotaxis system with signal-dependent sensitivities and logistic source

$$\begin{aligned} u_t &= \Delta u - \nabla \cdot (u\chi(v)\nabla v) + \nabla \cdot (u\xi(w)\nabla w) + \mu u(1-u), \\ v_t &= \Delta v - v + u, \\ w_t &= \Delta w - w + u, \end{aligned} \tag{1.1}$$

where χ, ξ are decreasing functions and $\mu > 0$. The existence of bounded global solution for (1.1) was obtained by using the effect of the logistic term. In light of this result, the following question is raised:

Does boundedness of solutions still hold without logistic term?

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To answer the above question, we study the fully parabolic attraction-repulsion chemotaxis system with signal-dependent sensitivities,

$$\begin{gathered} u_t = \Delta u - \nabla \cdot (u\chi(v)\nabla v) + \nabla \cdot (u\xi(w)\nabla w), \quad x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, \quad x \in \Omega,\ t > 0, \\ w_t = \Delta w - w + u, \quad x \in \Omega,\ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \quad x \in \Omega, \end{gathered} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$; ν is the outward normal vector to $\partial\Omega$; χ, ξ are positive known functions; u, v, w are unknown functions. The initial data u_0, v_0, w_0 are supposed to be nonnegative functions satisfying

$$u_0 \in C^0(\overline{\Omega}), \quad u_0 \not\equiv 0, \tag{1.3}$$

$$v_0, w_0 \in W^{1,\infty}(\Omega). \tag{1.4}$$

We now explain the background of (1.2). Chemotaxis is a property of cells to move in response to the concentration gradient of a chemical substance produced by the cells. The origin of the problem of describing such biological phenomena is the chemotaxis model proposed by Keller-Segel [9]:

$$u_t = \Delta u - \nabla \cdot (u\chi(v)\nabla v), \quad v_t = \Delta v - v + u.$$

Systems (1.1) and (1.2) describe a process by which cells move in response to a chemoattractant and a chemorepellent produced by the cells themselves. In particular, system (1.2) with constant sensitivities (i.e., $\chi(v) \equiv \chi$, $\xi(w) \equiv \xi$, where $\chi, \xi > 0$ are constants) represents the quorum sensing effect that cells keep away from a repulsive chemical substance [17] and describes the aggregation of microglial cells in Alzheimer's disease [12]. There are a lot of studies for such attraction-repulsion chemotaxis systems; we summarize some of them, by reducing parameters to 1, as follows.

For (1.1) with constant sensitivities (i.e., $\chi(v) \equiv \chi$, $\xi(w) \equiv \xi$, where $\chi, \xi > 0$ are constants), the existence of bounded global solutions was obtained in [5, 6, 7, 8]. More precisely, Jin-Wang [7] investigated the one-dimensional case. Also, when $\chi = \xi$, Jin-Liu [6] studied the two- and three-dimensional cases. Recently, Jin-Wang [8] obtained the existence, boundedness, and stabilization of global solutions under the condition $\frac{\xi}{\chi} \geq C > 0$ in the two-dimensional setting. In this way, boundedness is well established for system (1.1) with logistic term. This holds for the Keller-Segel system, that is, the system (1.1) with $w = 0$; see e.g., [20] for the parabolic-elliptic setting, and [23, 25, 27, 28, 29, 30] for the parabolic-parabolic setting.

On the other hand, system (1.2) with $\chi(v) \equiv \chi$, $\xi(w) \equiv \xi$ (positive constants) has been studied in [3, 10, 11]. In the two-dimensional setting, Liu-Tao [11] established existence and boundedness of global solutions under the condition $\chi < \xi$. In the case $\chi > \xi$, Fujie-Suzuki [3] obtained boundedness under the condition $\int_\Omega u_0 < \frac{4\pi}{\chi - \xi}$ ($\int_\Omega u_0 < \frac{8\pi}{\chi - \xi}$ for the radial case) in the two-dimensional setting, and they asserted finite-time blow-up in the higher-dimensional case. Lankeit [10] showed finite-time blow-up for the system having the second equation $v_t = \Delta v - \beta v + u$ and the third equation $w_t = \Delta w - \delta w + u$ with $\beta \neq \delta$ in the three-dimensional radial setting when $\chi > \xi$. Some related works which deal with the parabolic-elliptic-elliptic version of (1.1) can be found in [18, 19, 21, 31]. Specifically, Tao-Wang [19] derived global existence and boundedness under the condition $\chi < \xi$ in two or more space dimensions. While finite-time blow-up was proved in the two-dimensional setting when $\chi > \xi$ and the initial data satisfy the conditions that $\int_\Omega u_0 > \frac{8\pi}{\chi-\xi}$ and that $\int_\Omega u(x)|x-x_0|^2\,dx$ ($x_0 \in \Omega$) is sufficiently small. Salako-Shen [18] obtained global existence and boundedness when $\chi = 0$ (or $\mu > \chi - \xi + M$ with some $M > 0$ in (1.1)). Whereas in the two-dimensional setting, finite-time blow-up was shown by Yu-Guo-Zheng [31] and lower bound of blow-up time was given by Viglialoro [21]. Thus we see that if there is no logistic term, boundedness breaks down in some cases. A similar phenomenon occurs for the Keller-Segel system; see e.g., [15, 16] for the parabolic–elliptic type; and [4, 13, 22, 24, 26] for the parabolic-parabolic type.

As mentioned above, the logistic term seems helpful to derive boundedness in (1.1), whereas it is not clear whether boundedness in (1.2) without logistic term holds or not. The purpose of this article is to establish a result on boundedness for (1.2) with extra information about the outcome by our previous work.

Now we introduce conditions on the functions χ, ξ and then state the main theorem. We assume throughout this paper that χ, ξ satisfy the following conditions:

$$\chi \in C^{1+\theta_1}([\eta_1,\infty)) \cap L^1(\eta_1,\infty) \quad (0 < \exists\theta_1 < 1), \quad \chi > 0, \tag{1.5}$$

$$\xi \in C^{1+\theta_2}([\eta_2,\infty)) \cap L^1(\eta_2,\infty) \quad (0 < \exists\theta_2 < 1), \quad \xi > 0, \tag{1.6}$$

$$\exists\chi_0 > 0; \quad s\chi(s) \le \chi_0 \quad \forall s \ge \eta_1, \tag{1.7}$$

$$\exists\xi_0 > 0; \quad s\xi(s) \le \xi_0 \quad \forall s \ge \eta_2, \tag{1.8}$$

$$\exists\alpha > 0; \quad \chi'(s) + \alpha|\chi(s)|^2 \le 0 \quad \forall s \ge \eta_1, \tag{1.9}$$

$$\exists\beta > 0; \quad \xi'(s) + \beta|\xi(s)|^2 \le 0 \quad \forall s \ge \eta_2, \tag{1.10}$$

where $\eta_1, \eta_2 \ge 0$ are constants which will be fixed in Lemma 2.2; note that if $v_0, w_0 > 0$, then we can take $\eta_1, \eta_2 > 0$ (see also (2.1) with $z_0 > 0$). Moreover, we suppose that the α, β appearing in the conditions (1.9), (1.10) satisfy

$$\alpha > \frac{\frac{n}{2}(2\delta+1)\big((n-1)\delta+n\big)+\sqrt{D}}{2\delta(\beta-n)-\delta^2-\frac{n}{2}}, \quad \beta > n + \sqrt{n/2} \tag{1.11}$$

for some $\delta \in J := \big(\beta - n - \sqrt{(\beta-n)^2 - \frac{n}{2}}, \beta - n + \sqrt{(\beta-n)^2-\frac{n}{2}}\big)$, where

$$D := \frac{n\delta}{2}\big(2\beta + (2n-1)\delta\big)\Big[2\delta(\beta-n) + \frac{n}{2}\Big(2n(\delta+1)^2 - (2\delta+1)^2\Big)\Big].$$

The main result reads as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ ($n \ge 2$) be a bounded domain with smooth boundary. Assume that (u_0, v_0, w_0) satisfy (1.3), (1.4). Suppose that χ, ξ fulfill (1.5)–(1.10) with α, β which satisfy (1.11) for some $\delta \in J$. Then there exists a unique triplet (u, v, w) of nonnegative functions*

$$u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)),$$

$$v, w \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap L^\infty(0,\infty; W^{1,\infty}(\Omega)),$$

which solves (1.2) in the classical sense. Moreover, the solution (u, v, w) is bounded:

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C$$

for all $t > 0$ and some $C > 0$.

Remark 1.2. The above theorem provides the additional information.

- Boundedness still holds in the system without logistic damping under the identical condition in [1].
- The functions χ, ξ admits the singular case like χ_0/s^k with $\chi_0 > 0$, $k > 1$. This case was excluded from [1] because of the required regularity of $\chi, \xi \in L^1(0, \infty)$.

The strategy for the proof of Theorem 1.1 is to show the L^p-boundedness of u with some $p > n/2$. The key is to derive the differential inequality

$$\frac{d}{dt}\int_\Omega u^p f(x,t)\,dx \le c_1 \int_\Omega u^p f(x,t)\,dx - c_2\Big(\int_\Omega u^p f(x,t)\,dx\Big)^{1/\theta} + c_3 \tag{1.12}$$

with some constants $c_1, c_2, c_3 > 0$, $\theta \in (0,1)$ and some function f defined by using v, w (see (3.1)). In our previous work including the logistic term $\mu u(1-u)$, the differential inequality

$$\frac{d}{dt}\int_\Omega u^p f(x,t)\,dx \le c_1 \int_\Omega u^p f(x,t)\,dx - \mu p|\Omega|^{-1/p}\Big(\int_\Omega u^p f(x,t)\,dx\Big)^{1+\frac{1}{p}} \tag{1.13}$$

was established. The second term on the right-hand side of this inequality, which is important in proving boundedness, is derived from the effect of the logistic term. In other words, in the absence of a logistic source, the differential inequality (1.13) with $\mu = 0$ cannot derive boundedness which means the proof in the previous work fails. Thus we will show the differential inequality (1.12) without any help of the logistic term. More precisely, using the effect of the diffusion term, we will establish

$$\frac{d}{dt}\int_\Omega u^p f(x,t)\,dx \le c_1 \int_\Omega u^p f(x,t)\,dx - \varepsilon_0 p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 f(x,t)\,dx \tag{1.14}$$

with some small $\varepsilon_0 > 0$. Then, by applying the Gagliardo-Nirenberg inequality to the second term on the right-hand side of (1.14), we will obtain (1.12). This step is a difference between this paper and the previous one (see Lemma 3.6).

This article is organized as follows. In Section 2 we collect some preliminary facts about local existence of classical solutions to (1.2) and a lemma such that an L^p-estimate for u with some $p > n/2$ implies an L^∞-estimate for u. Section 3 is devoted to the proof of global existence and boundedness (Theorem 1.1).

2. Preliminaries

In this section we give some lemmas which will be used later. We first present the result obtained by a similar argument in [2, Lemma 2.2] (see also [14, Lemma 2.1 and Remark 2.2]), which will be applied to the second and third equations in (1.2).

Lemma 2.1. *Let $T > 0$. Let $u \in C^0(\overline{\Omega} \times [0,T))$ be a nonnegative function such that, with some $m > 0$, $\int_\Omega u(\cdot,t) = m$ for all $t \in [0,T)$. If $z_0 \in C^0(\overline{\Omega})$, $z_0 \ge 0$ in $\overline{\Omega}$ and $z \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ is a classical solution of*

$$\begin{gathered} z_t = \Delta z - z + u, \quad x \in \Omega,\ t > 0, \\ \nabla z \cdot \nu = 0, \quad x \in \partial\Omega,\ t > 0, \\ z(x,0) = z_0(x), \quad x \in \Omega, \end{gathered}$$

then for all $t \in (0,T)$,

$$\inf_{x\in\Omega} z(x,t) \geq \eta$$

with

$$\eta := \sup_{\tau>0} \Big(\min \big\{ e^{-2\tau} \min_{x\in\overline{\Omega}} z_0(x),\ c_0 m(1-e^{-\tau}) \big\} \Big) \geq 0, \tag{2.1}$$

where $c_0 > 0$ *is a lower bound for the fundamental solution of* $\varphi_t = \Delta\varphi - \varphi$ *with Neumann boundary condition.*

We next introduce a result on the existence of local classical solutions to (1.2).

Lemma 2.2. *Let* $n \geq 1$ *and let* (u_0, v_0, w_0) *fulfill* (1.3), (1.4). *Put* $m_0 := \int_\Omega u_0$ *and let* $\eta_1, \eta_2 \geq 0$ *be constants given by* (2.1) *with* $(z_0, m) = (v_0, m_0)$ *and* $(z_0, m) = (w_0, m_0)$, *respectively. Assume that* $\chi \in C^{1+\theta_1}([\eta_1, \infty))$, $\xi \in C^{1+\theta_2}([\eta_2,\infty))$ *with some* $\theta_1, \theta_2 \in (0,1)$. *Then there exists* $T_{\max} \in (0,\infty]$ *such that* (1.2) *admits a unique classical solution* (u,v,w) *such that*

$$u \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})),$$
$$v, w \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})) \cap L^\infty_{\rm loc}([0,T_{\max}); W^{1,\infty}(\Omega)),$$

and u *has positivity as well as the mass conservation property*

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \tag{2.2}$$

for all $t \in (0,T_{\max})$, *whereas* v *and* w *satisfy the lower estimates*

$$\inf_{x\in\Omega} v(x,t) \geq \eta_1, \quad \inf_{x\in\Omega} w(x,t) \geq \eta_2 \tag{2.3}$$

for all $t \in (0,T_{\max})$. *Moreover, if* $T_{\max} < \infty$, *then*

$$\limsup_{t\nearrow T_{\max}} \Big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \Big) = \infty. \tag{2.4}$$

Proof. Using a standard argument based on the contraction mapping principle as in [19, Lemma 3.1], we can show local existence and blow-up criterion (2.4). Note that the mass conservation property (2.2) can be obtained by integrating the first equation in (1.2) over $\Omega \times (0,t)$ for $t \in (0,T_{\max})$, and that the lower estimates (2.3) follow from Lemma 2.1. □

In the following we assume that $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded domain with smooth boundary, χ, ξ fulfill (1.5), (1.6), respectively, (u_0, v_0, w_0) satisfies (1.3), (1.4). Then we denote by (u,v,w) the local classical solution of (1.2) given in Lemma 2.2 and by $T_{\max}$ its maximal existence time.

We next give the following lemma which tells us a strategy to prove global existence and boundedness.

Lemma 2.3. *Assume that* χ, ξ *fulfill that* $\chi(s) \leq K_1$, $\xi(s) \leq K_2$ *for all* $s \geq 0$ *with some* $K_1, K_2 > 0$, *respectively. If there exist* $K_3 > 0$ *and* $p > n/2$ *satisfying*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \leq K_3$$

for all $t \in (0,T_{\max})$, *then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C$$

for all $t \in (0,T_{\max})$ *with some* $C > 0$.

For a proof of the above lemma see [1, Lemma 2.3]; note that it is rather easier to show it without the logistic term.

3. Proof of Theorem 1.1

Thanks to Lemma 2.3, it is sufficient to derive an L^p-estimate for u with some $p > n/2$. To establish the estimate for u we introduce the function $f = f(x,t)$ by

$$f(x,t) := \exp\Big(-r \int_{\eta_1}^{v(x,t)} \chi(s)\, ds - \sigma \int_{\eta_2}^{w(x,t)} \xi(s)\, ds \Big), \tag{3.1}$$

where $r, \sigma > 0$ are some constants which will be fixed later. Here the function f is finite valued, because integrability in (3.1) is assured by (1.5) and (1.6) together with (2.3).

Then we give the following lemma which was proved in [1, Lemma 3.2] with $\mu = 0$. Although in the literature we used the function f with $\eta_1 = \eta_2 = 0$, the conclusion of the following lemma does not depend on the choice of (η_1, η_2).

Lemma 3.1. *Let $r, \sigma > 0$. Then for all $p > 1$, we have*

$$\begin{aligned} \frac{d}{dt} \int_\Omega u^p f &= I_1 + I_2 + I_3 - r \int_\Omega u^p f \chi(v)(-v+u) \\ &\quad - \sigma \int_\Omega u^p f \xi(w)(-w+u) \end{aligned} \tag{3.2}$$

for all $t \in (0, T_{\max})$, where

$$\begin{gathered} I_1 := p \int_\Omega u^{p-1} f \nabla \cdot \Big(\nabla u - u\chi(v)\nabla v + u\xi(w)\nabla w \Big), \\ I_2 := -r \int_\Omega u^p f \chi(v) \Delta v, \\ I_3 := -\sigma \int_\Omega u^p f \xi(w) \Delta w. \end{gathered}$$

Next we state an estimate for $I_1 + I_2 + I_3$ in the following lemma.

Lemma 3.2. *Let $r, \sigma > 0$, $\varepsilon \in [0,1)$ and put*

$$\mathbf{x} := u^{-1}|\nabla u|, \quad \mathbf{y} := \chi(v)|\nabla v|, \quad \mathbf{z} := \xi(w)|\nabla w|.$$

Then for all $p > 1$, the following estimate holds:

$$\begin{aligned} I_1 + I_2 + I_3 &\le -\varepsilon p(p-1) \int_\Omega u^p f \mathbf{x}^2 \\ &\quad + \int_\Omega u^p f \cdot (a_1(\varepsilon)\mathbf{x}^2 + a_2 \mathbf{x}\mathbf{y} + a_3 \mathbf{x}\mathbf{z} + a_4 \mathbf{y}^2 + a_5 \mathbf{y}\mathbf{z} + a_6 \mathbf{z}^2) \end{aligned} \tag{3.3}$$

for all $t \in (0, T_{\max})$, where

$$\begin{gathered} a_1(\varepsilon) := -(1-\varepsilon)p(p-1), \quad a_2 := p(p+2r-1), \\ a_3 := p(p+2\sigma-1), \quad a_4 := -r(p+r+\alpha), \\ a_5 := pr + p\sigma + 2r\sigma, \quad a_6 := -\sigma(-p+\sigma+\beta). \end{gathered}$$

Proof. Noting that $-\varepsilon p(p-1) + a_1(\varepsilon) = -p(p-1)$ for all $\varepsilon \in (0,1)$, we see that the estimate (3.3) is almost the same as that in the case $\varepsilon = 0$ except multiplication by constants and is proved in [1, p. 10]. $\square$

We next give the following lemma, which is useful to show that the second term on the right-hand side of (3.3) is nonpositive.

Lemma 3.3. *Assume that α, β satisfy* (1.11). *Then there exist $p > n/2$ and $r, \sigma > 0$ such that*

$$A_1 := \begin{vmatrix} a_1(0) & \frac{a_3}{2} \\ \frac{a_3}{2} & a_6 \end{vmatrix} > 0 \quad \textit{and} \quad A_2 := \begin{vmatrix} a_1(0) & \frac{a_3}{2} & \frac{a_2}{2} \\ \frac{a_3}{2} & a_6 & \frac{a_5}{2} \\ \frac{a_2}{2} & \frac{a_5}{2} & a_4 \end{vmatrix} < 0. \tag{3.4}$$

Remark 3.4. In [1, Proof of Lemma 3.3], we showed that there exist $p > n/2$ and $r, \sigma > 0$ such that $A_1 > 0$, $A_2 \leq 0$. Here, $A_2 \leq 0$ can be refined as $A_2 < 0$ for some $p > n/2$ and $r, \sigma > 0$. More precisely, we set

$$\varphi_2(r) := c_1 r^2 + c_2 r + c_3$$

with $c_1, c_2, c_3 > 0$ and find $r > 0$ such that $\varphi_2(r) \leq 0$ by the following two conditions:

$$\begin{aligned} r_0 &> 0, \quad \text{where } r_0 \text{ is the axis of the parabola } \varphi_2, \\ D_2 &> 0, \quad \text{where } D_2 \text{ is the discriminant of } \varphi_2. \end{aligned}$$

These conditions show that $\varphi_2(r) < 0$ for some $r > 0$ which implies to $A_2 < 0$.

Combining the above three lemmas, we can derive the following important inequality which leads to the L^p-estimate for u.

Lemma 3.5. *Assume that χ, ξ satisfy* (1.5)–(1.10) *with α, β which fulfill* (1.11). *Then there exist $p > n/2$ and $r, \sigma > 0$ such that*

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^p f + \varepsilon_0 p(p-1)\int_\Omega u^{p-2} f|\nabla u|^2 \\ &\leq -r\int_\Omega u^p f\chi(v)(-v+u) - \sigma\int_\Omega u^p f\xi(w)(-w+u) \end{aligned}$$

for all $t \in (0, T_{\max})$ with some $\varepsilon_0 \in (0,1)$.

Proof. We put

$$A_1(\varepsilon) := \begin{vmatrix} a_1(\varepsilon) & \frac{a_3}{2} \\ \frac{a_3}{2} & a_6 \end{vmatrix} \quad \text{and} \quad A_2(\varepsilon) := \begin{vmatrix} a_1(\varepsilon) & \frac{a_3}{2} & \frac{a_2}{2} \\ \frac{a_3}{2} & a_6 & \frac{a_5}{2} \\ \frac{a_2}{2} & \frac{a_5}{2} & a_4 \end{vmatrix}$$

for $\varepsilon \in [0,1)$. Since $A_1(0) > 0$ and $A_2(0) < 0$ hold in view of (3.4) and the function $a_1 \colon \varepsilon \mapsto -(1-\varepsilon)p(p-1)$ is continuous at $\varepsilon = 0$, we can find $\varepsilon_0 \in (0,1)$ such that $A_1(\varepsilon_0) > 0$ and $A_2(\varepsilon_0) < 0$. By using the Sylvester criterion, we have

$$a_1(\varepsilon_0)\mathbf{x}^2 + a_2\mathbf{xy} + a_3\mathbf{xz} + a_4\mathbf{y}^2 + a_5\mathbf{yz} + a_6\mathbf{z}^2 \leq 0. \tag{3.5}$$

Combining (3.3) and (3.5) with (3.2), we arrive at the conclusion. □

We now show the desired L^p-estimate for u with some $p > n/2$.

Lemma 3.6. *Let $p > n/2$. Assume that χ, ξ satisfy* (1.5)–(1.10) *with α, β which fulfill* (1.11). *Then there exists $C > 0$ such that $\|u(\cdot,t)\|_{L^p(\Omega)} \leq C$ for all $t \in (0, T_{\max})$.*

Proof. By Lemma 3.5, we see from the positivity of χ, ξ and (1.7), (1.8) that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^p f + \varepsilon_0 p(p-1)\int_\Omega u^{p-2} f|\nabla u|^2 \\ &\le -r\int_\Omega u^p f\chi(v)(-v+u) - \sigma\int_\Omega u^p f\xi(w)(-w+u) \\ &\le r\chi_0\int_\Omega u^p f + \sigma\xi_0\int_\Omega u^p f \\ &= (r\chi_0+\sigma\xi_0)\int_\Omega u^p f \end{aligned} \tag{3.6}$$

for all $t \in (0, T_{\max})$ with some $\varepsilon_0 \in (0,1)$. Noting $f \le 1$ in view of (3.1) and then using the Gagliardo-Nirenberg inequality and the mass conservation property (2.2), we have

$$\begin{aligned} \int_\Omega u^p f \le \int_\Omega u^p &= \|u^{\frac{p}{2}}\|^2_{L^2(\Omega)} \\ &\le c_1\Big(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)} + \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}\Big)^{2\theta}\|u^{\frac{p}{2}}\|^{2(1-\theta)}_{L^{\frac{2}{p}}(\Omega)} \\ &= c_1\Big(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)} + \|u_0\|^{\frac{p}{2}}_{L^1(\Omega)}\Big)^{2\theta}\|u_0\|^{p(1-\theta)}_{L^1(\Omega)} \\ &\le c_2\|\nabla u^{\frac{p}{2}}\|^{2\theta}_{L^2(\Omega)} + c_3 \end{aligned} \tag{3.7}$$

for all $t \in (0, T_{\max})$ with $c_1, c_2, c_3 > 0$, where $\theta := \frac{\frac{pn}{2}-\frac{n}{2}}{\frac{pn}{2}+1-\frac{n}{2}} \in (0,1)$. Also, noticing from $\chi \in L^1(\eta_1,\infty)$, $\xi \in L^1(\eta_2,\infty)$ (see (1.5), (1.6)) that

$$f \ge c_4 := \exp\Big(-r\int_{\eta_1}^\infty \chi(s)\,ds - \sigma\int_{\eta_2}^\infty \xi(s)\,ds\Big) > 0 \quad \text{on } \Omega\times(0,T_{\max}), \tag{3.8}$$

we obtain

$$\frac{4c_4}{p^2}\|\nabla u^{\frac{p}{2}}\|^2_{L^2(\Omega)} = \frac{4c_4}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \le \int_\Omega u^{p-2} f|\nabla u|^2. \tag{3.9}$$

for all $t \in (0, T_{\max})$. Combining (3.7) and (3.9) with (3.6), we see that

$$\frac{d}{dt}\int_\Omega u^p f \le c_5\int_\Omega u^p f - c_6\Big(\int_\Omega u^p f\Big)^{1/\theta} + c_7$$

for all $t \in (0, T_{\max})$ with $c_5, c_6, c_7 > 0$. This provides a constant $c_8 > 0$ such that $\int_\Omega u^p f \le c_8$, which again by (3.8) implies

$$\int_\Omega u^p \le \frac{p^2 c_8}{4c_4}$$

for all $t \in (0, T_{\max})$ and thereby we arrive at the conclusion. $\square$

We are in a position to complete the proof of Theorem 1.1. If χ, ξ satisfy (1.5)–(1.10) with α, β fulfilling (1.11), then, according to the relations that $\chi(s) \le \chi(\eta_1)$ for all $s \ge \eta_1$ and $\xi(s) \le \xi(\eta_2)$ for all $s \ge \eta_2$ (see (1.9) and (1.10)), a combination of Lemmas 2.3 and 3.6, along with (2.4), leads to the end of the proof.

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