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EXISTENCE AND NONLINEAR STABILITY OF SOLITARY WAVE SOLUTIONS FOR COUPLED SCHRÖDINGER-KDV SYSTEMS

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ABSTRACT. In this article, we consider the existence and nonlinear stability of the solitary wave solutions to the coupled Schrödinger-KdV system. By using the undetermined coefficient method, we construct the exact solitary wave solutions. Furthermore, we prove the nonlinear stability of such solitary wave solutions with respect to small perturbations by applying the classical stability theory developed by Benjamin [8] and Bona [9], and the spectral analysis method.

1. INTRODUCTION

The interaction models between long waves and short waves play a fundamental role in a variety of physical settings, such as plasmas physics [19], diatomic lattice system [24], quantum mechanics [6] and fluid mechanics [20]. To describe the resonant interaction between gravity long wave and interface short wave on shallow water surface, when the group velocity of the short wave is close to the phase velocity of the long wave, Kawahara et al. [20] derived the coupled Schrödinger-KdV system

$$\begin{gathered} i(u_t + c_0 u_x) + \delta_1 u_{xx} = \alpha uv, \\ v_t + c_1 v_x + \delta_2 v_{xxx} + \beta (v^2)_x + \eta (|u|^2)_x = 0, \end{gathered} \tag{1.1}$$

where $c_0, c_1, \delta_1, \delta_2, \alpha, \beta, \eta$ are real constants, $u(x,t)$ is a complex value function describing interface short wave and $v(x,t)$ is a real value function describing gravity long wave.

It is obvious that, with the transformation $u \to u \cdot \exp\big(-\frac{c_0}{2\delta_1} i(x - \frac{c_0}{2}t)\big)$, system (1.1) can be reduced to

$$\begin{gathered} iu_t + \delta_1 u_{xx} = \alpha uv, \\ v_t + c_1 v_x + \delta_2 v_{xxx} + \beta (v^2)_x + \eta (|u|^2)_x = 0. \end{gathered} \tag{1.2}$$

During the past several decades, the coupled Schrödinger-KdV system has received extensive attention because of its important physical background. For the Cauchy problem of (1.2), please see [7, 12, 21, 23] and references therein. Tsutsumi [21] proved the global well-posedness in the space $H^{k+\frac{1}{2}}(\mathbb{R}) \times H^k(\mathbb{R}) (k \in Z^+)$ by using the conservation laws. Bekiranov et al. [7] used the Fourier restriction norm method to weaken the regularity assumptions on the initial data and obtained the

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local well-posedness in $H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ for any $s > 0$. Corcho and Linaves [12] improved the previous results of [7, 21] and obtained the local well-posedness in $L^2(\mathbb{R}) \times H^{-\frac{3}{4}+}(\mathbb{R})$ and a global result in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Wu [23] extended the result of [12] and obtained the global well-posedness in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ when $s > \frac{1}{2}$ whether the system is in the resonant case or in the non-resonant case by the I-method of Colliander et al. (see [13, 14] for examples).

Another issues of great concern for this model are the existence and stability of the solitary wave solutions. It is known that, due to the effect of nonlinearity and dispersion, the coupled Schrödinger-KdV system usually possesses such kind of solutions. Please see [1, 2, 4, 5, 11, 25] for the related results. Chen [11] considered a special model with $\delta_1 = \alpha = c_1 = \eta = 1$ in (1.2) and obtained the orbital stability of solitary wave solutions by using the abstract method of Grillakis et al. [16, 17]. Then, for system (1.2) with $\alpha = \eta = -\delta_1 = -1$, $c_1 = 0$, $\delta_2 = 2$ and a certain range of values of β, by using the concentration compactness method, Albert and Angulo [1] proved that the system has a nonempty set of ground state solutions which is stable. For system (1.2) with $\delta_1 = 1$ and $\beta = -\frac{3}{2}\alpha$, Angulo [2] also proved the existence and stability of a nonempty set of solitary wave solutions by using the stability theory developed by Cazenave and Lions in [10] and the concentration compactness method.

In this article, we consider the general model (1.2) and use the classical method of Benjamin [8] and Bona [9] to establish the results on the existence and orbital stability of solitary wave solutions. The results obtained in this paper can be regarded as a supplementary extension of [1, 2, 11]. The crucial idea of our proof is to show that solitary wave solutions is the local minimizer of the conserved functional for (1.2) via the detailed spectral analysis.

The remainder of his paper is organized as follows. In Section 2, we construct the exact solitary wave solutions of Schrödinger-KdV system (1.2). In Section 3, we give the spectral analysis which is needed to prove the stability of solitary wave solutions. In Section 4, we complete the proof of the orbital stability of the solitary wave solutions for (1.2).

Notation. The set of all real numbers is denoted by $\mathbb{R}$. The norm of $f \in L^p(\mathbb{R})$ is defined by $\|f\|_{L^p(\mathbb{R})} = (\int_{\mathbb{R}} |f|^p dx)^{1/p}$ for $1 \le p < \infty$, and $\|f\|_{L^\infty(\mathbb{R})}$ denotes the norm of $f \in L^\infty(\mathbb{R})$ which is defined as the essential supremum of f on $\mathbb{R}$. The inner product of two functions f, g in $L^2(\mathbb{R})$ is defined by $(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)}dx$. The Fourier transform of f is denoted by $\hat{f}$ which is defined as follows

$$\hat{f}(\tau) = \int_{\mathbb{R}} f(x)e^{-i\tau x}dx.$$

For $s \ge 0$, $H^s(\mathbb{R})$ denotes the Sobolev space with the norm

$$\|f\|_{H^s(\mathbb{R})} = \Big(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}|^2 d\xi \Big)^{1/2}.$$

It is obvious that $\|f\|^2_{H^1(\mathbb{R})} = \|f\|^2_{L^2(\mathbb{R})} + \|f'\|^2_{L^2(\mathbb{R})}$.

2. Existence of solitary wave solutions to system (1.2)

In this section, we seek the exact solitary wave solutions of system (1.2) of the form

$$\begin{aligned} u(x,t) &= e^{-i\omega t}\tilde{\phi}(\xi) = e^{-i\omega t}e^{iq(x-ct)}\phi(x-ct),\\ v(x,t) &= \varphi(\xi) = \varphi(x-ct), \end{aligned} \tag{2.1}$$

where $c, q, \omega \in \mathbb{R}$, $\xi = x - ct$, and $\phi(\xi), \varphi(\xi)$ are real functions satisfying $\phi(\xi) \to 0$ and $\varphi(\xi) \to 0$ as $|\xi| \to +\infty$.

Substituting (2.1) into (1.2), we obtain that $(\phi(\xi), \varphi(\xi))$ satisfies

$$\begin{gathered} \delta_1\phi'' + i(2\delta_1 q - c)\phi' + (\omega + qc - \delta_1 q^2 - \alpha\varphi)\phi = 0,\\ \delta_2\varphi'' + \beta\varphi^2 - (c - c_1)\varphi + \eta\phi^2 = 0. \end{gathered}$$

Noting that, both $\phi(\xi)$ and $\varphi(\xi)$ are real functions, so we need to require $q = \frac{c}{2\delta_1}$, which further reduces the above system to

$$\begin{gathered} \delta_1\phi'' + (\omega + \frac{c^2}{4\delta_1} - \alpha\varphi)\phi = 0,\\ \delta_2\varphi'' + \beta\varphi^2 - (c - c_1)\varphi + \eta\phi^2 = 0. \end{gathered} \tag{2.2}$$

Thus, the solitary wave solutions of system (1.2) can be constructed by solving system (2.2).

Theorem 2.1. *If $\omega, \alpha, \beta, c, c_1, \delta_1, \delta_2, \eta \in \mathbb{R}$ satisfy*

$$\delta_1\alpha\eta > 0, \quad 4\delta_1\omega + c^2 < 0, \quad c_1 - 4\delta_2(\frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2}) > c.$$

Then there exists a solitary wave solution of (1.2) *of the form* (2.1).

Proof. Assume $\phi = d_1 \operatorname{sech}(d_2\xi)$, where d_1 and d_2 will be determined in what follows. Then

$$\phi'' = (d_2^2 - 2d_2^2\operatorname{sech}^2(d_2\xi))d_1\operatorname{sech}(d_2\xi) = \Big(-\frac{\omega}{\delta_1} - \frac{c^2}{4\delta_1^2} + \frac{\alpha}{\delta_1}\varphi\Big)\phi. \tag{2.3}$$

By (2.2) and (2.3), we obtain

$$\frac{\alpha}{\delta_1}\varphi = -2d_2^2\operatorname{sech}^2(d_2\xi) + d_2^2 + \frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2} = -2d_2^2\operatorname{sech}^2(d_2\xi), \tag{2.4}$$

$$d_2^2 = -\frac{\omega}{\delta_1} - \frac{c^2}{4\delta_1^2}. \tag{2.5}$$

Substituting (2.3)–(2.5) into the second equation of (2.2), we have

$$\begin{aligned} &\frac{2\delta_1(c-c_1)d_2^2}{\alpha}\operatorname{sech}^2(d_2\xi) + \frac{4d_2^4\beta\delta_1^2}{\alpha^2}\operatorname{sech}^4(d_2\xi)\\ &+ \frac{4\delta_1\delta_2 d_2^4}{\alpha}(3\operatorname{sech}^4(d_2\xi) - 2\operatorname{sech}^2(d_2\xi)) + \eta d_1^2\operatorname{sech}^2(d_2\xi)\\ &= \Big(\frac{2\delta_1(c-c_1)d_2^2}{\alpha} - \frac{8\delta_1\delta_2 d_2^4}{\alpha} + \eta d_1^2\Big)\operatorname{sech}^2(d_2\xi)\\ &\quad + \Big(\frac{12\delta_1\delta_2 d_2^4}{\alpha} + \frac{4d_2^4\beta\delta_1^2}{\alpha^2}\Big)\operatorname{sech}^4(d_2\xi) = 0. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we obtain

$$q = \frac{c}{2\delta_1}, \quad \delta_2 = -\frac{\delta_1 \beta}{3\alpha},$$
$$d_1 = \sqrt{\frac{2\delta_1}{\alpha\eta}(-\frac{\omega}{\delta_1} - \frac{c^2}{4\delta_1^2})\Big(c_1 - c - 4\delta_2(\frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2})\Big)}, \quad d_2 = \sqrt{-\frac{\omega}{\delta_1} - \frac{c^2}{4\delta_1^2}}.$$

Thus, we have

$$\phi(\xi) = \sqrt{\frac{2\delta_1}{\alpha\eta}(-\frac{\omega}{\delta_1} - \frac{c^2}{4\delta_1^2})\big(c_1 - c - 4\delta_2(\frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2})\big)}\,\mathrm{sech}\,\big(\frac{\sqrt{-4\omega\delta_1 - c^2}}{2\delta_1}\xi\big),$$
$$\varphi(\xi) = \frac{4\omega\delta_1 + c^2}{2\alpha\delta_1}\,\mathrm{sech}^2\,\big(\frac{\sqrt{-4\omega\delta_1 - c^2}}{2\delta_1}\xi\big).$$

The proof is complete. □

3. Spectral analysis

By (2.2) and Theorem 2.1, we have

$$\begin{gathered}
\Big(-\frac{d^2}{d\xi^2} - (\frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2}) + \frac{3\alpha}{\delta_1}\varphi\Big)\phi' = 0, \\
\Big(-\frac{d^2}{d\xi^2} - (\frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2}) + \frac{\alpha}{\delta_1}\varphi\Big)\phi = 0, \\
\delta_2\varphi'' + \beta\varphi^2 - (c - c_1)\varphi + \eta\phi^2 = \Big(\delta_2\frac{d^2}{d\xi^2} + \frac{\delta_2(4\delta_1\omega + c^2)}{\delta_1^2} + \beta\varphi\Big)\varphi = 0.
\end{gathered} \tag{3.1}$$

Now, we define

$$\begin{gathered}
L_1 = -\frac{d^2}{d\xi^2} - (\frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2}) + \frac{3\alpha}{\delta_1}\varphi, \\
L_2 = -\frac{d^2}{d\xi^2} - (\frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2}) + \frac{\alpha}{\delta_1}\varphi, \\
L_3 = \delta_2\frac{d^2}{d\xi^2} + \frac{\delta_2(4\delta_1\omega + c^2)}{\delta_1^2} + \beta\varphi;
\end{gathered} \tag{3.2}$$

therefore $L_1\phi' = 0$, $L_2\phi = 0$, $L_3\varphi = 0$.

To prove the orbital stability of the solitary in next section, we study the spectra of the self-adjoint operators L_1, L_2 and L_3.

Theorem 3.1. *Let $\delta_2 < 0$, ϕ and φ be the solitaty wave solutions given by Theorem 2.1. Then*

(i) *operator L_1 in (3.2) defined in $H^2(\mathbb{R})$ whose domain is $L^2(\mathbb{R})$ has exactly one negative eigenvalue which is simple; zero is the second simple eigenvalue with eigenfunction ϕ'. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues;*

(ii) *operator L_2 in (3.2) defined in $H^2(\mathbb{R})$ whose domain is $L^2(\mathbb{R})$ has only non-negative eigenvalues and zero is the first one which is simple with eigenfunction ϕ. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues;*

(iii) *operator L_3 in* (3.2) *defined in $H^2(\mathbb{R})$ whose domain is $L^2(\mathbb{R})$ has only nonnegative eigenvalues and zero is the first one which is simple with eigenfunction φ. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.*

Proof. Since $x = 0$ is a unique zero point of ϕ', by using the Sturm-Liouville Theorem [15], we obtain that zero is the second eigenvalue of L_1. Hence, L_1 has a negative eigenvalue $-\sigma^2$ whose corresponding eigenfunction is χ, satisfying

$$L_1\chi = -\sigma^2\chi, \quad \langle \chi, \chi\rangle = 1.$$

Similarly, ϕ and φ have no zero point in $\mathbb{R}$, then zero is the first eigenvalue of L_2 and L_3 by the Sturm-Liouville Theorem. Furthermore, noting (3.2), we have

$$\begin{gathered}
\frac{3\alpha}{\delta_1}\varphi \to 0, \quad \text{as } |x| \to +\infty,\\
\frac{\alpha}{\delta_1}\varphi \to 0, \quad \text{as } |x| \to +\infty,\\
\beta\varphi \to 0, \quad \text{as } |x| \to +\infty.
\end{gathered}$$

Then by Weyl's essential spectral Theorem [18], we have

$$\begin{gathered}
\sigma_{\text{ess}}(L_1) = [-(\frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2}), +\infty),\\
\sigma_{\text{ess}}(L_2) = [-(\frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2}), +\infty),\\
\sigma_{\text{ess}}(L_3) = [\frac{\delta_2(4\delta_1\omega + c^2)}{\delta_1^2}, +\infty),
\end{gathered}$$

where $\frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2} < 0$ and $\delta_2 < 0$. The theorem is proved. □

Now let us do further study on the properties of operators L_1, L_2 and L_3, which will be used later in the proof of stability. To do so, we need the following lemma.

Lemma 3.2 ([22]). *Let L be a self-adjoint operator having exactly one negative eigenvalue λ_0 with corresponding ground state eigenfunction $f_0 \ge 0$. Define*

$$-\infty < \alpha \equiv \min_f (Lf, f), \quad \text{where } \|f\|_{L^2(\mathbb{R})} = 1 \text{ and } (f, R) = 0.$$

We assume $(R, f_0) \neq 0$ and $R \in N^\perp(L)$. Then $\alpha \ge 0$ if

$$(L^{-1}R, R) \le 0.$$

Theorem 3.3. *Under the conditions of Theorems 2.1 and 3.1, we have*

$$\inf\{(L_2\psi, \psi) : \psi \in H^1(\mathbb{R}), \|\psi\|_{L^2(\mathbb{R})} = 1, (\psi, \phi\varphi) = 0\} := \iota_1 > 0. \tag{3.3}$$

Proof. By Theorem 3.1, we know that L_2 is a nonnegative operator, so it is obvious that $\iota_1 \ge 0$.

In what follows, we suppose that $\iota_1 = 0$. Firstly, we prove that the infimum of (3.3) can be attained. Let $\{\psi_i\}$ be a sequence of $H^1(\mathbb{R})$-functions with $\|\psi_i\|_{L^2(\mathbb{R})} = 1, (\psi_i, \phi\varphi) = 0$ and $(L_2\psi_i, \psi_i) \to \iota_1$ as $i \to \infty$. It follows that $\|\psi_i\|_{H^1(\mathbb{R})}$ is bounded for any $i \ge 0$. Then there is a subsequence of $\{\psi_i\}$ which is still denoted by itself such that $\psi_i \rightharpoonup \Phi$ weakly in $H^1(\mathbb{R})$. Now, since the classical embedding

$H^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ is compact, we obtain that Φ satisfies $\|\Phi\|_{L^2(\mathbb{R})} = 1$ and $(\Phi, \phi\varphi) = 0$. Furthermore, since weak convergence is lower semi-continuous, it follows that

$$\iota_1 \leq (L_2\Phi, \Phi) < \liminf_{i\to\infty}(L_2\psi_i, \psi_i) = \iota_1.$$

Therefore, the infimum ι_1 of (3.3) is attained at some admissible function $\Phi \neq 0$. Thus, there exists a function Φ with $\|\Phi\|_{L^2(\mathbb{R})} = 1, (\Phi, \phi\varphi) = 0$ and $(L_2\Phi, \Phi) = 0$.

Next, from the theory of Lagrange multipliers, there are real constants k_1, k_2 such that

$$L_2\Phi = k_1\Phi + k_2\phi\varphi.$$

Because $(L_2\Phi, \Phi) = 0$ and $(\Phi, \phi\varphi) = 0$, we obtain $k_1 = 0$. And since $L_2\phi = 0$, we have

$$k_2 \int_{\mathbb{R}} \phi^2\varphi d\xi = (L_2\Phi, \phi) = 0,$$

which implies $k_2 = 0$. Then $L_2\Phi = 0$. There is a real constant $k_3 \neq 0$ such that $\Phi = k_3\phi$. But

$$0 = (\Phi, \phi\varphi) = k_3 \int_{\mathbb{R}} \phi^2\varphi d\xi \neq 0,$$

which is a contradiction. Therefore the minimum $\iota_1 > 0$. The proof is complete. $\square$

Remark 3.4. From Theorem 3.3 and the specific form of L_2, we have that if $f \in H^1(\mathbb{R})$ satisfies $(f, \phi\varphi) = 0$, then

$$(L_2f, f) \geq \delta_2\|f\|^2_{H^1(\mathbb{R})}.$$

Theorem 3.5. *Under the conditions of Theorems 2.1 and 3.1, if*

$$c_1 - 8\delta_2(\frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2}) > c, \tag{3.4}$$

then: (i)

$$\inf\big\{(L_1\psi, \psi) : \psi \in H^1(\mathbb{R}), \|\psi\|_{L^2(\mathbb{R})} = 1, (\psi, \phi) = 0\big\} := \iota_2 = 0;$$

and (ii)

$$\inf\big\{(L_1\psi, \psi) : \psi \in H^1(\mathbb{R}), \|\psi\|_{L^2(\mathbb{R})} = 1, (\psi, \phi) = 0, (\psi, (\phi\varphi)') = 0\big\} := \iota_3 > 0.$$

Proof. The solitary wave solution ϕ given by Theorem 2.1 is a bounded function which implies that ι_2 is finite. And since $(\phi', \phi) = 0$, $L_1\phi' = 0$, we have $\iota_2 \leq 0$.

Furthermore, we can obtain $\iota_2 = 0$ by proving $\iota_2 \geq 0$ in virtue of Lemma 3.2. According to Theorem 3.1, we obtain that the operator L_1 satisfies the condition of Lemma 3.2. So, we only need to find a function χ satisfying $L_1\chi = \phi$ and $(\chi, \phi) \leq 0$. In fact, we define the mapping $\mu \to \phi_\mu \in H^1(\mathbb{R})$, where $\mu = -(\frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2})$. By differentiating (3.1) with respect to μ, it yields

$$-\frac{\partial^2}{\partial x^2}\frac{d\phi}{d\mu} + \phi - (\frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2})\frac{d\phi}{d\mu} + \frac{3\alpha}{\delta_1}\varphi\frac{d\phi}{d\mu} = 0.$$

Thus $\chi = -\frac{d\phi}{d\mu}$ satisfies $L_1\chi = \phi$. Namely, $\chi = L_1^{-1}\phi$. Furthermore, we have

$$\begin{aligned}(\chi, \phi) &= (-\frac{d\phi}{d\mu}, \phi)\\ &= -\frac{1}{2}\frac{d}{d\mu}\int_{\mathbb{R}} \phi^2 d\xi\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\frac{d}{d\mu}\int_{\mathbb{R}} \frac{2\delta_1}{\alpha\eta}\mu(c_1 - c + 4\delta_2\mu)\operatorname{sech}^2(\xi)d\xi \\
&= -\frac{\delta_1}{\alpha\eta}(c_1 - c + 8\delta_2\mu)\int_{\mathbb{R}} \operatorname{sech}^2(\xi)d\xi.
\end{aligned}
$$

By (3.4) and the conditions of Theorem 2.1, we know $(\chi, \phi) < 0$. Then, according to Lemma 3.2, we obtain $\iota_2 \geq 0$. Therefore $\iota_2 = 0$. The proof of (i) is complete.

By (i), we have $\iota_3 \geq 0$. In what follows, we suppose that $\iota_3 = 0$. By using the similar proof of Theorem 3.3, we can obtain an admissible function Φ satisfying $\|\Phi\|_{L^2(\mathbb{R})} = 1, (\Phi, \phi) = 0, (\Phi, (\phi\varphi)') = 0$ and $(L_1\Phi, \Phi) = 0$.

Next, from the theory of Lagrange multipliers, there are real constants k_4, k_5, k_6 such that

$$
L_1\Phi = k_4\Phi + k_5\phi + k_6(\phi\varphi)'.
$$

From $(L_1\Phi, \Phi) = 0, (\Phi, \phi) = 0$ and $(\Phi, (\phi\varphi)') = 0$, we obtain $k_4 = 0$. Since $L_1\phi' = 0, (\phi, \phi') = 0$, we have

$$
k_6\int_{\mathbb{R}} \phi'(\phi\varphi)'d\xi = \frac{-3k_6\eta}{(c_1 - c + 4\delta_2(-\frac{\omega}{\delta_1} - \frac{c^2}{4\delta_1^2}))}\int_{\mathbb{R}}(\phi')^2\phi^2 d\xi = 0.
$$

By (3.4), we obtain $k_6 = 0$. Thus $L_1\Phi = k_5\phi$. Since $L_1\chi = \phi$ with $\chi = -\frac{d\phi}{d\mu}$, we have $L_1(\Phi - k_5\chi) = 0$. Therefore there exists a real constant $k_7 \neq 0$ such that $\Phi - k_5\chi = k_7\phi'$. Since $(\chi, \phi) \neq 0, (\phi', \phi) = 0$ and $(\Phi, \phi) = 0$, we obtain $k_5 = 0$. That is, $\Phi = k_7\phi'$. But

$$
0 = (\Phi, (\phi\varphi)') = k_7(\phi', (\phi\varphi)') = \frac{-3k_7\eta}{(c_1 - c + 4\delta_2(-\frac{\omega}{\delta_1} - \frac{c^2}{4\delta_1^2}))}\int_{\mathbb{R}}(\phi')^2\phi^2 d\xi \neq 0,
$$

which is a contradiction. Therefore $\iota_3 > 0$. The proof is complete. □

Remark 3.6. From (ii) in Theorem 3.5 and the specific form of L_1, we have that if $f \in H^1(\mathbb{R})$ satisfies $(f, \phi) = 0$ and $(f, (\phi\varphi)') = 0$, then

$$
(L_1 f, f) \geq \delta_1\|f\|^2_{H^1(\mathbb{R})}.
$$

4. Orbital stability

To obtain the stability of the solitary wave solutions, we rewrite (1.2) in the Hamiltonian form

$$
\frac{dU}{dt} = JE'(U), \quad U = (u, v) \in X,
$$

where $X = H^1_{complex}(\mathbb{R}) \times L^2_{real}(\mathbb{R})$, J is a skew-symmetrical matrix operator by

$$
J = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & -\frac{\eta}{\alpha}\frac{\partial}{\partial x} \end{pmatrix},
$$

$$
E(U) = \int_{\mathbb{R}} \Big(\delta_1|u_x|^2 + \alpha v|u|^2 + \frac{\alpha c_1}{2\eta}v^2 + \frac{\alpha\beta}{3\eta}v^3 - \frac{\alpha\delta_2}{2\eta}v_x^2\Big)dx, \tag{4.1}
$$

$$
E'(U) = \begin{pmatrix} -2\delta_1 u_{xx} + 2\alpha uv \\ \frac{\alpha\delta_2}{\eta}v_{xx} + \frac{\alpha c_1}{\eta}v + \frac{\alpha\beta}{\eta}v^2 + \alpha u^2 \end{pmatrix}.
$$

And the inner product in X is

$$
(\vec{u}, \vec{v}) = \operatorname{Re}\int_{\mathbb{R}} \Big(u_1\bar{v}_1 + u_{1x}\bar{v}_{1x} + u_2 v_2\Big)dx, \quad \vec{u} = (u_1, u_2), \vec{v} = (v_1, v_2) \in X. \tag{4.2}
$$

The dual space of X is $X^* = H^{-1}_{complex}(\mathbb{R}) \times L^{-2}_{real}(\mathbb{R})$. There exists a natural isomorphism $I : X \to X^*$, defined by

$$\langle I\vec{u}, \vec{v}\rangle = (\vec{u}, \vec{v}), \tag{4.3}$$

where

$$\langle \vec{u}, \vec{v}\rangle = \mathrm{Re}\int_{\mathbb{R}} (u_1\bar{v}_1 + u_2 v_2)\,dx, \quad \vec{u} = (u_1, u_2), \vec{v} = (v_1, v_2) \in X. \tag{4.4}$$

From (4.2)–(4.4), we obtain

$$I = \begin{pmatrix} 1 - \frac{\partial^2}{\partial x^2} & 0 \\ 0 & 1 \end{pmatrix}.$$

In the remainder of this paper, we will use the method of Benjamin [8] and Bona [9] to prove the orbital stability of the solitary wave solution $\Psi = (\tilde{\phi}(\xi), \varphi(\xi))$ with $\tilde{\phi}(\xi) = e^{i\frac{c}{2\delta_1}\xi}\phi(\xi)$ given by Theorem 2.1. First of all, let us give the definition of orbital stability.

Definition 4.1. We say that the orbit generated by $\Psi = (\tilde{\phi}, \varphi)$,

$$\Omega_\Psi := \{(e^{i\theta}\tilde{\phi}(\cdot + y), \varphi(\cdot + y)) : (y, \theta) \in \mathbb{R} \times [0, 2\pi)\} \tag{4.5}$$

is stable in $X = H^1_{complex}(\mathbb{R}) \times L^2_{real}(\mathbb{R})$ by the flow of (1.2), if for every $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that, for any $(u_0(x,t), v_0(x,t)) \in X$ satisfying

$$\|u_0 - \tilde{\phi}\|_{H^1(\mathbb{R})} < \delta, \quad \|v_0 - \varphi\|_{L^2(\mathbb{R})} < \delta,$$

the solution of the Schrödinger-KdV equations (1.2) with initial data $u(0) = u_0$, $v(0) = v_0$ exists globally and satisfies

$$\inf_{y\in\mathbb{R}, \theta\in[0,2\pi)} \|e^{i\theta}u(\cdot + y, t) - \tilde{\phi}\|_{H^1(\mathbb{R})} < \varepsilon, \quad \inf_{y\in\mathbb{R}} \|v(\cdot + y, t) - \phi\|_{L^2(\mathbb{R})} < \varepsilon,$$

for any $t \in \mathbb{R}$.

Otherwise, we say that $\Psi = (\tilde{\phi}, \varphi)$ is unstable in X.

For the proof of orbital stability, we need to introduce two energy functions. Let T_1 and T_2 be the one-parameter group of unitary operator on X defined by

$$\begin{gathered} T_1(s_1)U(\cdot) = U(\cdot - s_1), \forall s_1 \in \mathbb{R}, U(\cdot) = (u(\cdot), v(\cdot)) \in X, \\ T_2(s_2)U(\cdot) = (e^{-is_2}u(\cdot), v(\cdot)), \forall s_2 \in \mathbb{R}, U(\cdot) = (u(\cdot), v(\cdot)) \in X. \end{gathered} \tag{4.6}$$

From (4.6), we obtain

$$T_1'(0) = \begin{pmatrix} -\frac{\partial}{\partial x} & 0 \\ 0 & -\frac{\partial}{\partial x} \end{pmatrix}, \quad T_2'(0) = \begin{pmatrix} -i & 0 \\ 0 & 0 \end{pmatrix}.$$

By requiring $T_1'(0) = JB_1$ and $T_2'(0) = JB_2$, we can obtain

$$B_1 = \begin{pmatrix} -2i\frac{\partial}{\partial x} & 0 \\ 0 & -\frac{\alpha}{\eta} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, we define

$$Q_1(U) = \frac{1}{2}\langle B_1 U, U\rangle = \int_{\mathbb{R}} \mathrm{Im}(u_x\bar{u})dx + \frac{\alpha}{2\eta}\int_{\mathbb{R}} v^2 dx, \tag{4.7}$$

$$Q_2(U) = \frac{1}{2}\langle B_2 U, U\rangle = \int_{\mathbb{R}} |u|^2 dx, \tag{4.8}$$

where $U(\cdot) = (u(\cdot), v(\cdot)) \in X$. It is easy to verify that $E(U)$, $Q_1(U)$ and $Q_2(U)$ are invariant under the transformation of T_1 and T_2 (see [16, 17] for details), that is,

$$\begin{aligned} E(T_1(s_1)T_2(s_2)U) &= E(U), \\ Q_1(T_1(s_1)T_2(s_2)U) &= Q_1(U), \\ Q_2(T_1(s_1)T_2(s_2)U) &= Q_2(U), \end{aligned} \tag{4.9}$$

for any $s_1, s_2 \in \mathbb{R}$, where $U(t) = (u(t), v(t))$ is a flow of (1.2) with

$$\begin{aligned} E(u(t), v(t)) &= E(u(0), v(0)) = E(u_0, v_0), \\ Q_1(u(t), v(t)) &= Q_1(u(0), v(0)) = Q_1(u_0, v_0), \\ Q_2(u(t), v(t)) &= Q_2(u(0), v(0)) = Q_2(u_0, v_0). \end{aligned} \tag{4.10}$$

To investigate the orbital stability, we need to use some related results on the local and global well-posedness of the initial value problem of (1.2) which is actually studied extensively in [7, 12, 21, 23]. So we omit the details here and enter into the study of orbital stability directly.

Theorem 4.2. *Under the conditions of Theorem 2.1, if*

$$\delta_2 < 0, \quad \delta_1 > 0, \quad \beta > 0, \quad c_1 + 10\delta_2\Big(-\frac{\omega}{\delta_1} - \frac{c^2}{4\delta_1^2}\Big) > c, \tag{4.11}$$

then the orbit Ω_Ψ given by (4.5) *is orbitally stable in $X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with respect to the flow of the nonlinear Schrödinger-KdV system* (1.2).

Proof. The main idea of our proof is based on the method of Benjamin [8], Bona [9], and Weinstein [22]. Let us start with the declaration, for any initial data $(u_0, v_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, $(u(t), v(t))$ is the global solution of Schrödinger-KdV system (1.2) with initial value (u_0, v_0). If we define

$$\Omega_t(y, \theta) = \|e^{i\theta}(T_3u)'(\cdot + y, t) - \phi'\|^2_{L^2(\mathbb{R})} + \mu\|e^{i\theta}(T_3u)(\cdot + y, t) - \phi\|^2_{L^2(\mathbb{R})},$$

where $\mu = -(\frac{\omega}{\delta_1} + \frac{c^2}{4\delta_1^2})$ and $T_3u = e^{-i\frac{c}{2\delta_1}(x-ct)}u(x,t)$, then the error of the solution $(u(t), v(t))$ from Ω_Ψ is measured by

$$\rho((u(t), v(t)), \Omega_\Psi) = \sqrt{\inf_{(y,\theta)\in\mathbb{R}\times[0,2\pi)} \Omega_t(y, \theta)}.$$

So, by using the standard arguments in [8, 9], there is an interval $I = [0, T]$ such that the infimum of $\Omega_t(y, \theta)$ is reached in $(y, \theta) = (y(t), \theta(t))$ for any $t \in I$. Then we have

$$\Big(\rho((u(t), v(t)), \Omega_\Psi)\Big)^2 = \Omega_t(y(t), \theta(t)). \tag{4.12}$$

Now, let us consider the perturbation of the solitary wave solutions $\Psi = (\tilde{\phi}, \varphi)$ which can be written as

$$\begin{aligned} e^{i\theta}u(x + y, t) &= \tilde{\phi} + \tilde{\gamma}_1(x, t), \\ v(x + y, t) &= \varphi + \gamma_2(x, t), \end{aligned} \tag{4.13}$$

with $\tilde{\phi} = e^{i\frac{c}{2\delta_1}(x-ct)}\phi$, $y = y(t)$ and $\theta = \theta(t)$ are determined by (4.12). For ease of calculation, we denote $\tilde{\gamma}_1(x,t) = e^{i\frac{c}{2\delta_1}(x-ct)}\gamma_1(x,t) = e^{i\frac{c}{2\delta_1}(x-ct)}(p(x,t) + iq(x,t))$ with real functions $p(x,t), q(x,t)$.

Since the minimum of $\Omega_t(y,\theta)$ can be reached in $(y,\theta)=(y(t),\theta(t))$, we can obtain that $\frac{\partial\Omega_t}{\partial\theta}|_{\theta=\theta(t)}=0$ and $\frac{\partial\Omega_t}{\partial y}|_{y=y(t)}=0$. Hence,

$$\begin{gathered}\frac{\partial\Omega_t}{\partial\theta}|_{\theta=\theta(t)}=-2\int_{\mathbb{R}}(\phi''-\mu\phi)qdx=-2\int_{\mathbb{R}}(\phi\varphi)qdx=0,\\ \frac{\partial\Omega_t}{\partial y}|_{y=y(t)}=-2\int_{\mathbb{R}}(\phi'''-\mu\phi')pdx=-2\int_{\mathbb{R}}(\phi\varphi)'pdx=0.\end{gathered}$$

From the above equations, we obtain the following compatibility relation between $p(x,t)$ and $q(x,t)$

$$\int_{\mathbb{R}}(\phi(x)\varphi(x))q(x)dx=0,\quad \int_{\mathbb{R}}(\phi(x)\varphi(x))'p(x)dx=0. \tag{4.14}$$

We define the continuous functional in $X=H^1(\mathbb{R})\times H^1(\mathbb{R})$:

$$H(u,v)=E(u,v)-cQ_1(u,v)-\omega Q_2(u,v),$$

where E, Q_1 and Q_2 are the conserved functional given in (4.1), (4.7) and (4.8). According to (4.9) and (4.10), the values of E,Q_1 and Q_2 are invariant under translation and rotation. By (2.2), (4.13) and the classical embedding $H^1(\mathbb{R})\hookrightarrow L^p(\mathbb{R})$, for any $p\ge 2$, we have

$$\begin{aligned}
&\Delta H(u,v)\\
&=H(u,v)-H(\tilde\phi,\varphi)\\
&=\delta_1\langle L_1p,p\rangle+\delta_1\langle L_2q,q\rangle+2\delta_1\langle L_2\phi,p\rangle+\frac{\alpha}{\eta}\langle L_3\varphi,\gamma_2\rangle+\frac{\alpha}{2\eta}\langle L_3\gamma_2,\gamma_2\rangle\\
&\quad+\int_{\mathbb{R}}\frac{\alpha\beta}{2\eta}\varphi\gamma_2^2+\alpha\gamma_2(p^2+2p\phi+q^2)+\frac{\alpha}{2\eta}\Big(c_1-c-\frac{\delta_2(4\delta_1\omega+c^2)}{\delta_1^2}\Big)\gamma_2^2dx\\
&\quad+\int_{\mathbb{R}}\frac{c^2}{4\delta_1}\phi^2-2\alpha\varphi p^2+\frac{\alpha\beta}{3\eta}\gamma_2^3dx\\
&=\delta_1\langle L_1p,p\rangle+\delta_1\langle L_2q,q\rangle+\frac{\alpha}{2\eta}\langle L_3\gamma_2,\gamma_2\rangle+\int_{\mathbb{R}}\frac{c^2}{4\delta_1}\phi^2-2\alpha\varphi p^2+\frac{\alpha\beta}{3\eta}\gamma_2^3dx\\
&\quad+\int_{\mathbb{R}}m_1\gamma_2^2+2\gamma_2\frac{\alpha(p^2+2p\phi+q^2)}{2}+\frac{\alpha^2(p^2+2p\phi+q^2)^2}{4m_1}dx\\
&\quad+\int_{\mathbb{R}}m_1\gamma_2^2-\frac{\alpha^2(p^2+2p\phi+q^2)^2}{4m_1}dx\\
&=\delta_1\langle L_1p,p\rangle+\delta_1\langle L_2q,q\rangle+\frac{\alpha}{2\eta}\langle L_3\gamma_2,\gamma_2\rangle\\
&\quad+\int_{\mathbb{R}}\Big(\gamma_2\sqrt{m_1}+\frac{\alpha(p^2+2p\phi+q^2)}{\sqrt{4m_1}}\Big)^2dx\\
&\quad+\int_{\mathbb{R}}m_1\gamma_2^2-\frac{\alpha^2(p^2+2p\phi+q^2)^2}{4m_1}+\frac{c^2}{4\delta_1}\phi^2-2\alpha\varphi p^2+\frac{\alpha\beta}{3\eta}\gamma_2^3dx,
\end{aligned}\tag{4.15}$$

where

$$m_1:=\frac{\alpha}{4\eta}\Big(\beta\varphi+c_1-c-\frac{\delta_2(4\delta_1\omega+c^2)}{\delta_1^2}\Big).$$

Since $\varphi<0$, by (4.11), we have

$$\int_{\mathbb{R}}-2\alpha\varphi p^2dx>0,\quad m_1>0.$$

Thus, (4.15) can be reduced to

$$\begin{aligned}\Delta H(u,v) &\geq \delta_1\langle L_1 p,p\rangle + \delta_1\langle L_2 q,q\rangle + \frac{\alpha}{2\eta}\langle L_3\gamma_2,\gamma_2\rangle \\ &\quad + \int_{\mathbb{R}} \Big(\gamma_2\sqrt{m_1} + \frac{\alpha(p^2+2p\phi+q^2)}{\sqrt{4m_1}}\Big)^2 dx \\ &\quad - C_0\|\gamma_1\|^4_{H^1(\mathbb{R})} + C_1\|\gamma_2\|^2_{L^2(\mathbb{R})} - C_2\|\gamma_2\|^3_{L^2(\mathbb{R})},\end{aligned} \tag{4.16}$$

where C_1 and C_2 are positive constants. Now let us estimate the terms $\langle L_1 p,p\rangle$, $\langle L_2 q,q\rangle$ and $\langle L_3\gamma_2,\gamma_2\rangle$, where $p(x,t), q(x,t)$ satisfy the compatibility relation (4.14).

We first estimate $\langle L_1 p,p\rangle$. Since $Q_2(U)$ is invariant, we consider the normalization $\|u_0\|_{L^2(\mathbb{R})} = \|\phi\|_{L^2(\mathbb{R})}$ for every $t\in[0,T]$. According to (4.13), we have

$$\int_{\mathbb{R}} \phi^2 dx = \|u(t)\|^2_{L^2(\mathbb{R})} = \|\gamma_1(t)+\phi(t)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} (p+\phi)^2 + q^2 dx.$$

Thus, we obtain

$$\int_{\mathbb{R}} (p^2+q^2)dx = -2\int_{\mathbb{R}} p\phi dx.$$

That is

$$\|\gamma_1\|^2_{L^2(\mathbb{R})} = -2(p,\phi),$$

for any $t\geq 0$. Without loss of generality, we suppose that $\|\phi\|^2_{L^2(\mathbb{R})} = 1$. To estimate $\langle L_1 p,p\rangle$, we define the following two variables

$$p_{\|} = (p,\phi)\phi = -\frac{1}{2}[\|p\|^2_{L^2(\mathbb{R})} + \|q\|^2_{L^2(\mathbb{R})}]\phi, \quad p_{\perp} = p - p_{\|}.$$

By (4.14), it is easy to see that

$$\begin{aligned}(p_\perp,(\phi\varphi)') &= \int_{\mathbb{R}} p(\phi\varphi)' - \frac{1}{2}(\|p\|^2_{L^2(\mathbb{R})} + \|q\|^2_{L^2(\mathbb{R})})\phi(\phi\varphi)' dx \\ &= \frac{3\|\gamma_1\|^2_{L(\mathbb{R})}}{1-c-\frac{4}{3}\beta(-\omega-\frac{c^2}{4})}\int_{\mathbb{R}} \phi^3(x)\phi'(x)dx = 0,\end{aligned} \tag{4.17}$$

and

$$(p_\perp,\phi) = \int_{\mathbb{R}} p\phi + \frac{1}{2}(\|p\|^2_{L^2(\mathbb{R})} + \|q\|^2_{L^2(\mathbb{R})})\phi^2 dx = 0. \tag{4.18}$$

Combining (4.17), (4.18) with Theorem 3.5, we have

$$(L_1 p_\perp, p_\perp) \geq C_3\|p_\perp\|^2_{H^1(\mathbb{R})} \geq C_3\|p\|^2_{H^1(\mathbb{R})} - C_4\|\gamma_1\|^4_{H^1(\mathbb{R})}. \tag{4.19}$$

Then, noting that $(L_1\phi,\phi) < 0$, we can obtain

$$(L_1 p_{\|}, p_{\|}) \geq -C_5\|\gamma_1\|^4_{H^1(\mathbb{R})}. \tag{4.20}$$

Furthermore, by the Cauchy-Schwarz inequality and the definition of L_1, we have

$$\begin{aligned}(L_1 p_\perp, p_{\|}) = (p_\perp, L_1 p_{\|}) &= \frac{1}{2}\|\gamma_1\|^2_{L(\mathbb{R})}(p_\perp, L_1\phi) \\ &\geq -C_6\|\gamma_1\|^3_{H^1(\mathbb{R})} - C_7\|\gamma_1\|^4_{H^1(\mathbb{R})}.\end{aligned} \tag{4.21}$$

Hence, by (4.19)–(4.21), we obtain

$$(L_1 p,p) \geq D_1\|p\|^2_{H^1(\mathbb{R})} - D_2\|\gamma_1\|^3_{H^1(\mathbb{R})} - D_3\|\gamma_1\|^4_{H^1(\mathbb{R})}, \tag{4.22}$$

where $D_i > 0$ for $i=1,2,3$.

Next, according to Theorem 3.3, (4.14) and the specific form of L_2, there is a $D_4 > 0$ such that

$$(L_2 q, q) \geq D_4 \|q\|^2_{H^1(\mathbb{R})}. \tag{4.23}$$

Finally, by Theorem 3.1, we have

$$\langle L_3\gamma_2, \gamma_2\rangle \geq 0. \tag{4.24}$$

Thus substituting (4.22)–(4.24) into (4.16), we have

$$\begin{aligned} \Delta H(u,v) &\geq \tilde{C}_1\|\gamma_1\|^2_{H^1(\mathbb{R})} - \tilde{C}_2\|\gamma_1\|^3_{H^1(\mathbb{R})} - \tilde{C}_3\|\gamma_1\|^4_{H^1(\mathbb{R})} + C_1\|\gamma_2\|^2_{L^2(\mathbb{R})} \\ &\quad - C_2\|\gamma_2\|^3_{L^2(\mathbb{R})} \\ &\geq b_1\|\gamma_1\|^2_{1,\mu} - b_2\|\gamma_1\|^3_{1,\mu} - b_3\|\gamma_1\|^4_{1,\mu} + b_4\|\gamma_2\|^2_{L^2(\mathbb{R})} - b_5\|\gamma_2\|^3_{L^2(\mathbb{R})} \\ &= b_1\|\tilde{\gamma_1}\|^2_{1,\mu} - b_2\|\tilde{\gamma_1}\|^3_{1,\mu} - b_3\|\tilde{\gamma_1}\|^4_{1,\mu} + b_4\|\gamma_2\|^2_{L^2(\mathbb{R})} - b_5\|\gamma_2\|^3_{L^2(\mathbb{R})} \\ &:= g(\|\tilde{\gamma_1}\|_{1,\mu}, \|\gamma_2\|_{L^2(\mathbb{R})}), \end{aligned} \tag{4.25}$$

where $g(s,z) = b_1 s^2 - b_2 s^3 - b_3 s^4 + b_4 z^2 - b_5 z^3$ with $b_i > 0$ for $i = 1,2,3,4,5$ and $\|\tilde{\gamma_1}\|^2_{1,\mu} = \|\tilde{\gamma_1}'\|^2_{L^2(\mathbb{R})} + \mu\|\tilde{\gamma_1}\|^2_{L^2(\mathbb{R})}$.

Obviously, $g(0,0) = 0$ and $g(s,z) > 0$ for $(s,z) \neq (0,0)$ belonging to some sufficiently small neighborhood of $(0,0)$. From (4.25), we can immediately get the result of stability of Theorem 4.2. In fact, let $\varepsilon > 0$, from the continuity of $H(u,v)$ on $S = \{u_0 \in H^1(\mathbb{R}), v_0 \in L^2(\mathbb{R}) : \|u_0\|_{L^2(\mathbb{R})} = \|\phi\|_{L^2(\mathbb{R})}\}$ and the continuity of the mapping $\rho((u(t), v(t)), \Omega_\Psi)$ in time, there is a $\delta(\varepsilon) > 0$ such that if $(u_0, v_0) \in S$ and

$$\|u_0 - \tilde{\phi}\|_{H^1(\mathbb{R})} < \delta(\varepsilon), \quad \|v_0 - \varphi\|_{L^2(\mathbb{R})} < \delta(\varepsilon),$$

then

$$g(\|\tilde{\gamma_1}\|_{1,\mu}, \|\gamma_2\|_{L^2(\mathbb{R})}) \leq \Delta H(u(t), v(t)) = \Delta H(u_0, v_0) \leq g(\varepsilon, \varepsilon), \tag{4.26}$$

for all $t \in [0,T]$. By (4.26) and the continuity of $\inf_{(y,\theta)\in\mathbb{R}\times[0,2\pi)} \Omega_t(y,\theta)$ as a function of t, we have

$$\|\tilde{\gamma_1}\|_{1,\mu} < \varepsilon, \quad \|\gamma_2\|_{L^2(\mathbb{R})} < \varepsilon. \tag{4.27}$$

Similar to the proof of [3, Theorem 6.1], we obtain that (4.27) still holds for all $t > 0$. Thus we know that the orbit Ω_Ψ is stable in X for the perturbations which are small in H_1 and L^2-norm, respectively. The proof is complete. □

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