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DECAY ESTIMATES FOR SOLUTIONS OF EVOLUTIONARY DAMPED p-LAPLACE EQUATIONS

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Abstract. In this note, we study the asymptotic behavior, as t tends to infinity, of the solution u to the evolutionary damped p-Laplace equation

$$u_{tt} + au_t = \Delta_p u$$

with Dirichlet boundary conditions. Let u^* denote the stationary solution with same boundary values, then we prove the $W^{1,p}$-norm of $u(t) - u^*$ decays for large t like $t^{-\frac{1}{(p-1)p}}$, in the degenerate case $p > 2$.

1. Introduction and problem setting

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $p \geq 2$ and $g \in W^{1,p}(\Omega)$. Consider the minimization of the functional

$$\mathscr{E}(u) := \frac{1}{p} \int_\Omega |\nabla u|^p \,\mathrm{d}x, \tag{1.1}$$

over the class $\mathscr{C} := \{u : u - g \in W_0^{1,p}(\Omega)\}$. The minimizer denoted by $u^*(x)$ satisfies the following Euler-Lagrange-equation in the weak sense:

$$\begin{aligned} -\Delta_p u^* &= 0 \quad \text{in } \Omega, \\ u^* &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

The first order flow of $\mathscr{E}(v)$, i.e. $v_t + \partial_v \mathscr{E}(v) = 0$, can be considered as a classical steepest descent flow for solving the minimization problem (1.1). In the degenerate case $p > 2$ the authors of [9] obtained the sharp decay rate

$$\sup_{x\in\Omega} |v(t,x) - u^*(x)| = O\big(t^{-\frac{1}{p-2}}\big) \quad \text{as } t \to \infty.$$

Their proof is based on the Moser iteration applied to the difference $v(t,x) - u^*(x)$, which itself is not a solution, thus bounding the L^∞-norm in terms of the L^p-norm.

It is well known, that an improvement in the convergence rate may be gained by considering the corresponding second order damped problem, cf. [6, 10, 3] and references therein. Moreover, second order damped problems naturally appear in modeling mechanical systems. For instance, the motion of a material point with positive mass sliding on a profile defined by a function Φ under the action of

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the gravity force, the reaction force, and the friction force can asymptotically be approximated by the following second order dynamical system

$$\ddot{x}(t) + \lambda \dot{x}(t) + \nabla \Phi(x(t)) = 0 \tag{1.3}$$

called *heavy ball with friction system (HBF)*, cf. [2]. We refer to [7] and [4] to see numerical algorithms based on the HBF system for solving some special problems, e.g. large systems of linear equations, eigenvalue problems, nonlinear Schrödinger problems, inverse source problems, and ill-posed problems. In [4] the authors have shown advantages and superior convergence properties of such a dynamical functional particle method compared to a first order dynamical system, and also to several other iterative methods. So, it's hardly surprising that second order dynamical equations play an important role in acceleration for convergence to steady state solutions. In fact, the power of the use of the damped p-Laplace equation in image denoising was investigated in [3]. However, an analysis as in [9] of the asymptotic behavior, as $t \to \infty$, of the solutions to a damped p-Laplace equation was not done so far.

Our purpose here is to obtain the decay rate for large time of $u - u^*$ where u denotes a solution to the evolutionary damped p-Laplace equation (The question of existence of solutions will be the subject of a forthcoming note.), namely:

$$\begin{gathered} u_{tt} + a\, u_t = \Delta_p u \quad \text{in } (0,\infty) \times \Omega, \\ u(0,x) = u_0(x) \quad \text{in } \{0\} \times \Omega, \\ u_t(0,x) = 0 \quad \forall x \in \Omega, \\ u(t,x) = g(x) \quad \text{on } [0,\infty) \times \partial\Omega, \end{gathered} \tag{1.4}$$

where $a > 0$ is constant and $u_0 \in W^{1,p}(\Omega)$, such that $u_0 - g \in W_0^{1,p}(\Omega)$.

It is clear, that the solution of the damped equation (1.4) behaves for large time like the stationary solution of (1.2). Moreover, we show the following rate of decay for the $W^{1,p}$-norm of their difference.

Theorem 1.1. *Let $p \geq 2$, u^* denote a solution to* (1.2) *and u a solution to* (1.4)*. For large time we have*

$$\|u - u^*\|_{W^{1,p}(\Omega)} \leq C \cdot t^{-\frac{1}{(p-1)p}},$$

with a constant $C = C(p, \Omega, u_0, a) > 0$.

Our proof is based on a careful analysis of the error term

$$\mathrm{e}(t) := \int_\Omega \frac{a^2}{2} w^2 + a\, w\, w_t + w_t^2 + 2\Big(\frac{1}{p}|\nabla u|^p - \frac{1}{p}|\nabla u^*|^p\Big)\,\mathrm{d}x \tag{1.5}$$

where we have set $w = u - u^*$. Note that our error term is chosen in such a way that it is compatible to our problem and we can estimate the error in terms of its derivative. Moreover, the fact

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{p}|\nabla u|^p \,\mathrm{d}x = -\int_\Omega u_t \Delta_p u \,\mathrm{d}x,$$

cf. page 4, justifies the appearance of the last term in the error. It is worth mentioning that with our argumentation scheme we can improve the decay rate in the linear case $p = 2$ and obtain the classical result from [8], cf. the discussion in section 3.1.

2. Basic results

Let us briefly introduce the notation used throughout this work. The Euclidean norm in $\mathbb{R}^n$ is denoted by $|\cdot|$. A generic positive constant is represented by capital or lower case letter c, possibly varying from line to line. We often write $u(t)(x)$ for $u(t,x)$.

Given a real Banach space X, the (Banach) space $L^p(0,T;X)$ consists of all measurable functions $u:[0,T]\to X$ such that

$$\|u\|_{L^p(0,T;X)} = \Big(\int_0^T \|u(t)\|_X^p \,\mathrm{d}t\Big)^{1/p} < \infty\,, \quad 1\le p<\infty\,,$$

$L^\infty(0,T;X)$ is the space of all measurable $u:[0,T]\to X$ such that

$$\|u\|_{L^\infty(0,T;X)} = \operatorname{ess\,spuu}_{t\in[0,T]} \|u(t)\|_X < \infty.$$

The Banach space $W^{1,p}(0,T;X)$, for $1\le p\le\infty$, consists of all $u\in L^p(0,T;X)$ such that $\partial_t u$ exists in the weak sense and belongs to $L^p(0,T;X)$.

Recall that for $u\in W^{1,p}(0,T;X)$ we have $u\in C^0([0,T];X)$ and

$$\max_{0\le t\le T}\|u(t)\|_X \le c(T)\|u\|_{W^{1,p}(0,T;X)}.$$

For further reading and elaborated clarifications on spaces involving time, we refer the reader to [5, Sec. 5.9.2].

Throughout this work, we use the following inequalities:

- let $p\ge 2$. For all $a,b\in\mathbb{R}^n$ we have
$$2^{2-p}|a-b|^p \le \langle |a|^{p-2}a - |b|^{p-2}b, a-b\rangle, \tag{2.1}$$
- for $p\ge 2$ and with an adequate constant $c(p)\in(0,1]$:
$$|b|^p \ge |a|^p + p\langle |a|^{p-2}a, b-a\rangle + c(p)|b-a|^p, \tag{2.2}$$
- furthermore, for $\|f\|_{L^p(\Omega)}\le M$ and $\|g\|_{L^p(\Omega)}\le M$ the estimate
$$\int_\Omega \big||f|^p - |g|^p\big|\,\mathrm{d}x \le c(p,\Omega)M^{p-1}\|f-g\|_{L^p(\Omega)} \tag{2.3}$$
holds, cf. [11, p. 75].

Firstly, let us define the concept of weak solutions to the evolutionary damped p-Laplace equation.

Definition 2.1. We say that $u\in W^{1,p}_{\rm loc}(0,\infty;W^{1,p}(\Omega))$ is a solution to

$$u_{tt} + au_t = \Delta_p u$$

if

$$\int_0^\infty\int_\Omega -u_t\,\phi_t - au\phi_t + |\nabla u|^{p-2}\langle\nabla u,\nabla\phi\rangle\,\mathrm{d}x\,\mathrm{d}t = 0,$$

for each $\phi\in C_0^\infty((0,\infty)\times\Omega)$.

In the following, let us denote by u^* a solution to (1.2), and by u a solution to (1.4). Moreover, we set

$$E(t) := \int_\Omega \frac12\,u_t^2(t,x) + \frac1p\,|\nabla u(t,x)|^p\,\mathrm{d}x.$$

Corollary 2.2. *$E(\cdot)$ is non-increasing, or rather in the weak sense we have*

$$E'(t) = -a \int_\Omega u_t^2 \,\mathrm{d}x. \tag{2.4}$$

Proof. A multiplication of

$$u_{tt} + a\, u_t = \Delta_p u$$

by u_t followed by an integration over Ω gives

$$\int_\Omega u_{tt} u_t \,\mathrm{d}x - \int_\Omega (\Delta_p u) u_t \,\mathrm{d}x = -a \int_\Omega u_t^2 \,\mathrm{d}x. \tag{2.5}$$

Moreover, an integration by parts yields

$$-\int_\Omega u_t\, \Delta_p u \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{p} |\nabla u|^p \,\mathrm{d}x,$$

note that there is no time dependence of u_t on the boundary. In view of

$$\int_\Omega u_{tt}\, u_t \,\mathrm{d}x = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega {u_t}^2 \,\mathrm{d}x,$$

we combine the last two equalities with (2.5) to achieve the desired relation (2.4):

$$E'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} u_t^2 \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{p} |\nabla u|^p \,\mathrm{d}x = -a \int_\Omega u_t^2 \,\mathrm{d}x. \qquad \square$$

The above considerations were formal and can all be made rigorous, cf. [12, p. 156ff]. In view of (2.4), we show that the gradient of u (with respect to space) is bounded by the initial data and that u_t tends to zero for large times.

Corollary 2.3. *Let u be a solution to* (1.4)*. Then:*

(a) $\|u_t(T)\|_{L^2(\Omega)} \xrightarrow{T\to\infty} 0$.
(b) *For all $T \geq 0$ it holds* $\|\nabla u(T)\|_{L^p(\Omega)} \leq \|\nabla u_0\|_{L^p(\Omega)}$.

Proof. Integrating (2.4) over $(0, T)$ we obtain

$$\begin{aligned} &\int_\Omega \frac{1}{2} u_t^2(T,x) \,\mathrm{d}x + \int_\Omega \frac{1}{p} |\nabla u(T,x)|^p \,\mathrm{d}x + a \int_0^T \int_\Omega u_t^2(\tau,x) \,\mathrm{d}x \, d\tau \\ &\leq \int_\Omega \frac{1}{p} |\nabla u_0(x)|^p \,\mathrm{d}x. \end{aligned} \tag{2.6}$$

Note that the right-hand side of inequality (2.6) is independent of T, hence, the statement follows as $T \to \infty$. $\square$

Remark 2.4. *Taking the essential supremum with respect to time on both sides of* (2.6) *shows*

$$u_t \in L^\infty(0,\infty; L^2(\Omega)), \quad \textit{and} \quad u \in L^\infty(0,\infty; W^{1,p}(\Omega)).$$

Recall that u^* minimizes $\mathscr{E}(\cdot)$. Hence, Corollary 2.3 ensures the boundedness of the gradients of both u and u^*, more precisely

$$\|\nabla u^*\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)} \leq M \tag{2.7}$$

where we have set $M := \|\nabla u_0\|_{L^p(\Omega)}$.

Next, let us focus on the behavior of the energies. Since for large time the dependence of u on time shrinks, cf. Corollary 2.3, the convergence of energies should follow from the uniqueness of p-harmonic functions, and indeed, we have the following result.

Lemma 2.5. *Let u^* and u be solutions of* (1.2) *and* (1.4), *respectively. Then*

$$\mathscr{E}(u) \xrightarrow{t\to\infty} \mathscr{E}(u^*).$$

Proof. Since u^* is the unique minimizer of $\mathscr{E}(\cdot)$, it suffices to show that

$$\limsup_{t\to\infty} \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x \le \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x \tag{2.8}$$

for all v such that $v - g \in W_0^{1,p}(\Omega)$. For that purpose we will basically follow the proof of [1, Theorem 2.1]:

Let $v \in W^{1,p}(\Omega)$ with $v - g \in W_0^{1,p}(\Omega)$ be given. Consider the auxiliary function

$$\varphi(t) := \frac{1}{2}\int_\Omega \left(u(t,x) - v(x)\right)^2 \,\mathrm{d}x.$$

Then $\varphi \in W^{2,1}(0,\infty)$, cf. Remark 2.4, and, as u fulfills (1.4), we have

$$\begin{aligned}\varphi''(t) + a\varphi'(t) &= \int_\Omega (u-v)\Delta_p u + u_t^2 \,\mathrm{d}x = -\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla u - \nabla v\rangle + u_t^2 \,\mathrm{d}x \\ &\overset{(2.2)}{\le} \int_\Omega \frac{1}{p}|\nabla v|^p - \frac{1}{p}|\nabla u|^p + u_t^2 \,\mathrm{d}x \le \int_\Omega \frac{1}{p}|\nabla v|^p + \frac{3}{2}u_t^2 \,\mathrm{d}x - E(T)\end{aligned}$$

for all $t \in [0,T]$, where we have used that $E(\cdot)$ is non-increasing. A multiplication of both sides with e^{at}, followed by an integration yields

$$\begin{aligned}\varphi'(t) \le e^{-at}\varphi'(0) &+ \frac{1}{a}(1 - e^{-at})\Big(\int_\Omega \frac{1}{p}|\nabla v|^p \,\mathrm{d}x - E(T)\Big) \\ &+ \frac{3}{2}\int_0^t \int_\Omega e^{-a(t-\tau)} u_t^2(\tau,x) \,\mathrm{d}x \,\mathrm{d}\tau.\end{aligned}$$

Integrating once more and using the fact that $E(T) \ge \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x$, implies

$$\begin{aligned}&\varphi(T) + \frac{1}{a^2}\left(aT - 1 + e^{-aT}\right)\frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x \\ &\le \frac{1}{a^2}\left(aT - 1 + e^{-aT}\right)\frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x \ + \varphi(0) + \frac{1}{a}(1 - e^{-aT})\,\varphi'(0) + h(T)\end{aligned} \tag{2.9}$$

where we have set

$$\begin{aligned}h(T) :=& \frac{3}{2}\int_0^T \int_0^t \int_\Omega e^{-a(t-\tau)} u_t^2(\tau,x) \,\mathrm{d}x \, d\tau \,\mathrm{d}t \\ =& \frac{3}{2a}\int_0^T \int_\Omega u_t^2(\tau,x)(1 - e^{-a(T-\tau)}) \,\mathrm{d}x \, d\tau.\end{aligned}$$

From Remark 2.4, the term $h(T)$ is bounded. Hence, dividing (2.9) by $\frac{1}{a^2}\big(aT - 1 + e^{-aT}\big)$ and letting $T \to \infty$ gives the desired estimate (2.8). □

On account of the convergence of the energies, we obtain the $W^{1,p}$ convergence of u to u^*.

Corollary 2.6. *Let u and u^* be as before, then we have*

$$\|u - u^*\|_{W^{1,p}(\Omega)} \xrightarrow{t\to\infty} 0.$$

Proof. By Poincaré's inequality we have

$$\int_\Omega |u-u^*|^p \,\mathrm{d}x \le \tilde{c}(p,\Omega)\int_\Omega |\nabla u - \nabla u^*|^p \,\mathrm{d}x. \tag{2.10}$$

For the p-harmonic function u^* it holds

$$\int_\Omega |\nabla u^*|^{p-2}\langle \nabla u^*, \nabla u - \nabla u^*\rangle = 0,$$

so that, by (2.2) we obtain

$$c(p)\int_\Omega |\nabla u - \nabla u^*|^p \,\mathrm{d}x \le \int_\Omega |\nabla u|^p - |\nabla u^*|^p \,\mathrm{d}x = p\cdot(\mathscr{E}(u)-\mathscr{E}(u^*)). \tag{2.11}$$

Thus, combining the above estimates we arrive at

$$\|u-u^*\|_{W^{1,p}(\Omega)} \le c(p,\Omega)\cdot|\mathscr{E}(u)-\mathscr{E}(u^*)| \xrightarrow{t\to\infty} 0$$

by Lemma 2.5. □

3. Proof of decay rates

We are now prepared to prove our main result.

Proof of Theorem 1.1. A multiplication of

$$u_{tt} + a\, u_t = \Delta_p u - \Delta_p u^*$$

by $w = w(t,x) := u(t,x) - u^*(x)$, and integrating by parts (note that $w\big|_{\partial\Omega} = 0$) yields

$$\begin{aligned}\int_\Omega w_{tt}\, w + a\, w_t\, w \,\mathrm{d}x &= -\int_\Omega \langle |\nabla u|^{p-2}\nabla u - |\nabla u^*|^{p-2}\nabla u^*, \nabla u - \nabla u^*\rangle \,\mathrm{d}x \\ &\overset{(2.1)}{\le} -2^{2-p}\int_\Omega |\nabla w|^p \,\mathrm{d}x.\end{aligned}$$

Hence, multiplying both sides of the last inequality by $a>0$ and adding $\int_\Omega a w_t^2$ we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{a^2}{2}w^2 + a w w_t \,\mathrm{d}x \le \int_\Omega a w_t^2 - 2^{2-p}a|\nabla w|^p \,\mathrm{d}x. \tag{3.1}$$

Recall the definition of our error term

$$\mathrm{e}(t) := \int_\Omega \frac{a^2}{2}w^2 + a\, w\, w_t + w_t^2 + 2\Big(\frac{1}{p}|\nabla u|^p - \frac{1}{p}|\nabla u^*|^p\Big) \,\mathrm{d}x. \tag{1.5}$$

So, $\mathrm{e} \in W^{1,1}(0,\infty)$ and due to the minimizing properties of $u^* = u^*(x)$, we have that $\mathrm{e}(t) \ge 0$ for all $t>0$. Since $w_t = u_t$ we obtain

$$\begin{aligned}\mathrm{e}'(t) &\overset{(3.1)}{\le} \int_\Omega a\, w_t^2 - 2^{2-p}a|\nabla w|^p \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_t^2 + 2\frac{1}{p}|\nabla u|^p \,\mathrm{d}x \\ &\overset{(2.4)}{=} -a\int_\Omega w_t^2 + 2^{2-p}|\nabla w|^p \,\mathrm{d}x \le 0.\end{aligned} \tag{3.2}$$

Moreover, again with $w_t = u_t$ we have

$$\begin{aligned}\frac{\mathrm{e}'(t)}{a} &= \int_\Omega a\,w\,w_t + w\,w_{tt} + w_t^2\,\mathrm{d}x + \frac{2}{a}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}w_t^2 + \frac{1}{p}|\nabla u|^p\,\mathrm{d}x \\ &\overset{(2.4)}{=} \int_\Omega a\,w\,w_t + w\,w_{tt} - w_t^2\,\mathrm{d}x = \int_\Omega w(\Delta_p u - \Delta_p u^*) - w_t^2\,\mathrm{d}x \\ &= \int_\Omega w\Delta_p u - w_t^2\,\mathrm{d}x,\end{aligned} \tag{3.3}$$

since $\Delta_p u^* = 0$ in Ω. Using integration by parts (note that $w\big|_{\partial\Omega} = 0$) we obtain

$$\begin{aligned}\Big|\int_\Omega w\Delta_p u\,\mathrm{d}x\Big| &= \Big|\int_\Omega |\nabla u|^{p-2}\langle\nabla u, \nabla w\rangle\,\mathrm{d}x\Big| \le \int_\Omega |\nabla u|^{p-1}|\nabla w|\,\mathrm{d}x \\ &\le \|\nabla u\|_{L^p(\Omega)}^{p-1}\|\nabla w\|_{L^p(\Omega)}\end{aligned} \tag{3.4}$$

by Hölder's inequality. Using the boundedness of the gradient (2.7) we conclude that

$$\begin{aligned}\Big|\frac{\mathrm{e}'(t)}{a}\Big| &\overset{(3.3)}{=} \Big|\int_\Omega w\Delta_p u - w_t^2\,\mathrm{d}x\Big| \\ &\overset{(3.4)}{\le} \|\nabla u\|_{L^p(\Omega)}^{p-1}\|\nabla w\|_{L^p(\Omega)} + \|w_t\|_{L^2(\Omega)}^2 \\ &\overset{(2.7)}{\le} M^{p-1}\|\nabla w\|_{L^p(\Omega)} + \|w_t\|_{L^2(\Omega)}^2 \xrightarrow{t\to\infty} 0,\end{aligned} \tag{3.5}$$

by Corollaries 2.6 and 2.3, respectively.

Our next goal is to estimate the error in terms of its derivative. From (2.3) we arrive at

$$\mathrm{e}(t) \le \int_\Omega \Big(\frac{a^2}{2} + a\Big)w^2 + \big(\frac{a}{4} + 1\big)w_t^2\,\mathrm{d}x + c(p, \Omega, u_0)\|\nabla w\|_{L^p(\Omega)}\,.$$

Using Lebesgue embedding and Poincaré's inequality for the first term we obtain

$$\mathrm{e}(t) \le c_1(p, \Omega, a)\|\nabla w\|_{L^p(\Omega)}^2 + \big(\frac{a}{4} + 1\big)\int_\Omega w_t^2\,\mathrm{d}x + c(p, \Omega, u_0)\|\nabla w\|_{L^p(\Omega)}\,.$$

Furthermore, in (3.2) we already aimed

$$\int_\Omega w_t^2 + 2^{2-p}|\nabla w|^p\,\mathrm{d}x \le -\frac{\mathrm{e}'(t)}{a}.$$

All in all, we obtain

$$\mathrm{e}(t) \le c_2(p, \Omega, a)\Big(-\frac{\mathrm{e}'(t)}{a}\Big)^{2/p} + \big(\frac{a}{4} + 1\big)\Big(-\frac{\mathrm{e}'(t)}{a}\Big) + c_3(p, \Omega, u_0)\Big(-\frac{\mathrm{e}'(t)}{a}\Big)^{1/p}.$$

Since

$$-\frac{\mathrm{e}'(t)}{a} \xrightarrow{t\to\infty} 0,$$

cf. (3.5), the error term $\mathrm{e}(t) \ge 0$ satisfies for large time a differential inequality

$$\mathrm{e}(t) \le c_4(p, \Omega, u_0, a)(-\mathrm{e}'(t))^{\frac{1}{p}},$$

and we may rewrite this as

$$\mathrm{e}'(t) \le -c_5(p, \Omega, u_0, a)\cdot \mathrm{e}(t)^p,$$

respectively, so by [8, Lemma 1.6] we obtain

$$\mathrm{e}(t) \le c_6(p, \Omega, u_0, a)t^{-\frac{1}{p-1}}. \tag{3.6}$$

By (3.6), (2.11) and Poincaré inequality we finally arrive at

$$\|u-u^*\|^p_{W^{1,p}(\Omega)} \le c_7(p,\Omega,u_0,a)t^{-\frac{1}{p-1}}\,. \qquad \square$$

3.1. **Enhancement of the decay rate for $p=2$.** A crucial ingredient in our proof of the decay rate was inequality (2.3) which we applied to estimate the difference of the energies. In fact, for $p=2$ this relation can be improved to the *equality*

$$\int_\Omega |\nabla u|^2 - |\nabla u^*|^2 \,\mathrm{d}x = \int_\Omega |\nabla u - \nabla u^*|^2 \,\mathrm{d}x$$

where we used the harmonicity of u^*. Hence, we obtain

$$\begin{aligned}
\mathrm{e}(t) &\le \int_\Omega \Big(\frac{a^2}{2}+a\varepsilon\Big)w^2 + \Big(\frac{a}{4\varepsilon}+1\Big)w_t^2 + |\nabla w|^2 \,\mathrm{d}x \\
&\le \int_\Omega \Big(\Big(\frac{a^2}{2}+a\varepsilon\Big)\tilde c(\Omega)+1\Big)|\nabla w|^2 + \Big(\frac{a}{4\varepsilon}+1\Big)w_t^2 \,\mathrm{d}x \\
&= c(a,\Omega)\int_\Omega w_t^2 + |\nabla w|^2 \,\mathrm{d}x \\
&\le c(a,\Omega)\Big(-\frac{\mathrm{e}'(t)}{a}\Big)
\end{aligned} \tag{3.7}$$

where in the intermediate steps we used the Poincaré inequality, and $\varepsilon > 0$ was chosen in such a way that the prefactors coincided. Relation (3.7) may be rewritten as

$$\mathrm{e}'(t) \le -\frac{a}{c(a,\Omega)}\,\mathrm{e}(t) \quad \text{for all } t>0,$$

so, by Gronwall's inequality, the error term fulfills

$$\mathrm{e}(t) \le c\cdot \exp\Big(-\frac{a}{c(a,\Omega)}\,t\Big)$$

and for the decay rate we arrive at

$$\|u-u^*\|^2_{W^{1,2}(\Omega)} \le C\cdot\exp\Big(-\frac{a}{c(a,\Omega)}t\Big) \quad \text{for all } t>0,$$

which is a well known result, cf. [8, Theorem 2.1 a)].

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References

[1] F. Alvarez; *On the minimizing property of a second order dissipative system in Hilbert spaces,* SIAM J. Control Optim., **38** (2000), no. 4, pp. 1102–1119.

[2] H. Attouch, X. Goudou, P. Redont; *The heavy ball with friction method. I. The continuous dynamical system,* Commun. Contemp. Math., **2** (2000), no. 1, pp. 1–34.

[3] G. Baravdish, O. Svensson, M. Gulliksson, Y. Zhang; *A damped flow for image denoising,* arXiv:1806.06732 [math.NA]

[4] S. Edvardsson, M. Neuman, P. Edström, H. Olin; *Solving equations through particle dynamics,* Comput. Phys. Commun., **197** (2015), pp. 169–181.

[5] L. C. Evans, *Partial Differential Equations,* 2nd ed., Graduate Studies in Mathematics, **19**, AMS Providence, 2010.
[6] S. P. Frankel; *Convergence rates of iterative treatments of partial differential equations,* Math. Tables and Other Aids to Computation, 4 (1950), pp. 65–75.
[7] M. Gulliksson, M. Ögren, A. Oleynik, Y. Zhang; *Damped Dynamical Systems for Solving Equations and Optimization Problems,* Handbook of the Mathematics of the Arts and Sciences, Springer, 1–44, 2018.
[8] A. Hauraux, E. Zuazua; *Decay estimates for some semilinear damped hyperbolic problems,* Arch. Rational Mech. Anal., **100** (1988), no. 2, pp. 191–206.
[9] P. Juutinen and P. Lindqvist; *Pointwise decay for the solutions of degenerate and singular parabolic equations,* Adv. Differential Equations **14** (2009), no. 7-8, pp. 663–684.
[10] B. T. Polyak; *Some methods of speeding up the convergence of iteration methods,* Zh. Vychisl. Mat. Mat. Fiz., **4** (1964), no. 5, pp. 791–803 (Russian); engl. trans. in USSR. Comput. Math. Math. Phys., **4** (1964), no. 5, pp. 1–17.
[11] W. Rudin; *Real and Complex Analysis,* 3rd ed., McGraw-Hill Book Co., New York, 1987.
[12] Z. Wu, J. Zhao, J. Yin, H. Li; *Nonlinear diffusion equations,* Singapore: World Scientific, 2001.

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