ESTIMATES OF CHARACTERISTICS OF A MICROPOLAR FLOW PASSING THROUGH AN AXIALLY SYMMETRIC CELL

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ABSTRACT. We study a model for the filtration of micropolar fluid in the framework of a cell model technique. A porous medium is presented as an assemblage of axially symmetric cells of an arbitrarily geometry. Each cell consists of a solid core, porous layer and liquid shell. The influence of the neighboring cells is taken into account via Cunningham's-type boundary condition. We derive a priori estimates for flow characteristics which show the behavior of the velocity filed. The boundedness of velocity filed is justified by the derived estimates.

1. INTRODUCTION

Modern technologies and devices utilize microscale processes and knowledge of the material structure for their adequate description and modeling. Regarding hydrodynamics, these two items are demanded for fluid flows in constrained conditions. One of the examples of such flows is a filtration through membranes, which represent porous media of an irregular structure. A widely spread method of the membrane flow simulations is the cell model technique [13]. According to this method the chaotic membrane structure is replaced by a set of identical cells. Each cell may be constructed of solid or porous, or liquid core and liquid shell. In the most general case the core can be a solid-porous composite. The influence of the neighboring particles is taken into account by setting the appropriate boundary conditions at the outer cell surfaces. So, the consideration of the flow in the membrane is reduced to the solution of the flow problem in a single cell. Originally the cell shape was cubical, and this led to a computational instability in the corners. A substantial progress occurred in applying the cell model when the shape of the cell and its core were taken to be the same, namely, cylindrical and spherical [14]-[21], with the inter-cell space being neglected. These papers dealt with a simple cell with solid core and used various conditions at the outer cell surface. Happel [14, 15] used vanishing of shear stress, Kuwabara [20] supposed vanishing of rotation, Mehta-Morse [24] applied flow homogeneity originally introduced by Cunningham [5], Kvashnin [21] exploited the symmetry of the velocity profile. Analytical solutions were obtained in all of the mentioned papers, and the specifics of the cell geometry was substantially used. A spherical cell with concentric porous spherical core was considered in [26]. The obtained explicit analytical solution was

used for the calculation of the hydrodynamic permeability of a set of porous particles (membrane). The latter is an important macro characteristic of a membrane measured in an experiment. The flow problem in a cell with partially porous core consisting of a solid inner sphere, covered with a concentric porous hydrodynamically uniform layer, was solved in [28]. Later, the developed method was applied to the filtration modeling in fibrous membranes [29]. The membrane material was simulated as an ensemble of identical cylindrical cells. The core of each cell consisted of a solid inner cylinder, covered by a coaxial porous layer. The flows along and across the cell axis were studied. The obtained solutions were composed to simulate the chaotic orientation of fibers with respect to the flow direction. Along with the aforementioned four boundary conditions at the outer cell surface various conditions are known at the porous-liquid interface. A stress jump condition was applied at this boundary and its influence on the membrane hydrodynamic permeability was analyzed in [6, 30]. The flow of a fluid with variable viscosity was considered in [9] for composite cylindrical cells, and in [10] for composite spherical cells.

Nevertheless, classical hydrodynamics does not take into account effects which may occur at the micro-level because it treats fluids as an ensemble of material points. As a result, one observes the loss of solution similarity for micro flow domains or the unexpected behavior of viscosities in the vicinity of boundaries. Numerous non-Newtonian liquid models confirm the fact that none of them is satisfactory and adequate enough to describe all the known effects, despite each of them works fine in the particular case it was designed for.

The simplest sophistication of the classical model of liquid is the micropolar fluids. They treat liquid particles as infinitesimal rigid bodies, which can perform independent translational and rotational motion. The original idea of the micro structured continua belongs to the Cosserat brothers [4], but its active application began after the development of the mathematical theory of simple micro fluids by Eringen [7, 8]. Among basic applications of the micropolar fluids theory one can mention flows of suspensions, lubricants, blood, synovial liquid, drilling solutions. A review of the known analytical solutions and applications of the micropolar fluids was given by Khanukaeva and Filippov [16].

Field equations for such liquids include the continuity equation, momentum equation, and moment of momentum equation. They contain additional viscosity coefficients characterizing microstructural properties of the liquid. The main peculiarity of the governing equations for the micropolar fluids is that they contain a characteristic flow scale. So, the solution does not possess a similarity property and depends on the microscale of the flow. Nevertheless, in the limiting case the system of governing equations reduces to the classical Navier-Stokes equations, which describe the flow of Newtonian liquids.

Consideration of liquids with microstructure is especially relevant to the filtration problems, when the most intensive interaction of flow with the surrounding medium takes place. The close position of boundaries makes ineffective the Darcy law for the filtration modeling in the framework of the cell model. Darcy law requires some additional assumptions on the mechanism of the flow interaction with solid surfaces. At the same time, Newtonian liquid flows in porous media are successfully described by the Brinkman equation [1]. The extension of this equation to the micropolar fluids was made in [12]. The cell model technique was applied to the micropolar

fluid flows in fibrous and globular membranes in [17]-[19]. The cells of cylindrical and spherical shapes were considered and the cell shape certainty allows to obtain a solution in an explicit analytical form. Here we extend the study to a cell of arbitrary shape with axial symmetry. This property of the cell geometry plays a key role in the formulation of the field equations in the form allowing for their integration. The solution to our problem is treated in the weak sense. We derive a priori estimates for the behavior of flow. The obtained inequalities allow making easy estimates of linear and angular velocities at micro scale in filtration flows of micropolar fluids. Moreover, with the help of the derived estimates we have shown the boundedness of weakly defined velocity fields. The exact analytical formulae solving our problem for the case of spherical particle are given in [19].

2. Statement of the problem

Assume that a porous cell $\Omega = \Omega_1 \cup \Omega_2 \in \mathbb{R}^3$ is bounded and has an arbitrary shape with axial symmetry (see Figure 1). Its boundary is assumed to be smooth and consists of the parts $\partial\Omega_1 = \Gamma_1 \cup \Gamma_2$, $\partial\Omega_2 = \Gamma_2 \cup \Gamma_3$. The direction of the uniform incoming flow of velocity $\widetilde{\mathbf{U}}$ coincides with the symmetry axis. Here and further we use symbol $\widetilde{}$ for the dimensioned flow characteristics.
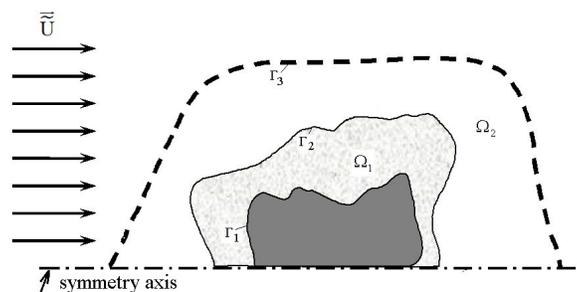

Figure 1. Cell domain

The flow domain Ω is split into two regions Ω_1 and Ω_2 in order to represent separately porous and liquid parts of the cell. Micropolar fluid occupies domain Ω_2, while domain Ω_1 represents porous material with pore space fully saturated with the fluid. The filtration flow in domain Ω_1 is governed by the system of equations which differs from the system governing the free flow in domain Ω_2. Now we proceed to the discussion of these systems.

Micropolar media principally differ from Newtonian liquids by the construction of their elements, which can rotate in addition to the translation motion. The angular velocity of microrotation $\widetilde{\boldsymbol{\omega}}$ contributes to the deformation rate tensor $\hat{\gamma}$, which becomes a non-symmetric and equals to $\hat{\gamma} = (\nabla\widetilde{\mathbf{v}})^T - \hat{\varepsilon} \cdot \widetilde{\boldsymbol{\omega}}$, where $\nabla\widetilde{\mathbf{v}}$ is the gradient of linear velocity $\widetilde{\mathbf{v}}$, $\hat{\varepsilon}$ is the Levi-Civita tensor, superscript T implies transposed values. Apart from the volumetric, shear and rotational deformations described by tensor $\hat{\gamma}$, an element of a micropolar medium may twist and bend due to the dependence of $\widetilde{\boldsymbol{\omega}}$ on the spatial coordinate. The rate of these deformations are characterized by the curvature-twist rate tensor $\hat{\chi} = (\nabla\widetilde{\boldsymbol{\omega}})^T$, which is also non-symmetric due to the definition of the angular velocity gradient $\nabla\widetilde{\boldsymbol{\omega}}$. Hence, the

stress tensor $\hat{t}$ and couple stress tensor $\hat{m}$ of a micropolar fluid are both non symmetric. Meanwhile, strictly speaking micropolar media cannot be included in the list of non-Newtonian liquids. The reason is that the theory of simple micro fluids uses linear relation of stresses and couple stresses with the deformation rate tensors, as accepted in classical hydrodynamics. Besides, in the limiting case the state equations should transfer to the classical Navier-Stokes law, which linearly relates the symmetric stress tensor with the symmetric deformation rate tensor, dynamic viscosity μ being the coefficient of proportionality. We use the state equations of simple micro fluids in the form traditional to non symmetric theory of elasticity [25], namely

$$\hat{t} = (-\widetilde{p} + \lambda \operatorname{tr} \hat{\gamma})\hat{G} + 2\mu\hat{\gamma}^{(S)} + 2\kappa\hat{\gamma}^{(A)},$$
$$\hat{m} = \alpha(\operatorname{tr} \hat{\chi})\hat{G} + 2\delta\hat{\chi}^{(S)} + 2\varsigma\hat{\chi}^{(A)},$$

where $\widetilde{p}$ is the hydrostatic pressure, λ is the second classical viscosity coefficient, $\hat{G}$ is a metric tensor, κ is the rotational viscosity coefficient, $\alpha, \delta, \varsigma$ are the angular viscosity coefficients, superscripts (S) and (A) denote symmetric and skew symmetric parts of tensors correspondingly. The advantage of this form of the state equations is that in the limiting case of $\kappa = 0$ one obtains the classical expression for the stress tensor with coefficient μ coinciding with the dynamic viscosity of the Newtonian liquid.

In the further notations subscripts 1 or 2 are applied to show that the flow is considered in the corresponding subdomain Ω_1, Ω_2. For example, $\widetilde{\mathbf{v}}_i$ denotes the dimensioned linear velocity of flow passing through $\Omega_i : \widetilde{\mathbf{v}}_i = \widetilde{\mathbf{v}}|_{\Omega_i}$. The notations $\widetilde{\boldsymbol{\omega}}_i$, $\widetilde{p}_i$ are analogous.

Thus, the field equations in the domain Ω_2 are the following [7]. The mass conservation law gives the continuity equation for an incompressible liquid

$$\operatorname{div} \widetilde{\mathbf{v}}_2 = 0. \tag{2.1}$$

Combined with the definition of tensor $\hat{\gamma}$ equation (2.1) yields $\operatorname{tr} \hat{\gamma} = 0$ that excludes coefficient λ from the problem statement. The momentum conservation law with substituted stress tensor reads

$$\nabla \widetilde{p}_2 = (\mu + \kappa)\Delta \widetilde{\mathbf{v}}_2 + 2\kappa \operatorname{curl} \widetilde{\omega}_2. \tag{2.2}$$

The moment of the momentum balance with substituted couple stress tensor gives the differential form of the moment of momentum equation

$$(\alpha + \delta - \zeta)\nabla \operatorname{div} \widetilde{\boldsymbol{\omega}}_2 + (\delta + \zeta)\Delta \widetilde{\boldsymbol{\omega}}_2 + 2\kappa \operatorname{curl} \widetilde{\mathbf{v}}_2 - 4\kappa \widetilde{\boldsymbol{\omega}}_2 = 0. \tag{2.3}$$

Governing equations for the stationary filtration of the micropolar fluid in porous media were derived in [12] with the use of a standard averaging technique. So, for region Ω_1 they look as follows:

$$\operatorname{div} \widetilde{\mathbf{v}}_1 = 0, \tag{2.4}$$

$$\nabla \widetilde{p}_1 = \frac{\mu + \kappa}{\varepsilon}\Delta \widetilde{\mathbf{v}}_1 + \frac{2\kappa}{\varepsilon} \operatorname{curl} \widetilde{\boldsymbol{\omega}}_1 - \frac{\mu + \kappa}{k}\widetilde{\mathbf{v}}_1, \tag{2.5}$$

$$(\alpha + \delta - \zeta)\nabla \operatorname{div} \widetilde{\boldsymbol{\omega}}_1 + (\delta + \zeta)\Delta \widetilde{\boldsymbol{\omega}}_1 + 2\kappa \operatorname{curl} \widetilde{\mathbf{v}}_1 - 4\kappa \widetilde{\boldsymbol{\omega}}_1 = 0, \tag{2.6}$$

where k is the permeability and ε is the porosity of the porous medium (given constants). It is worth mentioning, that the values of linear and angular velocities $\widetilde{\mathbf{v}}_1, \widetilde{\boldsymbol{\omega}}_1$ and pressure $\widetilde{p}_1$ in equations (2.4)-(2.5) are averaged. Hence, they can be directly compared with the corresponding values measured in experiments.

Owing to the axial symmetry of the problem, all the unknown functions are independent of the angle in the plane perpendicular to the axis of revolution. The linear velocity lies in the plane of any cross-section containing the axis of revolution. That is for the section, shown in Figure 1 it lies in the plane of the figure. The angular velocity is always perpendicular to the linear velocity. This fact leads to the divergence-free property of the spin field

$$\operatorname{div}\widetilde{\boldsymbol{\omega}}_1 = 0, \quad \operatorname{div}\widetilde{\boldsymbol{\omega}}_2 = 0. \tag{2.7}$$

Thus, the first member in equation (2.3) and (2.6) vanishes. Hence, we can rewrite (2.3) and (2.6) as

$$(\delta + \zeta)\Delta\widetilde{\boldsymbol{\omega}}_2 + 2\kappa\operatorname{curl}\widetilde{\mathbf{v}}_2 - 4\kappa\widetilde{\boldsymbol{\omega}}_2 = 0, \tag{2.8}$$
$$(\delta + \zeta)\Delta\widetilde{\boldsymbol{\omega}}_1 + 2\kappa\operatorname{curl}\widetilde{\mathbf{v}}_1 - 4\kappa\widetilde{\boldsymbol{\omega}}_1 = 0. \tag{2.9}$$

We set the no-slip and no-spin boundary conditions on Γ_1:

$$\widetilde{\mathbf{v}}_1|_{\Gamma_1} = 0, \quad \widetilde{\boldsymbol{\omega}}_1|_{\Gamma_1} = 0. \tag{2.10}$$

We suppose the continuity of both linear and angular velocity vectors on Γ_2:

$$\widetilde{\mathbf{v}}_1\big|_{\Gamma_2} = \widetilde{\mathbf{v}}_2\big|_{\Gamma_2}, \quad \widetilde{\boldsymbol{\omega}}_1\big|_{\Gamma_2} = \widetilde{\boldsymbol{\omega}}_2\big|_{\Gamma_2}. \tag{2.11}$$

Finally, we set conditions at the boundary Γ_3, which is responsible for the cell interactions. A rather wide but finite variety of types of conditions at this boundary is considered in the literature. Since the outer cell surface is hypothetical, the conditions at this boundary should represent some physical processes. Happel's condition [14] assumes the outer cell boundary is a rigid envelope providing the absence of energy exchange between the cell and the environment. Kuwabara [20], on the opposite, used an analog of a free surface model expressed as the absence of vorticity at the cell outer surface. Kvashnin [21] generalized the condition of the velocity profile symmetry which takes place at the points where the cells come into contact with each other and applied it to the whole boundary. Cunningham's condition [5] considers each cell as if placed in a uniform free stream. The conservation of the flow uniformity condition, that is the free stream velocity is set at the outer cell boundary. For a cell of an arbitrary shape this condition seems to be the most appropriate:

$$\widetilde{\mathbf{v}}_2|_{\Gamma_3} = \widetilde{\mathbf{U}}. \tag{2.12}$$

One more condition deals with the angular velocity or couple stress. Either the former or the latter can be set to be equal to zero. There is no any physical evidences in favor of any of them. So, we can use the no-spin condition

$$\widetilde{\boldsymbol{\omega}}_2\big|_{\Gamma_3} = 0. \tag{2.13}$$

Passing to non-dimensional variables

$$x = \frac{\widetilde{x}}{d_{\Omega_2}}, \quad \mathbf{v} = \frac{\widetilde{\mathbf{v}}}{|\widetilde{\mathbf{U}}|}, \quad \boldsymbol{\omega} = \frac{\widetilde{\omega}d_{\Omega_2}}{|\widetilde{\mathbf{U}}|}, \quad p = \frac{\widetilde{p}d_{\Omega_2}}{\mu|\widetilde{\mathbf{U}}|},$$

where d_{Ω_2} is the diameter of Ω_2, we rewrite the original system in the non-dimensional form

$$\operatorname{div} \mathbf{v}_1 = 0,$$

$$\varepsilon\Big(\frac{1}{N^2} - 1\Big)\nabla p_1 = \frac{1}{N^2}\Delta\mathbf{v}_1 + 2\operatorname{curl}\boldsymbol{\omega}_1 - \frac{\varepsilon\sigma^2}{N^2}\mathbf{v}_1,$$

$$L^2\Delta\boldsymbol{\omega}_1 + \frac{1}{2}\frac{N^2}{1 - N^2}\operatorname{curl}\mathbf{v}_1 - \frac{N^2}{1 - N^2}\boldsymbol{\omega}_1 = 0. \tag{2.14}$$

$$\operatorname{div} \mathbf{v}_2 = 0,$$

$$\Big(\frac{1}{N^2} - 1\Big)\nabla p_2 = \frac{1}{N^2}\Delta\mathbf{v}_2 + 2\operatorname{curl}\boldsymbol{\omega}_2,$$

$$L^2\Delta\boldsymbol{\omega}_2 + \frac{1}{2}\frac{N^2}{1 - N^2}\operatorname{curl}\mathbf{v}_2 - \frac{N^2}{1 - N^2}\boldsymbol{\omega}_2 = 0. \tag{2.15}$$

Here we use three non-dimensional parameters N, L and σ:

$$N^2 = \frac{\kappa}{\mu + \kappa}, \quad L^2 = \frac{\delta + \zeta}{4\mu d_{\Omega_2}^2}, \quad \sigma = \frac{d_{\Omega_2}}{\sqrt{k}}. \tag{2.16}$$

Coupling parameter N^2 - demonstrates the fraction of rotational viscosity in the sum of rotational and translational viscosities, L^2 represents the relation between the micro scale of the problem and its macro scale, the parameter σ represents the ratio of the macro scale of the cell to the micro scale of the porous layer. Observe that $0 < N < 1$ due to its definition. The boundary conditions in non-dimensional form are as follows:

$$\mathbf{v}_1|_{\Gamma_1} = 0, \quad \boldsymbol{\omega}_1|_{\Gamma_1} = 0, \tag{2.17}$$

$$\mathbf{v}_1|_{\Gamma_2} = \mathbf{v}_2|_{\Gamma_2}, \quad \boldsymbol{\omega}_1|_{\Gamma_2} = \boldsymbol{\omega}_2|_{\Gamma_2}, \tag{2.18}$$

$$\mathbf{v}_2|_{\Gamma_3} = \frac{\widetilde{\mathbf{U}}}{|\widetilde{\mathbf{U}}|} = \mathbf{U}, \tag{2.19}$$

$$\boldsymbol{\omega}_2|_{\Gamma_3} = 0. \tag{2.20}$$

3. Weak solution

We suppose that the incoming velocity $\mathbf{U}$ was a smooth given vector-function. All unknown vector-functions shall be considered in the weak sense, i.e. as general functions from Sobolev space H^1 satisfying a corresponding integral identity. The notation $\mathbf{f} \in H^1(\Omega; \mathbb{R}^3)$ means that each component of the vector-function $\mathbf{f} = (f_1, f_2, f_3)$ belongs to the same space: $f_i \in H^1(\Omega)$. The notation $H^1(\Omega, \Gamma; \mathbb{R}^3)$ is reserved for all functions from the space $H^1(\Omega; \mathbb{R}^3)$ having a zero trace on the set $\Gamma \subset \partial\Omega$. The space H^{-1} is the dual one to H^1. Let us denote further the outer normal vector to the boundary Γ by $\mathbf{n}_\Gamma = (n_\Gamma^1, n_\Gamma^2, n_\Gamma^3)$. In particular, $\mathbf{n}_{\partial\Omega_i}$ means the outer normal vector to the domain Ω_i. Define the spaces

$$A_1 = \{\mathbf{u} \in H^1(\Omega; \mathbb{R}^3) : \operatorname{div}\mathbf{u} = 0\},$$

$$A_2 = \{\mathbf{u} \in H^1(\Omega; \mathbb{R}^3) : \operatorname{div}\mathbf{u} = 0, \ \mathbf{u} = \mathbf{U} \text{ on } \Gamma_3\}.$$

Now we can formulate the definition of the weak solution.

Definition 3.1. The triplet $(\mathbf{v}_1, \boldsymbol{\omega}_1, p_1) \in (H^1(\Omega_1, \Gamma_1; \mathbb{R}^3), H^1(\Omega_1, \Gamma_1; \mathbb{R}^3), H^{-1}(\Omega_1))$ is the weak solution to equations (2.14) and boundary conditions (2.17)-(2.18) if

the following integral equalities hold

$$\varepsilon\Big(\frac{1}{N^2}-1\Big)\int_{\Gamma_2}\mathbf{u}p_1\cdot\mathbf{n}_{\Gamma_2}\,dS = -\frac{1}{N^2}\int_{\Omega_1}\nabla\mathbf{v}_1\nabla\mathbf{u}\,dx + \frac{1}{N^2}\int_{\Gamma_2}\mathbf{u}\frac{\partial\mathbf{v}_1}{\partial\mathbf{n}_{\Gamma_2}}\,dS$$
$$+ 2\int_{\Omega_1}\operatorname{curl}\boldsymbol{\omega}_1\mathbf{u}\,dx - \frac{\varepsilon\sigma^2}{N^2}\int_{\Omega_1}\mathbf{u}\mathbf{v}_1\,dx, \tag{3.1}$$

for any test-function $\mathbf{u} = (u_1, u_2, u_3) \in H^1(\Omega_1, \Gamma_1; \mathbb{R}^3) \cap A_1$;

$$-L^2\int_{\Omega_1}\nabla\boldsymbol{\varphi}\nabla\boldsymbol{\omega}_1\,dx + L^2\int_{\Gamma_2}\boldsymbol{\varphi}\frac{\partial\boldsymbol{\omega}_1}{\partial\mathbf{n}_{\Gamma_2}}\,dS$$
$$+\frac{1}{2}\frac{N^2}{1-N^2}\int_{\Omega_1}\boldsymbol{\varphi}\operatorname{curl}\mathbf{v}_1\,dx - \frac{N^2}{1-N^2}\int_{\Omega_1}\boldsymbol{\varphi}\boldsymbol{\omega}_1\,dx = 0 \tag{3.2}$$

for any test-function $\boldsymbol{\varphi} \in H^1(\Omega_1, \Gamma_1; \mathbb{R}^3) \cap A_1$;

$$\int_{\Omega_1}\mathbf{v}_1\nabla q_1\,dx = \int_{\Gamma_2}\mathbf{v}_2 q_1\cdot\mathbf{n}_{\Gamma_2}\,dS, \quad \text{for all } q_1 \in L_2(\Omega_1); \tag{3.3}$$

$$\int_{\Omega_1}\boldsymbol{\omega}_1\nabla q_2\,dx = \int_{\Gamma_2}\boldsymbol{\omega}_2 q_2\cdot\mathbf{n}_{\Gamma_2}\,dS, \quad \text{for all } q_2 \in L_2(\Omega_1). \tag{3.4}$$

Definition 3.2. The triplet $(\mathbf{v}_2, \boldsymbol{\omega}_2, p_2) \in (H^1(\Omega_2; \mathbb{R}^3), H^1(\Omega_2, \Gamma_3; \mathbb{R}^3), H^{-1}(\Omega_2))$ is the weak solution to equations (2.15) and (2.7)-(2.20) if the following integral identities hold:

$$-\Big(\frac{1}{N^2}-1\Big)\int_{\Gamma_2}\mathbf{g}p_2\cdot\mathbf{n}_{\Gamma_2}\,dS + \Big(\frac{1}{N^2}-1\Big)\int_{\Gamma_3}\mathbf{U}p_2\cdot\mathbf{n}_{\Gamma_3}\,dS$$
$$= -\frac{1}{N^2}\int_{\Omega_2}\nabla\mathbf{v}_2\nabla\mathbf{g}\,dx - \frac{1}{N^2}\int_{\Gamma_2}\mathbf{g}\frac{\partial\mathbf{v}_2}{\partial\mathbf{n}_{\Gamma_2}}\,dS + \frac{1}{N^2}\int_{\Gamma_3}\mathbf{U}\frac{\partial\mathbf{v}_2}{\partial\mathbf{n}_{\Gamma_3}}\,dS \tag{3.5}$$
$$+ 2\int_{\Omega_2}\operatorname{curl}\boldsymbol{\omega}_2\mathbf{g}\,dx,$$

for any test-function $\mathbf{g} = (g_1, g_2, g_3) \in H^1(\Omega_2; \mathbb{R}^3) \cap A_2$;

$$-L^2\int_{\Omega_2}\nabla\boldsymbol{\psi}\nabla\boldsymbol{\omega}_2\,dx + L^2\int_{\Gamma_2}\boldsymbol{\psi}\frac{\partial\boldsymbol{\omega}_2}{\partial\mathbf{n}_{\Gamma_2}}\,dS$$
$$+\frac{1}{2}\frac{N^2}{1-N^2}\int_{\Omega_2}\boldsymbol{\psi}\operatorname{curl}\mathbf{v}_2\,dx - \frac{N^2}{1-N^2}\int_{\Omega_2}\boldsymbol{\psi}\boldsymbol{\omega}_2\,dx = 0 \tag{3.6}$$

for any test-function $\boldsymbol{\psi} \in H^1(\Omega_2, \Gamma_3; \mathbb{R}^3) \cap A_1$;

$$\int_{\Omega_2}\mathbf{v}_2\nabla q_3\,dx = -\int_{\Gamma_2}\mathbf{v}_1 q_3\cdot\mathbf{n}_{\Gamma_2}\,dS + \int_{\Gamma_3}\mathbf{U}q_3\cdot\mathbf{n}_{\Gamma_3}\,dS, \quad \text{for all } q_3 \in L_2(\Omega_2), \tag{3.7}$$

$$\int_{\Omega_2}\boldsymbol{\omega}_2\nabla q_4\,dx = \int_{\Gamma_2}\boldsymbol{\omega}q_4\cdot\mathbf{n}_{\Gamma_2}\,dS, \quad \text{for all } q_4 \in L_2(\Omega_2) \tag{3.8}$$

4. A PRIORI ESTIMATES FOR THE WEAK SOLUTION

In this section we derive a priori estimates for the velocity functions $\mathbf{v}_1, \mathbf{v}_2$ and $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ solving the original problem in the weak sense. We recall the definition of L_2 and H^1 norms in the Sobolev space $H^1(\Omega)$ which will be used in our analysis:

$$\|\mathbf{v}\|_{L_2(\Omega)}^2 = \int_\Omega \mathbf{v}^2\,dx,$$

$$\|\mathbf{v}\|^2_{H^1(\Omega)} = \int_\Omega \mathbf{v}^2 \, dx + \int_\Omega |\nabla \mathbf{v}|^2 \, dx.$$

We will estimate the unknown velocities with respect to L_2 and H^1 norms. The pressure p_1, p_2 are obviously defined up to the constant. First we state an a priori estimate of Friedrichs-type valid for functions vanishing on a part of a boundary. The technique of the proof in case when functions vanishes not on the whole boundary, was demonstrated in [2, 3].

Lemma 4.1 (Friedrichs inequality)**.** *Let Ω be a bounded domain with boundary $\partial\Omega$ satisfying a local Lipschitz condition. Consider $f \in H^1(\Omega)$ such that $f = 0$ on $\Gamma \subset \partial\Omega$, where $\mathrm{meas}(\Gamma) \neq 0$. Then there exists a constant $C > 0$ which depends on diameter of the domain Ω such that*

$$\int_\Omega f^2 \, dx \le C \int_\Omega |\nabla f|^2 \, dx. \tag{4.1}$$

Moreover, following to the proof of [23, Lemma 2.2.2] it is possible to show that the sharp constant $C \le d_\Omega^2$ even for the case when $\Gamma \neq \partial\Omega$, see also [22, p. 22]. Boundary conditions (2.17) and (2.20) imply the validity of (4.1) for $\mathbf{v}_1, \boldsymbol\omega_1$ in the domain Ω_1 as well as for $\boldsymbol\omega_2$ in the domain Ω_2:

$$\int_{\Omega_1} \mathbf{v}_1^2 \, dx \le d_{\Omega_1}^2 \int_{\Omega_1} |\nabla \mathbf{v}_1|^2 \, dx, \tag{4.2}$$

$$\int_{\Omega_1} \boldsymbol\omega_1^2 \, dx \le d_{\Omega_1}^2 \int_{\Omega_1} |\nabla \boldsymbol\omega_1|^2 \, dx, \tag{4.3}$$

$$\int_{\Omega_2} \boldsymbol\omega_2^2 \, dx \le d_{\Omega_2}^2 \int_{\Omega_2} |\nabla \boldsymbol\omega_2\|^2 \, dx, \tag{4.4}$$

where d_{Ω_i} is the diameter of domain Ω_i, $i = 1, 2$. A similar inequality for $\mathbf{v}_2$ is slightly different due to the non-zero trace $\mathbf{v}_2|_{\partial\Omega_2}$ on the boundary (see [11, 27]):

$$\int_{\Omega_2} \mathbf{v}_2^2 \, dx \le d_{\Omega_2}^2 \Big(\int_{\Omega_2} |\nabla \mathbf{v}_2|^2 \, dx + \frac{1}{|\partial\Omega_2|} \int_{\partial\Omega_2} |\mathbf{v}_2|^2 \, dS \Big). \tag{4.5}$$

Here $|\partial\Omega_2|$ is measure of the boundary $\partial\Omega_2$. We use this fact to derive estimates for the weak solution in the next theorem. Observe that $d_{\Omega_2} = 1$ since we have passed to non-dimensional values.

Theorem 4.2. *The solution to* (3.1)–(3.4) *and* (3.5)–(3.8) *satisfies the following estimates:*

$$\|\mathbf{v}_1\|^2_{L_2} < (2\varepsilon)^{-1}\sigma^{-2} + \varepsilon^{-1}\sigma^{-2} \max\{2N^2, 4L^2(1 - N^2)\}(\|\boldsymbol\omega_1\|^2_{H^1} + \|\boldsymbol\omega_2\|^2_{H^1}), \tag{4.6}$$

$$\|\mathbf{v}_2\|^2_{L_2}$$
$$< \Big(\frac{1 + \frac{d_{\Omega_1}^2}{2}}{1 - \frac{d_{\Omega_1}^2}{2}} \Big) \Big(\frac{1}{2} + \max\{2N^2, 4L^2(1 - N^2)\}(\|\boldsymbol\omega_1\|^2_{H^1} + \|\boldsymbol\omega_2\|^2_{H^1}) \Big) + 1. \tag{4.7}$$

$$\Big(1 - \frac{d_{\Omega_1}^2}{2}\Big) \|\mathbf{v}_1\|^2_{H^1} + \|\nabla \mathbf{v}_2\|^2_{L_2}$$
$$< \frac{1}{2} + \max\{2N^2, 4L^2(1 - N^2)\}(\|\boldsymbol\omega_1\|^2_{H^1} + \|\boldsymbol\omega_2\|^2_{H^1}), \tag{4.8}$$

$$\|\boldsymbol{\omega}_1\|_{H^1}^2 + \|\boldsymbol{\omega}_2\|_{H^1}^2 \leq \max\{\frac{N^2}{4L^2(1-N^2)}, 1\} \left(\|\operatorname{curl}\mathbf{v}_1\|_{L_2}^2 + \|\operatorname{curl}\mathbf{v}_2\|_{L_2}^2\right) \quad (4.9)$$

$$\|\boldsymbol{\omega}_1\|_{H^1}^2 + \|\boldsymbol{\omega}_2\|_{H^1}^2 \leq \max\{\frac{N^2}{2L^2(1-N^2)}, 2\} \left(\|\mathbf{v}_1\|_{H^1}^2 + \|\mathbf{v}_2\|_{H^1}^2\right) \quad (4.10)$$

$$\|\boldsymbol{\omega}_1\|_{H_1}^2 \leq \frac{1}{8(1-d_{\Omega_1}^2)^2} \frac{N^4}{L^4(1-N^2)^2} \|\operatorname{curl}\mathbf{v}_1\|_{L_2}^2, \quad (4.11)$$

$$\|\boldsymbol{\omega}_2\|_{H_1}^2 \leq \max\{\frac{1}{3}, \frac{1}{3}\frac{N^2}{(1-N^2)L^2}\}\|\operatorname{curl}\mathbf{v}_2\|_{L_2}^2, \quad (4.12)$$

$$\|\mathbf{v}_1\|_{H^1}^2 + \|\mathbf{v}_2\|_{H^1}^2 < \frac{1}{C}, \quad (4.13)$$

where

$$C = \begin{cases} 2 - d_{\Omega_1}^2 + 16L^2N^2 - 16L^2, & \text{if } \frac{N^2}{L^2(1-N^2)} < 2, \\ 2 - d_{\Omega_1}^2 - 8N^2, & \text{if } 2 \leq \frac{N^2}{L^2(1-N^2)} < 4, \\ 2 - d_{\Omega_1}^2 - \frac{2N^4}{L^2(1-N^2)}, & \text{if } \frac{N^2}{L^2(1-N^2)} \geq 4. \end{cases}$$

Proof. We substitute $\mathbf{u} = \mathbf{v}_1 \equiv (v_1^1, v_1^2, v_1^3)$ in (3.1), $\boldsymbol{\varphi} = \boldsymbol{\omega}_1$ in (3.2) and take into the account $q_1 = p_1$ in (3.3). One obtains

$$\varepsilon\left(\frac{1}{N^2} - 1\right) \sum_{i=1}^3 \int_{\Gamma_2} v_1^i p_1 n_{\Gamma_2}^i \, dS + \frac{\varepsilon\sigma^2}{N^2} \int_{\Omega_1} |\mathbf{v}_1|^2 \, dx + \frac{1}{N^2} \int_{\Omega_1} |\nabla\mathbf{v}_1|^2 \, dx$$
$$= \frac{1}{N^2} \int_{\Gamma_2} \mathbf{v}_1 \frac{\partial\mathbf{v}_1}{\partial\mathbf{n}_{\Gamma_2}} \, dS + 2 \int_{\Omega_1} \mathbf{v}_1 \operatorname{curl}\boldsymbol{\omega}_1 \, dx, \quad (4.14)$$

$$L^2 \int_{\Omega_1} |\nabla\boldsymbol{\omega}_1|^2 \, dx + \frac{N^2}{1-N^2} \int_{\Omega_1} |\boldsymbol{\omega}_1|^2 \, dx$$
$$= \frac{1}{2}\frac{N^2}{1-N^2} \int_{\Omega_1} \boldsymbol{\omega}_1 \operatorname{curl}\mathbf{v}_1 \, dx + \frac{L^2}{2} \int_{\Gamma_2} \boldsymbol{\omega}_1 \frac{\partial\boldsymbol{\omega}_1}{\partial\mathbf{n}_{\Gamma_2}} \, dS. \quad (4.15)$$

Observing that

$$\mathbf{v}_1 \frac{\partial\mathbf{v}_1}{\partial\mathbf{n}_{\Gamma_2}} = \frac{1}{2}\frac{\partial(\mathbf{v}_1)^2}{\partial\mathbf{n}_{\Gamma_2}},$$

we rewrite (4.14) and (4.15) equivalently as

$$\varepsilon\left(\frac{1}{N^2} - 1\right) \sum_{i=1}^3 \int_{\Gamma_2} v_1^i p_1 n_{\Gamma_2}^i \, dS + \frac{\varepsilon\sigma^2}{N^2} \int_{\Omega_1} |\mathbf{v}_1|^2 \, dx + \frac{1}{N^2} \int_{\Omega_1} |\nabla\mathbf{v}_1|^2 \, dx$$
$$= \frac{1}{2N^2}|\mathbf{v}_1|_{\Gamma_2}^2 + 2 \int_{\Omega_1} \mathbf{v}_1 \operatorname{curl}\boldsymbol{\omega}_1 \, dx, \quad (4.16)$$

$$L^2 \int_{\Omega_1} |\nabla\boldsymbol{\omega}_1|^2 \, dx + \frac{N^2}{1-N^2} \int_{\Omega_1} |\boldsymbol{\omega}_1|^2 \, dx$$
$$= \frac{1}{2}\frac{N^2}{1-N^2} \int_{\Omega_1} \boldsymbol{\omega}_1 \operatorname{curl}\mathbf{v}_1 \, dx + \frac{L^2}{2}|\boldsymbol{\omega}_1|_{\Gamma_2}^2. \quad (4.17)$$

Multiplying (4.16) by $1/2$, and (4.17) by $-2(1-N^2)/N^2$, after the summation one obtains

$$\frac{\varepsilon}{2}\Big(\frac{1}{N^2}-1\Big)\sum_{i=1}^{3}\int_{\Gamma_2} v_1^i p_1 n_{\Gamma_2}^i \, dS + \frac{\varepsilon\sigma^2}{2N^2}\int_{\Omega_1}|\mathbf{v}_1|^2\,dx + \frac{1}{2N^2}\int_{\Omega_1}|\nabla\mathbf{v}_1|^2\,dx$$

$$+ \frac{L^2}{N^2}(1-N^2)|\boldsymbol{\omega}_1|_{\Gamma_2}^2$$

$$= \frac{1}{2N^2}|\mathbf{v}_1|_{\Gamma_2}^2 + \int_{\Omega_1}\operatorname{div}[\mathbf{v}_1\times\boldsymbol{\omega}_1]\,dx$$

$$+ \int_{\Omega_1}|\boldsymbol{\omega}_1|^2\,dx + \frac{2L^2}{N^2}(1-N^2)\int_{\Omega_1}|\nabla\boldsymbol{\omega}_1|^2\,dx. \tag{4.18}$$

Here we used the formula

$$\operatorname{div}[\mathbf{v}_1\times\boldsymbol{\omega}_1] = \mathbf{v}_1\operatorname{curl}\boldsymbol{\omega}_1 - \boldsymbol{\omega}_1\operatorname{curl}\mathbf{v}_1.$$

Substituting $\mathbf{g} = \mathbf{v}_2$ in (3.5), $\boldsymbol{\psi} = \boldsymbol{\omega}_2$ in (3.6) and using (3.7) with $q_3 = p_2$ one gets the following integral identity after integration by parts,

$$\Big(\frac{1}{N^2}-1\Big)\sum_{i=1}^{3}\int_{\partial\Omega_2} v_2^i p_2 n_{\Gamma_2}^i \, dS + \frac{1}{N^2}\int_{\Omega_2}|\nabla\mathbf{v}_2|^2\,dx$$

$$= \frac{1}{N^2}\int_{\partial\Omega_2}\mathbf{v}_2\frac{\partial\mathbf{v}_2}{\partial\mathbf{n}_{\Omega_2}}\,dS + 2\int_{\Omega_2}\mathbf{v}_2\operatorname{curl}\boldsymbol{\omega}_2\,dx, \tag{4.19}$$

$$L^2\int_{\Omega_2}|\nabla\boldsymbol{\omega}_2|^2\,dx + \frac{N^2}{1-N^2}\int_{\Omega_2}|\boldsymbol{\omega}_2|^2\,dx$$

$$= \frac{1}{2}\frac{N^2}{1-N^2}\int_{\Omega_2}\boldsymbol{\omega}_2\operatorname{curl}\mathbf{v}_2\,dx + L^2\int_{\partial\Omega_2}\boldsymbol{\omega}_2\frac{\partial\boldsymbol{\omega}_2}{\partial\mathbf{n}_{\Omega_2}}\,dS. \tag{4.20}$$

Analogously, observing that

$$\mathbf{v}_2\frac{\partial\mathbf{v}_2}{\partial n_{\Omega_2}} = \frac{1}{2}\frac{\partial(\mathbf{v}_2)^2}{\partial\mathbf{n}_{\Omega_2}},$$

we can simplify (4.19) to the form

$$\Big(\frac{1}{N^2}-1\Big)\sum_{i=1}^{3}\int_{\partial\Omega_2} v_2^i p_2 n_{\Gamma_2}^i \, dS + \frac{1}{2N^2}|\mathbf{v}_2|_{\Gamma_2}^2 + \frac{1}{N^2}\int_{\Omega_2}|\nabla\mathbf{v}_2|^2\,dx$$

$$= \frac{1}{2N^2}|\mathbf{v}_2|_{\Gamma_3}^2 + 2\int_{\Omega_2}\mathbf{v}_2\operatorname{curl}\boldsymbol{\omega}_2\,dx. \tag{4.21}$$

From the boundary conditions (2.18)-(2.19) we have

$$-\Big(\frac{1}{N^2}-1\Big)\sum_{i=1}^{3}\int_{\Gamma_2} v_2^i p_2 n_{\Gamma_2}^i \, dS + \Big(\frac{1}{N^2}-1\Big)\int_{\Gamma_3}\sum_{i=1}^{3} U_i p_2 n_{\Gamma_3}^i \, dS$$

$$+ \frac{1}{2N^2}|\mathbf{v}_2|_{\Gamma_2}^2 + \frac{1}{N^2}\int_{\Omega_2}|\nabla\mathbf{v}_2|^2\,dx$$

$$= \frac{1}{2N^2} + 2\int_{\Omega_2}\mathbf{v}_2\operatorname{curl}\boldsymbol{\omega}_2\,dx. \tag{4.22}$$

In a similar way, (4.20) is equivalent to

$$L^2 \int_{\Omega_2} |\nabla \boldsymbol{\omega}_2|^2 \, dx + \frac{N^2}{1 - N^2} \int_{\Omega_2} |\boldsymbol{\omega}_2|^2 \, dx + \frac{L^2}{2} |\boldsymbol{\omega}_2|_{\Gamma_2}^2$$
$$= \frac{1}{2} \frac{N^2}{1 - N^2} \int_{\Omega_2} \boldsymbol{\omega}_2 \operatorname{curl} \mathbf{v}_2 \, dx. \tag{4.23}$$

Multiplying (4.22) by $1/2$, and (4.23) by $\frac{-2(1-N^2)}{N^2}$, after its summation one gets

$$-\frac{1}{2} \Big(\frac{1}{N^2} - 1 \Big) \sum_{i=1}^{3} \int_{\Gamma_2} v_2^i p_2 n_{\Gamma_2}^i \, dS + \frac{1}{2} \Big(\frac{1}{N^2} - 1 \Big) \int_{\Gamma_3} \sum_{i=1}^{3} U_i p_2 n_{\Gamma_3}^i \, dS$$
$$+ \frac{1}{4N^2} |\mathbf{v}_2|_{\Gamma_2}^2 + \frac{1}{2N^2} \int_{\Omega_2} |\nabla \mathbf{v}_2|^2 \, dx$$
$$= \frac{1}{4N^2} + \int_{\Omega_2} \operatorname{div}[\mathbf{v}_2 \times \boldsymbol{\omega}_2] \, dx + \frac{2L^2}{N^2} (1 - N^2) \int_{\Omega_2} |\nabla \boldsymbol{\omega}_2|^2 \, dx \tag{4.24}$$
$$+ \int_{\Omega_2} |\boldsymbol{\omega}_2|^2 \, dx + \frac{L^2}{N^2} (1 - N^2) |\boldsymbol{\omega}_2|_{\Gamma_2}^2.$$

Integrating by parts $\int_{\Omega_i} \operatorname{div}[\mathbf{v}_i \times \boldsymbol{\omega}_i] \, dx$ for $i = 1, 2$. Applying boundary conditions, we obtain

$$\int_{\Omega_1} \operatorname{div}[\mathbf{v}_1 \times \boldsymbol{\omega}_1] \, dx = \int_{\Gamma_2} [\mathbf{v}_1 \times \boldsymbol{\omega}_1] \cdot \mathbf{n}_{\Gamma_2} \, dS,$$
$$\int_{\Omega_2} \operatorname{div}[\mathbf{v}_2 \times \boldsymbol{\omega}_2] \, dx = - \int_{\Gamma_2} [\mathbf{v}_2 \times \boldsymbol{\omega}_2] \cdot \mathbf{n}_{\Gamma_2} \, dS.$$

Taking this into account, we sum (4.18), (4.24) and use the continuity of linear and angular velocities on the boundary Γ_2:

$$\Big(-\frac{1}{2} + \frac{\varepsilon}{2} \Big) \Big(\frac{1}{N^2} - 1 \Big) \sum_{i=1}^{3} \int_{\Gamma_2} v_2^i p_2 n_{\Gamma_2}^i \, dS + \frac{1}{2} \Big(\frac{1}{N^2} - 1 \Big) \int_{\Gamma_3} \sum_{i=1}^{3} U_i p_2 n_{\Gamma_3}^i \, dS$$
$$+ \frac{1}{2N^2} \int_{\Omega_2} |\nabla \mathbf{v}_2|^2 \, dx + \frac{\varepsilon \sigma^2}{2N^2} \int_{\Omega_1} |\mathbf{v}_1|^2 \, dx + \frac{1}{2N^2} \int_{\Omega_1} |\nabla \mathbf{v}_1|^2 \, dx$$
$$= \frac{1}{4N^2} |\mathbf{v}_2|_{\Gamma_2}^2 + \int_{\Gamma_2} [\mathbf{v}_1 \times \boldsymbol{\omega}_1] \cdot \mathbf{n}_{\Gamma_2} \, dS + \int_{\Omega_1} |\boldsymbol{\omega}_1|^2 \, dx$$
$$+ \frac{2L^2}{N^2} (1 - N^2) \int_{\Omega_2} |\nabla \boldsymbol{\omega}_2|^2 \, dx + \frac{1}{4N^2} - \int_{\Gamma_2} [\mathbf{v}_2 \times \boldsymbol{\omega}_2] \cdot \mathbf{n}_{\Gamma_2} \, dS$$
$$+ \frac{2L^2}{N^2} (1 - N^2) \int_{\Omega_2} |\nabla \boldsymbol{\omega}_2|^2 \, dx + \int_{\Omega_2} |\boldsymbol{\omega}_2|^2 \, dx.$$

This equality implies

$$\varepsilon \sigma^2 \|\mathbf{v}_1\|_{L_2}^2 + \|\nabla \mathbf{v}_1\|_{L_2}^2 + \|\nabla \mathbf{v}_2\|_{L_2}^2$$
$$< \frac{1}{2} |\mathbf{v}_2|_{\Gamma_2}^2 + \frac{1}{2} + \max\{2N^2, 4L^2(1 - N^2)\}(\|\boldsymbol{\omega}_1\|_{H^1}^2 + \|\boldsymbol{\omega}_2\|_{H^1}^2). \tag{4.25}$$

The trace $|\mathbf{v}_2|_{\Gamma_2}^2$ on the boundary can be estimated as follows

$$|\mathbf{v}_2|_{\Gamma_2}^2 = |\mathbf{v}_1|_{\Gamma_2}^2 \le d_{\Omega_1}^2 \|\nabla \mathbf{v}_1\|_{L_2}^2;$$

therefore,

$$\varepsilon\sigma^2\|\mathbf{v}_1\|_{L_2}^2 + (1 - \frac{d_{\Omega_1}^2}{2})\|\nabla\mathbf{v}_1\|_{L_2}^2 + \|\nabla\mathbf{v}_2\|_{L_2}^2$$
$$< \frac{1}{2} + \max\{2N^2, 4L^2(1 - N^2)\}(\|\boldsymbol{\omega}_1\|_{H^1}^2 + \|\boldsymbol{\omega}_2\|_{H^1}^2), \tag{4.26}$$

$$\min\left\{\varepsilon\sigma^2, \left(1 - \frac{d_{\Omega_1}^2}{2}\right)\right\}\|\mathbf{v}_1\|_{H^1}^2 + \|\nabla\mathbf{v}_2\|_{L_2}^2$$
$$< \frac{1}{2} + \max\{2N^2, 4L^2(1 - N^2)\}(\|\boldsymbol{\omega}_1\|_{H^1}^2 + \|\boldsymbol{\omega}_2\|_{H^1}^2). \tag{4.27}$$

In practical applications the parameter $\varepsilon\sigma^2 > 1$, while $d_{\Omega_1}^2 < 1$, therefore one arrives at

$$\left(1 - \frac{d_{\Omega_1}^2}{2}\right)\|\mathbf{v}_1\|_{H^1}^2 + \|\nabla\mathbf{v}_2\|_{L_2}^2$$
$$< \frac{1}{2} + \max\{2N^2, 4L^2(1 - N^2)\}(\|\boldsymbol{\omega}_1\|_{H^1}^2 + \|\boldsymbol{\omega}_2\|_{H^1}^2). \tag{4.28}$$

An estimate for $\mathbf{v}_2$ in L_2 follows from the Friedrichs inequality (4.5) and (4.26). Namely,

$$\|\mathbf{v}_2\|_{L_2}^2 \le \|\nabla\mathbf{v}_2\|_{L_2}^2 + |\mathbf{v}_1|_{\Gamma_2}^2 + 1$$
$$\le \left(\frac{1 + \frac{d_{\Omega_1}^2}{2}}{1 - \frac{d_{\Omega_1}^2}{2}}\right)\left(\frac{1}{2} + \max\{2N^2, 4L^2(1 - N^2)\}(\|\boldsymbol{\omega}_1\|_{H^1}^2 + \|\boldsymbol{\omega}_2\|_{H^1}^2)\right) + 1.$$

Hence, inequalities (4.6), (4.7) and (4.8) are proved.

Let us derive estimates for norms of angular velocities. An analysis of identity (4.17) leads to the restriction

$$L^2 \int_{\Omega_1} |\nabla\boldsymbol{\omega}_1|^2 \, dx \le \frac{1}{2}\frac{N^2}{1 - N^2} \int_{\Omega_1} \boldsymbol{\omega}_1 \operatorname{curl} \mathbf{v}_1 \, dx + \frac{L^2}{2}|\boldsymbol{\omega}_1|_{\Gamma_2}^2.$$

Applying Hölder's inequality, Friedrichs inequality (4.3) and the estimate

$$|\boldsymbol{\omega}_1|_{\Gamma_2}^2 \le d_{\Omega_1}^2 \int_{\Omega_1} |\nabla\boldsymbol{\omega}_1|^2 \, dx,$$

one deduces that

$$L^2 \int_{\Omega_1} |\nabla\boldsymbol{\omega}_1|^2 \, dx$$
$$\le \frac{1}{4\alpha_1^2}\frac{N^2}{1 - N^2} \int_{\Omega_1} |\boldsymbol{\omega}_1|^2 \, dx + \frac{\alpha_1^2}{4}\frac{N^2}{1 - N^2} \int_{\Omega_1} (\operatorname{curl} \mathbf{v}_1)^2 \, dx + \frac{L^2 d_{\Omega_1}^2}{2} \int_{\Omega_1} |\nabla\boldsymbol{\omega}_1|^2 \, dx$$
$$\le d_{\Omega_1}^2\left(\frac{1}{4\alpha_1^2}\frac{N^2}{1 - N^2} + \frac{L^2}{2}\right) \int_{\Omega_1} |\nabla\boldsymbol{\omega}_1|^2 \, dx + \frac{\alpha_1^2}{4}\frac{N^2}{1 - N^2} \int_{\Omega_1} (\operatorname{curl} \mathbf{v}_1)^2 \, dx.$$

Finally, we obtain

$$\int_{\Omega_1} |\nabla\boldsymbol{\omega}_1|^2 \, dx \le \left(L^2 - d_{\Omega_1}^2\left(\frac{1}{4\alpha_1^2}\frac{N^2}{1 - N^2} + \frac{L^2}{2}\right)\right)^{-1}\frac{\alpha_1^2}{4}\frac{N^2}{1 - N^2} \int_{\Omega_1} (\operatorname{curl} \mathbf{v}_1)^2 \, dx,$$

where an arbitrary constant α_1 is chosen such that the coefficient in the right-hand side of the derived inequality was positive. To prove the estimate (4.11) for H^1

norm one needs to apply the Friedrichs inequality (4.3) to express the bound for L_2 norm of function $\boldsymbol{\omega}_1$ via L_2 norm of its gradient,

$$\|\boldsymbol{\omega}_1\|_{H^1}^2 \leq \max\{1, d_{\Omega_1}\} \Big(L^2 - d_{\Omega_1}^2\Big(\frac{1}{4\alpha_1^2}\frac{N^2}{1-N^2} + \frac{L^2}{2}\Big)\Big)^{-1}\frac{\alpha_1^2}{4}\frac{N^2}{1-N^2}\int_{\Omega_1}(\operatorname{curl}\mathbf{v}_1)^2\,dx.$$

We shall choose the constant α_1 such that $\frac{1}{4\alpha_1^2}\frac{N^2}{1-N^2} = \frac{L^2}{2}$. Observing that $d_{\Omega_1} < 1$ from the geometry of our problem, we simplify the obtained inequality to the form

$$\|\boldsymbol{\omega}_1\|_{H^1}^2 \leq \frac{1}{8(1-d_{\Omega_1}^2)^2}\frac{N^4}{L^4(1-N^2)^2}\int_{\Omega_1}(\operatorname{curl}\mathbf{v}_1)^2\,dx$$

and prove (4.11).

Observe that left-hand side of (4.23) is non-negative, therefore its right-hand side is also nonnegative and each term in (4.23) can be estimated by $\frac{N^2}{1-N^2}\int_{\Omega_2}\boldsymbol{\omega}_2\operatorname{curl}\mathbf{v}_2\,dx$. In particular,

$$\frac{N^2}{1-N^2}\int_{\Omega_2}|\boldsymbol{\omega}_2|^2\,dx \leq \frac{1}{2}\frac{N^2}{1-N^2}\int_{\Omega_2}\boldsymbol{\omega}_2\operatorname{curl}\mathbf{v}_2\,dx$$

if and only if

$$\int_{\Omega_2}|\boldsymbol{\omega}_2|^2\,dx \leq \frac{1}{2}\int_{\Omega_2}\boldsymbol{\omega}_2\operatorname{curl}\mathbf{v}_2\,dx \leq \frac{1}{4}\int_{\Omega_2}|\boldsymbol{\omega}_2|^2\,dx + \frac{1}{4}\int_{\Omega_2}|\operatorname{curl}\mathbf{v}_2|^2\,dx.$$

Thus,

$$\int_{\Omega_2}|\boldsymbol{\omega}_2|^2\,dx \leq \frac{1}{3}\int_{\Omega_2}|\operatorname{curl}\mathbf{v}_2|^2\,dx. \tag{4.29}$$

Similarly, applying (4.29) and Hölder's inequality we can estimate

$$L^2\int_{\Omega_2}|\nabla\boldsymbol{\omega}_2|^2\,dx \leq \frac{1}{2}\frac{N^2}{1-N^2}\int_{\Omega_2}\boldsymbol{\omega}_2\operatorname{curl}\mathbf{v}_2\,dx$$

$$\leq \frac{1}{4}\frac{N^2}{1-N^2}\int_{\Omega_2}|\boldsymbol{\omega}_2|^2\,dx + \frac{1}{4}\frac{N^2}{1-N^2}\int_{\Omega_2}|\operatorname{curl}\mathbf{v}_2|^2\,dx$$

$$\leq \frac{1}{3}\frac{N^2}{1-N^2}\int_{\Omega_2}|\operatorname{curl}\mathbf{v}_2|^2\,dx,$$

one obtains

$$\int_{\Omega_2}|\nabla\boldsymbol{\omega}_2|^2\,dx \leq \frac{1}{3}\frac{N^2}{(1-N^2)L^2}\int_{\Omega_2}|\operatorname{curl}\mathbf{v}_2|^2\,dx. \tag{4.30}$$

Applying again the Friedrichs inequality (4.4), we derive the estimate

$$\int_{\Omega_2}|\boldsymbol{\omega}_2|^2\,dx \leq \frac{1}{3}\frac{N^2}{(1-N^2)L^2}\int_{\Omega_2}|\operatorname{curl}\mathbf{v}_2|^2\,dx, \tag{4.31}$$

thus we can improve the constant in the bound (4.29):

$$\int_{\Omega_2}|\boldsymbol{\omega}_2|^2\,dx \leq C\int_{\Omega_2}|\operatorname{curl}\mathbf{v}_2|^2\,dx, \tag{4.32}$$

where $C = \max\{\frac{1}{3}, \frac{1}{3}\frac{N^2}{(1-N^2)L^2}\}$. Estimates (4.30) and (4.32) directly imply

$$\|\boldsymbol{\omega}_2\|_{H^1}^2 \leq \max\{\frac{1}{3}, \frac{1}{3}\frac{N^2}{(1-N^2)L^2}\}\|\operatorname{curl}\mathbf{v}_2\|_{L_2}^2.$$

Summing integral identities (4.23), (4.20), using the boundary conditions (2.18), we obtain

$$L^2\Big(\int_{\Omega_1}|\nabla\boldsymbol{\omega}_1|^2\,dx + \int_{\Omega_2}|\nabla\boldsymbol{\omega}_2|^2\,dx\Big) + \frac{N^2}{1-N^2}\Big(\int_{\Omega_1}|\boldsymbol{\omega}_1|^2\,dx + \int_{\Omega_2}|\boldsymbol{\omega}_2|^2\,dx\Big)$$
$$= \frac{N^2}{2(1-N^2)}\Big(\int_{\Omega_1}\boldsymbol{\omega}_1\operatorname{curl}\mathbf{v}_1\,dx + \int_{\Omega_2}\boldsymbol{\omega}_2\operatorname{curl}\mathbf{v}_2\,dx\Big).$$

This equality and the inequality $\int fg\,dx \leq \frac{1}{2\alpha^2}\int f^2\,dx + \frac{\alpha^2}{2}\int g^2\,dx$ with nonzero free coefficient α imply

$$\frac{2L^2(1-N^2)}{N^2}\Big(\int_{\Omega_1}|\nabla\boldsymbol{\omega}_1|^2\,dx + \int_{\Omega_2}|\nabla\boldsymbol{\omega}_2|^2\,dx\Big)$$
$$+ \Big(1-\frac{1}{2\alpha^2}\Big)\Big(\int_{\Omega_1}|\boldsymbol{\omega}_1|^2\,dx + \int_{\Omega_2}|\boldsymbol{\omega}_2|^2\,dx\Big) \qquad (4.33)$$
$$\leq \frac{\alpha^2}{2}\Big(\int_{\Omega_1}(\operatorname{curl}\mathbf{v}_1)^2\,dx + \int_{\Omega_2}(\operatorname{curl}\mathbf{v}_2)^2\,dx\Big),$$

what can be reduced to the estimate

$$\|\boldsymbol{\omega}_1\|_{H^1}^2 + \|\boldsymbol{\omega}_2\|_{H^1}^2$$
$$\leq \frac{\alpha^2}{2}\max\{\frac{N^2}{2L^2(1-N^2)}, \frac{2\alpha^2}{2\alpha^2-1}\}\left(\|\operatorname{curl}\mathbf{v}_1\|_{L_2}^2 + \|\operatorname{curl}\mathbf{v}_2\|_{L_2}^2\right). \qquad (4.34)$$

Assuming $\alpha = 1$, (4.34) gives

$$\|\boldsymbol{\omega}_1\|_{H^1}^2 + \|\boldsymbol{\omega}_2\|_{H^1}^2 \leq \max\{\frac{N^2}{4L^2(1-N^2)}, 1\}\left(\|\operatorname{curl}\mathbf{v}_1\|_{L_2}^2 + \|\operatorname{curl}\mathbf{v}_2\|_{L_2}^2\right). \quad (4.35)$$

Applying the definition of curl operator, one obtains the inequality

$$\int_{\Omega_i}(\operatorname{curl}\mathbf{v}_i)^2\,dx \leq 2\int_{\Omega_i}|\nabla\mathbf{v}_i|^2\,dx \leq 2\|\mathbf{v}_i\|_{H^1}^2, \quad i = 1,2.$$

Thus, (4.35) becomes

$$\|\boldsymbol{\omega}_1\|_{H^1}^2 + \|\boldsymbol{\omega}_2\|_{H^1}^2 \leq \max\{\frac{N^2}{2L^2(1-N^2)}, 2\}\left(\|\mathbf{v}_1\|_{H^1}^2 + \|\mathbf{v}_2\|_{H^1}^2\right), \qquad (4.36)$$

and we have obtained (4.10). Finally, combining (4.8) and (4.36), one obtains (4.13). The boundedness of norms for angular velocity fields is the consequence of inequalities (4.10) and (4.13). □

The estimates obtained here show that the weakly defined velocity fields $\mathbf{v}_1$, $\mathbf{v}_2$, $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ are bounded.

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