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HILLE-NEHARI TYPE NON-OSCILLATION CRITERIA FOR HALF-LINEAR DYNAMIC EQUATIONS WITH MIXED DERIVATIVES ON A TIME SCALE

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Dedicated to Professor Jitsuro Sugie on his 65th birthday

Abstract. This article deals with half-linear dynamic equations that have two types of derivatives, and obtains sufficient conditions for all solutions to be non-oscillatory. The obtained results extend a previous Hille-Nehari type theorems for problems of dynamic equations. To prove our main result, we use a generalized Riccati inequality. As an application, we apply the main result to self-adjoint Euler type linear differential and difference equations with a changing sign coefficient. The equation selected for this application is of Mathieu type.

1. Introduction

Equations of continuous type are often used for modeling natural science phenomena in physics and chemistry. While discrete models are used for approximating the continuous models. However, since the beginning of this century, with the development of basic theories of difference equations, many phenomena have been modeled directly with discrete type models and excellent reports have been obtained. On the other hand, the idea of a theory that can unify continuous type models (differential equations) and discrete type model (difference equations) was initiated by Stefan Hilger [12], and it is known as the theory of time scales [6, 7]. Here, a time scale $\mathbb{T}$ is defined as a nonempty closed subset of the real numbers. In time scales, operators such as σ, ρ, μ, and ν are often used, and are defined in Section 5.

This article concerns the non-oscillation of solutions to the half-linear dynamic equation with mixed derivatives

$$\big(r(t)\Phi_p(x^{\Delta})\big)^{\nabla} + c(t)\Phi_p(x) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.1}$$

where $t_0 \in \mathbb{T}$; the function $r{:}\mathbb{T} \to \mathbb{R}$ is continuous and $r(t) > 0$ for all $t \in \mathbb{T}$; the function $c{:}\mathbb{T} \to \mathbb{R}$ is real and left-dense continuous; p is a parameter greater than

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1; Φ_p is the so-called one dimensional p-Laplacian defined for $s \in \mathbb{R}$ by

$$\Phi_p(s) = \begin{cases} |s|^{p-2}s & s \neq 0 \\ 0 & s = 0\,. \end{cases}$$

For simplicity, let q be the conjugate number of p; namely, $1/p+1/q = 1$. Then, Φ_q is the inverse function of Φ_p. Here, the term mixed derivatives indicates the use of Δ-derivative and ∇-derivative (see Section 5 for details). Note that if $\mathbb{T} = \mathbb{R}$, then (1.1) becomes a half-linear differential equation (see [9]). If $\mathbb{T} = \mathbb{Z}$, then (1.1) becomes a half-linear difference equation. Some results from half-linear differential equations have bee generalized to elliptic partial differential equations, see for example [11].

When $p = 2$, equation (1.1) becomes the Sturm-Liouville linear dynamic equation

$$\big(r(t)x^{\Delta}\big)^{\nabla} + c(t)x = 0. \tag{1.2}$$

Many papers have been devoted to finding conditions which guarantee that all non-trivial solutions of (1.2) (and more general nonlinear equations) are oscillatory, and non-oscillatory. See for example [1, 2, 3, 4, 10] and the references cited therein. In particular, the definitions of oscillatory and non-oscillatory for dynamic equations with mixed derivatives are given by Messer [16].

Definition 1.1. A non-trivial solution x of (1.1) is said to have a *generalized zero* at t if $x(t) = 0$, or if t is left-scattered and $x(\rho(t))x(t) < 0$. Here $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ which is the backward jump operator.

Definition 1.2. Let $t^* = \sup \mathbb{T}$ and $a \in \mathbb{T}$. When $t^* < \infty$, assume $\rho(t^*) = t^*$. A non-trivial solution x of (1.1) is said to be *oscillatory* on $[a, t^*)$ if every non-trivial solution has infinitely many generalized zeros in $[a, t^*)$. Otherwise, it is said to be *non-oscillatory* on $[a, t^*)$.

Looking back on the history, the use of mix derivatives as in (1.2) was considered by Messer [16] for oscillation problems (see also [7, Chap. IV]). In extension, Došlý and Marek [8] studied the half-linear dynamic equation (1.1) and its oscillatory properties (for example, Sturm's comparison theorem and Hille-Nehari type oscillation). In this article, we prove the following the Hille-Nehari type non-oscillation theorem of the type studied by Došlý and Marek [8, Theorem 4.5].

Theorem 1.3. *Let*

$$A_p(\rho(t)) = \Big(\int_{t_0}^{\rho(t)} \big(r(\rho(s))\big)^{1-q}\nabla s\Big)^{p-1}\Big(\int_{\rho(t)}^{\infty} c(s)\nabla s\Big).$$

Assume that $\int_{t_0}^{\infty} \big(r(\rho(t))\big)^{1-q}\nabla t = \infty$, $\int_{t_0}^{\infty} c(t)\nabla t < \infty$, *and*

$$\lim_{t\to\infty} \frac{\nu(t)\big(r(\rho(t))\big)^{1-q}}{\int_{t_0}^{\rho(t)} \big(r(\rho(s))\big)^{1-q}\nabla s} = 0. \tag{1.3}$$

If there exists $A_p(\rho)$ *such that*

$$\liminf_{t\to\infty} A_p(\rho(t)) > -\frac{2p-1}{p}\Big(\frac{p-1}{p}\Big)^{p-1}$$

and

$$\limsup_{t\to\infty} A_p(\rho(t)) < \frac{1}{p}\Big(\frac{p-1}{p}\Big)^{p-1},$$

then all non-trivial solutions of (1.1) *are non-oscillatory.*

In Theorem 1.3, Došlý and Marek [8, Theorem 4.2] established the Hille-Nehari type nonoscillation criterion by considering the lower limit condition

$$\liminf_{t\to\infty} A_p(\rho(t)) > -\frac{2p-1}{p}\Big(\frac{p-1}{p}\Big)^{p-1}$$

and some other conditions. For the case $p=2$, the lower limit condition is

$$\liminf_{t\to\infty} \Big(\int_{t_0}^{\rho(t)} \frac{1}{r(\rho(s))}\nabla s\Big)\Big(\int_{\rho(t)}^{\infty} c(s)\nabla s\Big) > -\frac{3}{4}\,.$$

The purpose of this article is to extended result of Theorem 1.3, by finding conditions on the lower limit value. The new Hille-Nehari type nonoscillation criterion for (1.1) is as follows.

Theorem 1.4. *Assume that* $\int_{t_0}^{\infty}\big(r(\rho(t))\big)^{1-q}\nabla t=\infty$, $\int_{t_0}^{\infty} c(t)\nabla t<\infty$, *and* (1.3) *holds. Let* $h{:}\mathbb{T}\to\mathbb{R}$ *be a positive function such that* $h^{\nabla}(t)\le 0$ *for large* t. *If there exists* $h(\rho)$ *such that*

$$\liminf_{t\to\infty} A_p(\rho(t)) > -(h(\rho(t)))^{1/q} - h(\rho(t)), \text{ and} \tag{1.4}$$

$$\limsup_{t\to\infty} A_p(\rho(t)) < (h(\rho(t)))^{1/q} - h(\rho(t)), \tag{1.5}$$

then all non-trivial solutions of (1.1) *are non-oscillatory, where* $A_p(\rho)$ *is the function given by Theorem 1.3.*

In Section 2 we shall show that Theorem 1.4 includes Theorem 1.3. Note that both Theorem 1.3 and Theorem 1.4 assume the integral condition

$$\int_{t_0}^{\infty}\big(r(\rho(t))\big)^{1-q}\nabla t=\infty\,.$$

Therefore, Theorems 1.3 and 1.4 cannot be applied when

$$\int_{t_0}^{\infty}\big(r(\rho(t))\big)^{1-q}\nabla t<\infty\,. \tag{1.6}$$

Under this condition, the Hille-Nehari type non-oscillation theorem of (1.1) is not given. Under assumption (1.6), this article shows a non-oscillation condition corresponding to Theorem 1.4. A new Hille-Nehari type non-oscillation criterion for (1.1) reads as follows.

Theorem 1.5. *Let*

$$B_p(\rho(t)) = \Big(\int_{\rho(t)}^{\infty}\big(r(\rho(s))\big)^{1-q}\nabla s\Big)^{p-1}\Big(\int_{t_0}^{\rho(t)} c(s)\nabla s\Big).$$

Assume (1.6) *holds, and*

$$\lim_{t\to\infty}\frac{\nu(t)\big(r(\rho(t))\big)^{1-q}}{\int_{\rho(t)}^{\infty}\big(r(\rho(s))\big)^{1-q}\nabla s}=0. \tag{1.7}$$

Let $h{:}\mathbb{T}\to\mathbb{R}$ *be a positive function such that* $h^{\nabla}(t)\ge 0$ *for large* t. *If there exists* $h(\rho)$ *such that*

$$\liminf_{t\to\infty} B_p(\rho(t)) > -(h(\rho(t)))^{1/q} - h(\rho(t)), \text{ and} \tag{1.8}$$

$$\limsup_{t\to\infty} B_p(\rho(t)) < (h(\rho(t)))^{1/q} - h(\rho(t)), \tag{1.9}$$

then all non-trivial solutions of (1.1) *are non-oscillatory.*

In Theorem 1.5 we investigated the same bound with distinct conditions. Under these conditions, all non-trivial solutions of (1.1) are also non-oscillatory. However, since it is not the same conditions as in Theorem 1.3, Theorem 1.4 and Theorem 1.5 can be considered as new results. Moreover, the main results extend Moore's results [17] (see Section 2). In Section 4, as an application of Theorem 1.5, we give a non-oscillation theorem for the linear differential and difference equation with a changing sign coefficient. For these equations we cannot use Theorem 1.4 directly. By using Theorem 1.5, we show that the linear differential and the difference equations have similar non-oscillation results.

2. Remarks about Theorems 1.4 and 1.5

Let us compare Theorem 1.4 with Theorem 1.3. In the case that $h(\rho) \equiv ((p-1)/p)^p$, by using $p/q = p-1$, we have the upper limit value of

$$(h(\rho(t)))^{1/q} - h(\rho(t)) = \Big(\frac{p-1}{p}\Big)^{p-1}\Big(1 - \frac{p-1}{p}\Big) = \frac{1}{p}\Big(\frac{p-1}{p}\Big)^{p-1}$$

and the lower limit value of

$$-(h(\rho(t)))^{1/q} - h(\rho(t)) = -\Big(\frac{p-1}{p}\Big)^{p-1}\Big(1 + \frac{p-1}{p}\Big) = -\frac{2p-1}{p}\Big(\frac{p-1}{p}\Big)^{p-1}.$$

Hence, the condition of Theorem 1.4 becomes the one of Theorem 1.3. For the case $p = 2$, from Theorem 1.3, we have

$$\liminf_{t\to\infty} A_p(\rho(t)) > -\frac{3}{4} = -0.75 \quad \text{and} \quad \limsup_{t\to\infty} A_p(\rho(t)) < \frac{1}{4} = 0.25.$$

In the case $p = 2$, from Theorem 1.4 ((1.4) and (1.5)), we assume that there exists $h(\rho) \equiv k$ (positive constant) such that

$$\liminf_{t\to\infty} A_p(\rho(t)) > -\sqrt{k} - k \quad \text{and} \quad \limsup_{t\to\infty} A_p(\rho(t)) < \sqrt{k} - k \le \frac{1}{4}.$$

Notice that the parameter k gives us opportunity to obtain the desired values. If $k = 1/4$, then we have the same result as the one from Došlý and Marek. If we set $k = 1/3$, then

$$\liminf_{t\to\infty} A_p(\rho(t)) > -\frac{1}{\sqrt{3}} - \frac{1}{3} \approx -0.91068\cdots,$$
$$\limsup_{t\to\infty} A_p(\rho(t)) < \frac{1}{\sqrt{3}} - \frac{1}{3} \approx 0.24401\cdots.$$

Thus we decreased the the lower limit from -0.75 to $-0.91068\cdots$. Therefore, we can conclude that by setting the parameter k, Theorem 1.4 can extend the lower limit. Under these conditions, all non-trivial solutions of (1.1) are also non-oscillatory. However, since it is not the same conditions with Theorem 1.3, Theorem 1.4 can be considered as a new result.

Let $\mathbb{T} = \mathbb{R}$ and $p = 2$. Then (1.1) becomes the linear differential equation

$$(r(t)x')' + c(t)x = 0. \tag{2.1}$$

Decreasing the lower limit $-3/4$ for the Hille-Nehari type non-oscillation result has been actively studied by Moore [17], Wray [22] and Wu and Sugie [23]. In fact, Moore [17] gave the following two theorems.

Theorem 2.1. *Suppose that* $\int_{t_0}^{\infty}(r(t))^{-1}dt = \infty$ *and* $\int_{t_0}^{\infty} c(t)dt < \infty$. *If there exists a constant* $k > 0$ *such that*

$$\Big(1 + \int_{t_0}^{t} \frac{1}{r(s)}ds\Big)\Big(\int_{t}^{\infty} c(s)ds\Big) \ge -\sqrt{k} - k,$$
$$\Big(1 + \int_{t_0}^{t} \frac{1}{r(s)}ds\Big)\Big(\int_{t}^{\infty} c(s)ds\Big) \le \sqrt{k} - k \le \frac{1}{4},$$

then all nontrivial solutions of (2.1) *are non-oscillatory.*

Theorem 2.2. *Suppose that* $\int_{t_0}^{\infty}(r(t))^{-1}dt < \infty$. *If there exists a constant* $k > 0$ *such that*

$$\Big(1 + \int_{t}^{\infty} \frac{1}{r(s)}ds\Big)\Big(\int_{t_0}^{t} c(s)ds\Big) \ge -\sqrt{k} - k,$$
$$\Big(1 + \int_{t}^{\infty} \frac{1}{r(s)}ds\Big)\Big(\int_{t_0}^{t} c(s)ds\Big) \le \sqrt{k} - k \le \frac{1}{4},$$

then all nontrivial solutions of (2.1) *are non-oscillatory.*

Theorems 1.4 and 1.5 are generalization to Theorems 2.1 and 2.2. Indeed, we assume that $\mathbb{T} = \mathbb{R}$, $p = 2$ and $h(\rho(t)) \equiv k$ (positive constant) for Theorems 1.4 and 1.5. Then, we have the upper bound $\sqrt{k} - k$ and the lower bound $-\sqrt{k} - k$.

Recently, Wu, She and Ishibashi [24] gave the Moore-type nonoscillation theorem for half-linear difference equations. Theorems 1.4 and 1.5 also extend their results, when $\mathbb{T} = \mathbb{N}$.

3. Proof of Theorems 1.4 and 1.5

First we show some preliminary results that are used for proving the main results. The readers can find more preliminaries that support the proof in [8].

Lemma 3.1. *Let* $f{:}\mathbb{R} \to \mathbb{R}$ *be a differentiable function and let* $g{:}\mathbb{T} \to \mathbb{R}$ *be* ∇-*differentiable function. Then we have*

$$[f(g(t))]^{\nabla} = f'(\xi)g^{\nabla}(t),$$

where $g(\rho(t)) \le \xi(t) \le g(t)$.

Lemma 3.2. *Equation* (1.1) *is non-oscillatory if and only if there exists a* ∇-*differentiable function* w *satisfying* (3.1) *such that* $\mathcal{R}[w] \le 0$, *where*

$$\mathcal{R}[w] := \begin{cases} w^{\nabla}(t) + c(t) + (p-1)\frac{|w(t)|^q}{\Phi_q(r(t))} & \text{if } \rho(t) = t, \\ w^{\nabla}(t) + c(t) + \frac{w(\rho(t))}{\nu(t)}\Big(1 - \frac{r(\rho(t))}{\Phi_p\big(\Phi_q(r(\rho(t))) + \nu(t)\Phi_q(w(\rho(t)))\big)}\Big) & \text{if } \rho(t) < t \end{cases}$$

for large t.

Remark 3.3. Let x be a non-oscillatory solution of (1.1). Then, in $\mathcal{R}[w]$, we see that

$$\Phi_q(r(\rho(t))) + \nu(t)\Phi_q(w(\rho(t))) > 0 \tag{3.1}$$

for large t. In other words, we need only one function w and establish $\mathcal{R}[w] \le 0$ for each case (left scattered case, and left dense case) to prove our main theorems.

Proof of Theorem 1.4. For simplicity, let

$$\hat{r}(t) := r(\rho(t)), \quad \hat{w}(t) := w(\rho(t)),$$

$$A_p(t) := \Big(\int_{t_0}^{t} (\hat{r}(s))^{1-q} \nabla s \Big)^{p-1} \Big(\int_{t}^{\infty} c(s) \nabla s \Big).$$

Also, put

$$w(t) = h(t) \Big(\int_{t_0}^{t} (\hat{r}(s))^{1-q} \nabla s \Big)^{1-p} + \int_{t}^{\infty} c(s) \nabla s.$$

Using Lemma 3.1, we can calculate

$$\Big[\Big(\int_{t_0}^{t} (\hat{r}(s))^{1-q} \nabla s \Big)^{1-p} \Big]^{\nabla} = (1-p)(\hat{r}(s))^{1-q} \, (\theta(t))^{-p},$$

where

$$\int_{t_0}^{\rho(t)} (\hat{r}(s))^{1-q} \nabla s \le \theta(t) \le \int_{t_0}^{t} (\hat{r}(s))^{1-q} \nabla s.$$

Also, using the Lagrange mean value, we have

$$\begin{aligned}
&\frac{\hat{w}(t)}{\nu(t)} \Big(1 - \frac{\hat{r}(t)}{\Phi_p\big(\Phi_q(\hat{r}(t)) + \nu(t)\Phi_q(\hat{w}(t))\big)} \Big) \\
&= \frac{\hat{w}(t)}{\nu(t)} \Big(\frac{\Phi_p\big(\Phi_q(\hat{r}(t)) + \nu\Phi_q(\hat{w}(t))\big) - \Phi_p(\Phi_q(\hat{r}(t)))}{\Phi_p\big(\Phi_q(\hat{r}(t)) + \nu\Phi_q(\hat{w}(t))\big)} \Big) \\
&= (p-1) \frac{|\eta(t)|^{p-2} |\hat{w}(t)|^q}{\Phi_p\big(\Phi_q(\hat{r}(t)) + \nu(t)\Phi_q(\hat{w}(t))\big)},
\end{aligned}$$

where

$$\Phi_q(\hat{r}(t)) \le \eta(t) \le \Phi_q(\hat{r}(t)) + \nu\Phi_q(\hat{w}(t)).$$

From (1.4) and (1.5), there exists $\varepsilon > 0$ such that

$$|A_p(\rho(t)) + h(\rho(t))|^q (1+\varepsilon) < h(\rho(t)).$$

We also need to calculate

$$|\hat{w}(t)|^q = \Big(\int_{t_0}^{\rho(t)} (\hat{r}(s))^{1-q} \nabla s \Big)^{-p} |A_p(\rho(t)) + h(\rho(t))|^q.$$

Next we consider two cases: $t > \rho(t)$ and $t = \rho(t)$.

Case (i): $t > \rho(t)$. Since $h^{\nabla}(t) \le 0$ for large t, we have

$$\begin{aligned}
\mathcal{R}[w] &= w^{\nabla}(t) + c(t) + \frac{\hat{w}(t)}{\nu(t)} \Big(1 - \frac{\hat{r}(t)}{\Phi_p\big(\Phi_q(\hat{r}(t)) + \nu(t)\Phi_q(\hat{w}(t))\big)} \Big) \\
&= -(p-1) h(\rho(t)) (\theta(t))^{-p} (\hat{r}(t))^{1-q} + h^{\nabla}(t) \Big(\int_{t_0}^{t} (\hat{r}(s))^{1-q} \nabla s \Big)^{1-p} - c(t) \\
&\quad + c(t) + (p-1) \frac{|\eta(t)|^{p-2} |\hat{w}(t)|^q}{\Phi_p\big(\Phi_q(\hat{r}(t)) + \nu(t)\Phi_q(\hat{w}(t))\big)} \\
&\le (p-1)(\hat{r}(t))^{1-q} \Big[- h(\rho(t)) \Big(\int_{t_0}^{t} (\hat{r}(s))^{1-q} \nabla s \Big)^{-p} \\
&\quad + \Big(\int_{t_0}^{\rho(t)} (\hat{r}(s))^{1-q} \nabla s \Big)^{-p} \frac{|\eta(t)|^{p-2} (\hat{r}(t))^{q-1}}{\Phi_p\big(\Phi_q(\hat{r}(t)) + \nu(t)\Phi_q(\hat{w}(t))\big)} |A_p(\rho(t)) + h(\rho(t))|^q \Big]
\end{aligned}$$

$$= \frac{(p-1)(\hat{r}(t))^{1-q}}{\big(\int_{t_0}^{t}(\hat{r}(s))^{1-q}\nabla s\big)^{p}}[-h(\rho(t)) + Z_1(t)|A_p(\rho(t)) + h(\rho(t))|^q],$$

where

$$Z_1(t) := \Big(\frac{\int_{t_0}^{t}(\hat{r}(s))^{1-q}\nabla s}{\int_{t_0}^{\rho(t)}(\hat{r}(s))^{1-q}\nabla s}\Big)^{p}\frac{|\eta(t)|^{p-2}(\hat{r}(t))^{1-q}}{\Phi_p\big(\Phi_q(\hat{r}(t)) + \nu(t)\Phi_q(\hat{w}(t))\big)}.$$

We can see that

$$\begin{aligned}
&\nu(t)\Big|\frac{\hat{w}(t)}{\hat{r}(t)}\Big|^{q-1}\\
&= \nu(t)\frac{\big|h(\rho(t))\big(\int_{t_0}^{\rho(t)}(\hat{r}(s))^{1-q}\nabla s\big)^{1-p} + \int_{\rho(t)}^{\infty}c(s)\nabla s\big|^{q-1}}{(\hat{r}(t))^{q-1}}\\
&= \frac{\nu(t)(\hat{r}(t))^{1-q}}{\int_{t_0}^{\rho(t)}(\hat{r}(s))^{1-q}\nabla s}\Big|h(\rho(t)) + \Big(\int_{t_0}^{\rho(t)}(\hat{r}(s))^{1-q}\nabla s\Big)^{p-1}\Big(\int_{\rho(t)}^{\infty}c(s)\nabla s\Big)\Big|^{q-1} \to 0
\end{aligned}$$

as $t \to \infty$ because of (1.3). Therefore, we can estimate

$$\begin{aligned}
|Z_1(t)| &= \Big(\frac{\int_{t_0}^{\rho(t)}(\hat{r}(s))^{1-q}\nabla s + \nu(t)(\hat{r}(t))^{1-q}}{\int_{t_0}^{\rho(t)}(\hat{r}(s))^{1-q}\nabla s}\Big)^{p}\frac{|\Phi_q(\hat{r}(t)) + \nu\Phi_q(\hat{w}(t))|^{p-2}(\hat{r}(t))^{1-q}}{\Phi_p(\Phi_q(\hat{r}(t)) + \nu(t)\Phi_q(\hat{w}(t)))}\\
&= \Big(1 + \frac{\nu(t)(\hat{r}(t))^{1-q}}{\int_{t_0}^{\rho(t)}(\hat{r}(s))^{1-q}\nabla s}\Big)^{p}\frac{(\hat{r}(t))^{(q-1)(p-1)}|1 + \nu(t)\Phi_q(\hat{w}(t)/\hat{r}(t))|^{p-2}}{\hat{r}(t)\Phi_p(1 + \nu(t)\Phi_q(\hat{w}(t)/\hat{r}(t)))}\\
&= \Big(1 + \frac{\nu(t)(\hat{r}(t))^{1-q}}{\int_{t_0}^{\rho(t)}(\hat{r}(s))^{1-q}\nabla s}\Big)^{p}\frac{1}{1 + \nu(t)\Phi_q(\hat{w}(t)/\hat{r}(t))} \to 1
\end{aligned}$$

as $t \to \infty$. Summarizing all estimates, we see that

$$\mathcal{R}[w] \le \frac{(p-1)(\hat{r}(t))^{1-q}}{\big(\int_{t_0}^{t}(\hat{r}(s))^{1-q}\nabla s\big)^{p}}[-h(\rho(t)) + Z_1(t)|A_p(\rho(t)) + h(\rho(t))|^q(1+\varepsilon)] < 0$$

for large t.

Case (ii): $t = \rho(t)$. In this case $\hat{r} = r$ and $\hat{w} = w$. Hence, the Riccati-type expression is

$$\begin{aligned}
\mathcal{R}[w] &= w^{\nabla}(t) + c(t) + (p-1)\frac{|w(t)|^q}{\Phi_q(r(t))}\\
&= -(p-1)h(\rho(t))\Big(\int_{t_0}^{t}(r(s))^{1-q}\nabla s\Big)^{-p}(r(t))^{1-q}\\
&\quad + h^{\nabla}(t)\Big(\int_{t_0}^{t}(r(s))^{1-q}\nabla s\Big)^{1-p} - c(t) + c(t)\\
&\quad + (p-1)\frac{\big(\int_{t_0}^{t}(r(s))^{1-q}\nabla s\big)^{-p}|A_p(\rho(t)) + h(\rho(t))|^q}{\Phi_q(r(t))}\\
&= (p-1)\Big(\int_{t_0}^{t}(r(s))^{1-q}\nabla s\Big)^{-p}(r(t))^{1-q}[-h(\rho(t)) + |A_p(\rho(t)) + h(\rho(t))|^q] < 0
\end{aligned}$$

for large t. From Lemma 3.2, we can complete the proof. $\square$

Proof of Theorem 1.5. From (1.6), we define

$$
\begin{aligned}
w(t) &= -h(t)\Big(\int_t^\infty (\hat{r}(s))^{1-q}\nabla s\Big)^{1-p} - \int_{t_0}^t c(s)\nabla s\\
&= -\Big(\int_t^\infty (\hat{r}(s))^{1-q}\nabla s\Big)^{1-p}[B_p(t)+h(t)],
\end{aligned}
$$

where $\hat{r}(t) = r(\rho(t))$ and

$$
B_p(t) = \Big(\int_t^\infty (\hat{r}(s))^{1-q}\nabla s\Big)^{p-1}\Big(\int_{t_0}^t c(s)\nabla s\Big).
$$

By using Lemma 3.1, we need to calculate

$$
\begin{aligned}
\Big[\Big(\int_t^\infty (\hat{r}(s))^{1-q}\nabla s\Big)^{1-p}\Big]^\nabla &= \Big[\Big(\int_T^\infty (\hat{r}(s))^{1-q}\nabla s - \int_T^t (\hat{r}(s))^{1-q}\nabla s\Big)^{1-p}\Big]^\nabla\\
&= (p-1)(\hat{r}(t))^{1-q}(\theta(t))^{-p},
\end{aligned}
$$

where

$$
\int_t^\infty (\hat{r}(s))^{1-q}\nabla s \le \theta(t) \le \int_{\rho(t)}^\infty (\hat{r}(s))^{1-q}\nabla s
$$

for sufficiently large T. Then, by using the product rule, we obtain

$$
w^\nabla(t) = -h(\rho(t))(p-1)(\hat{r}(s))^{1-q}(\theta(t))^{-p} - k^\nabla(t)\Big(\int_t^\infty (\hat{r}(s))^{1-q}\nabla s\Big)^{1-p} - c(t).
$$

While

$$
|\hat{w}(t)|^q = \Big(\int_{\rho(t)}^\infty (\hat{r}(s))^{-p}\nabla s\Big)^{-p}[B_p(\rho(t)) + h(\rho(t))]^q,
$$

where $\hat{w}(t) = w(\rho(t))$. Here, we consider two cases: $t > \rho(t)$ and $t = \rho(t)$.

Case (i): $t > \rho(t)$. Since $h^\nabla(t) \ge 0$ for large t, we have

$$
\begin{aligned}
\mathcal{R}[w] &= w^\nabla(t) + c(t) + \frac{\hat{w}(t)}{\nu(t)}\Big(1 - \frac{\hat{r}(t)}{\Phi_p\big(\Phi_q(\hat{r}(t)) + \nu(t)\Phi_q(\hat{w}(t))\big)}\Big)\\
&= -(p-1)h(\rho(t))(\theta(t))^{-p}(\hat{r}(t))^{1-q} - h^\nabla(t)\Big(\int_t^\infty (\hat{r}(s))^{1-q}\nabla s\Big)^{1-p} - c(t)\\
&\quad + c(t) + (p-1)\frac{|\eta(t)|^{p-2}|\hat{w}(t)|^q}{\Phi_p\big(\Phi_q(\hat{r}(t)) + \nu(t)\Phi_q(\hat{w}(t))\big)}\\
&\le (p-1)(\hat{r}(t))^{1-q}\Big[-h(\rho(t))\Big(\int_{\rho(t)}^\infty (\hat{r}(s))^{1-q}\nabla s\Big)^{-p}\\
&\quad + \Big(\int_{\rho(t)}^\infty (\hat{r}(s))^{1-q}\nabla s\Big)^{-p} Z_2(t)|B_p(\rho(t)) + h(\rho(t))|^q\Big]\\
&= \frac{(p-1)(\hat{r}(t))^{1-q}}{\big(\int_{\rho(t)}^\infty (\hat{r}(s))^{1-q}\nabla s\big)^p}\Big[-h(\rho(t)) + Z_2(t)|B_p(\rho(t)) + h(\rho(t))|^q\Big],
\end{aligned}
$$

where

$$
Z_2(t) := \frac{|\eta(t)|^{p-2}(\hat{r}(t))^{q-1}}{\Phi_p\big(\Phi_q(\hat{r}(t)) + \nu(t)\Phi_q(\hat{w}(t))\big)}.
$$

In a similar way to the proof of Theorem 1.4, by using (1.7) one can show that $Z_2(t) \to 1$ as $t \to \infty$. Therefore, from (1.8) and (1.9), we have

$$\mathcal{R}[w] \le \frac{(p-1)(\hat{r}(t))^{1-q}}{\big(\int_{\rho(t)}^{\infty}(\hat{r}(s))^{1-q}\nabla s\big)^p}\Big[-h(\rho(t)) + |B_p(\rho(t)) + h(\rho(t))|^q\Big] < 0$$

for large t. Thus, the proof of Case (i) is complete.

Case (ii): $t = \rho(t)$**.** We follow the proof of Theorem 1.4 Csae (ii) to have

$$\mathcal{R}[w] = w^{\nabla}(t) + c(t) + (p-1)\frac{|w(t)|^q}{\Phi_q(r(t))} < 0$$

for large t. □

4. Linear differential and difference equations with a changing sign coefficient

As a special case of (1.1), we consider the linear dynamic equation with a changing sign coefficient

$$\big(\sigma(t)\sigma(\sigma(t))x^{\Delta}\big)^{\nabla} + \Big(-\alpha + \beta\cos\Big(\log t - \frac{\pi}{4}\Big)\Big)x = 0, \quad t \in [1,\infty)_{\mathbb{T}}, \tag{4.1}$$

where α and β are real numbers. Since the coefficient is $-\alpha + \beta\cos(\log t - \pi/4)$, equation (4.1) is a non-periodic Mathieu type dynamic equation; see [13, 14, 15, 19, 21] for the Mathieu type differential equations. In the case $\mathbb{T} = \mathbb{R}$, equation (4.1) becomes the new self-adjoint Euler type linear differential equation with a changing sign coefficient

$$(t^2x')' + \Big(-\alpha + \beta\cos\Big(\log t - \frac{\pi}{4}\Big)\Big)x = 0, \quad t \ge t_0 = 1. \tag{4.2}$$

See [18, 20] for the oscillation of the usual self-adjoint Euler type differential equations.

On the other hand, in the case $\mathbb{T} = \mathbb{N}$, equation (4.1) becomes the linear difference equation

$$\Delta\big(t(t+1)\Delta x(t-1)\big) + \Big(-\alpha + \beta\cos\Big(\log t - \frac{\pi}{4}\Big)\Big)x(t) = 0, \quad t \ge t_0 = 1. \tag{4.3}$$

Note that equation (4.3) for $\mathbb{T} = \mathbb{N}$ and $\nabla(r(t)\Delta x(t)) = \Delta(r(t-1)\Delta x(t-1))$, we see that $\nabla((t+1)(t+2)\Delta x(t)) = \Delta(t(t+1)\Delta x(t-1))$.

Note that Theorems 1.3 and (1.4) cannot be applied to equations (4.2) and (4.3). In this section, we present an example of which all non-trivial solutions of (4.2) and (4.3) are non-oscillatory.

In Theorem 1.5, we assume that $\mathbb{T} = \mathbb{R}$ (or $\mathbb{T} = \mathbb{N}$), $p = 2$ and $h(\rho(t)) \equiv k$, a positive constant. Then, we have the following corollaries.

Corollary 4.1. *Assume that* $\int_{t_0}^{\infty}\big(r(t)\big)^{-1}dt < \infty$. *If there exists a constant* $k > 0$ *such that*

$$\liminf_{t\to\infty} B_2(t) > -\sqrt{k} - k, \ \textit{and} \tag{4.4}$$

$$\limsup_{t\to\infty} B_2(t) < \sqrt{k} - k \le \frac{1}{4}, \tag{4.5}$$

then all non-trivial solutions of second-order linear differential equation (2.1) *are non-oscillatory, where*

$$B_2(t) = \int_t^\infty \frac{1}{r(s)} ds \int_{t_0}^t c(s) ds.$$

Corollary 4.2. *Assume that*

$$\sum_{t=1}^\infty \frac{1}{r(t-1)} < \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{\frac{1}{r(t-1)}}{\sum_{j=t}^\infty \frac{1}{r(j-1)}} = 0.$$

If there exists a constant $k > 0$ *such that*

$$\liminf_{t\to\infty} B_2(t-1) > -\sqrt{k} - k, \text{ and} \tag{4.6}$$

$$\limsup_{t\to\infty} B_2(t-1) < \sqrt{k} - k \le \frac{1}{4}, \tag{4.7}$$

then all non-trivial solutions of second-order difference equation

$$\Delta(r(t-1)\Delta x(t-1)) + c(t)x(t) = 0 \tag{4.8}$$

are non-oscillatory, where

$$B_2(t-1) = \sum_{j=t}^\infty \frac{1}{r(j-1)} \sum_{j=1}^{t-1} c(j).$$

We expand Corollary 4.1 (or Corollary 4.2) in order to apply it to the equation (4.2) (or equation (4.3)). This result is obtained as follows.

Theorem 4.3. *If there exists a constant* $k > 0$ *such that*

$$|\beta| < \sqrt{2}(\sqrt{k} + k - \alpha), \tag{4.9}$$

$$|\beta| < \sqrt{2}(\sqrt{k} - k + \alpha), \tag{4.10}$$

then all non-trivial solutions of (4.2) *and* (4.3) *are non-oscillatory.*

All non-trivial solutions of (4.2) and (4.3) are non-oscillatory if a pair of coordinates (α, β) is contained in the grey part and the dark part of Figure 1. The grey part is the region

$$R_1 := \{(\alpha,\beta) : k - \sqrt{k} < \alpha \le k,\ |\beta| < \sqrt{2}(\sqrt{k} - k + \alpha)\}.$$

On the other hand, the dark part is the region

$$R_2 := \{(\alpha,\beta) : k \le \alpha < \sqrt{k} + k,\ |\beta| < \sqrt{2}(\sqrt{k} + k - \alpha)\}.$$

Thus, the union of areas R_1 and R_2 is represent the when conditions (4.9) or (4.10) are satisfied. As an example, let $k = 1.5$. Then, from numerical calculations

$$\begin{gathered} k - \sqrt{k} = 1.5 - \sqrt{1.5} \approx 0.275255\cdots, \\ \sqrt{k} + k = \sqrt{1.5} + 1.5 \approx 2.72474\cdots, \\ \sqrt{2}(\sqrt{k} - k + \alpha) = \sqrt{2}(\sqrt{1.5} - 1.5) + \sqrt{2}\alpha \approx -0.38927\cdots + \sqrt{2}\alpha, \\ \sqrt{2}(\sqrt{k} + k - \alpha) = \sqrt{2}(\sqrt{1.5} + 1.5) - \sqrt{2}\alpha \approx 3.85337\cdots + \sqrt{2}\alpha, \end{gathered}$$

we see that the nonoscillation regions

$$\tilde{R}_1 := \{(\alpha,\beta) : 0.275255\cdots < \alpha \le 1.5,\ |\beta| < -0.38927\cdots + \sqrt{2}\alpha\},$$

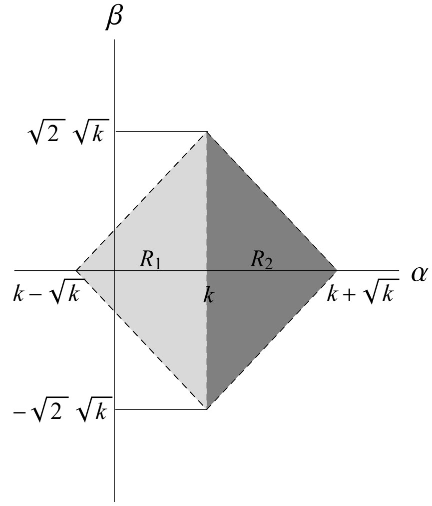

FIGURE 1. Nonoscillation region given by conditions (4.9) and (4.10)

$$\tilde{R}_2 := \{(\alpha,\beta) : 1.5 \le \alpha < 2.72474\cdots,\ |\beta| < 3.85337\cdots - \sqrt{2}\alpha\}.$$

For example, if $\alpha = 2$ and $\beta = 1$, then a point $(\alpha,\beta) = (2,1) \in \tilde{R}_1 \cup \tilde{R}_2$. In fact, from numerical simulation, we draw a solution curve of (4.2) and (4.3) for $(\alpha,\beta) = (2,1)$ (see, Figures 2 and 3). In Figures 2 and 3, the non-oscillation curve of (4.2) and (4.3) starting at the point $(0,1)$. The solution curves for (4.2) and (4.3) are very similar. Hence, from numerical simulations, we see that the linear dynamic equation (4.1) unifies linear differential equation (4.2) and linear difference equation (4.3).

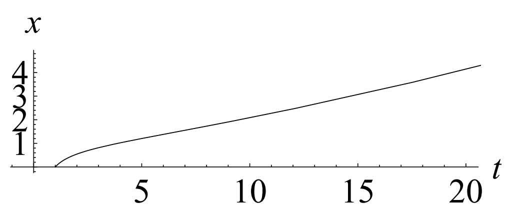

FIGURE 2. A non-oscillatory solution of (4.2) when $(\alpha,\beta) = (2,1)$

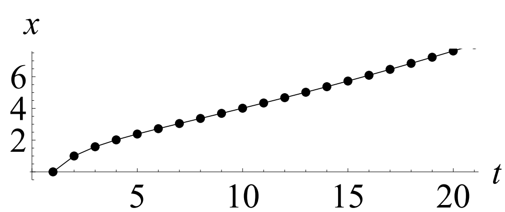

FIGURE 3. A non-oscillatory solution of (4.3) when $(\alpha,\beta) = (2,1)$

Finally, by using Corollaries 4.1 and 4.2, we prove Theorem 4.3.

Proof of Theorem 4.3. We split the proof into two cases: Linear differential equation (4.2), and Linear difference equation (4.3).

Case (i): Equation (4.2)**.** Let $t_0 = 1$. To show that (4.2) is non-oscillatory, we utilize Corollary 4.1. Comparing equation (4.2) with equation (2.1), we see that $r(t) = t^2$ and

$$c(t) = -\alpha + \beta \cos\Big(\log t - \frac{\pi}{4}\Big) = -\alpha + \frac{\beta\sqrt{2}}{2}\Big(\cos(\log t) + \sin(\log t)\Big).$$

Since

$$\lim_{t\to\infty} \int_1^t \frac{1}{r(s)} ds = \lim_{t\to\infty} \int_1^t \frac{1}{s^2} ds = \lim_{t\to\infty} \big[-\frac{1}{s}\big]_1^t = \lim_{t\to\infty} \Big(-\frac{1}{t} + 1\Big) = 1,$$

we see that $\int_1^\infty \big(r(t)\big)^{-1} dt < \infty$ is satisfied. In addition, we obtain

$$\begin{aligned}
B_2(t) &= \int_t^\infty \frac{1}{r(s)} ds \int_{t_0}^t c(s) ds \\
&= \big[-\frac{1}{s}\big]_t^\infty \Big[-\alpha s + \frac{\beta\sqrt{2}}{2} s \sin(\log s)\Big]_1^t \\
&= \frac{1}{t}\Big(-\alpha t + \frac{\beta\sqrt{2}}{2} t \sin(\log t) + \alpha\Big) \\
&= -\alpha + \frac{\beta\sqrt{2}}{2} \sin(\log t) + \frac{\alpha}{t}.
\end{aligned}$$

Hence, from conditions (4.4) and (4.5), we see that conditions (4.9) and (4.10) hold.

Case (ii): Equation (4.3)**.** To show that (4.3) is non-oscillatory, we utilize Corollary 4.2. Comparing equation (4.3) with equation (4.8), we see that $r(t-1) = t(t+1)$ and

$$c(t) = -\alpha + \beta \cos\Big(\log t - \frac{\pi}{4}\Big) = -\alpha + \frac{\beta\sqrt{2}}{2}\Big(\cos(\log t) + \sin(\log t)\Big).$$

From $r(t-1)$, it is easy to check that

$$\begin{gathered}
\sum_{t=1}^\infty \frac{1}{r(t-1)} = \sum_{t=1}^\infty \Big(\frac{1}{t(t+1)}\Big) = \sum_{t=1}^\infty \Big(\frac{1}{t} - \frac{1}{t+1}\Big) = 1 < \infty, \\
\sum_{j=t}^\infty \frac{1}{r(j-1)} = \sum_{j=t}^\infty \Big(\frac{1}{j(j+1)}\Big) = \sum_{j=t}^\infty \Big(\frac{1}{j} - \frac{1}{j+1}\Big) = \frac{1}{t}, \\
\lim_{t\to\infty} \frac{\frac{1}{r(t-1)}}{\sum_{j=t}^\infty \frac{1}{r(j-1)}} = \lim_{t\to\infty} \frac{\frac{1}{t(t+1)}}{\frac{1}{t}} = \lim_{t\to\infty} \frac{1}{t+1} = 0.
\end{gathered}$$

Hence, conditions (4.2) are satisfied. By a straightforward calculation, it follows that

$$\begin{aligned}
\sum_{j=1}^{t-1} c(j) &= -\sum_{j=1}^{t-1} \alpha + \sum_{j=1}^{t-1} \frac{\beta\sqrt{2}}{2}\big(\cos(\log j) + \sin(\log j)\big) \\
&= -\alpha(t-1) + \frac{\beta\sqrt{2}}{2} t \sum_{j=1}^{t} \frac{1}{t}\Big[\cos\Big(\log(\frac{j}{t} t)\Big) + \sin\Big(\log(\frac{j}{t} t)\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{\beta\sqrt{2}}{2}\left(\cos(\log t) + \sin(\log t)\right) \\
&= - \alpha(t-1) + \frac{\beta\sqrt{2}}{2} t \sum_{j=1}^{t} \frac{1}{t}\Big[\cos\Big(\log\big(\frac{j}{t}\big) + \log t\Big) + \sin\Big(\log\big(\frac{j}{t}\big) + \log t\Big)\Big] \\
&\quad - \frac{\beta\sqrt{2}}{2}\left(\cos(\log t) + \sin(\log t)\right).
\end{aligned}
$$

By using addition theorem of trigonometric functions, we have

$$
\begin{aligned}
\sum_{j=1}^{t-1} c(j) &= -\alpha(t-1) + \frac{\beta\sqrt{2}}{2} t \sum_{j=1}^{t} \frac{1}{t}\Big[\cos(\log t)\Big\{\sin\Big(\log\big(\frac{j}{t}\big)\Big) + \cos\Big(\log\big(\frac{j}{t}\big)\Big)\Big\}\Big] \\
&\quad + \frac{\beta\sqrt{2}}{2} t \sum_{j=1}^{t} \frac{1}{t}\Big[\sin(\log t)\Big\{\cos\Big(\log\big(\frac{j}{t}\big)\Big) - \sin\Big(\log\big(\frac{j}{t}\big)\Big)\Big\}\Big] \\
&\quad - \frac{\beta\sqrt{2}}{2}\left(\cos(\log t) + \sin(\log t)\right).
\end{aligned}
$$

Hence, we see that

$$
\begin{aligned}
\lim_{t\to\infty} B_2(t-1) &= \lim_{t\to\infty} \sum_{j=t}^{\infty} \frac{1}{r(j-1)} \sum_{j=1}^{t-1} c(j) \\
&= - \lim_{t\to\infty} \frac{\alpha(t-1)}{t} \\
&\quad + \lim_{t\to\infty} \frac{\beta\sqrt{2}}{2} \cos(\log t) \sum_{j=1}^{t} \frac{1}{t}\Big[\sin\Big(\log\big(\frac{j}{t}\big)\Big) + \cos\Big(\log\big(\frac{j}{t}\big)\Big)\Big] \\
&\quad + \lim_{t\to\infty} \frac{\beta\sqrt{2}}{2} \sin(\log t) \sum_{j=1}^{t} \frac{1}{t}\Big[\cos\Big(\log\big(\frac{j}{t}\big)\Big) - \sin\Big(\log\big(\frac{j}{t}\big)\Big)\Big] \\
&\quad - \lim_{t\to\infty} \frac{\beta\sqrt{2}}{2t}\left(\cos(\log t) + \sin(\log t)\right).
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
&\lim_{t\to\infty} \sum_{j=1}^{t} \frac{1}{t}\Big[\sin\Big(\log\big(\frac{j}{t}\big)\Big) + \cos\Big(\log\big(\frac{j}{t}\big)\Big)\Big] \\
&= \int_0^1 (\sin(\log\delta) + \cos(\log\delta)) d\delta \\
&= \lim_{\varepsilon\to 0^+} \Big[\delta\sin(\log\delta)\Big]_{\varepsilon}^{1} = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{t\to\infty} \sum_{j=1}^{t} \frac{1}{t}\Big[\cos\Big(\log\big(\frac{j}{t}\big)\Big) - \sin\Big(\log\big(\frac{j}{t}\big)\Big)\Big] \\
&= \int_0^1 (\cos(\log\delta) - \sin(\log\delta)) d\delta \\
&= \lim_{\varepsilon\to 0^+} \Big[\delta\cos(\log\delta)\Big]_{\varepsilon}^{1} = 1,
\end{aligned}
$$

we can check that

$$\liminf_{t\to\infty} B_2(t-1) = -\alpha - \frac{|\beta|\sqrt{2}}{2} \quad \text{and} \quad \limsup_{t\to\infty} B_2(t-1) = -\alpha + \frac{|\beta|\sqrt{2}}{2}.$$

Hence, from Corollary 4.2, conditions (4.6) and (4.7) hold. □

5. Basic definitions on a time scales

The Δ-derivative is defined as

$$x^{\Delta}(t) := \lim_{s\to t} \frac{x(\sigma(t)) - x(s)}{\sigma(t) - s}$$

which was introduced by Bohner and Peterson [6]. This is one of the mixed derivatives of (1.1). The ∇-derivative, is defined as

$$x^{\nabla}(t) := \lim_{s\to t} \frac{x(\rho(t)) - x(s)}{\rho(t) - s}$$

which was introduced by Atici and Guseinov[5]. This the another mixed derivative of (1.1). Here, $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ is the *forward jump operator*, $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ is the *backward jump operator*. The functions $\mu, \nu{:}\mathbb{T} \to [0,\infty)$ are called *forward graininess* and *backward graininess* respectively, and are defined by

$$\mu(t) = \sigma(t) - t \quad \text{and} \quad \nu(t) = t - \rho(t).$$

A point $t \in \mathbb{T}$ is said to be *right-dense* if $\mu(t) = 0$, and it is said to be *right-scattered* if $\mu(t) > 0$. Similarly, a point $t \in \mathbb{T}$ is said to be *left-dense* if $\nu(t) = 0$, and it is said to be *left-scattered* if $\nu(t) > 0$. We will use abbreviations rd, rs, ld and ls respectively. When $\mathbb{T} = \mathbb{R}$, we have

$$x^{\Delta}(t) = x'(t) = x^{\nabla}(t)\,.$$

When $\mathbb{T} = \mathbb{Z}$, we have

$$x^{\Delta}(t) = \Delta x(t) = x(t+1) - x(t) \quad \text{and} \quad x^{\nabla}(t) = \nabla x(t) = x(t) - x(t-1)\,.$$

A function $u{:}\mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* if it is right continuous at all rd points and the left limit at ld points exists. If u is rd-continuous, then there exists a Δ-differentiable function U such that $U^{\Delta}(t) = u(t)$. While a function $v{:}\mathbb{T} \to \mathbb{R}$ is said to be *ld-continuous* if it is left continuous at all ld points and the right limit at rd points exists. If v is ld-continuous, then there exists a ∇-differentiable function V such that $V^{\nabla}(t) = v(t)$. The Δ-integral and the ∇-integral are defined by

$$\int_a^b u(t)\Delta t = U(b) - U(a) \quad \text{and} \quad \int_a^b v(t)\nabla t = V(b) - V(a)$$

for $a < b$. In particular, if $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b u(t)\Delta t = \int_a^b u(t)dt \quad \text{and} \quad \int_a^b v(t)\nabla t = \int_a^b v(t)dt,$$

while if $\mathbb{T} = \mathbb{Z}$, we have

$$\int_a^b u(t)\Delta t = \sum_{t=a}^{b-1} u(t) \quad \text{and} \quad \int_a^b v(t)\nabla t = \sum_{t=a+1}^{b} v(t).$$

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