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CUBIC DIFFERENTIAL SYSTEMS WITH INVARIANT STRAIGHT LINES OF TOTAL MULTIPLICITY SEVEN AND FOUR REAL DISTINCT INFINITE SINGULARITIES

CRISTINA BUJAC, DANA SCHLOMIUK, NICOLAE VULPE

Abstract. In this article we consider the class $\mathrm{CSL}_7^{4s\infty}$ of non-degenerate real planar cubic vector fields possessing four distinct real infinite singularities and invariant straight lines, including the line at infinity of total multiplicity 7. We prove that there are exactly 93 distinct configurations of invariant straight lines for this class, and present corresponding examples for the realization of each one of the detected configurations.

1. Introduction and statement of main results

In this article we consider the real polynomial differential system

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \tag{1.1}$$

where P and Q are polynomials in x, y with real coefficients, i.e. $P, Q \in \mathbb{R}[x,y]$. The degree of a system is defined as $\max(\deg(P), \deg(Q))$, so that a cubic system is a system of degree three.

We also consider the vector field corresponding to (1.1):

$$\mathbf{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

Darboux [12] introduced the notion of an algebraic invariant curve for differential equations on the complex plane. An algebraic curve $f(x,y) = 0$ with $f(x,y) \in \mathbb{C}[x,y]$ is an invariant curve of a system of the form (1.1) where $P(x,y), Q(x,y) \in \mathbb{C}[x,y]$ if and only if there exists $K[x,y] \in \mathbb{C}[x,y]$ such that

$$\mathbf{X}(f) = P(x,y)\frac{\partial f}{\partial x} + Q(x,y)\frac{\partial f}{\partial y} = f(x,y)K(x,y)$$

is an identity in $\mathbb{C}[x,y]$. Since $\mathbb{R} \subset \mathbb{C}$, any system (1.1) over $\mathbb{R}$ generates a system of differential equation over $\mathbb{C}$. Using the embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}_2(\mathbb{C})$, $(x,y) \mapsto [x : y : 1] = [X : Y : Z]$, $(x = X/Z, y = Y/Z$ and $Z \neq 0)$, we can compactify the differential equation $Q(x,y)dy - P(x,y)dx = 0$ to an associated differential

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equation over the complex projective plane. In fact the theory of Darboux in [12] is done for differential equations on the complex projective plane.

We compactify the space of all the polynomial differential system (1.1) of degree n on $\mathbb{S}^{N-1}$ with $N=(n+1)(n+2)$ by multiplying the coefficients of each system with $1/(\sum(a_{ij}^2+b_{ij}^2))^{1/2}$, where a_{ij} and b_{ij} are the coefficients of the polynomials $P(x,y)$ and $Q(x,y)$, respectively.

Definition 1.1 ([31])**.** (1) We say that an invariant curve $\mathcal{L}: f(x,y)=0$, $f\in\mathbb{C}[x,y]$ for a polynomial system (S) of degree n has *multiplicity* m if there exists a sequence of real polynomial system (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}$, $N=(n+1)(n+2)$, such that each (S_k) has m distinct invariant curves $\mathcal{L}_{1,k}: f_{1,k}(x,y)=0,\dots,\mathcal{L}_{m,k}: f_{m,k}(x,y)=0$ over $\mathbb{C}$, $\deg(f)=\deg(f_{i,k})=r$, converging to $\mathcal{L}$ as $k\to\infty$, in the topology of $P_{R-1}(\mathbb{C})$, with $R=(r+1)(r+2)/2$ and this does not occur for $m+1$.

(2) We say that the line at infinity $\mathcal{L}_\infty: Z=0$ of a polynomial system (S) of degree n has *multiplicity* m if there exists a sequence of real polynomial system (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}$, $N=(n+1)(n+2)$, such that each (S_k) has $m-1$ distinct invariant lines $\mathcal{L}_{1,k}: f_{1,k}(x,y)=0,\dots,\mathcal{L}_{m,k}: f_{m-1,k}(x,y)=0$ over $\mathbb{C}$, converging to the line at infinity $\mathcal{L}_\infty$ as $k\to\infty$, in the topology of $P_2(\mathbb{C})$ and this does not occur for m.

In this work we consider a particular case of invariant algebraic curves, namely the invariant straight lines of system (1.1). A straight line over $\mathbb{C}$ is the locus $\{(x,y)\in\mathbb{C}^2 | f(x,y)=0\}$ of an equation $f(x,y)=ux+vy+w=0$ with $(u,v)\neq(0,0)$ and $(u,v,w)\in\mathbb{C}^3$. We note that by multiplying the equation by a non-zero complex number λ, the locus of the equation does not change. So that we have a bijection between the lines in $\mathbb{C}^2$ and the points in $\mathbb{P}_2(\mathbb{C})\backslash\{[0:0:1]\}$. This bijection induces a topology on the set of lines in $\mathbb{C}^2$ from the topology of $\mathbb{P}_2(\mathbb{C})$ and hence we can talk about a sequence of lines convergent to a line in $\mathbb{C}^2$.

For an invariant line $f(x,y)=ux+vy+w=0$ we denote $\hat{a}=(u,v,w)\in\mathbb{C}^3$ and by $[\hat{a}]=[u:v:w]$ the corresponding point in $\mathbb{P}_2(\mathbb{C})$. We say that a sequence of straight lines $f_i(x,y)=0$ converges to a straight line $f(x,y)=0$ if and only if the sequence of points $[\hat{a}_i]$ converges to $[\hat{a}]=[u:v:w]$ in the topology of $\mathbb{P}_2(\mathbb{C})$.

In view of the above definition of an invariant algebraic curve of a system (1.1), a line $f(x,y)=ux+vy+w=0$ over $\mathbb{C}$ is an invariant line if and only if it there exists $K(x,y)\in\mathbb{C}[x,y]$ which satisfies the following identity in $\mathbb{C}[x,y]$:

$$\mathbf{X}(f)=uP(x,y)+vQ(x,y)=(ux+vy+w)K(x,y).$$

We point out that if we have an invariant line $f(x,y)=0$ over $\mathbb{C}$ it could happen that multiplying the equation by a number $\lambda\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$, the coefficients of the new equation becomes real, i.e. $(u\lambda,v\lambda,w\lambda)\in\mathbb{R}^3$. In this case, along with the line $f(x,y)=0$ sitting in $\mathbb{C}^2$ we also have an associated real line, sitting in $\mathbb{R}^2$ defined by $\lambda f(x,y)=0$.

Note that, since a system (1.1) is with real coefficients, if its associated complex system has a complex invariant straight line $ux+vy+w=0$, then its conjugate complex invariant straight line $\bar{u}x+\bar{v}y+\bar{w}=0$ is also invariant.

A line in $\mathbb{P}_2(\mathbb{C})$ is the locus in $\mathbb{P}_2(\mathbb{C})$ of an equation $F(X,Y,Z)=uX+vY+wZ=0$ where $(u,v,w)\in\mathbb{C}^3$ and $F(X,Y,Z)\in\mathbb{C}[X,Y,Z]$. The line $Z=0$ in $\mathbb{P}_2(\mathbb{C})$ is called the line at infinity of the affine plane $\mathbb{C}^2$. This line is an invariant manifold of the complex differential equation on $\mathbb{P}_2(\mathbb{C}$. Clearly the lines in $\mathbb{P}_2(\mathbb{C})$ are in a one-to-one correspondence with points $[u : v : w] \in \mathbb{P}_2(\mathbb{C})$ and thus we have a topology on the set of lines in $\mathbb{P}_2(\mathbb{C})$. We can thus talk about a sequence of lines in $\mathbb{P}_2(\mathbb{C})$ convergent to a line in $\mathbb{P}_2(\mathbb{C})$.

To a line $f(x, y) = ux + vy + w = 0$, $(u, v) \neq (0, 0)$, $f \in \mathbb{C}[x, y]$, we associate its projective completion $F(X, Y, Z) = uX + vY + wZ = 0$ under the embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}_2(\mathbb{C})$, $(x, y) \mapsto [x : y : 1] = [X, Y, Z]$ indicated above.
We first remark that in the above definition we made an abuse of language. Indeed, we talk about complex invariant lines of real system. However we already said that to a real system one can associate a complex system and to a differential equation $Q(x, y)dy - P(x, y)dx = 0$ corresponds a differential equation in $\mathbb{P}_2(\mathbb{C})$.

We remark that the above definition is a particular case of the definition of geometric multiplicity given in [11], and namely the "strong geometric multiplicity" with the restriction, that the corresponding perturbations are cubic system.

The set $\mathbb{CS}$ of cubic differential system depends on 20 parameters and for this reason people began by studying particular subclasses of $\mathbb{CS}$. Some of these subclasses are on cubic system having invariant straight lines.

We mention here some papers on polynomial differential system possessing invariant straight lines. For quadratic system see [13, 27, 26, 31, 32, 33, 34, 35] and [36]; for cubic system see [18, 21, 22, 20, 28, 39, 40, 4, 5, 6, 7, 8, 9, 10] and [29]; for quartic system see [38] and [42].

The existence of sufficiently many invariant straight lines of planar polynomial system could be used for proving the integrability of such system. During the past 15 years several articles were published on this theme (see for example [32, 34]).

According to [1], for a non-degenerate polynomial differential system of degree m, the maximum number of invariant straight lines including the line at infinity and taking into account their multiplicities is $3m$. This bound is always reached (see [11]).

In particular, the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems with a finite number of infinite singularities is 9. In [20] the authors classified all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities according to their *configurations of invariant lines*. The notion of configuration of invariant lines for a polynomial differential system was first introduced in [31].

Definition 1.2 ([35])**.** Consider a real planar polynomial differential system (1.1). We call *configuration of invariant straight lines* of this system, the set of (complex) invariant straight lines (which may have real coefficients) including the line at infinity of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

In [20] the authors used a weaker notion, not taking into account the multiplicities of real singularities. They detected 23 such configurations. Moreover, in [20] the necessary and sufficient conditions for the realization of each one of 23 configurations detected, are determined using invariant polynomials with respect to the action of *the group of affine transformations* ($Aff(2, \mathbb{R})$) *and time rescaling* (i.e. $Aff(2, \mathbb{R}) \times \mathbb{R}^*$)). In [4] the author detected another class of cubic system whose configuration of invariant lines was not detected in [20].

If two polynomial systems are equivalent under the action of the affine group and time rescaling, clearly they must have the same kinds of configurations of invariant lines. But it could happen that two distinct polynomial systems which are non-equivalent modulo the action of the affine group and time rescaling have “the same kind of configurations” of straight lines. We need to say when two configurations are considered equivalent.

Definition 1.3. Suppose we have two cubic systems $(S),(S')$ both with a finite number of singularities, finite and infinite, a finite set of invariant straight lines $\mathcal{L}_i : f_i(x,y)=0$, $i=1,\dots,k$, of (S) (respectively $\mathcal{L}'_i : f'_i(x,y)=0$, $i=1,\dots,k'$, of (S')). We say that the two configurations C,C' of invariant lines, including the line at infinity, of these systems are equivalent if there is a one-to-one correspondence ϕ between the lines of C and C' such that:

(i) ϕ sends an affine line (real or complex) to an affine line and the line at infinity to the line at infinity conserving the multiplicities of the lines and also sends a invariant line with coefficients in $\mathbb{R}$ to an invariant line with coefficients in $\mathbb{R}$;

(ii) for each line $\mathcal{L} : f(x,y)=0$ we have a one-to-one correspondence between the real singular points on $\mathcal{L}$ and the real singular points on $\phi(\mathcal{L})$ conserving their multiplicities and their order on these lines;

(iii) we have a one-to-one correspondence ϕ_∞ between the real singular points at infinity on the (real) lines at infinity of (S) and (S') such that when we list in a counterclock wise sense the real singular points at infinity on (S) starting from a point p on the Poincaré disk, $p_1=p,\dots,p_k$, ϕ_∞ preserves the multiplicities of the singular points and preserves or reverses the orientation;

(iv) consider the total curves

$$\mathcal{F} : \prod F_j(X,Y,Z)^{m_i} Z^m = 0, \mathcal{F}' : \prod F'_j(X,Y,Z)^{m'_i} Z^m = 0$$

where $F_i(X,Y,Z)=0$ (respectively $F'_i(X,Y,Z)=0$) are the projective completions of $\mathcal{L}_i$ (respectively $\mathcal{L}'_i$) and m_i, m'_i are the multiplicities of the curves $F_i=0, F'_i=0$ and m,m' are respectively the multiplicities of $Z=0$ in the first and in the second system. Then, there is a one-to-one correspondence ψ between the real singularities of the curves $\mathcal{F}$ and $\mathcal{F}'$ conserving their multiplicities as singular points of the total curves.

Remark 1.4. To describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [31]. Thus we denote by “(a,b)” the maximum number a (respectively b) of infinite (respectively finite) singularities which can be obtained by perturbation of a multiple infinite singular point.

The configurations of invariant straight lines which were detected for some families of system (1.1), were instrumental for determining the phase portraits of those families. For example, in [32, 34] it was proved that we have a total of 57 distinct configurations of invariant lines for quadratic system with invariant lines total multiplicity greater than or equal to 4. These 57 configurations lead to the existence of 135 topologically distinct phase portraits. In [28, 39, 40, 29] it was proved that cubic system with invariant lines of total parallel multiplicity six or seven (the notion of ”parallel multiplicity” could be found in [40]) have 113 topologically distinct phase portraits. This was done by using the various possible configurations of invariant lines of these system.

In what follows we define some algebraic-geometric notions which will be needed in order to describe the invariants used for distinguishing configurations and phase portraits.

Let V be an irreducible algebraic variety of dimension n over a field K.

Definition 1.5. A cycle of dimension r or r-cycle on V with coefficients in an Abelian group G is a formal sum $\Sigma_W n_W W$, where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $n_W \in G$, and only a finite number of n_W are non-zero. The support of a cycle C is the set $\mathrm{Supp}(C) = \{W | n_W \neq 0\}$. An $(n-1)$-cycle is called a divisor $\mathcal{D}$.

Definition 1.6. We call type of a divisor $\mathcal{D}$ the set of all ordered couples (m, s_m) where m is an integer appearing as a coefficient in the divisor $\mathcal{D}$ and s_m is the number of occurrences in $\mathcal{D}$ of the coefficient m.

Clearly the notion of *type of a divisor* is an affine invariant.

These notions (see [16]) which occur frequently in algebraic geometry, were used for classification purposes of planar quadratic differential system by Pal and Schlomiuk [24], [30] and by Llibre and Schlomiuk in [19]. They are also helpful here as we indicate below.

We apply the preceding notions to planar polynomial differential system (1.1). We denote by $\mathrm{PSL}_{n,\mathfrak{L}}$ the class of all non-degenerate planar polynomial differential system of degree n with a finite number of infinite singularities and possessing invariant lines, including the line at infinity, of total multiplicity $\mathfrak{L}$.

We define here below an important divisor which is used in this work and which we call *the parallelism divisor.* Consider a system in $(S) \in \mathrm{PSL}_{n,\mathfrak{L}}$. Let $p_1, p_2, \dots, p_s$ be the set of all the real singular points at infinity of (S). Let j_k, $k \in \{1, \dots, s\}$ be the total multiplicity of all invariant affine lines which cut the line at infinity at p_k. Let i_k, $k \in \{1, \dots, s\}$ be the maximum number of distinct invariant affine lines which can appear from the line at infinity in a perturbation of (S) in the class $\mathrm{PSL}_{n,\mathfrak{L}}$ and which cut the line at infinity at p_k.

Definition 1.7. We call parallelism divisor on $Z = 0$ with coefficients in $\mathbb{Z}^2$ the divisor $D_L(S; Z)$ defined as follows:

$$D_L(S; Z) = \sum_{k=1}^{s} \binom{i_k}{j_k} p_k.$$

In this definition we spell out the affine part j_k (the finite parallelism index) as well as the infinite part expressed by i_k (the infinite parallelism index). We could form another divisor on the line at infinity, namely $\sum_{k=1}^{s} \ (i_k + j_k) p_k$ whose coefficients are the total parallelism indices.

Definition 1.8. We define the parallelism type of the configuration (or simply type of the configuration) of invariant lines occurring for a cubic polynomial system (S), the sequence of non-zero numbers, $\tau_k = i_k + j_k$, $k \in \{1, \dots, s\}$ attached to $D_L(S; Z)$, listed according to descending magnitudes:

$$\mathfrak{T} = (\tau_1, \tau_2, \dots, \tau_l), \quad 1 \leq l \leq s.$$

Clearly $\mathfrak{T}$ is an affine invariant of system in the class $\mathrm{PSL}_{n,\mathfrak{L}}$ and of their configurations of invariant lines.

Notation 1.9. We shall denote by $\mathrm{CSL}_7^{4s\infty}$ the class of cubic system with invariant lines of total multiplicity seven which have four distinct real singularities at infinity.

In this article we classify the family $\mathrm{CSL}_7^{4s\infty}$ according to the relation of equivalence of configurations. Our main result is the following one.

Theorem 1.10. *The class* $\mathrm{CSL}_7^{4s\infty}$ *has a total of 93 non-equivalent configurations of invariant lines, only 20 of which have complex invariant lines and these are always simple, 10 of which are with only one couple of complex conjugate invariant lines and 10 with two couples of complex conjugate invariant lines. The remaining 74 configurations have only invariant lines whose coefficients could be made real, of multiplicities at most three and in anyone of the configurations, there is at most one line of multiplicity three. There is a total of five configurations with a triple invariant line, in two of them this being the line at infinity. The 93 configurations Config. 7.1 – Config. 7.93 of invariant straight lines are given in Figure 1. The configurations split into subclasses according to the value of the invariant* $\mathfrak{T}$ *as follows:*

- *14 configurations Config. 7.1 – Config. 7.14 are of the type* $\mathfrak{T} = (3,3)$*;*
- *26 configurations Config. 7.15 – Config. 7.40 are of the type* $\mathfrak{T} = (3,2,1)$*;*
- *25 configurations Config. 7.41 – Config. 7.65 are of the type* $\mathfrak{T} = (3,1,1,1)$*;*
- *1 configuration Config. 7.66 is of the type* $\mathfrak{T} = (2,2,2)$*;*
- *27 configurations Config. 7.67 – Config. 7.93 are of the type* $\mathfrak{T} = (2,2,1,1)$.

We prove that each one of these configurations is realizable within $\mathrm{CSL}_7^{4s\infty}$ by constructing examples for each one of the configurations Config. 7.1–Config. 7.93. The proof that all these 93 configurations are non-equivalent, according to our definition of equivalence is done in Subsection 3.6.

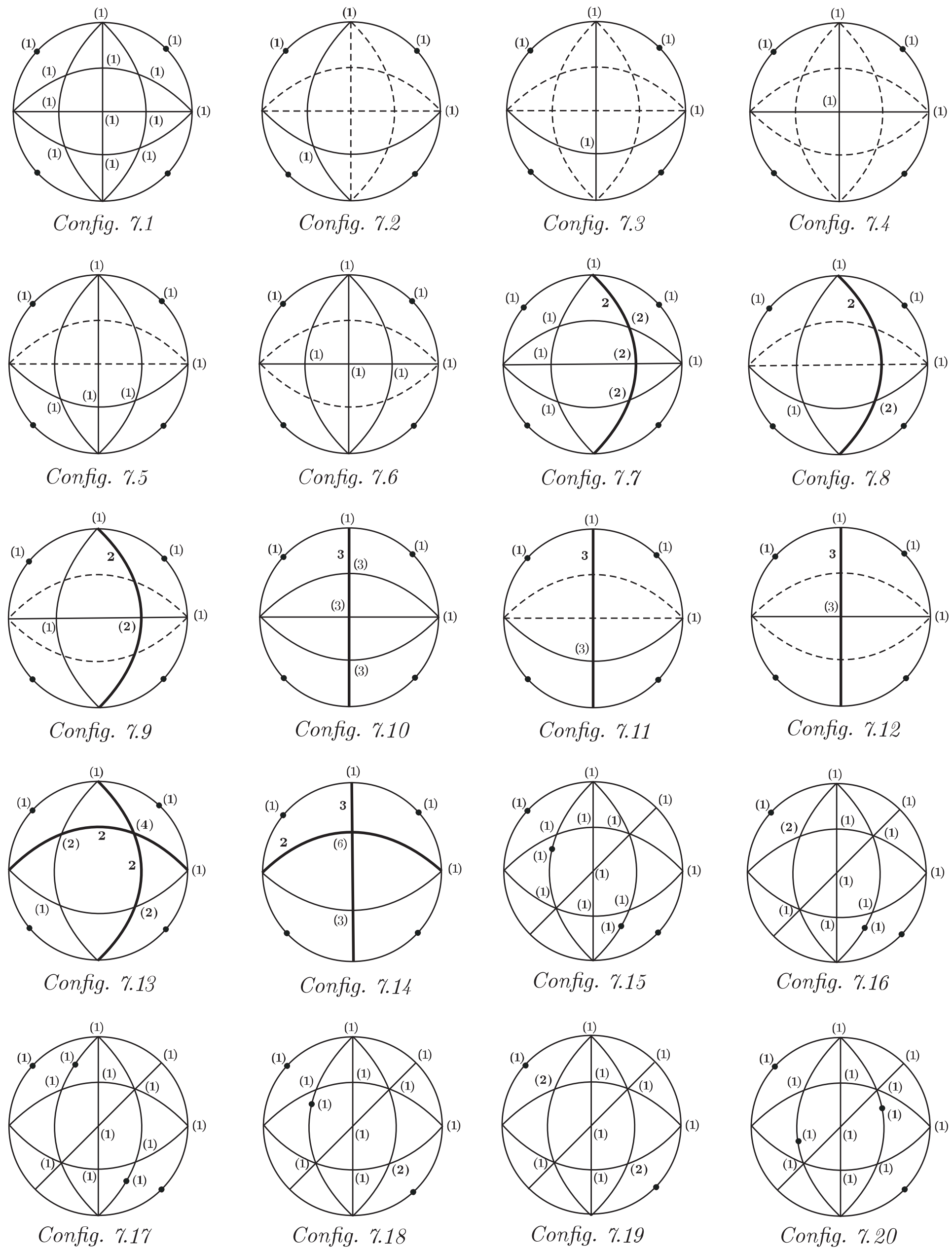
Notation 1.11. We explain here how to read the pictures representing the configurations. An invariant line with multiplicity $k > 1$ will appear in a configuration in bold face and will have next to it the number k. Real invariant straight lines are represented by continuous lines, whereas complex invariant straight lines are represented by dashed lines. The multiplicities of the real singular points of the system located on the invariant lines, will be indicated next to the singular points. The maximum number of parallel invariant straight lines will be shown to be three. Whenever we have three parallel lines, clearly at least for one of these will be real. Due to an affine transformation we can assume this line to be $x = 0$ and after this transformation the system will be of the form:

$$\dot{x} = x(a + 2bx + cx^2), \quad \dot{y} = Q(\hat{a}, x, y).$$

Here $Q(\hat{a}, x, y) = a_0 + a_{10}x + a_{01}y + a_{20}x^2 + 2a_{11}xy + a_{02}y^2 + a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3$ and $\hat{a} = (a_0, a_{10}, \dots, a_{03})$. If two invariant lines of the triplet are complex, then the condition $b^2 - ac < 0$ must hold and this implies that $c \neq 0$ and due to time rescaling we may assume $c = 1$. Setting $b^2 - a = -u^2$ ($a = b^2 + u^2$) we obtain the system

$$\begin{aligned} \dot{x} &= x[(x + b)^2 + u^2], \\ \dot{y} &= Q(\hat{a}, x, y). \end{aligned} \tag{1.2}$$

which has the triplet of invariant lines: $x = 0$, $x = -b + iu$, $x = -b - iu$. In case $b \neq 0$ we place both complex invariant lines on one side of the real line. If $b = 0$ we make the convention to place this line between the two complex lines.



Config. 7.1 Config. 7.2 Config. 7.3 Config. 7.4

Config. 7.5 Config. 7.6 Config. 7.7 Config. 7.8

Config. 7.9 Config. 7.10 Config. 7.11 Config. 7.12

Config. 7.13 Config. 7.14 Config. 7.15 Config. 7.16

Config. 7.17 Config. 7.18 Config. 7.19 Config. 7.20

FIGURE 1. Configurations of invariant lines of total multiplicity 7 for cubic system with 4 distinct real infinite singularities (to be continued)

The work is organized as follows. In Section 2 we give some preliminary results needed for this paper. In Section 3 we prove our Main Theorem restricting ourselves to cubic system with exactly four distinct singularities at infinity. In Subsection 3.1–3.5 we examine step by step each one of the five possible types of the configurations

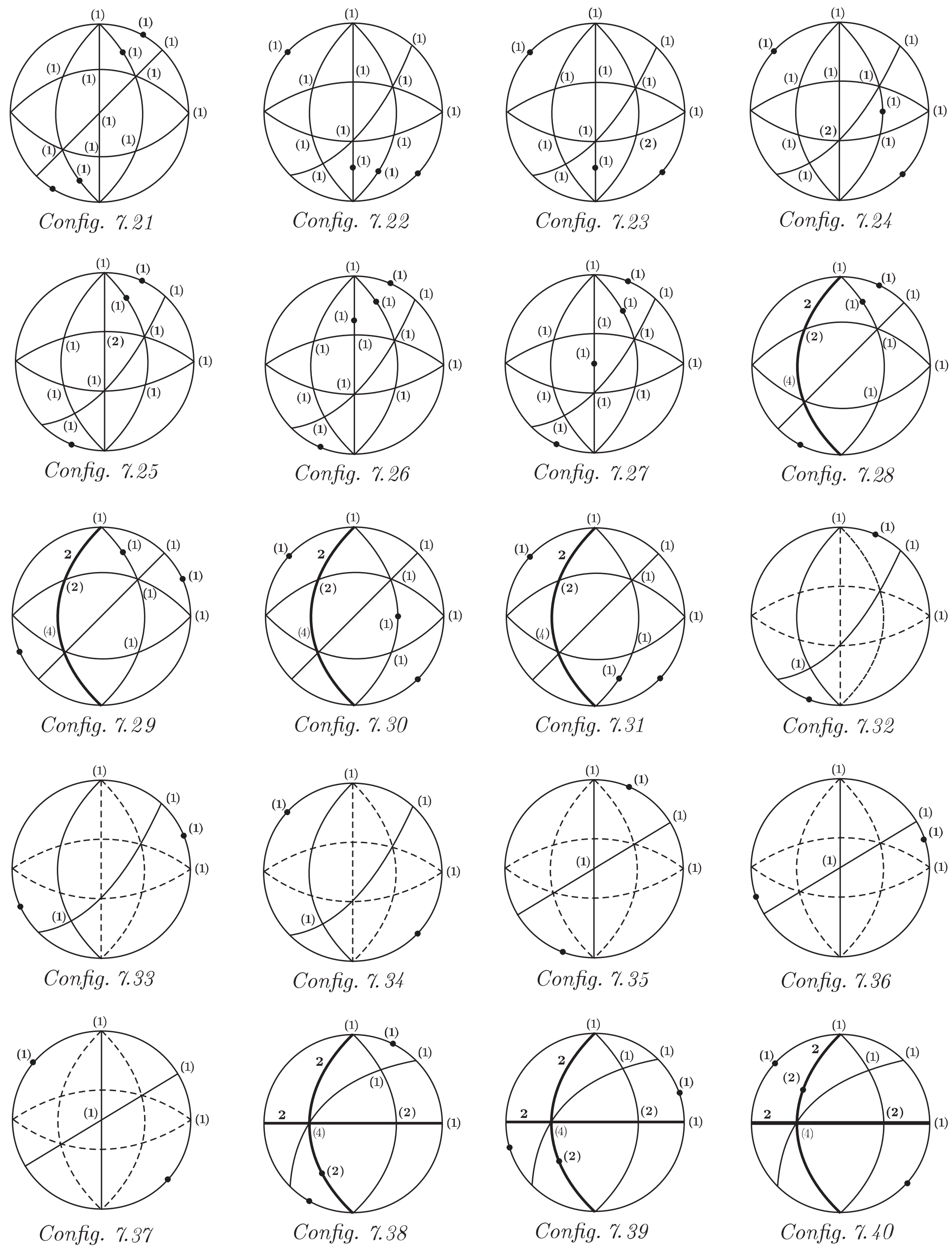


Figure 1. (cont.) Configurations of invariant lines of total multiplicity 7 for cubic system with 4 distinct real infinitesingularities (to be continued)

defined above (see Definition 1.8). In Subsection 3.6, using the geometric invariants, we prove that all the 93 detected configurations of invariant lines for the class of cubic system we considered are distinct according to Definition 1.3.

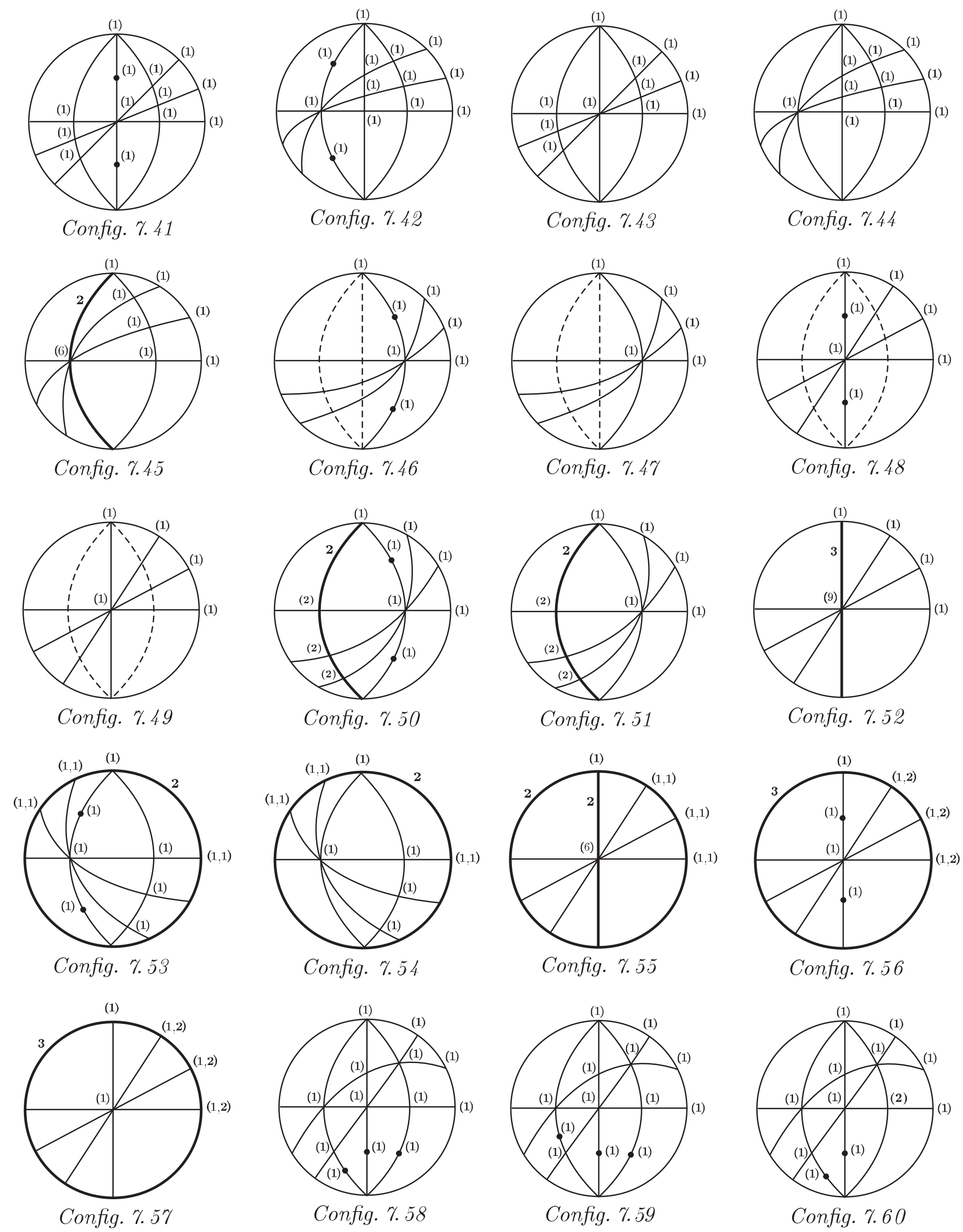
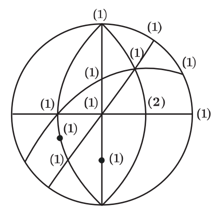


FIGURE 1. (cont.) Configurations of invariant lines of total multiplicity 7 for cubic system with 4 distinct real infinite singularities (to be continued)

We note that the construction of the affine invariant necessary and sufficient conditions for the distinction of the configurations as well as for the realization of each one of them will be the subject of a new article which is in progress.



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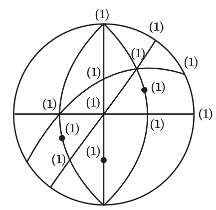

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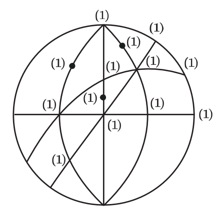

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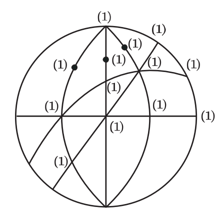

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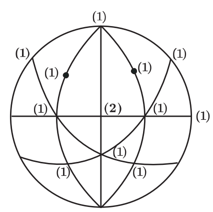

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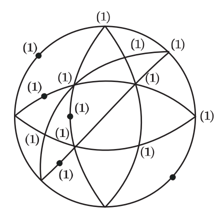

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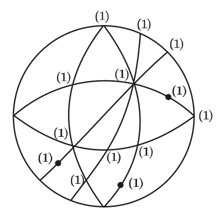

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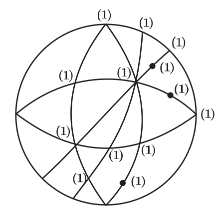

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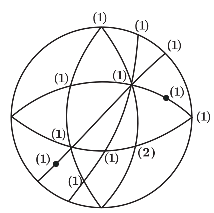

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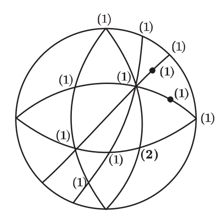

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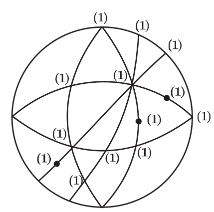

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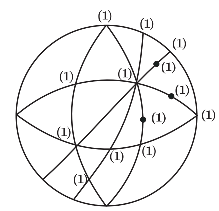

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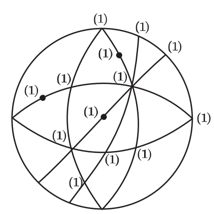
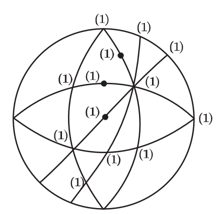
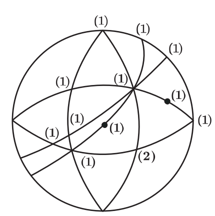


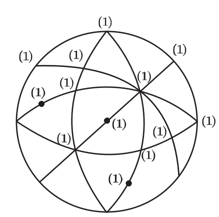
Config. 7.73



Config. 7.74



Config. 7.75



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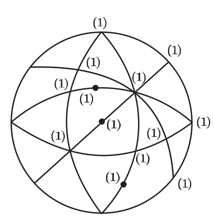

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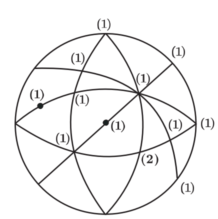

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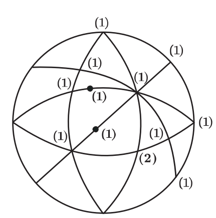
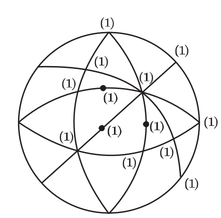
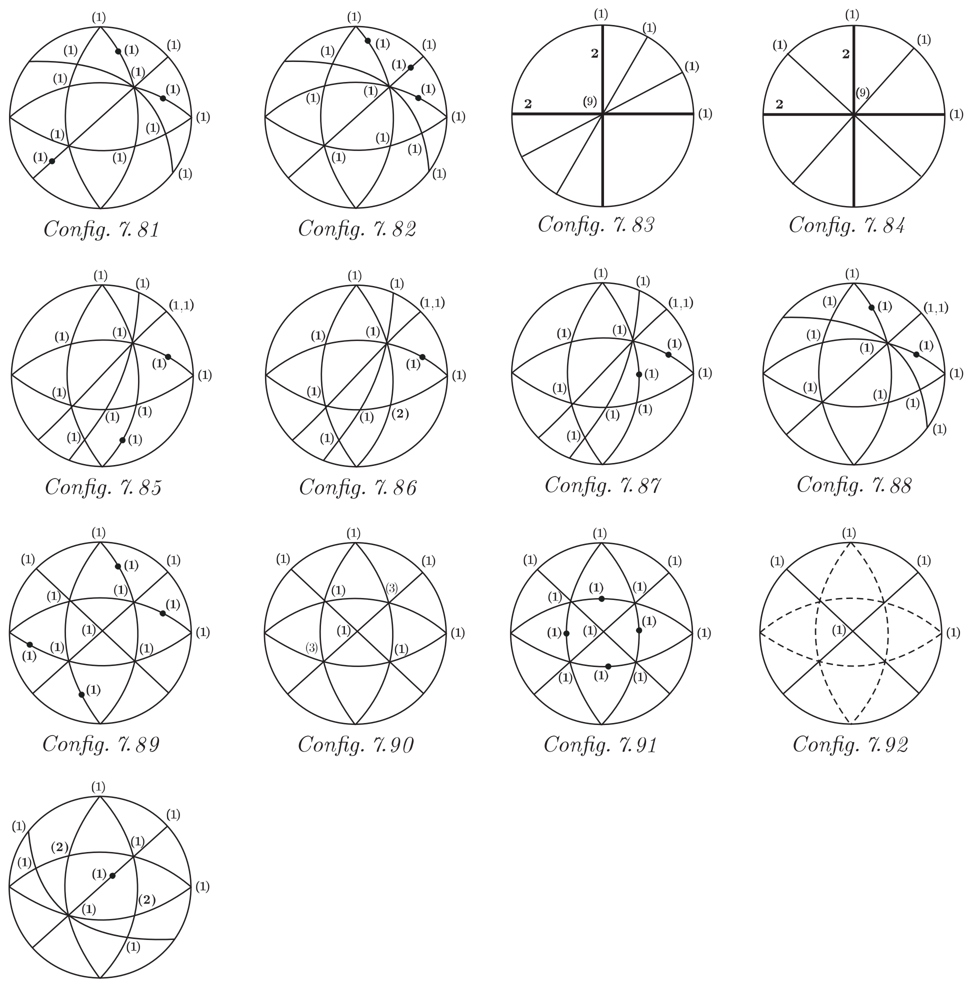


Config. 7.79



Config. 7.80

FIGURE 1. (cont.) Configurations of invariant lines of total multiplicity 7 for cubic system with 4 distinct real infinite singularities (to be continued)



Config. 7.81 *Config. 7.82* *Config. 7.83* *Config. 7.84*
Config. 7.85 *Config. 7.86* *Config. 7.87* *Config. 7.88*
Config. 7.89 *Config. 7.90* *Config. 7.91* *Config. 7.92*
Config. 7.93

FIGURE 1. (cont.) Configurations of invariant lines of total multiplicity 7 for cubic system with 4 distinct real infinite singularities

2. PRELIMINARIES

Consider real cubic system, i.e. system of the form:

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv P(a,x,y), \\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv Q(a,x,y) \end{aligned} \tag{2.1}$$

with variables x and y and real coefficients. The polynomials p_i and q_i $(i = 0,1,2,3)$ are homogeneous polynomials of degree i in x and y:

$$\begin{gathered} p_0 = a_{00}, \quad p_3(x,y) = a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3, \\ p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 = b_{00}, \quad q_3(x,y) = b_{30}x^3 + 3b_{21}x^2y + 3b_{12}xy^2 + b_{03}y^3, \\ q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{gathered}$$

Let $a=(a_{00},a_{10},a_{01},\dots,a_{03},b_{00},b_{10},b_{01},\dots,b_{03})$ be the 20-tuple of the coefficients of system (2.1) and denote

$$\mathbb{R}[a,x,y]=\mathbb{R}[a_{00},a_{10},a_{01},\dots,a_{03},b_{00},b_{10},b_{01},\dots,b_{03},x,y].$$

It is known that on the set of polynomial systems (1.1), in particular on the set CS of all cubic differential system (2.1), acts the group $\mathrm{Aff}(2,\mathbb{R})$ of affine transformation on the plane [35]. For every subgroup $G\subseteq \mathrm{Aff}(2,\mathbb{R})$ we have an induced action of G on CS. We can identify the set CS of system (2.1) with a subset of $\mathbb{R}^{20}$ via the map $CS\to\mathbb{R}^{20}$ which associates to each system (2.1) the 20-tuple $a=(a_{00},a_{10},a_{01},\dots,a_{03},\ b_{00},b_{10},b_{01},\dots,b_{03})$ of its coefficients.

For the definitions of an affine or GL-comitant or invariant as well as for the definition of a T-comitant and CT-comitant we refer the reader to [31]. Here we shall only construct the necessary invariant polynomials (T-comitants) which are needed to detect the existence of invariant lines for the class of cubic system with four real distinct infinite singularities and with invariant straight lines with total multiplicity seven, including the line at infinity (with its own multiplicity).

Let us consider the polynomials

$$\begin{gathered}
C_i(a,x,y)=yp_i(a,x,y)-xq_i(a,x,y)\in\mathbb{R}[a,x,y],\ \ i=0,1,2,3,\\
D_i(a,x,y)=\frac{\partial}{\partial x}p_i(a,x,y)+\frac{\partial}{\partial y}q_i(a,x,y)\in\mathbb{R}[a,x,y],\ \ i=1,2,3.
\end{gathered}$$

As it was shown in [37] the polynomials

$$\big\{C_0,C_1,C_2,C_3,D_1,D_2,D_3\big\} \tag{2.2}$$

of degree one in the coefficients of system (2.1) are GL-comitants of these system.

Notation 2.1. Let $f,\ g\in\mathbb{R}[a,x,y]$ and

$$(f,g)^{(k)}=\sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h\partial y^{k-h}}.$$

$(f,g)^{(k)}\in\mathbb{R}[a,x,y]$ is called the transvectant of index k of (f,g) (cf. [15], [23]).

Theorem 2.2 ([41])**.** *Any GL-comitant of system* (2.1) *can be constructed from the elements of the set* (2.2) *by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(f,g)^{(k)}$.*

Applying the translation $x=x'+x_0$, $y=y'+y_0$ to the $P(a,x,y)$ and $Q(a,x,y)$, we obtain $\tilde P(\tilde a(a,x_0,y_0),x',y')=P(a,x'+x_0,y'+y_0)$ and $\tilde Q(\tilde a(a,x_0,y_0),x',y')=Q(a,x'+x_0,y'+y_0)$. We construct the following polynomials

$$\begin{gathered}
\Omega_i(a,x_0,y_0)\equiv \mathrm{Res}_{x'}\Big(C_i\big(\tilde a(a,x_0,y_0),x',y'\big),C_0\big(\tilde a(a,x_0,y_0),x',y'\big)\Big)/(y')^{i+1},\\
\Omega_i(a,x_0,y_0)\in\mathbb{R}[a,x_0,y_0],\ \ (i=1,2,3)
\end{gathered}$$

and we denote

$$\tilde{\mathcal{G}}_i(a,x,y)=\Omega_i(a,x_0,y_0)\big|_{\{x_0=x,\ y_0=y\}}\in\mathbb{R}[a,x,y]\quad (i=1,2,3).$$

Remark 2.3. We note that the polynomials $\tilde{\mathcal{G}}_1(a,x,y)$, $\tilde{\mathcal{G}}_2(a,x,y)$ and $\tilde{\mathcal{G}}_3(a,x,y)$ are affine comitants of system (2.1) and are homogeneous polynomials in the coefficients $a_{00},\dots,b_{03}$ and non-homogeneous in x,y and

$$\deg_a\tilde{\mathcal{G}}_1=3,\quad \deg_a\tilde{\mathcal{G}}_2=4,\quad \deg_a\tilde{\mathcal{G}}_3=5,$$

$$\deg_{(x,y)} \tilde{\mathcal{G}}_1 = 8, \quad \deg_{(x,y)} \tilde{\mathcal{G}}_2 = 10, \quad \deg_{(x,y)} \tilde{\mathcal{G}}_3 = 12.$$

Notation 2.4. Let $\mathcal{G}_i(a, X, Y, Z)$ $(i = 1, 2, 3)$ be the homogenization of $\tilde{\mathcal{G}}_i(a, x, y)$, i.e.

$$\begin{aligned}
\mathcal{G}_1(a, X, Y, Z) &= Z^8 \tilde{\mathcal{G}}_1(a, X/Z, Y/Z), \\
\mathcal{G}_2(a, X, Y, Z) &= Z^{10} \tilde{\mathcal{G}}_2(a, X/Z, Y/Z), \\
\mathcal{G}_3(a, X, Y, Z) &= Z^{12} \tilde{\mathcal{G}}_3(a, X/Z, Y/Z), \\
\mathcal{H}(a, X, Y, Z) &= \gcd\big(\mathcal{G}_1(a, X, Y, Z), \mathcal{G}_2(a, X, Y, Z), \mathcal{G}_3(a, X, Y, Z)\big)
\end{aligned}$$

in $\mathbb{R}[a, X, Y, Z]$.

The geometrical meaning of these affine comitants is given by the two following lemmas (see [20]).

Lemma 2.5. *The straight line $\mathcal{L}(x, y) \equiv ux + vy + w = 0$, $u, v, w \in \mathbb{C}$, $(u, v) \neq (0, 0)$ is an invariant line for a cubic system (2.1) if and only if the polynomial $\mathcal{L}(x, y)$ is a common factor of the polynomials $\tilde{\mathcal{G}}_1(x, y)$, $\tilde{\mathcal{G}}_2(x, y)$ and $\tilde{\mathcal{G}}_3(x, y)$ over $\mathbb{C}$, i.e.*

$$\tilde{\mathcal{G}}_i(x, y) = (ux + vy + w)\widetilde{W}_i(x, y) \quad (i = 1, 2, 3),$$

where $\widetilde{W}_i(x, y) \in \mathbb{C}[x, y]$.

Lemma 2.6. *Consider a cubic system (2.1) and let $\mathbf{a} \in \mathbb{R}^{20}$ be its 20-tuple of coefficients.*

(1) If $\mathcal{L}(x, y) \equiv ux + vy + w = 0$, $u, v, w \in \mathbb{C}$, $(u, v) \neq (0, 0)$ is an invariant straight line of multiplicity k for a system (2.1) then $[\mathcal{L}(x, y)]^k \mid \gcd(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2, \tilde{\mathcal{G}}_3)$ in $\mathbb{C}[x, y]$, i.e. there exist $W_i(a, x, y) \in \mathbb{C}[x, y]$ $(i = 1, 2, 3)$ such that

$$\tilde{\mathcal{G}}_i(\mathbf{a}, x, y) = (ux + vy + w)^k W_i(a, x, y), \quad i = 1, 2, 3.$$

(2) If the line $l_\infty : Z = 0$ is of multiplicity $k > 1$ then $Z^{k-1} \mid \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$, i.e. we have $Z^{k-1} \mid H(\mathbf{a}, X, Y, Z)$.

Consider the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ constructed in [3] and acting on $\mathbb{R}[a, x, y]$, where

$$\begin{aligned}
\mathbf{L}_1 = {} & 3a_{00}\frac{\partial}{\partial a_{10}} + 2a_{10}\frac{\partial}{\partial a_{20}} + a_{01}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{02}\frac{\partial}{\partial a_{12}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{21}} + a_{20}\frac{\partial}{\partial a_{30}} \\
& + 3b_{00}\frac{\partial}{\partial b_{10}} + 2b_{10}\frac{\partial}{\partial b_{20}} + b_{01}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{02}\frac{\partial}{\partial b_{12}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{21}} + b_{20}\frac{\partial}{\partial b_{30}}, \\
\mathbf{L}_2 = {} & 3a_{00}\frac{\partial}{\partial a_{01}} + 2a_{01}\frac{\partial}{\partial a_{02}} + a_{10}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{20}\frac{\partial}{\partial a_{21}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{12}} + a_{02}\frac{\partial}{\partial a_{03}} \\
& + 3b_{00}\frac{\partial}{\partial b_{01}} + 2b_{01}\frac{\partial}{\partial b_{02}} + b_{10}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{20}\frac{\partial}{\partial b_{21}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{12}} + b_{02}\frac{\partial}{\partial b_{03}}.
\end{aligned}$$

Using this operator and the affine invariant

$$\mu_0 = \text{Resultant}_x\big(p_3(a, x, y), q_3(a, x, y)\big)/y^9$$

we construct the polynomials

$$\mu_i(a, x, y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0), \ i = 1, .., 9,$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0) = \mu_0$.

These polynomials are in fact comitants of system (2.1) with respect to the group $GL(2,\mathbb{R})$ (see [3]). The polynomial $\mu_i(a,x,y)$, $i\in\{0,1,\dots,9\}$ is homogeneous of degree 6 in the coefficients of system (2.1) and homogeneous of degree i in the variables x and y. The geometrical meaning of these polynomial is revealed in the next lemma.

Lemma 2.7 ([2, 3]). *Assume that a cubic system (S) with coefficients $\mathbf{a}\in\mathbb{R}^{20}$ belongs to the family* (2.1)*. Then:*

(i) The total multiplicity of all finite singularities of this system equals $9-k$ if and only if for every $i\in\{0,1,\dots,k-1\}$ we have $\mu_i(\mathbf{a},x,y)=0$ in the ring $\mathbb{R}[x,y]$ and $\mu_k(\mathbf{a},x,y)\neq 0$. In this case the factorization $\mu_k(\mathbf{a},x,y)=\prod_{i=1}^{k}(u_ix-v_iy)\neq 0$ over $\mathbb{C}$ indicates the coordinates $[v_i:u_i:0]$ of singularities at infinity which in perturbations generate finite singularities of the system (S). Moreover the number of distinct factors in this factorization is less than or equal to four (the maximum number of infinite singularities of a cubic system) and the multiplicity of each one of the factors u_ix-v_iy gives us the number of the finite singularities of the system (S) which have coalesced with the infinite singular point $[v_i:u_i:0]$.

(ii) The point $M_0(0,0)$ is a singular point of multiplicity k $(1\leq k\leq 9)$ for the cubic system (S) if and only if for every i such that $0\leq i\leq k-1$ we have $\mu_{9-i}(\mathbf{a},x,y)=0$ in $\mathbb{R}[x,y]$ and $\mu_{9-k}(\mathbf{a},x,y)\neq 0$.

(iii) The system (S) is degenerate (i.e. $\gcd(p,q)\neq \text{const}$) if and only if $\mu_i(\mathbf{a},x,y)=0$ in $\mathbb{R}[x,y]$ for every $i=0,1,\dots,9$.

To define the invariant polynomials we need, we first construct the following comitants of second degree with respect to the coefficients of initial system (2.1):

$$
\begin{aligned}
&S_1=(C_0,C_1)^{(1)}, && S_{10}=(C_1,C_3)^{(1)}, && S_{19}=(C_2,D_3)^{(1)},\\
&S_2=(C_0,C_2)^{(1)}, && S_{11}=(C_1,C_3)^{(2)}, && S_{20}=(C_2,D_3)^{(2)},\\
&S_3=(C_0,D_2)^{(1)}, && S_{12}=(C_1,D_3)^{(1)}, && S_{21}=(D_2,C_3)^{(1)},\\
&S_4=(C_0,C_3)^{(1)}, && S_{13}=(C_1,D_3)^{(2)}, && S_{22}=(D_2,D_3)^{(1)},\\
&S_5=(C_0,D_3)^{(1)}, && S_{14}=(C_2,C_2)^{(2)}, && S_{23}=(C_3,C_3)^{(2)},\\
&S_6=(C_1,C_1)^{(2)}, && S_{15}=(C_2,D_2)^{(1)}, && S_{24}=(C_3,C_3)^{(4)},\\
&S_7=(C_1,C_2)^{(1)}, && S_{16}=(C_2,C_3)^{(1)}, && S_{25}=(C_3,D_3)^{(1)},\\
&S_8=(C_1,C_2)^{(2)}, && S_{17}=(C_2,C_3)^{(2)}, && S_{26}=(C_3,D_3)^{(2)},\\
&S_9=(C_1,D_2)^{(1)}, && S_{18}=(C_2,C_3)^{(3)}, && S_{27}=(D_3,D_3)^{(2)}.
\end{aligned}
$$

We shall use here the following invariant polynomials constructed in [20] and [6] in order to determine the necessary conditions for the existence and the numbers of triplets and/or couples of parallel invariant straight lines which a cubic system could have (see Theorem 2.10):

$$
\begin{aligned}
\mathcal{V}_1(a,x,y)=&S_{23}+2D_3^2,\quad \mathcal{V}_2(a,x,y)=S_{26},\quad \mathcal{V}_3(a,x,y)=6S_{25}-3S_{23}-2D_3^2,\\
\mathcal{V}_4(a,x,y)=&C_3\left[(C_3,S_{23})^{(4)}+36\,(D_3,S_{26})^{(2)}\right],\\
\mathcal{V}_5(a,x,y)=&6C_3(9A_5-7A_6)+2D_3(4T_{16}-T_{17})-3T_3(3A_1+5A_2)+3A_2T_4
\end{aligned}
$$

$$+36T_5^2-3T_{44},$$

where

$$A_1=S_{24}/288,\quad A_2=S_{27}/72,\quad A_3=\Big(72D_1A_2+\big(S_{22},D_2\big)^{(1)}\Big)/24,$$
$$A_5=\big(S_{23},C_3\big)^{(4)}/2^7/3^5,\quad A_6=\big(S_{26},D_3\big)^{(2)}/2^5/3^3$$

are affine invariants, whereas the polynomials

$$T_3=S_{23}/18,\quad T_4=S_{25}/6,\quad T_5=S_{26}/72,\quad T_{16}=\big(S_{23},D_3\big)^{(2)}/2^6/3^3,$$
$$T_{17}=\big(S_{26},D_3\big)^{(1)}/2^5/3^3,\quad T_{44}=\big((S_{23},C_3)^{(1)},\,D_3\big)^{(2)}/5/2^6/3^3,$$

are T-comitants of cubic system (2.1) (see [31] for the definition of a T-comitant). We remark that in the above invariant polynomials we keep the notations introduced in [6].

To determine the degree of the common factor of the polynomials $\tilde{\mathcal{G}}_i(a,x,y)$ for $i=1,2,3$, we shall use the notion of the k^{th} *subresultant* of two polynomials with respect to a given indeterminate (see for instance, [17], [23]).

Following [20] we consider two polynomials

$$f(z)=a_0z^n+a_1z^{n-1}+\dots+a_n,\quad g(z)=b_0z^m+b_1z^{m-1}+\dots+b_m,$$

in the variable z of degree n and m, respectively.

We say that the k-th *subresultant* (see for example, [23]) with respect to variable z of the two polynomials $f(z)$ and $g(z)$ is the $(m+n-2k)\times(m+n-2k)$ determinant

$$R_z^{(k)}(f,g)=\begin{vmatrix} a_0 & a_1 & a_2 & \dots & \dots & a_{m+n-2k-1}\\ 0 & a_0 & a_1 & \dots & \dots & a_{m+n-2k-2}\\ 0 & 0 & a_0 & \dots & \dots & a_{m+n-2k-3}\\ \dots & \dots & \dots & \dots & \dots & \dots\,\dots\,\dots\\ 0 & 0 & b_0 & \dots & \dots & b_{m+n-2k-3}\\ 0 & b_0 & b_1 & \dots & \dots & b_{m+n-2k-2}\\ b_0 & b_1 & b_2 & \dots & \dots & b_{m+n-2k-1} \end{vmatrix}\begin{array}{l}\left.\begin{array}{l}\\ \\ \\ \end{array}\right\}(m-k)-\text{times}\\ \\ \left.\begin{array}{l}\\ \\ \\ \end{array}\right\}(n-k)-times\end{array}\tag{2.3}$$

in which there are $m-k$ rows of a's and $n-k$ rows of b's, and $a_i=0$ for $i>n$, and $b_j=0$ for $j>m$.

For $k=0$ we obtain the standard resultant of two polynomials. In other words we can say that the k–th subresultant with respect to the variable z of the two polynomials $f(z)$ and $g(z)$ can be obtained by deleting the first and the last k rows and the first and the last k columns from its resultant written in the form (2.3) when $k=0$.

The geometrical meaning of the subresultants is based on the following lemma.

Lemma 2.8 (see [17, 23])**.** *Polynomials $f(z)$ and $g(z)$ have precisely k roots in common (considering their multiplicities) if and only if the following conditions hold:*

$$R_z^{(0)}(f,g)=R_z^{(1)}(f,g)=R_z^{(2)}(f,g)=\dots=R_z^{(k-1)}(f,g)=0\neq R_z^{(k)}(f,g).$$

For the polynomials in more than one variables it is easy to deduce from Lemma 2.8 the following result.

Lemma 2.9. *Two polynomials $\tilde{f}(x_1, x_2, \dots, x_n)$ and $\tilde{g}(x_1, x_2, \dots, x_n)$ have a common factor of degree k with respect to the variable x_j if and only if the following conditions are satisfied:*

$$R^{(0)}_{x_j}(\tilde{f}, \tilde{g}) = R^{(1)}_{x_j}(\tilde{f}, \tilde{g}) = R^{(2)}_{x_j}(\tilde{f}, \tilde{g}) = \dots = R^{(k-1)}_{x_j}(\tilde{f}, \tilde{g}) = 0 \neq R^{(k)}_{x_j}(\tilde{f}, \tilde{g}),$$

where $R^{(i)}_{x_j}(\tilde{f}, \tilde{g}) = 0$ in $\mathbb{R}[x_1, \dots x_{j-1}, x_{j+1}, \dots, x_n]$.

In [20] all the possible configurations of invariant lines are determined in the case, when the total multiplicity of these line (including the line at infinity) equals nine. All possible configurations of invariant lines in the case when the total multiplicity of these line (including the line at infinity) equals eight, are determined in [5, 6, 7, 8, 9].

In the above mentioned articles several lemmas are proved concerning the number of triplets and/or couples of parallel invariant straight lines which could have a cubic system. Taking together these lemmas produce the following theorem.

Theorem 2.10. *If a cubic system* (2.1) *possesses a given number of triplets or/and couples of invariant parallel lines real or/and complex, then the following conditions are satisfied, respectively:*

- (i) *two triplets imply $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$;*
- (ii) *one triplet and one couple imply $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$;*
- (iii) *one triplet imply $\mathcal{V}_4 = \mathcal{U}_2 = 0$;*
- (iv) *3 couples imply $\mathcal{V}_3 = 0$;*
- (v) *(v) 2 couples imply $\mathcal{V}_5 = 0$.*

Remark 2.11. The above conditions depend only on the coefficients of the cubic homogeneous parts of the system (2.1).

We rewrite the system (2.1) differently:

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3 \\ &\equiv P(x, y), \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + ny^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3 \\ &\equiv Q(x, y). \end{aligned} \tag{2.4}$$

Let $L(x, y) = Ux + Vy + W = 0$ be an invariant straight line of this family of cubic system. Then, we have

$$UP(x, y) + VQ(x, y) = (Ux + Vy + W)(Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F),$$

and this identity provides the following 10 relations:

$$\begin{gathered} Eq_1 = (p - A)U + tV = 0, \quad Eq_6 = (2h - E)U + (2m - D)V - 2BW = 0, \\ Eq_2 = (3q - 2B)U + (3u - A)V = 0, \quad Eq_7 = kU + (n - E)V - CW = 0, \\ Eq_3 = (3r - C)U + (3v - 2B)V = 0, \quad Eq_8 = (c - F)U + eV - DW = 0 \\ Eq_4 = (s - C)U + Vw = 0, \quad Eq_9 = dU + (f - F)V - EW = 0, \\ Eq_5 = (g - D)U + lV - AW = 0, \quad Eq_{10} = aU + bV - FW = 0. \end{gathered} \tag{2.5}$$

It is well known that the infinite singularities (real or complex) of system (2.4) are determined by the linear factors of the polynomial

$$C_3 = yp_3(x, y) - xq_3(x, y).$$

Remark 2.12. Let $C_3 = \prod_{i=1}^4(\alpha_i x + \beta_i y)$, $i = 1, 2, 3, 4$. Since among the infinite singularities we have the points at infinity of invariant lines, for an invariant line $Ux + Vy + W = 0$ we must have $[U : V] = [\alpha_i : \beta_i]$ for some i and we may assume $(U, V) = (\alpha_i, \beta_i)$. In this case, considering W as a fixed parameter, six equations among (2.5) become linear with respect to the parameters $\{A, B, C, D, E, F\}$ (with the corresponding non-zero determinant) and we can determine their values, which annihilate some of the equations (2.5)). So in what follows we will examine only the non-trivial equations containing the last parameter W.

For the proof of the Main Theorem it is useful to consider the following homogeneous cubic system associated to system (2.4):

$$dx/dt = P_3(x, y), \quad dy/dt = Q_3(x, y). \tag{2.6}$$

Clearly in the case of four real distinct infinite singularities the polynomial $C_3(x, y)$ has four distinct real linear factors. The following remark concerning the associated homogeneous cubic system (2.6) is useful.

Remark 2.13. Assume that a cubic system (2.4) possesses invariant lines of total multiplicity three (respectively two) in a real direction. Then the corresponding associated homogeneous cubic system (2.4) has one invariant line of total multiplicity at least three (respectively two) in the same direction.

Indeed, if a system (2.4) possesses a triplet of parallel invariant lines (distinct or coinciding) in a real direction then via an affine transformation this system could be brought to the form

$$\dot{x} = x[(x + b)^2 + u], \quad \dot{y} = Q(a, x, y).$$

It is clear that if $u < 0$ (respectively $u > 0$) then we have three real (respectively one real and two complex) all distinct invariant lines. In the case $u = 0$ we either have one simple and one double invariant lines if $b \neq 0$, or one triple invariant line if $b = 0$. It remains to observe that in all four cases the corresponding associated homogeneous cubic systems possess the invariant line $x = 0$ of total multiplicity at least three. The case of a couple of parallel invariant lines ca be examined similarly.

According to [20,] (see also [25]) we have the following result.

Lemma 2.14. *If a cubic system* (2.4) *has 4 real distinct infinite singularities then its associated homogeneous cubic system* (2.6) *could be brought via a linear transformation to the canonical form* (S_I)*:*

$$\begin{aligned} x' &= (p + r)x^3 + (s + v)x^2y + qxy^2, \quad C_3 = xy(x - y)(rx + sy), \\ y' &= px^2y + (r + v)xy^2 + (q + s)y^3, \quad rs(r + s) \neq 0 \end{aligned} \tag{2.7}$$

3. Proof of the main theorem

The proof proceeds in three steps:

Firstly we construct the cubic homogeneous parts $(\tilde{P}_3, \tilde{Q}_3)$ of system (2.7) for which the corresponding necessary conditions provided by Theorem 2.10 in order to have the specified number of triplets or/and couples of invariant parallel lines in the corresponding directions are satisfied.

Secondly we consider the system (2.4) with the cubic homogeneous parts $(\tilde{P}_3, \tilde{Q}_3)$ and generic homogeneities of lower degrees. For these system using the equations

(2.5) we determine the coefficients of these homogeneities in order to get the required number of invariant lines (of total multiplicity 7) in the required configuration. Thus the second step ends with the construction of the canonical system possessing the required configurations.

Thirdly, we prove that all the constructed configurations are distinct in view of the Definition 1.3. To do this we apply some geometric invariants defined in Subsection 3.6.

Assuming that cubic systems in the family (2.4) possess four distinct real infinite singularities, according to Lemma 2.14 via a linear transformations they could be brought to the family of systems

$$\begin{aligned}\dot{x} =&a + cx + dy + gx^2 + 2hxy + ky^2 + (p + r)x^3 + (s + v)x^2y + qxy^2,\\ \dot{y} =&b + ex + fy + lx^2 + 2mxy + ny^2 + px^2y + (r + v)xy^2 + (q + s)y^3\end{aligned} \tag{3.1}$$

with $C_3 = xy(x - y)(rx + sy)$ and $rs(r + s) \neq 0$. As we have four real infinite singularities and the total multiplicity of the invariant lines (including the line at infinity) must be 7, then the above system could only have one of the following five possible types of configurations of invariant lines:

$$\begin{gathered}(i)\ \mathfrak{T} = (3, 3);\quad (ii)\ \mathfrak{T} = (3, 2, 1);\quad (iii)\ \mathfrak{T} = (3, 1, 1, 1);\\ (iv)\ \mathfrak{T} = (2, 2, 2);\quad (v)\ \mathfrak{T} = (2, 2, 1, 1).\end{gathered} \tag{3.2}$$

3.1. **Systems with the type of the configuration $\mathfrak{T} = (3, 3)$.** Since we have two triplets of parallel invariant lines, according to Theorem 2.10 the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ are necessary for system (3.1). In [20, Section 5.1] it was proved that in this case via a linear transformation and time rescaling the cubic homogeneities of these system could be brought to the forms x^3, y^3 and we consider the homogeneous system

$$\dot{x} = x^3, \quad \dot{y} = y^3. \tag{3.3}$$

So applying a translation we may assume $g = n = 0$ in the quadratic parts of system (3.1) with the cubic homogeneities x^3, y^3. In such a way we obtain the family of systems

$$\begin{aligned}\dot{x} &= a + cx + dy + 2hxy + ky^2 + x^3,\\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + y^3,\end{aligned} \tag{3.4}$$

for which we have $C_3(x, y) = xy(x - y)(x + y)$.

To find out the directions of two triplets, according to Remark 2.13, we determine the multiplicity of the invariant lines of system (3.3). For this system we calculate (see the definition of the polynomial $H(X, Y, Z)$ on the page 13, Notation 2.4):

$$H(X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 3X^3(X - Y)Y^3(X + Y).$$

So system (3.3) has two triple invariant lines $x = 0$ and $y = 0$ and by Remark 2.13, system (3.4) could have triplets of parallel invariant lines only in these two directions.

(i) The direction $x = 0$. Considering (2.5) and Remark 2.12 we obtain

$$Eq_7 = k, \quad Eq_9 = d - 2hW, \quad Eq_{10} = a - cW - W^3$$

and obviously we can have a triplet of parallel invariant lines (which could coincide) in the direction $x = 0$ if and only if $k = d = h = 0$. Assuming that these conditions hold we consider another direction for the second triplet.

(ii)The direction $y = 0$. In this case we have

$$Eq_5 = l, \quad Eq_8 = e - 2mW, \quad Eq_{10} = b - fW - W^3$$

and again we conclude that for the existence of a triplet of parallel invariant lines for system (3.4) the conditions $e = l = m = 0$ have to be satisfied.

It remains to examine the directions $y = x$ and $y = -x$ to determine the conditions for the non-existence of an additional invariant line in one of these two directions.

For the direction $y = \pm x$ considering (2.5) and Remark 2.12 we have

$$Eq_7 = -3W; \quad Eq_9 = \mp(c - f + 3W^2); \quad Eq_{10} = a \pm b - cW - W^3.$$

We observe that in each one of the cases we could have only one invariant line. Moreover the necessary and sufficient conditions for the existence of such a line are $c - f = a + b = 0$ for the direction $y = x$ and $c - f = a - b = 0$ for the direction $y = -x$.

Thus we conclude that for the non-existence of an invariant line in additions to the two triplets the following conditions are necessary and sufficient:

$$(c - f)^2 + (a^2 - b^2)^2 \neq 0.$$

We arrive at the family of systems

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + fy + y^3 \tag{3.5}$$

for which the above conditions must be satisfied. These system possess the invariant lines defined by the equations

$$x^3 + cx + a = 0, \quad y^3 + fy + b = 0.$$

We observe that the number of distinct invariant lines and their kinds (real and/or complex) depend on the discriminants of the cubic polynomials $x^3 + cx + a$ and $y^3 + fy + b$, i.e.

$$\xi_1 = -(27a^2 + 4c^3), \quad \xi_2 = -(27b^2 + 4f^3),$$

respectively. Moreover, we observe that the polynomial $x^3 + cx + a$ (respectively $y^3 + fy + b$) has a triple root if and only if $\nu_1 = a^2 + c^2 = 0$ (respectively $\nu_2 = b^2 + f^2 = 0$).

Remark 3.1. Note that for system (3.5) we could not have simultaneously $\nu_1 = \nu_2 = 0$, otherwise we obtain the homogeneous cubic system $\dot{x} = x^3,\ \dot{y} = y^3$ which possesses invariant lines of total multiplicity nine.

In what follows we examine the possibilities provided by the discriminants ξ_1 and ξ_2.

(1) *Case* $\xi_1\xi_2 > 0$ *and* $\xi_1 + \xi_2 > 0$. Then each one of the mentioned cubic polynomials factorizes in three distinct real factors, i.e. we obtain the system

$$\dot{x} = (x - \alpha_1)(x - \beta_1)(x - \delta_1), \quad \dot{y} = (y - \alpha_2)(y - \beta_2)(y - \delta_2), \tag{3.6}$$

where $\alpha_i, \beta_i, \delta_i \in \mathbb{R}$, $i = 1, 2$. As all the lines are distinct then via the transformation

$$(x, y, t) \mapsto \big(\alpha_1 - (\alpha_1 - \beta_1)x, \quad \alpha_2 - (\alpha_1 - \beta_1)y, \quad t/(\alpha_1 - \beta_1)^2\big) \tag{3.7}$$

we arrive at the following 3-parameter family of systems

$$\dot{x} = x(x - 1)(x - a), \quad \dot{y} = y(y - b)(y - c), \quad a(a + 1)bc(b - c) \neq 0, \tag{3.8}$$

where

$$a = \frac{\alpha_1 - \delta_1}{\alpha_1 - \beta_1}, \quad b = \frac{\alpha_2 - \beta_2}{\alpha_1 - \beta_1}, \quad c = \frac{\alpha_2 - \delta_2}{\alpha_1 - \beta_1}.$$

This system has 9 finite real singularities which are located at the intersections of these two triplets of invariant lines. As a result we obtain Config. 7.1.

(2) *Case* $\xi_1\xi_2 > 0$ *and* $\xi_1 + \xi_2 < 0$. Then in each one of the directions $x = 0$ and $y = 0$, system (3.5) has one real and two complex invariant lines. After the translation of the origin of coordinates at the intersections of the real invariant lines we arrive at the system

$$\dot{x} = x(x^2 + 2\beta_1 x + \delta_1), \quad \dot{y} = y(y^2 + 2\beta_2 y + \delta_2),$$

where $\beta_1^2 - \delta_1 < 0$ and $\beta_2^2 - \delta_2 < 0$. So we can set $\beta_1^2 - \delta_1 = -u^2 \neq 0$ and $\beta_2^2 - \delta_2 = -v^2 \neq 0$ respectively, the above system becomes

$$\dot{x} = x\big[u^2 + (x + \beta_1)^2\big], \quad \dot{y} = y\big[v^2 + (y + \beta_2)^2\big].$$

Since $u \neq 0$ then we may assume $u = 1$ due to the rescaling $(x, y, t) \mapsto (ux,\ uy, t/u^2)$ and we obtain the following 3-parameter family of systems

$$\dot{x} = x\big[(x + a)^2 + 1\big], \quad \dot{y} = y\big[(y + b)^2 + c^2\big], \quad c \neq 0. \tag{3.9}$$

These systems possess 1 real and 8 complex finite singularities which are located at the intersections of these two triplets of invariant lines (real and complex).

As a result, considering Notation 1.11 we obtain Config. 7.2 if $ab \neq 0$, Config. 7.3 if $ab = 0$ and $a + b \neq 0$, and Config. 7.4 if $a = 0 = b$.

(3) *Case* $\xi_1\xi_2 < 0$. Without loss of generality we may assume $\xi_1 > 0$ and $\xi_2 < 0$ due to the change $(x, y, t, a, b, c, f) \mapsto (y, x, t, b, a, f, c)$ which conserves system (3.5). Then we have the following factorization of the right hand parts of these system

$$\dot{x} = (x - \alpha_1)(x - \beta_1)(x - \delta_1), \quad \dot{y} = (y - \alpha_2)(y^2 + 2\beta_2 y + \delta_2), \tag{3.10}$$

where $\alpha_i, \beta_i, \delta_i \in \mathbb{R}$, $i = 1, 2$ and $\beta_2^2 - \delta_2 < 0$. So we can set $\beta_2^2 - \delta_2 = -u^2 \neq 0$ and then applying the transformation (3.7) we obtain the following 3-parameter family of systems

$$\dot{x} = x(x - 1)(x - a), \quad \dot{y} = y\big[(y + b)^2 + c^2\big], \quad a(a - 1)c \neq 0. \tag{3.11}$$

These systems possess 3 real and 6 complex finite singularities which are located at the intersections of these two triplets of invariant lines (real and complex).

So considering Notation 1.11 we obtain Config. 7.5 if $b \neq 0$, and Config. 7.6 if $b = 0$.

(4) *Case* $\xi_1\xi_2 = 0$, $\xi_1 + \xi_2 > 0$, $\nu_1\nu_2 \neq 0$. As it was mentioned earlier due to the change $(x, y, t, a, b, c, f) \mapsto (y, x, t, b, a, f, c)$ (which conserves system (3.5)) we may assume $\xi_1 = 0$ and $\xi_2 > 0$. Following the same arguments as before, system (3.5) can be written in the form (3.6) with $\beta_1 = \alpha_1 \neq \delta_1$. Then applying the transformation (3.7) in which we substitute β_1 by $\alpha_1 \neq \delta_1$ we arrive at the following 2-parameter family of systems

$$\dot{x} = x^2(x - 1), \quad \dot{y} = y(y - b)(y - c), \quad bc(b - c) \neq 0. \tag{3.12}$$

These systems possess three double real singularities (located at the intersections of the double invariant line $x = 0$ with three simple lines) and 3 simple real singularities, located at the intersections of the simple invariant line $x = 1$ with the triplet in the direction $y = 0$. As a result we obtain Config. 7.7.

(5) *Case* $\xi_1\xi_2 = 0$, $\xi_1 + \xi_2 < 0$, $\nu_1\nu_2 \neq 0$. We may assume again $\xi_1 = 0$, $\xi_2 < 0$ and $\nu_1 \neq 0$. In this case we consider system (3.10) with $\beta_1 = \alpha_1 \neq \delta_1$ and following the same steps and the corresponding similar transformation we obtain the following 2-parameter family of systems

$$\dot{x} = x^2(x-1), \quad \dot{y} = y\big[(y+b)^2 + c^2\big], \quad c \neq 0. \tag{3.13}$$

Clearly these systems possess three double singularities (one real and two complex) on the double line $x = 0$ and three simple singularities (one real and two complex) located on the simple invariant line $x = 1$.

So considering Notation 1.11 we obtain the configuration of invariant lines given by Config. 7.8 if $b \neq 0$, and Config. 7.9 if $b = 0$.

(6) *Case* $\xi_1\xi_2 = 0$, $\xi_1 + \xi_2 > 0$, $\nu_1\nu_2 = 0$. As it was mentioned above we may consider $\xi_1 = 0$ which implies $\xi_2 > 0$. Then $\nu_2 \neq 0$ and hence we have $\nu_1 = 0$. In this case we have a triple line in the direction $x = 0$ and after a translation we obtain the system

$$\dot{x} = x^3, \quad \dot{y} = y(y-b)(y-c)$$

with $bc \neq 0$. Then applying the rescaling $(x, y, t) \mapsto (cx,\, cy, t/c^2)$ we force $c = 1$ and we arrive at the following 1-parameter family of systems

$$\dot{x} = x^3, \quad \dot{y} = y(y-1)(y-b), \quad b(b-1) \neq 0. \tag{3.14}$$

It is easy to determine that these systems possess three triple real singularities located on the triple invariant line $x = 0$. This leads to the configuration Config. 7.10.

(7) *Case* $\xi_1\xi_2 = 0$, $\xi_1 + \xi_2 < 0$, $\nu_1\nu_2 = 0$. So similarly as before, we may consider $\xi_1 = \nu_1 = 0$ and $\xi_2 < 0$. In this case we have a triple line in the direction $x = 0$ and after a translation setting some new parameters (see the second equation of system (3.13)) we obtain the system

$$\dot{x} = x^3, \quad \dot{y} = y\big[c^2 + (y+b)^2\big]$$

with $c \neq 0$. Then applying the rescaling $(x, y, t) \mapsto (cx, cy, t/c^2)$ we arrive at the following 1-parameter family of systems

$$\dot{x} = x^3, \quad \dot{y} = y\big[1 + (y+b)^2\big]. \tag{3.15}$$

These systems possess three triple singularities (one real and two complex) located on the triple invariant line $x = 0$. Considering Notation 1.11 this leads to the configuration Config. 7.11 if $b \neq 0$ and Config. 7.12 if $b = 0$.

(8) *Case* $\xi_1 = \xi_2 = 0$, $\nu_1\nu_2 \neq 0$. Then we have two double real invariant lines (one in the direction $x = 0$ and the second in the direction $y = 0$). Due to $\nu_1\nu_2 \neq 0$ none of them could be triple. So after a translation which moves the origin of coordinates at the intersection of the double lines we arrive at the system

$$\dot{x} = x^2(x-a), \quad \dot{y} = y^2(y-b),$$

where $ab \neq 0$. Then applying the rescaling $(x, y, t) \mapsto (ax,\, ay, t/a^2)$ we obtain the following 1-parameter family of systems

$$\dot{x} = x^2(x-1), \quad \dot{y} = y^2(y-b), \quad b \neq 0. \tag{3.16}$$

It is not difficult to determine that these systems possess four distinct real finite singularities: one of multiplicity four (located at the intersection of the double lines) two double and one simple. As a result we obtain the configuration Config. 7.13.

(9) *Case* $\xi_1 = \xi_2 = 0$, $\nu_1\nu_2 = 0$. According to Remark 3.1 the condition $\nu_1^2+\nu_2^2 \neq 0$ is necessary and by the same reasons as above we may assume $\nu_1 = 0$ and $\nu_2 \neq 0$. Therefore we have a triple invariant line in the direction $x = 0$ and a double one in the direction $y = 0$. As a result via a translation we obtain the system

$$\dot{x} = x^3, \quad \dot{y} = y^2(y-a),$$

with $a \neq 0$. Then we may assume $a = 1$ due to the rescaling $(x, y, t) \mapsto (ax,\ ay, t/a^2)$ and we arrive at the system

$$\dot{x} = x^3, \quad \dot{y} = y^2(y-1). \tag{3.17}$$

We observe that this system has only two distinct finite singularities: one of the multiplicity six and one triple both located at the invariant line $x = 0$. So we obtain Config. 7.14. Thus we have proved the following lemma.

Lemma 3.2. *Assume that for a system* (3.5) *the conditions in terms of the polynomials* ξ_1, ξ_2, ν_1 *and* ν_2 *and indicated below in the first column are satisfied. Then this system could be brought via an affine transformation and time rescaling to one of the corresponding canonical system, indicated in the second column. Moreover this system possesses one of the configurations Config. 7.1 – 7.14 if and only if the conditions on the parameters a and b of the corresponding canonical system wherever they are indicated, are satisfied, respectively:*

$$\xi_1\xi_2 > 0, \xi_1+\xi_2 > 0 \ \Rightarrow\ (3.8)\ \Leftrightarrow\ \textit{Config. 7.1};$$

$$\xi_1\xi_2 > 0, \xi_1+\xi_2 < 0 \ \Rightarrow\ (3.9)\ \textit{with}\ \begin{cases} ab \neq 0 & \Leftrightarrow \textit{Config. 7.2}; \\ ab = 0, a+b \neq 0 & \Leftrightarrow \textit{Config. 7.3}; \\ a = b = 0 & \Leftrightarrow \textit{Config. 7.4}; \end{cases}$$

$$\xi_1\xi_2 < 0 \ \Rightarrow\ (3.11)\ \textit{with}\ \begin{cases} b \neq 0 & \Leftrightarrow \textit{Config. 7.5}; \\ b = 0 & \Leftrightarrow \textit{Config. 7.6}; \end{cases}$$

$$\xi_1\xi_2 = 0, \xi_1+\xi_2 > 0, \nu_1\nu_2 \neq 0 \ \Rightarrow\ (3.12)\ \Leftrightarrow \textit{Config. 7.7};$$

$$\xi_1\xi_2 = 0, \xi_1+\xi_2 < 0, \nu_1\nu_2 \neq 0 \ \Rightarrow; (3.13)\ \textit{with}\ \begin{cases} b \neq 0 & \Leftrightarrow \textit{Config. 7.8}; \\ b = 0 & \Leftrightarrow \textit{Config. 7.9}; \end{cases}$$

$$\xi_1\xi_2 = 0, \xi_1+\xi_2 > 0, \nu_1\nu_2 = 0 \ \Rightarrow\ (3.14)\ \Leftrightarrow\ \textit{Config. 7.10};$$

$$\xi_1\xi_2 = 0, \xi_1+\xi_2 < 0, \nu_1\nu_2 = 0 \ \Rightarrow\ (3.15)\ \textit{with}\ \begin{cases} b \neq 0 & \Leftrightarrow \textit{Config. 7.11}; \\ b = 0 & \Leftrightarrow \textit{Config. 7.12}; \end{cases}$$

$$\xi_1\xi_2 = 0, \xi_1+\xi_2 = 0, \nu_1\nu_2 \neq 0 \ \Rightarrow\ (3.16)\ \Leftrightarrow\ \textit{Config. 7.13};$$

$$\xi_1\xi_2 = 0, \xi_1+\xi_2 = 0, \nu_1\nu_2 = 0 \ \Rightarrow\ (3.17).\ \Leftrightarrow\ \textit{Config. 7.14}.$$

3.2. **Systems with configuration type** $\mathfrak{T} = (3, 2, 1)$**.** In this subsection we construct the cubic system with 4 real infinite singular points which has 6 invariant affine straight lines (counted with multiplicities) with configuration of type $(3, 2, 1)$, having total multiplicity 7 including the line at infinity.

Since we have one triplet and one couple of parallel invariant lines, according to Theorem 2.10 the conditions $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$ are necessary for system (3.1). In [6, Subsection 3.3.1] it was proved that in this case via a linear transformation and time rescaling the associated cubic homogeneous system could be brought to the form

$$\dot{x} = rx^3, \quad \dot{y} = (r-1)xy^2 + y^3, \quad r(r+1) \neq 0. \tag{3.18}$$

Consider the generic cubic system with cubic homogeneities as indicated in system (3.18). Since $r \neq 0$ via a translation we may assume $g = n = 0$ in system (3.1), i.e. a system possessing invariant lines in the configuration $(3, 2, 1)$ can be brought to the following family:

$$\begin{aligned} \dot{x} &= a + cx + dy + 2hxy + ky^2 + rx^3, \quad r(r+1) \neq 0, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + (r-1)xy^2 + y^3. \end{aligned} \tag{3.19}$$

Remark 3.3. We observe that due to a rescaling, the directions $y = -rx$ and $y = x$ can be interchanged without changing the form of the system (3.19).

First of all we recall that the directions for the potential invariant lines of system (3.44) are defined by the factors of the invariant polynomial $C_3(x, y) = xy(x - y)(rx+y)$. Since all non-zero coefficients of the polynomials of degree less than three of the right-hand parts of system (3.19) are free parameters, we can assume that any rescaling does not affect them. So we consider only the cubic homogeneous system associated to the system (3.19). Due to $r \neq 0$ applying the rescaling $(x, y, t) \mapsto (-x_1, ry_1, t_1/r^2)$ we obtain the homogeneous cubic system

$$\dot{x}_1 = r_1 x_1^3, \quad \dot{y}_1 = (r_1 - 1)x_1 y_1^2 + y_1^3,$$

where $r_1 = 1/r$. Since $x_1 = -x$ and $y_1 = y/r$ for these system we have

$$\begin{aligned} C_3(x_1, y_1) &= x_1 y_1 (x_1 - y_1)(r_1 x_1 + y_1) \\ &= -x(y/r)(-x - y/r)(-x/r + y/r) \\ &= -\frac{1}{r^3} xy(rx + y)(x - y). \end{aligned}$$

So we conclude that the direction $rx+y = 0$ (respectively $x-y = 0$) corresponding to system (3.19) passes to the direction $x_1 - y_1 = 0$ (respectively $rx_1 + y_1 = 0$) corresponding to the above system and hence the claim of Remark 3.3 is valid.

In what follows we determine necessary and sufficient conditions for a system (3.19) to have configuration of the type $\mathfrak{T} = (3,\ 2,\ 1)$.

According to [6, Remark 3.3] for system (3.19) the following remark is valid:

Remark 3.4. A cubic system (3.19) can possess: (i) a triplet of parallel invariant lines either in the direction $x = 0$, or $y = 0$ and in the second case the condition $r = 1$ holds; (ii) a couple of parallel invariant lines either in the direction $y = 0$, or $y = x$ (if $r = -1/2$), or $y = 2x$ (if $r = -2$).

So we have to examine the two cases: $(r-1)(1+2r)(2+r) \neq 0$ and $(r-1)(1+2r)(2+r) = 0$ (which splits in tree subcases).

3.2.1. *Case* $(r-1)(1+2r)(2+r) \neq 0$. Then by Remark 3.4 system (3.19) could have a triplet (respectively a couple) of parallel invariant lines only in the direction $x = 0$ (respectively $y = 0$). In [6] (see Subsection 3.2.1.1) these two directions were examined and the following result was obtained:

Lemma 3.5. *In the case* $r - 1 \neq 0$ *system* (3.19) *has a triplet of parallel invariant lines (which could be real or complex, distinct or coinciding) in the direction* $x =$

0 *and a couple of such lines in the direction* $y = 0$ *if and only if the following conditions hold:*

$$\begin{gathered} k = d = h = l = 0, \quad e = \big[4m^2 + f(r-1)^2\big]/(r-1), \\ b = -2m\big[4m^2 + f(r-1)^2\big]/(r-1)^3. \end{gathered} \tag{3.20}$$

Clearly, in addition we need exactly one invariant line, which could be in one of the directions $y = x$ or $rx + y = 0$. However according to Remark 3.3 it is sufficient to only consider the direction $y = x$.

Considering the conditions (3.20) in [6, Subsection 3.3.2] was proved that a system (3.19) possesses one invariant line in this direction if and only if beside the conditions (3.20) the following conditions are satisfied:

$$c = fr + \frac{12m^2r(2+r)}{(r-1)(1+2r)^2}, \quad a = -\frac{6fmr}{(r-1)(1+2r)} - \frac{72m^3r(1+r+r^2)}{(r-1)^3(1+2r)^3}. \tag{3.21}$$

It can be checked directly that system (3.19) with the conditions (3.20) and (3.21), possess the following invariant lines with configuration of the type $\mathfrak{T} = (3, 2, 1)$:

$$\begin{gathered} L_1 = (r-1)(2r+1)x - 6m, \quad L_2 = (1+2r)(x-y) - 2m, \\ L_{3,4} = (r-1)^2(2r+1)^2x^2 + 6m(r-1)(2r+1)x \\ + f(1+r-2r^2)^2 + 12m^2(1+r+r^2), \\ L_{5,6} = (r-1)^2y^2 + 2m(r-1)y + f(r-1)^2 + 4m^2. \end{gathered}$$

To determine if the invariant lines $L_{3,4} = 0$ and $L_{5,6} = 0$ are real or complex as well as if the invariant line $L_1 = 0$ coincides with one of the lines $L_{3,4} = 0$ we calculate:

$$\begin{gathered} \operatorname{Discrim}[L_{3,4}, x] = -4(r-1)^2(1+2r)^4\big[3m^2 + f(r-1)^2\big], \\ \operatorname{Discrim}[L_{5,6}, y] = -4(r-1)^2\big[3m^2 + f(r-1)^2\big], \\ \operatorname{Res}_x(L_1, L_{3,4}) = (r-1)^2(1+2r)^2\big[f(r-1)^2(1+2r)^2 + 12m^2(7+r+r^2)\big], \end{gathered}$$

Setting the notation

$$\xi = -\big[3m^2 + f(r-1)^2\big], \quad \nu = f(r-1)^2(1+2r)^2 + 12m^2(7+r+r^2) \tag{3.22}$$

we observe that

$$\operatorname{sign}\big(\operatorname{Discrim}[L_{3,4}, x]\big) = \operatorname{sign}\big(\operatorname{Discrim}[L_{5,6}, y]\big) = \operatorname{sign}(\xi).$$

Moreover, if $\xi = 0$ then $\nu \neq 0$, otherwise the condition $\xi = 0 = \nu$ imply $m = f = 0$ and we arrive at the homogeneous system possessing invariant lines of total multiplicity 8. So we examine the possibilities provided by the polynomials ξ and ν.

(a) *Case* $\xi > 0$ *and* $\nu \neq 0$. This means that $3m^2 + f(r-1)^2 < 0$, i.e. all 6 invariant lines are real. Setting $3m^2 + f(r-1)^2 = -u^2 \neq 0$ we obtain $f = -\frac{3m^2+u^2}{(r-1)^2}$ and we arrive at the following system

$$\begin{aligned} \dot{x} =& r\big[x - \frac{6m}{(r-1)(1+2r)}\big]\big[x + \frac{3m-u-2ru}{(r-1)(1+2r)}\big]\big[x + \frac{3m+u+2ru}{(r-1)(1+2r)}\big], \\ \dot{y} =& \big[y + \frac{m-u}{(r-1)}\big]\big[y + \frac{m+u}{(r-1)}\big]\big[(r-1)x + y - \frac{2m}{(r-1)}\big] \end{aligned} \tag{3.23}$$

For $f=-\frac{3m^2+u^2}{(r-1)^2}$ we obtain

$$\nu=(9m-u-2ru)(9m+u+2ru)\neq 0.$$

Then considering the condition $u(r-1)(1+2r)(2+r)\neq 0$ via the transformation

$$(x,y,t)\mapsto\Big(\frac{6m+(u-9m+2ru)x}{(r-1)(1+2r)},\ \frac{(u-m)(1+2r)+(u-9m+2ru)y}{(r-1)(1+2r)},\ \frac{(r-1)^2(1+2r)^2}{(u-9m+2ru)^2}t\Big),$$

after setting a new parameter

$$a=\frac{9m+u+2ru}{9m-u-2ru}\quad\big(\text{i.e. } m=\frac{(1+a)(1+2r)u}{9(a-1)}\big),$$

system (3.23) can be brought to the 2-parameter family of systems:

$$\begin{gathered}\dot{x}=rx(x-1)(x-a),\quad a(a-1)r(r-1)\neq 0\\ \dot{y}=y(y+1-a)\big[(r-1)x+y+1\big].\end{gathered}\tag{3.24}$$

These systems possess the invariant lines

$$\begin{gathered}L_1: x=0,\quad L_2: x=1,\quad L_3: x=a,\\ L_4: y=0,\quad L_5: y=a-1,\quad L_6: x-y=1\end{gathered}$$

and the nine finite singularities:

$$\begin{gathered}M_1(0,0),\quad M_2(0,-1),\quad M_3(0,a-1),\quad M_4(1,0),\quad M_5(1,-r),\\ M_6(1,a-1),\quad M_7(a,0),\quad M_8(a,a-1),\quad M_9\big(a,a(1-r)-1\big).\end{gathered}$$

It is easy to determine that 7 of these singularities are located at the intersections of the above invariant lines, more precisely the singular points M_i for $i\in\{1,2,3,4,6,7,8\}$. The singular point M_5 (respectively M_9) is located on the invariant line L_2 (respectively L_3). Moreover we have two singular points located at the intersections of three invariant lines: L_2, L_4 and L_6 which intersect at the point M_4, whereas L_3, L_5 and L_6 intersect at the point M_8. To determine all possible configurations for system (3.24) we have to examine the positions of the invariant lines as well as of the singularities M_5 and M_9 depending on the parameters a and r.

Let us first examine the position of the invariant lines. We observe that four of the lines are fixed and only the positions of the lines L_3 and L_5 depend on the parameter a. More exactly, if $a<0$ then the non-fixed invariant line $x=a$ is located on the left of the fixed invariant lines $x=0$ and $x=1$. If $0<a<1$ then the invariant line $x=a$ is located between the invariant lines $x=0$ and $x=1$. And finally, if $a>1$ then the invariant line $x=a$ is located on the right of the invariant lines $x=0$ and $x=1$.

Notation 3.6. Assume that the two finite real singular points $\widetilde{M}_1(x_1,y_1)$ and $\widetilde{M}_2(x_2,y_2)$ of a cubic system are located on the real invariant line $ax+by+c=0$ of this system. Then: *(α)* in the case $a\neq 0$ we say that the singular point $\widetilde{M}_1$ is located *below* (respectively *above*) or coincides with the singularity $\widetilde{M}_2$ if $y_1\leq y_2$ (respectively $y_2<y_1$) and we denote this position by $\widetilde{M}_1\preceq\widetilde{M}_2$ (respectively $\widetilde{M}_2\prec\widetilde{M}_1$);

(β) in the case $a=0$ (then $y_1=y_2$) we say that the singular point $\widetilde{M}_1$ is located *on the left* or coincides with (respectively *on the right*) the singularity $\widetilde{M}_2$ if $x_1\leq x_2$

(respectively $x_2 < x_1$) and we again denote this position by $\widetilde{M}_1 \preceq \widetilde{M}_2$ (respectively $\widetilde{M}_2 \prec \widetilde{M}_1$).

Next we consider the position of the finite singularity $M_5(1, y_5)$ with $y_5 = -r$ (respectively $M_9(a, y_9)$ with $y_9 = a(1-r)-1$ on the invariant line $x = 1$ (respectively $x = a$). We observe that on the line $x = 1$ (respectively $x = a$) two real singularities $M_4(1, 0)$ and $M_6(1, a-1)$ (respectively $M_7(a, 0)$) and $M_8(a, a-1)$) are located and their reciprocal position depends on the sign of $a - 1 \neq 0$. So we distinguish two possibilities: (i) $a < 1$ and (ii) $a > 1$.

It is clear that the positions of these singularities in two different cases (i) and (ii) could be distinct and therefore we examine each one of these cases. Since $ar \neq 0$ we deduce that $y_5 \neq 0$ (respectively $y_9 \neq a - 1$), but we could have $y_5 = a - 1$ (respectively $y_9 = 0$) and in this case the singularity M_5 (respectively M_9) coalesced with the singularity M_6 (respectively M_7).

Case (i): $a < 1$. Then we have $M_6 \prec M_4$ and $M_8 \prec M_7$ and considering the coordinates $y_5 = -r$ and $y_9 = a(1 - r) - 1$ and Notation 3.6 we have the following implications.
(I) For the singular point M_5:

$$y_5 \leq a - 1 \Rightarrow M_5 \preceq M_6 \prec M_4; \quad a - 1 < y_5 < 0 \Rightarrow M_6 \prec M_5 \prec M_4;$$
$$y_5 > 0 \Rightarrow M_6 \prec M_4 \prec M_5.$$

(II) For the singular point M_9:

$$y_9 < a - 1 \Rightarrow M_9 \prec M_8 \prec M_7; \quad a - 1 < y_9 < 0 \Rightarrow M_8 \prec M_9 \prec M_7;$$
$$y_9 \geq 0 \Rightarrow M_8 \prec M_7 \preceq M_9.$$

Case (ii): $a > 1$. Then we have $M_4 \prec M_6$ and $M_7 \prec M_8$, and considering the coordinates y_5 and y_9 we have the following implications.
(I) For the singular point M_5:

$$y_5 < 0 \Rightarrow M_5 \prec M_4 \prec M_6; \quad 0 < y_5 < a - 1 \Rightarrow M_4 \prec M_5 \prec M_6;$$
$$y_5 \geq a - 1 \Rightarrow M_4 \prec M_6 \preceq M_5.$$

(II) For the singular point M_9:

$$y_9 \leq 0 \Rightarrow M_9 \preceq M_7 \prec M_8; \quad 0 < y_9 < a - 1 \Rightarrow M_7 \prec M_9 \prec M_8;$$
$$y_9 > a - 1 \Rightarrow M_7 \prec M_8 \prec M_9.$$

It is clear that not all the possibilities described above are realizable and examining the compatibilities of the conditions it is not too hard to convince ourselves (using, for example, the tools "'FindInstance" or "'Reduce" of computer algebra system Mathematica) that the following lemma is valid.

Lemma 3.7. *Consider the family of systems* (3.19) *with the conditions* (3.20), (3.21), $(r - 1)(1 + 2r)(2 + r) \neq 0$ *and* $\xi > 0$ *and* $\nu \neq 0$. *Then via an affine transformation and time rescaling, after introducing some new parameters, system* (3.19) *can be brought to the 2-parameter family of systems* (3.24). *Moreover this family of systems possesses the following configurations of invariant lines when the corresponding conditions indicated below, are satisfied (examples are given in the* *last column):*

$$Config.\ 7.15\ \Leftrightarrow; \begin{cases} a<0, 1-a<r<1-1/a, & (a=-1/2, r=2),\\ or\ a<0, 1-1/a<r<1-a & (a=-2, r=2); \end{cases}$$

$$Config.\ 7.16\ \Leftrightarrow\ \begin{cases} -1<a<0, r=1-1/a & (a=-1/2, r=3),\\ or\ a<-1, r=1-a & (a=-2, r=3); \end{cases}$$

$$Config.\ 7.17\ \Leftrightarrow\ a<0, r>1-a, r>1-1/a \quad (a=-2, r=4);$$

$$Config.\ 7.18\ \Leftrightarrow\ \begin{cases} -1<a<0, r=1-a & (a=-1/4, r=5/4),\\ or\ a<-1, r=1-1/a & (a=-2, r=3/2), \end{cases}$$

$$Config.\ 7.19\ \Leftrightarrow\ a=-1, r=2;$$

$$Config.\ 7.20\ \Leftrightarrow\ a<0, 0<r<1-a, r<1-1/a; \quad (a=-2, r=1/2);$$

$$Config.\ 7.21\ \Leftrightarrow\ a<0, r<0; \quad (a=-1, r=-3);$$

$$Config.\ 7.22\ \Leftrightarrow\ \begin{cases} 0<a<1, r>1-a, & (a=1/2, r=3/4),\\ or\ a>1, r>1-1/a & (a=5/4, r=1/2); \end{cases}$$

$$Config.\ 7.23\ \Leftrightarrow\ \begin{cases} 0<a<1, r=1-a & (a=1/2, r=1/2),\\ or\ a>1, r=1-1/a, & (a=2, r=1/2); \end{cases}$$

$$Config.\ 7.24\ \Leftrightarrow'\ \begin{cases} 0<a<1, 0<r<1-a & (a=1/2, r=1/4),\\ or\ a>1, 0<r<1-1/a & (a=3, r=1/2); \end{cases}$$

$$Config.\ 7.25\ \Leftrightarrow\ \begin{cases} 0<a<1, r=1-1/a & (a=5/32, r=-27/5),\\ or\ a>1, r=1-a & (a=4, r=-3); \end{cases}$$

$$Config.\ 7.26\ \Leftrightarrow\ \begin{cases} 0<a<1, r<1-1/a & (a=-1/4, r=-4),\\ or\ a>1, r<1-a; & (a=3, r=-3); \end{cases}$$

$$Config.\ 7.27\ \Leftrightarrow\ \begin{cases} 0<a<1, 1-1/a<r<0, & (a=1/4, r=-5/2),\\ or\ a>1, 1-a<r<0(a=5, r=-3). \end{cases}$$

(b) *Case* $\xi>0$ *and* $\nu=0$. Since $(r-1)(2r+1)\neq 0$, considering (3.22) the condition $\nu=0$ gives $f=-\frac{12m^2(7+r+r^2)}{(r-1)^2(1+2r)^2}$ and we arrive at the system

$$\begin{aligned} \dot x =& r\big[x-\frac{6m}{(r-1)(1+2r)}\big]^2\big[x+\frac{12m}{(r-1)(1+2r)}\big],\\ \dot y =& \big[y+\frac{2m(-4+r)}{(r-1)(1+2r)}\big]\big[y+\frac{2m(5+r)}{(r-1)(1+2r)}\big]\big[(r-1)x+y-\frac{2m}{(r-1)}\big] \end{aligned}$$

We observe that $m\neq 0$ otherwise we obtain $f=0$ and this implies $\xi=0$, i.e. we arrive at a contradiction. Then considering the condition $m(r-1)(1+2r)(2+r)\neq 0$ via the transformation

$$(x,y,t)\mapsto\Big(\frac{6m(1-3x)}{(r-1)(1+2r)}, \frac{2m(4-r-3y)}{(r-1)(1+2r)}, \frac{(r-1)^2(1+2r)^2}{324m^2}t\Big)$$

we arrive at the following 1-parameter family of systems

$$\dot x = rx^2(x-1), \quad \dot y = y(y-1)\big[(r-1)x+y\big] \tag{3.25}$$

These system possess the invariant lines

$$L_{1,2}: x=0 \text{ (double)}, \quad L_3: x=1, \quad L_4: y=0, \quad L_5: y=1, \quad L_6: y=x$$

and the following five finite singularities of total multiplicity 9:

$$M_{1,2,3,4}(0,0), \quad M_{5,6}(0,1), \quad M_7(1,0), \quad M_8(1,1), \quad M_9(1,1-r).$$

As we can see the positions of the invariant lines are fixed, as well as the positions of the finite singularities, except for the singularity $M_9(1,1-r)$ located on the invariant line $x=1$. It is not too difficult to detect that depending on the position of M_9 with respect to the singularities M_7 and M_8 as well as of the position of the s-points ("smooth", see Definition 3.29 on page 103) infinite singularity we obtain the following four distinct singularities: Config. 7.28 if $r<-1$; Config. 7.29 if $-1<r<0$; Config. 7.30 if $0<r<1$; and Config. 7.31 if $r>1$.

(c) *Case* $\xi<0$. This means that $3m^2+f(r-1)^2>0$, i.e. the invariant lines $L_{3,4}$ and $L_{5,6}$ are complex, whereas L_1 and L_2 remain real. Setting $3m^2+f(r-1)^2=u^2\neq 0$ we obtain $f=-\frac{3m^2-u^2}{(r-1)^2}$ and we arrive at the following system

$$\begin{aligned}\dot{x}=&r\big[x-\frac{6m}{(r-1)(1+2r)}\big]\Big[\big(x+\frac{3m}{(r-1)(1+2r)}\big)^2+\frac{u^2}{(r-1)^2}\Big],\\ \dot{y}=&\Big[\big(y+\frac{m}{(r-1)}\big)^2+\frac{u^2}{(r-1)^2}\Big]\big[(r-1)x+y-\frac{2m}{(r-1)}\big]\end{aligned}$$

For $f=-\frac{3m^2-u^2}{(r-1)^2}$ we obtain $\nu=81m^2+u^2(1+2r)^2\neq 0$ because of $u(1+2r)\neq 0$.

So considering the condition $u(r-1)(1+2r)(2+r)\neq 0$ via the transformation

$$(x,y,t)\mapsto\Big(-\frac{u(2a-3x)}{3(r-1)(1+2r)},\ \frac{u(a+2ar+9x)}{9(r-1)},\ \frac{(r-1)^2}{u^2}t\Big)$$

and setting a new parameter $a=-\frac{9m}{u(1+2r)}$ (i.e. $m=-a(1+2r)u/9$) we arrive at the following 2-parameter family of systems:

$$\dot{x}=rx\big[(x-a)^2+1\big], \quad \dot{y}=(1+y^2)\big[(r-1)x+y+a\big]. \tag{3.26}$$

These systems possess the following six invariant affine lines (two real and four complex):

$$L_1: x=0, \quad L_{2,3}: x=a\pm i, \quad L_{4,5}: y=\pm i, \quad L_6: x-y=a$$

and the following nine finite singularities:

$$\begin{gathered}M_1(0,-a),\ M_{2,3}(0,\pm i),\ M_{4,5}(a\pm i,i),\ M_{6,7}(a\pm i,-i),\\ M_8(a-i,-ar+i(r-1)),\ M_9(a+i,-ar-i(r-1)).\end{gathered}$$

We observe that one of the infinite singularities is located at the end of the affine line $y=-rx$ and since the other three fixed infinite singularities ale located at the intersections of the line at infinity with the lines $x=0$, $y=0$ and $y=x$ it is clear that we need to distinguish three possibilities: $r<-1$, $-1<r<0$ and $r>0$, respectively. As a result, considering Notation 1.11, in the case $a\neq 0$ we obtain Config. 7.32 if $r<-1$; Config. 7.33 if $-1<r<0$; and Config. 7.34 if $r>0$.

On the other hand if $a=0$ we obtain Config. 7.35 if $r<-1$; Config. 7.36 if $-1<r<0$ and Config. 7.37 if $r>0$.

(d) *Case* $\xi=0$. Considering (3.22) we obtain $f=-3m^2/(r-1)^2$ and then $\nu=81m^2$. We observe that $m\neq 0$ otherwise we obtain the system (3.18) possessing invariant lines of total multiplicity 8, because for these system we have (see Notation 2.4):

$$H(X,Y,Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = -rX^3(X-Y)Y^2(rX+Y).$$

So for $f = -3m^2/(r-1)^2$ we arrive at the family of systems

$$\begin{aligned}\dot{x} =& r\big[x + \frac{3m}{(r-1)(1+2r)}\big]^2\big[x - \frac{6m}{(r-1)(1+2r)}\big],\\ \dot{y} =& \big[y + \frac{m}{r-1}\big]^2\big[(r-1)x + y - \frac{2m}{(r-1)}\big].\end{aligned}$$

Since $m \neq 0$ we can apply the transformation

$$(x,y,t) \mapsto \Big(\frac{3m(3x-1)}{(r-1)(1+2r)},\ -\frac{m(1+2r-9y)}{(r-1)(1+2r)},\ \frac{(r-1)^2(1+2r)^2}{81m^2}t\Big)$$

we arrive at the 1-parameter family of systems

$$\dot{x} = rx^2(x-1), \quad \dot{y} = y^2\big[(r-1)x + y - r\big].$$

These systems possess the invariant lines

$$L_{1,2} : x = 0 \text{ (double)}, \quad L_3 : x = 1, \quad L_{4,5} : y = 0, \text{ (double)}, \quad L_6 : y = x$$

and the following four finite singularities of total multiplicity 9:

$$M_{1,2,3,4}(0,0), \quad M_{5,6}(0,r), \quad M_{7,8}(1,0), \quad M_9(1,1).$$

As we can see, the positions of the invariant lines are fixed, as well as the positions of the finite singularities, except for the double singularity $M_{5,6}(0,r)$ located on the invariant line $x = 0$. It is not too difficult to detect that depending on the position of $M_{5,6}$ with respect to the singularity M_1 as well as the position of the s-points at infinity, due to $r(r+1) \neq 0$, we obtain the following three distinct configurations: Config. 7.38 if $r < -1$; Config. 7.39 if $-1 < r < 0$ and Config. 7.40 if $r > 0$.

3.2.2. *Case* $(r-1)(1+2r)(2+r) = 0$. We examine each one of the three subcases given by these factors. However we observe that the subcase $r = -2$ could be brought to the subcase $r = -1/2$ via the rescaling $x \to x/2$. Therefore we consider only two subcases: $r = 1$ and $r = -1/2$.

For system (3.19), considering the equations (2.5) and Remark 2.12 we obtain

$$Eq_7 = k, \quad Eq_9 = d - 2hW, \quad Eq_{10} = a - cW - rW^3, \tag{3.27}$$

for the direction $x = 0$,

$$Eq'_5 = l, \quad Eq'_8 = e - 2mW + (r-1)W^2, \quad Eq'_{10} = b - fW - W^3 \tag{3.28}$$

for the direction $y = 0$ and

$$\begin{aligned}Eq''_6 &= l - 2h - k + 2m - (1+2r)W,\\ Eq''_9 &= -c - d + e + f + 2(l - h + m)W - 3rW^2,\\ Eq''_{10} &= -a + b + (e-c)W + lW^2 - rW^3\end{aligned} \tag{3.29}$$

for the direction $y = x$.

1. *Subcase* $r = 1$. According to Remark 3.4 system (3.19) could have the following parallel invariant lines: *(i)* a triplet in the direction $x = 0$ and a couple in the direction $y = 0$; *(ii)* a triplet in the direction $y = 0$ and a couple in the direction $x = 0$.

However for $r = 1$ from (3.27) and (3.28) it follows that in each one of these directions we could have either one, or three parallel invariant lines. So in the case $r = 1$ we could not have a configuration with the type $\mathfrak{T} = (3, 2, 1)$.

2. *Subcase* $r = -1/2$. According to Remark 3.4 system (3.19) could have a triplet of parallel invariant lines in the direction $x = 0$ and a couple of parallel invariant lines either in the direction $y = 0$ or $y = x$.

Forcing to have a triplet in the direction $x = 0$, according to (3.27), for system (3.19) the conditions $k = d = h = 0$ must be satisfied. So it remains to examine two possibilities for the existence of a couple of invariant lines: (i) in the direction $y = 0$ and (ii) in the direction $y = x$.

2.1. *Possibility: a couple in the direction* $y = 0$. Considering (3.28) we have $l = 0$ and taking into account the condition $r = -1/2$ we calculate

$$R^{(1)}_W(Eq'_8, Eq'_{10}) = -(6e + 9f + 16m^2)/4 = 0 \Rightarrow e = -(9f + 16m^2)/6$$

and then we obtain

$$R^{(0)}_W(Eq'_8, Eq'_{10}) = -(27b - 36fm - 64m^3)^2/216 = 0, \Rightarrow b = 4m(9f + 16m^2)/27.$$

So we determine the conditions

$$r = -1/2,\ k = d = h = l = 0,\ e = -(9f + 16m^2)/6,\ b = 4m(9f + 16m^2)/27 \quad (3.30)$$

which lead to the family of systems

$$\begin{aligned} \dot{x} &= (2a + 2cx - x^3)/2, \\ \dot{y} &= (8m - 9x + 6y)(9f + 16m^2 - 12my + 9y^2)/54. \end{aligned} \quad (3.31)$$

We observe that these systems possess one triplet in the direction $x = 0$ and one couple of parallel invariant lines in the direction $y = 0$. So we need one more invariant affine line, which could be either in the direction $y = x$ or in the direction $y = x/2$ (since we have $r = -1/2$). However according to Remark 3.3 it is sufficient to only examine the case $y = x$.

Considering (3.29) and the conditions (3.30) we obtain $Eq''_6 = 2m = 0$, i.e. $m = 0$ and then we calculate

$$R^{(0)}_W(Eq''_9, Eq''_{10}) = \big[27a^2 - 4(2c + f)(c + 2f)^2\big]/8 \equiv \Psi''(a, c, f).$$

In order for the equation $\Psi'' = 0$ to have real solutions we clearly must have either $2c + f \geq 0$ or $c + 2f = 0$. We consider each one of these cases.

2.1.1. *Case* $2c + f \geq 0$. So setting $2c + f = 3u^2 \geq 0$ we obtain $f = -2c + 3u^2$ and then we have

$$\Psi'' = 27(a + 2cu - 4u^3)(a - 2cu + 4u^3) = 0.$$

So because of the change $u \to -u$ we may assume $a - 2cu + 4u^3 = 0$ and we obtain the condition $a = 2u(c - 2u^2)$. Considering the conditions (3.30) we arrive at the family of systems

$$\dot{x} = (2u + x)(2c - 4u^2 + 2ux - x^2)/2, \quad \dot{y} = (3x - 2y)(2c - 3u^2 - y^2)/2. \quad (3.32)$$

These systems possess the following invariant lines with the configuration of the type $\mathfrak{T} = (3, 2, 1)$:

$$\begin{aligned} L_1 = x + 2u, &\quad L_2 = x - y - u, \\ L_{3,4} = (u - x)^2 - (2c - 3u^2), &\quad L_{5,6} = y^2 - (2c - 3u^2). \end{aligned} \quad (3.33)$$

Clearly the invariant lines $L_{3,4} = 0$ as well as $L_{5,6} = 0$ are real (respectively coinciding) if $2c - 3u^2 > 0$ (respectively $2c - 3u^2 = 0$) and they are complex if $2c - 3u^2 < 0$. Moreover since

$$\mathrm{Res}_x(L_1, L_{3,4}) = -2(c - 6u^2),$$

we conclude, that the invariant line $L_1 = 0$ coincides with one of the lines $L_{3,4} = 0$ if and only if $c = 6u^2$ and clearly this implies $2c - 3u^2 = 3u^2 > 0$, i.e. the invariant lines $L_{3,4} = 0$ must be real. So we examine the subcases provided by the polynomials $\xi'' = 2c - 3u^2$ and $\nu'' = c - 6u^2$.

2.1.1.1. *Subcase $\xi'' > 0$ and $\nu'' \neq 0$.* Then all the invariant lines (3.33) are real and distinct. Setting $2c - 3u^2 = v^2 \neq 0$ we obtain $c = (3u^2 + v^2)/2$ and we arrive at the system

$$\dot{x} = (2u - x)(u - v + x)(u + v + x)/2, \quad \dot{y} = (3x - 2y)(v - y)(v + y)/2. \tag{3.34}$$

Then since $v \neq 0$ via the transformation

$$(x, y, t) \mapsto \big(-u + v - 2vx,\ -v(1 + 2y),\ t/(2v^2)\big)$$

and introducing a new parameter $b = (v - 3u)/(2v)$ (i.e. $u = v(1 - 4a)/3$) system (3.34) can be brought to the system

$$\dot{x} = x(x - 1)(b - x), \quad \dot{y} = y(y + 1)(2 + b - 3x + 2y).$$

We observe that in system (3.24) the condition $a - 1 \neq 0$ holds. Then setting $b = a/(a - 1)$ via the following affine transformation and time rescaling:

$$x_1 = (1 - a)x + a, \quad y_1 = (1 - a)y, \quad t_1 = \frac{2}{(1 - a)^2}t$$

the above system could be brought to the system (3.24) for which $r = -1/2$. This family was investigated earlier (see Lemma 3.7) and since for $r = -1/2$ we do not have bifurcation points for the family (3.24) we deduce that there are no new configurations.

2.1.1.2. *Subcase $\xi'' > 0$ and $\nu'' = 0$.* Then $c = 6u^2$ which implies $\xi'' = 6u^2 > 0$ and we obtain the family of systems

$$\dot{x} = -(2u - x)^2(4u + x)/2, \quad \dot{y} = (3x - 2y)(3u - y)(3u + y)/2.$$

Due to $u \neq 0$, the above system can be brought via the transformation

$$(x, y, t) \mapsto \big(2u(1 - 3x),\ 3u(2y - 1),\ t/(18u^2)\big)$$

to the system

$$\dot{x} = -x^2(x - 1), \quad \dot{y} = y(y - 1)(3x + 2y - 2).$$

After the additional change $(x, y, t) \mapsto (x,\ 1-y,\ t/2)$ this system can be transformed to the system (3.25) with $r = -1/2$. This system was already investigated and hence no new configuration could be obtained.

2.1.1.3. *Subcase $\xi'' < 0$.* Setting $2c - 3u^2 = -v^2 < 0$ we obtain $c = (3u^2 + v^2)/2$ and we arrive at the following system

$$\dot{x} = (2u - x)\big[(x + u)^2 + v^2\big]/2, \quad \dot{y} = (3x - 2y)(y^2 + v^2)/2. \tag{3.35}$$

Since $v \neq 0$ the above system could be brought via the transformation $(x, y, t) \mapsto \big(2u + vx,\ vy,\ -2t/v^2\big)$ to the system

$$\dot{x} = x\big[(x - a)^2 + 1\big], \quad \dot{y} = (1 + y^2)(3x - 2y - 2a).$$

We observe that applying the time rescaling $t \to -t/2$, the above system becomes the system (3.26) with $r = -1/2$ and hence, no new configuration could be obtained.

2.1.2. *Case* $c + 2f = 0$. Then the equation $\Psi'' = 0$ implies $a = 0$ and we arrive at the family of systems

$$\dot{x} = -x(4f + x^2)/2, \quad \dot{y} = -(3x - 2y)(f + y^2)/2. \tag{3.36}$$

These systems possess the following 8 invariant affine lines

$$x = 0, \quad 4f + x^2 = 0, \quad x = 2y, \quad f + y^2 = 0, \quad f + (x - y)^2 = 0,$$

i.e. we are out of the class of system studied in this article.

2.2. *Possibility: a couple in the direction* $y = x$. Considering (3.29) and the conditions $r = -1/2$ and $k = d = h = 0$ (in order to have a triplet in the direction $x = 0$) we obtain:

$$Eq_7'' = l + 2m, \quad Eq_9'' = -c - d + e + f + 2lW - 3W^2/2,$$
$$Eq_{10}'' = -a + b + (e - c)W + lW^2 + W^3/2.$$

So the condition $Eq_7'' = 0$ yields $l = -2m$ which implies

$$R_W^{(1)}(Eq_9'', Eq_{10}'') = (-6c + 6e - 3f - 16m^2)/4 = 0 \;\Rightarrow\; e = (6c + 3f + 16m^2)/6$$

and then we obtain

$$R_W^{(0)}(Eq_9'', Eq_{10}'') = (27a - 27b - 36fm - 64m^3)^2/216 = 0,$$
$$\Rightarrow b = (27a - 36fm - 64m^3)/27.$$

So we found that under the conditions

$$\begin{gathered} r = -1/2, \quad k = d = h = 0, \quad l = -2m, \\ e = (6c + 3f + 16m^2)/6, \quad b = (27a - 36fm - 64m^3)/27, \end{gathered} \tag{3.37}$$

system (3.19) has one triplet in the direction $x = 0$ and one couple of parallel invariant lines in the direction $y = x$. In this case we arrive at the system

$$\begin{aligned} \dot{x} &= (2a + 2cx - x^3)/2, \\ \dot{y} &= \frac{1}{27}(27a - 36fm - 64m^3) + \frac{1}{6}(6c + 3f + 16m^2)x - 2mx^2 + fy \\ &\quad + 2mxy - \frac{3}{2}xy^2 + y^3. \end{aligned}$$

However we detect that these system can be brought via the change $(x, y, t) \mapsto (x, x - y, t)$ to the system (3.31) with exactly the same parameters a, c, f and m. Since the system (3.31) were already investigated we deduce that no new configurations could be obtained.

3.3. **Systems with the configuration of the type $\mathfrak{T} = (3, 1, 1, 1)$.** In this subsection we construct a cubic system with 4 real infinite singular points which has 6 invariant affine straight lines, with configuration of type $\mathfrak{T} = (3, 1, 1, 1)$, having total multiplicity 7, as always the invariant straight line at infinity included.

According to Theorem 2.10 if a cubic system possesses 6 invariant affine straight lines in the configuration of type $\mathfrak{T} = (3, 1, 1, 1)$, then necessarily condition $\mathcal{V}_4 = \mathcal{U}_2 = 0$ holds.

3.3.1. *Construction of the associated homogeneous cubic system.* As a first step we construct the cubic homogeneous parts of system (3.1) for which the above condition is fulfilled. Since we have 4 real infinite distinct singularities, according to Lemma 2.14 we consider the family of systems

$$\begin{aligned}\dot{x} =&(p+r)x^3+(s+v)x^2y+qxy^2,\\ \dot{y} =&px^2y+(r+v)xy^2+(q+s)y^3, \quad rs(r+s)\neq 0,\end{aligned} \tag{3.38}$$

and we force the condition $\mathcal{V}_4=\mathcal{U}_2=0$ to be satisfied.

A straightforward computation of the value of $\mathcal{V}_4$ for system (3.38) yields $\mathcal{V}_4 = -9216\,\widehat{\mathcal{V}}_4\,C_3(x,y)$, where

$$\widehat{\mathcal{V}}_4=r^2(3q+s+v)+r(2pq-s^2+3qv+v^2)-s(2pq+3ps+3pv+sv+v^2).$$

As for system (3.38) we have $C_3=xy(x-y)(rx+sy)\neq 0$, we conclude that for these systems the condition $\mathcal{V}_4=0$ is equivalent to $\widehat{\mathcal{V}}_4=0$.

We observe that the invariant polynomial $\mathcal{U}_2$ is a homogeneous polynomial of degree four in x and y. So we shall use the following notations:

$$\mathcal{U}_2=\sum_{j=0}^{4}\mathcal{U}_{2j}x^{4-j}y^j.$$

Calculating the value of the polynomial $\mathcal{U}_2$ for system (3.38) we obtain

$$\mathcal{U}_{20}=p\big[q(2p+3r)^2+(s+v)(2pr+3r^2-3ps-3rs-pv)\big]\equiv p\widehat{U}_{20}$$

and we consider two cases: $p\neq 0$ and $p=0$.

1. *Case* $p\neq 0$. Then we must have $\widehat{U}_{20}=0$ and as this polynomial is linear with respect to the parameter q, we examine two subcases: $2p+3r\neq 0$ and $2p+3r=0$.

1.1. *Subcase* $2p+3r\neq 0$. In this case the condition $\widehat{U}_{20}=0$ gives

$$q=-\frac{(s+v)(2pr+3r^2-3ps-3rs-pv)}{(2p+3r)^2}$$

and then we calculate

$$\mathcal{U}_{21}=\frac{3p(s+v)(2p+3r+s+v)(2ps+2rs-rv)}{2p+3r}=-(2p+3r)\widehat{\mathcal{V}}_4.$$

So because of $2p+3r\neq 0$ the condition $\mathcal{U}_{21}=0$ implies $\widehat{\mathcal{V}}_4=0$.

On the other hand since $p\neq 0$ the condition $\mathcal{U}_{20}=0$ gives

$$(s+v)(2p+3r+s+v)(2ps+2rs-rv)=0$$

and we consider the three possibilities provided by this condition.

1.1.1. *Possibility* $s+v=0$. Then $v=-s$ (this implies $q=0$) and we obtain $\mathcal{U}_2=0=\widehat{\mathcal{V}}_4$. In this case we arrive at the system

$$\dot{x}=(p+r)x^3, \quad \dot{y}=px^2y+(r-s)xy^2+sy^3, \quad rs(r+s)\neq 0. \tag{3.39}$$

1.1.2. *Possibility* $s+v\neq 0$ *and* $2p+3r+s+v=0$. Then $v=-(2p+3r+s)$ and calculations yield $\mathcal{U}_2=0$. So we arrive at the family of systems

$$\begin{aligned}\dot{x} =&(p+r)x^3-(2p+3r)x^2y+(p+r-s)xy^2,\\ \dot{y} =&px^2y-(2p+2r+s)xy^2+(p+r)y^3.\end{aligned} \tag{3.40}$$

These systems, via the transformation

$$x_1 = -\frac{r}{s}(x-y), \quad y_1 = \frac{r}{s^2}(rx+sy), \quad t_1 = \frac{s^3 t}{r^2(r+s)},$$

could be brought to the systems

$$\dot{x}_1 = (p_1+r_1)x_1^3, \quad \dot{y}_1 = p_1 x_1^2 y_1 + (r_1-s_1)x_1 y_1^2 + s_1 y_1^3, \tag{3.41}$$

where

$$p_1 = -(pr+r^2+ps+2rs)/s, \quad r_1 = r, \quad s_1 = s \;\Rightarrow\; r_1 s_1(r_1+s_1) = rs(r+s) \neq 0.$$

In other words we arrive at systems (3.39).

1.1.3. *Possibility* $(s+v)(2p+3r+s+v) \neq 0$ *and* $2ps+2rs-rv=0$. In this case we have $v = 2(ps+rs)/r$ (this implies $q = s(ps-r^2+rs)/r^2$) and we again get $\mathcal{U}_2 = 0$. This leads to the family of systems

$$\begin{aligned} \dot{x} =&(p+r)x^3 + s(2p+3r)x^2y/r - s(r^2-ps-rs)xy^2/r^2, \\ \dot{y} =&px^2y + (r^2+2ps+2rs)xy^2/r + (p+r)s^2y^3/r^2. \end{aligned} \tag{3.42}$$

We claim that these systems could be brought to system (3.39) via a linear transformation and a time rescaling. Indeed, since $rs(r+s) \neq 0$ we can apply the change

$$x_1 = rx+sy, \quad y_1 = sy, \quad t_1 = t/(rs)$$

and we obtain systems (3.41), where

$$p_1 = ps/r, \quad r_1 = s, \quad s_1 = -(r+s) \;\Rightarrow\; r_1 s_1(r_1+s_1) = rs(r+s) \neq 0.$$

So we arrive at systems (3.39) and our claim is proved.

1.2. *Subcase* $2p+3r=0$. We have $p=-3r/2$ and then we obtain $U_{20} = -9r^2(s+v)^2/4$. Since $r \neq 0$ the condition $U_{20}=0$ implies $v=-s$ and then we calculate

$$U_{24} = -3q(2q+r+2s)(2qr+2rs+s^2)/2 = 0.$$

So we consider the three possibilities provided by this condition.

1.2.1. *Possibility* $q=0$. Then we obtain $\mathcal{U}_2 = 0 = \widehat{\mathcal{V}}_4$ and we arrive at the family of systems

$$\dot{x} = -rx^3/2, \quad \dot{y} = -3rx^2y/2 + (r-s)xy^2 + sy^3,$$

which is a subfamily of systems (3.39) defined by the condition $p=-3r/2$.

1.2.2. *Possibility* $q \neq 0$ *and* $2q+r+2s=0$. In this case $q = -(r+2s)/2$ and we obtain the family of systems

$$\dot{x} = -rx^3/2 - (r+2s)xy^2/2, \quad \dot{y} = -3rx^2y/2 + (r-s)xy^2 + sy^3,$$

which is a subfamily of systems (3.40) defined by the condition $p=-3r/2$ and which could be brought via an affine transformation to systems (3.39).

1.2.3. *Possibility* $q(2q+r+2s) \neq 0$ *and* $2qr+2rs+s^2=0$. In this case we obtain $q = -s(2r+s)/(2r)$ and we arrive at the family of systems

$$\begin{aligned} \dot{x} =& -rx^3/2 - s(2r+s)xy^2/(2r), \\ \dot{y} =& -3rx^2y/2 + (r-s)xy^2 - s^2y^3/(2r). \end{aligned}$$

It remains to observe that this family of systems is a subfamily of systems (3.42) defined by the condition $p=-3r/2$.

Thus the case $p \neq 0$ is completely investigated.

2. *Case* $p=0$. Then we calculate

$$\mathcal{U}_{21}=3r(r+v)(3qr+rs-s^2+rv-sv)=3r\widehat{\mathcal{V}}_4.$$

So because of $r\neq 0$ the condition $\mathcal{U}_{21}=0$ implies $\widehat{\mathcal{V}}_4=0$. Since we have to impose the condition $\mathcal{U}_{21}=0$ to be satisfied we examine two subcases: $r+v=0$ and $r+v\neq 0$.

2.1. *Subcase* $r+v=0$. Hence we have $p=0$ and $v=-r$ (this implies $\mathcal{U}_2=0$) and we obtain the system

$$\dot{x}=rx^3+(s-r)x^2y+qxy^2,\quad \dot{y}=(q+s)y^3. \tag{3.43}$$

Because of $rs\neq 0$ we can apply the transformation

$$x_1=y,\quad y_1=-rx/s,\quad t_1=st/r$$

and we obtain systems (3.41) with

$$p_1=qr/s,\quad r_1=r,\quad s_1=r\ \Rightarrow\ r_1s_1(r_1+s_1)=rs(r+s)\neq 0.$$

So by an additional transformation we arrive at systems (3.39) again.

2.2. *Subcase* $r+v\neq 0$. Then the condition $\mathcal{U}_{21}=0$ implies $3qr+rs-s^2+rv-sv=0$ and we obtain $q=-\frac{(r-s)(s+v)}{3r}$. Then calculations yield

$$\mathcal{U}_{22}=(2s-v)(r+v)(s+v)(3r+s+v)=0$$

and since $r+v\neq 0$ we have three possibilities.

2.2.1. *Possibility* $2s-v=0$. Then $v=2s$ (this implies $q=-s(r-s)/r$) and considering the condition $p=0$ we arrive at the family of systems

$$\dot{x}=rx^3+3sx^2y-s(r-s)xy^2/r,\quad \dot{y}=(r+2s)xy^2+s^2y^3/r.$$

We observe that this family of systems is a subfamily of systems (3.42) defined by the condition $p=0$, and therefore could be brought to systems of the form (3.39) via a linear transformation and time rescaling.

2.2.2. *Possibility* $s+v=0$. Then $v=-s$ and this implies $q=0$. Evidently for $p=q=0$ and $v=-s$ system (3.38) becomes a subfamily of systems (3.39) defined by the condition $p=0$.

2.2.3. *Possibility* $3r+s+v=0$. In this case we have $v=-(3r+s)$ (which implies $q=-(r-s)^2/(3r)$) and considering the condition $p=0$ we obtain the family of systems

$$\dot{x}=rx^3-3rx^2y+(r-s)xy^2,\quad \dot{y}=-(2r+s)xy^2+ry^3$$

which evidently is a subfamily of the family (3.40) defined by the condition $p=0$.

So all the cases are examined and we arrive at the following remark.

Remark 3.8. Consider a homogeneous system (3.38) for which the condition $\mathcal{U}_2=0$ is satisfied. Then this system could be brought via a linear transformation and a time rescaling to the form (3.39).

Consider the generic cubic system (3.1) with cubic homogeneities as indicated in system (3.39). Since $s\neq 0$ via a translation we may assume $n=0$ in system (3.1). Moreover we may assume $s=1$ due to the time rescaling $t\to t/s$. So we obtain the following cubic system

$$\begin{aligned}&\dot{x}=a+cx+dy+gx^2+2hxy+ky^2+(p+r)x^3,\\ &\dot{y}=b+ex+fy+lx^2+2mxy+px^2y+(r-1)xy^2+y^3,\quad r(r+1)\neq 0\end{aligned} \tag{3.44}$$

which we consider here below.

3.3.2. *Construction of the cubic system possessing invariant lines with configuration of type* $\mathfrak{T} = (3,1,1,1)$. In what follows we shall determine necessary and sufficient conditions for a system (3.44) to have a configuration of the type $(3,1,1,1)$.

Considering Remark 2.13 for the homogeneous system (3.39) (with $s = 1$), associated to the system (3.44) we calculate

$$H(X,Y,Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = -(p+r)X^3(X-Y)Y(rX+Y).$$

So each one of the invariant lines $y = 0$ or $y = x$ or $y = -rx$ of system (3.39) is simple, whereas the invariant line $x = 0$ is of multiplicity three.

Remark 3.9. We observe that due to a rescaling, the directions $y = -rx$ and $y = x$ can be interchanged without changing the form of the system (3.44).

First of all we recall that the directions for the potential invariant lines of system (3.44) are defined by the factors of the invariant polynomial $C_3(x,y) = xy(x-y)(rx+y)$. Since all non-zero coefficients of the polynomials of degree less than three of the right-hand parts of system (3.44) are free parameters, we can assume that any rescaling does not affect them. So we consider only the cubic homogeneous system associated to the system (3.44). Due to $r \neq 0$ applying the rescaling $(x,y,t) \mapsto (x_1, -ry_1, t_1/r^2)$ we obtain the homogeneous cubic system

$$\dot{x}_1 = (p_1 + r_1)x_1^3, \quad \dot{y}_1 = p_1 x^2 y_1 + (r_1 - 1)x_1 y_1^2 + y_1^3,$$

where $p_1 = p/r^2$ and $r_1 = 1/r$. Since $x_1 = x$ and $y_1 = -y/r$ for these systems we have

$$\begin{aligned} C_3(x_1,y_1) &= x_1 y_1 (x_1 - y_1)(r_1 x_1 + y_1) = x(-y/r)(x + y/r)(x/r - y/r) \\ &= -\frac{1}{r^3} xy(rx+y)(x-y). \end{aligned}$$

So we conclude that the direction $rx + y = 0$ (respectively $x - y = 0$) corresponding to systems (3.44) passes to the direction $x_1 - y_1 = 0$ (respectively $rx_1 + y_1 = 0$) corresponding to the above systems and hence the claim of Remark 3.9 is valid.

Considering Remark 3.9 we examine three possibilities for the direction of a triplet of parallel invariant lines for systems (3.44): (i) direction $x = 0$, (ii) direction $y = 0$, and (iii) direction $y = x$.

Existence of a triplet in the direction $x = 0$. Consider system (3.44). Taking into account the equations (2.5) and Remark 2.12 for the direction $x = 0$ we obtain:

$$Eq_7 = k, \quad Eq_9 = d - 2hW, \quad Eq_{10} = a + gW^2 - cW - (p+r)W^3.$$

Evidently we have a triplet of parallel invariant lines in this direction if and only if the condition $k = d = h = 0$ holds.

Next we ask for the existence of exactly one invariant line in each one of the remaining three directions.

1. *Direction* $y = 0$. In this case considering the equations (2.5) and Remark 2.12 for systems (3.44) we have

$$Eq'_5 = l - pW, \quad Eq'_8 = e - 2mW + (r-1)W^2, \quad Eq'_{10} = b - fW - W^3. \tag{3.45}$$

2. *Direction* $y = x$. Considering the equations (2.5) and Remark 2.12 in this case we calculate

$$\begin{aligned} Eq_6'' &= l - g - 2h - k + 2m - (1 + p + 2r)W, \\ Eq_9'' &= -c - d + e + f + (l - g + k)W + (1 - p - r)W^2, \\ Eq_{10}'' &= -a + b + (e - c)W + (l - g)W^2 - (p + r)W^3. \end{aligned} \tag{3.46}$$

3. *Direction* $y = -rx$. In this case considering the equations (2.5) and Remark 2.12 we have

$$\begin{aligned} Eq_6''' &= 2m - g + 2hr - kr^2 - l/r + (p + 2r + r^2)W/r, \\ Eq_9''' &= f - c - e/r + (l + gr - kr^3)W/r^2 - (p + r - r^2)W^2/r^2, \\ Eq_{10}''' &= b + ar - (e + cr)W/r + (l + gr)W^2/r^2 - (p + r)W^3/r^2. \end{aligned} \tag{3.47}$$

Thus in what follows we have to impose the existence of exactly one invariant line in each one of the directions $y = 0$, $y = x$ and $y = -rx$ providing that the condition $k = d = h = 0$ is satisfied.

We observe, that the equation $Eq_5' = 0$ (respectively $Eq_6'' = 0$; $Eq_6''' = 0$) depends on the parameter W (linearly) if and only if $p \neq 0$ (respectively $(1 + p + 2r) \neq 0$; $(p + 2r + r^2) \neq 0$). So we consider two cases: $p(1 + p + 2r)(p + 2r + r^2) \neq 0$ and $p(1 + p + 2r)(p + 2r + r^2) = 0$.

Case $p(1+p+2r)(p+2r+r^2) \neq 0$. Considering (3.45) the condition $Eq_5' = 0$ gives $W = l/p$ and then we calculate

$$Eq_8' = \big[ep^2 - 2lmp + l^2(r - 1)\big]/p^2 = 0, \quad Eq_{10}' = (bp^3 - l^3 - lfp^2)/p^3 = 0.$$

Since $p \neq 0$ we obtain the relations $e = l(l + 2mp - lr)/p^2$ and $b = l(l^2 + fp^2)/p^3$. These conditions guarantee the existence of an invariant line in the direction $y = 0$.

Next we consider equations (3.46) associated to the direction $y = x$ providing that the conditions

$$k = d = h = 0, \quad e = l(l + 2mp - lr)/p^2, \quad b = l(l^2 + fp^2)/p^3 \tag{3.48}$$

hold. So considering these conditions the equation $Eq_6'' = 0$ gives $W = (l - g + 2m)/(1 + p + 2r) \equiv W_0''$ and then the equation $Eq_9''|_{\{W=W_0''\}} = 0$ yields

$$f = c - \frac{l(l + 2mp - lr)}{p^2} - \frac{(l - g)^2 - 4m^2}{2(1 + p + 2r)} + \frac{(p - 3)(l - g + 2m)^2}{2(1 + p + 2r)^2}. \tag{3.49}$$

Considering this value of the parameter f and (3.48) the equation $Eq_{10}''|_{\{W=W_0''\}} = 0$ implies

$$\begin{aligned} a = \frac{l + gp - 2mp + 2lr}{p^3(1 + p + 2r)^3} \Big[&cp^2(1 + p + 2r)^2 + (l + gp - 2mp + 2lr)\big[l(p + r)(1 + 2r) \\ &- pg(1 + r) + 2mp(p + r)\big]\Big]. \end{aligned} \tag{3.50}$$

Thus conditions (3.48), (3.49) and (3.50) guarantee the existence of 5 affine invariant lines of system (3.44): a triplet in the direction $x = 0$, one line in the direction $y = 0$ and one in the direction $y = x$. It remains now to find the additional conditions for the existence of one common solution in W of equations $Eq_6''' = 0$, $Eq_9''' = 0$ and $Eq_{10}''' = 0$ given in (3.47), providing that the conditions (3.48), (3.49) and (3.50) hold.

Equation $Eq_6''' = 0$ gives $W = (l + gr - 2mr)/(p + 2r + r^2) \equiv W_0'''$ and then we calculate

$$
\begin{aligned}
Eq_9'''|_{\{W=W_0'''\}} &= \frac{(1+r)G_1G_2}{p^2(1+p+2r)^2(p+2r+r^2)^2},\\
Eq_{10}'''|_{\{W=W_0'''\}} &= -\frac{(1+r)G_1G_3}{p^3(1+p+2r)^3(p+2r+r^2)^3},
\end{aligned}
$$

where

$$
G_1(l,g,m,p,r) = p(g-2m)(1-r) - l(2+3p+5r+2r^2) \tag{3.51}
$$

and $G_2(l,g,m,p,r)$ and $G_3(c,l,g,m,p,r)$ are the following polynomials:

$$
\begin{aligned}
G_2 =& 2mp(p+r)(1+3p+7r+r^2) - l(-1+r)(2+r)(p+r)(1+2r)\\
&- gp\,(2p^2 - r + 6pr + 2r^2 - r^3),
\end{aligned}
$$

$$
\begin{aligned}
G_3 =& cp^2r(1+p+2r)^2(p+2r+r^2)^2 + l^2(p+r)\big[r^3(2+r)^2(1+2r)^2\\
&+ p(2+r)(1+2r)(-1+3r^2+4r^3) + p^2(-2+6r^2+5r^3)\big]\\
&- lp\big[2m(p+r)(p+4p^2+3p^3-2r+8pr+16p^2r-r^2+30pr^2\\
&+ 7p^2r^2+20r^3+32pr^3+38r^4+10pr^4+22r^5+4r^6) - g(2p^3\\
&+ 2p^4-3pr+3p^2r+12p^3r-2r^2-6pr^2+24p^2r^2+4p^3r^2-5r^3\\
&+ 16pr^3+21p^2r^3+30pr^4+6p^2r^4+5r^5+15pr^5+2r^6+2pr^6)\big]\\
&+ (g-2m)p^2r\big[-2m(p+r)(1+3p+3p^2+6r+12pr+13r^2\\
&+ 3pr^2+6r^3+r^4) + g(p^3-r-3pr+3p^2r-7r^2-3pr^2-11r^3\\
&- 3pr^3-7r^4-r^5)\big]\\
\equiv& cp^2r(1+p+2r)^2(p+2r+r^2)^2 + \Psi(l,g,m,p,r).
\end{aligned} \tag{3.52}
$$

Therefore we conclude that an invariant line exists in the direction $y = -rx$ if and only if either $G_1 = 0$, or $G_2 = G_3 = 0$. So we examine two subcases: (i) $G_1 = 0$, (ii) $G_1 \neq 0$ and $G_2 = G_3 = 0$.

1. *Subcase* $G_1 = 0$. We observe that the polynomial G_1 is linear with respect to the parameter g with the coefficient $p(1-r)$ and since $p \neq 0$ we consider two possibilities $1 - r \neq 0$ and $1 - r = 0$.

1.1. *Possibility* $1 - r \neq 0$. Then the equation $G_1 = 0$ yields

$$
g = \frac{2mp(r-1) - l(2+3p+5r+2r^2)}{p(r-1)}. \tag{3.53}
$$

It can be checked directly that this condition together with (3.48), (3.49) and (3.50) lead to system (3.44) which has the invariant lines $\tilde{L}_i = 0$ $i = i, \dots, 6$ with configuration of type $\mathfrak{T} = (3,1,1,1)$, where

$$
\begin{aligned}
\tilde{L}_1 &= p(r-1)x - 3l, \quad \tilde{L}_2 = p(r-1)(x-y) - l(2+r),\\
\tilde{L}_{3,4} &= p^2(r-1)^2(p+r)x^2 - 2p(r-1)\big[l(1+r+r^2) - mp(r-1)\big]x\\
&\quad + \big[cp^2(r-1)^2 - 6l^2(1+r+r^2) + 6lmp(r-1)\big],\\
\tilde{L}_5 &= py + l, \quad \tilde{L}_6 = p(r-1)(rx+y) - l(1+2r).
\end{aligned} \tag{3.54}
$$

To determine if the invariant lines $\tilde{L}_{3,4} = 0$ are real or complex as well as if the invariant line $\tilde{L}_1 = 0$ coincides with one of the lines $\tilde{L}_{3,4} = 0$ we calculate:

$$\begin{aligned}\operatorname{Discrim}[\tilde{L}_{3,4}, x] &= 4p^2(r-1)^2\lambda(c,l,m,p,r),\\ \operatorname{Res}_x(\tilde{L}_1, \tilde{L}_{3,4}) &= p^2(r-1)^2\mu(c,l,m,p,r),\end{aligned}$$

where

$$\begin{aligned}\lambda = &\big[mp(r-1) - l(1+r+r^2)\big]\big[mp(r-1) - l(1+6p+7r+r^2)\big]\\ &- cp^2(r-1)^2(p+r),\\ \mu =\ & cp^2(r-1)^2 + 12lmp(r-1) + 3l^2(3p-4-r-4r^2)\end{aligned} \tag{3.55}$$

We observe that

$$\operatorname{sign}\big(\operatorname{Discrim}[\tilde{L}_{3,4}, x]\big) = \operatorname{sign}(\lambda),$$

i.e. the invariant lines $\tilde{L}_{3,4} = 0$ are real (respectively complex; coinciding) if $\lambda > 0$ (respectively $\lambda < 0$; $\lambda = 0$). And the invariant line $\tilde{L}_1 = 0$ coincides with one of the lines $\tilde{L}_{3,4} = 0$ if and only if $\mu = 0$.

On the other hand we observe that the equation $\lambda = 0$ is linear with respect to the parameter c with the coefficient $p^2(r-1)^2(p+r)$ and since $p(r-1) \neq 0$ we examine two cases: $p + r \neq 0$ and $p + r = 0$.

1.1.1. *Case* $p + r \neq 0$. In what follows we examine the possibilities provided by the polynomials λ and μ.

1.1.1.1. *Subcase* $\lambda > 0$. Then we use a new parameter u setting $\lambda = u^2$ we obtain

$$c = \frac{\big[mp(r-1) - l(1+r+r^2)\big]\big[mp(r-1) - l(1+6p+7r+r^2)\big] - u^2}{p^2(r-1)^2(p+r)}. \tag{3.56}$$

This leads to the system

$$\begin{aligned}\dot{x} =&(p+r)\Big[x - \frac{3l}{p(r-1)}\Big]\Big[x + \frac{mp(r-1) - l(1+r+r^2) + u}{p(r-1)(p+r)}\Big]\\ &\times\Big[x + \frac{mp(r-1) - l(1+r+r^2) - u}{p(r-1)(p+r)}\Big],\\ \dot{y} =&(y + l/p)\Big[\frac{l^2(r^2-1)^2 - 2lmp(r-1)(r^2+r+1) - u^2}{p^2(r-1)^2(p+r)} + \frac{m^2}{p+r}\\ &- \frac{l^2(r+2)(2r+1)}{p(r-1)^2(p+r)} + \frac{l+2mp-lr}{p}x - \frac{l}{p}y + px^2 + y^2 + (r-1)xy\Big]\end{aligned} \tag{3.57}$$

which has six real invariant lines.

On the other hand for the value of c given in (3.56) we calculate

$$\mu = \frac{1}{p+r}(\gamma^2 - u^2), \quad \gamma = mp(r-1) + l\big[3p - (r-1)^2\big] \tag{3.58}$$

and since the condition $\mu = 0$ leads to the coincidence of two invariant lines of the triplet, we examine two possibilities: $\mu \neq 0$ and $\mu = 0$.

1.1.1.1.1. *Possibility* $\mu \neq 0$. Then $(\gamma - u)(\gamma + u) \neq 0$ and via the transformation

$$x_1 = \alpha x - \frac{3l(p+r)}{\gamma - u}, \quad y_1 = \alpha y + \frac{l(r-1)(p+r)}{\gamma - u},$$

$$t_1 = \frac{(\gamma-u)^2}{p^2(r-1)^2(p+r)^2}t, \quad \alpha = \frac{p(-1+r)(p+r)}{\gamma-u},$$

system (3.57) could be brought to the following family of systems (we keep the old notation for the variables):

$$\begin{aligned} \dot{x} =& (p+r)x(x-1)(x-v), \\ \dot{y} =& y\big[(p+r)v-(p+r)(1+v)x+px^2+(r-1)xy+y^2\big], \end{aligned} \tag{3.59}$$

where $v=(\gamma+u)/(\gamma-u)\neq 0$. Moreover $v\neq 1$ because $v-1=2u/(\gamma-u)\neq 0$. So, considering the conditions on the parameters p and r, which we impose in order to obtain the above systems, we conclude that for systems (3.59) the following conditions must hold:

$$p(1+p+2r)(p+2r+r^2)r(r^2-1)v(v-1)\neq 0. \tag{3.60}$$

We detect that systems (3.59) have six distinct invariant affine straight lines

$$L_1: x=0, \quad L_2: x=1, \quad L_3: x=v, \quad L_4: y=0, \quad L_5: y=x, \quad L_6: rx+y=0$$

and the following nine finite singularities:

$$\begin{gathered} M_1(0,0), \quad M_{2,3}\big(0,\pm\sqrt{-(p+r)v}\big), \quad M_4(1,0), \quad M_5(1,1), \quad M_6(1,-r), \\ M_7(v,0), \quad M_8(v,-rv), \quad M_9(v,v). \end{gathered} \tag{3.61}$$

We observe that the singular points M_2 and M_3 could be real (if $v(p+r)<0$) or complex (if $v(p+r)>0$), but they could not coincide due to $v(p+r)\neq 0$.

Assume first $v(p+r)<0$. Then we use a new parameter w setting $v(p+r)=-w^2<0$. We obtain $p=-r-w^2/v$ and then the singular points M_2 and M_3 from (3.61) become $M_{2,3}(0,\pm w)$. In this case the condition (3.60) becomes

$$r(1+r)vw(v-1)(v+rv-w^2)(rv+r^2v-w^2)(rv+w^2)\neq 0$$

and we observe that due to $r(1+r)v(v-1)\neq 0$ all the finite singularities are distinct.

We observe that all the singularities (3.61) are located at the intersection of the invariant lines, except for $M_{2,3}(0,\pm w)$ which lie on the line $x=0$ and are symmetric with respect to the origin of coordinates. Since $w\neq 0$ we deduce that fixing the position of all invariant straight lines and moving only the singularities $M_{2,3}(0,\pm w)$ we could not obtain new configurations. So the distinct configurations depend only on the position of the invariant lines.

We remark that only two lines are not fixed: $L_3: x=v$ and $L_6: y=-rx$. Moreover four of them (namely L_1,L_4, L_5 and L_6) intersect at the same point $M_1(0,0)$. Since this point lies on the invariant line L_1, considering the triplet of parallel invariant lines (L_1, L_2 and L_3) we deduce that we could get different configurations if L_3 is located on the right of L_1 (if $v>0$) or on the left of it (if $v<0$).

In the case $v(p+r)>0$ we obtain complex singularities $M_{2,3}(0,\pm\sqrt{-(p+r)v})$ whose location does not change the configuration, i.e. the arguments we underlined above are valid in this case too. So we arrive at the following result.

Lemma 3.10. *The family of systems* (3.59) *with the conditions* (3.60) *possesses the following configurations of invariant lines when the corresponding conditions indicated below are satisfied (examples are given in the last column):*

$$Config.\ 7.41 \;\Leftrightarrow\; v(p+r)<0, v<0, \quad (p=5/2, r=-2, v=-2);$$

$$\begin{aligned}
&Config.\ 7.42 \;\Leftrightarrow\; v(p+r)<0, v>0, \quad (p=3/2, r=-2, v=2);\\
&Config.\ 7.43 \;\Leftrightarrow\; v(p+r)>0, v<0, \quad (p=3/2, r=-2, v=-2);\\
&Config.\ 7.44 \;\Leftrightarrow\; v(p+r)>0, v>0, \quad (p=5/2, r=-2, v=2);
\end{aligned}$$

1.1.1.1.2. *Possibility* $\mu = 0$. Then $(\gamma-u)(\gamma+u)=0$ and we may assume $\gamma+u=0$ because of the change $u \to -u$. So setting $u=-\gamma \neq 0$ in system (3.57) and applying the transformation

$$\begin{gathered}
x_1 = \alpha x + \frac{3l(p+r)}{2\gamma}, \quad y_1 = \alpha y - \frac{l(r-1)(p+r)}{2\gamma},\\
t_1 = \frac{4\gamma^2}{p^2(r-1)^2(p+r)^2}t, \quad \alpha = -\frac{p(r-1)(p+r)}{2\gamma}
\end{gathered}$$

we obtain the 2-parameter family of systems

$$\dot{x} = (p+r)x^2(x-1), \quad \dot{y} = y\big[-(p+r)x + px^2 + (r-1)xy + y^2\big]. \tag{3.62}$$

We observe that this family of systems is a subfamily of (3.59) defined by the condition $v=0$. So considering invariant lines of the system (3.59) for $v=0$ we obtain

$$L_{1,3}: x=0, \quad L_2: x=1, \quad L_4: y=0, \quad L_5: y=x, \quad L_6: rx+y=0.$$

Moreover examining the singularities (3.61) we observe that 6 of them coalesced. More exactly the singular points $M_{2,3}$, M_7, M_8 and M_9 coalesced with $M_1(0,0)$ and we obtain a singularity of multiplicity six. As a result we only obtain one configuration given by Config. 7.45 (providing that the condition $p+r\neq 0$ is satisfied).

1.1.1.2. *Subcase* $\lambda<0$. This means that two invariant lines in the triplet in the direction $x=0$ are complex. We set $\lambda=-u^2<0$ and we obtain

$$c = \frac{1}{p^2(r-1)^2(p+r)}\Big\{\big[mp(r-1)-l(1+r+r^2)\big]\big[mp(r-1)-l(1+6p+7r+r^2)\big]+u^2\Big\}.$$

This leads to the following system

$$\begin{aligned}
\dot{x} =& -\frac{3l+p(1-r)x}{p^3(r-1)^3(p+r)}\Big[\big[p(r-1)(p+r)x-3l(p+r)+\gamma\big]^2+u^2\Big],\\
\dot{y} =&(y+l/p)\Big[\frac{l^2(r^2-1)^2-2lmp(r-1)(r^2+r+1)+u^2}{p^2(r-1)^2(p+r)} + \frac{m^2}{p+r}\\
&-\frac{l^2(r+2)(2r+1)}{p(r-1)^2(p+r)} + \frac{l+2mp-lr}{p}x - \frac{l}{p}y + px^2 + y^2 + (r-1)xy\Big]
\end{aligned} \tag{3.63}$$

and we consider two possibilities: $\gamma\neq 0$ and $\gamma=0$ (γ is given in (3.58)).

1.1.1.2.1. *Possibility* $\gamma \neq 0$. Then via the transformation

$$\begin{gathered}
x_1 = \alpha x - \frac{3l(p+r)}{\gamma}, \quad y_1 = \alpha y - \frac{l(r-1)(p+r)}{\gamma},\\
t_1 = \frac{\gamma^2}{p^2(r-1)^2(p+r)^2}t, \quad \alpha = \frac{p(r-1)(p+r)}{\gamma}
\end{gathered}$$

system (3.63) could be brought to the following family of systems (we keep the old notations for the variables):

$$\begin{aligned}\dot{x} =& (p+r)x\big[(x+1)^2+v^2\big],\\ \dot{y} =& y\big[(p+r)(1+v^2)+2(p+r)x+px^2+(r-1)xy+y^2\big],\end{aligned} \tag{3.64}$$

where $v=u/\gamma\neq 0$. So, considering the conditions on the parameters p and r which we impose in order to obtain the above system, we conclude that for the system (3.59) the following conditions must hold:

$$p(1+p+2r)(p+2r+r^2)r(r+1)(p+r)v\neq 0. \tag{3.65}$$

We detect that system (3.64) has six distinct invariant affine straight lines

$$L_1: x=0,\quad L_{2,3}: x=-1\pm iv,\quad L_4: y=0,\quad L_5: y=x,\quad L_6: rx+y=0$$

and the following nine finite singularities:

$$\begin{aligned}&M_1(0,0),\quad M_{2,3}(0,\pm\sqrt{-(p+r)(1+v^2)}),\quad M_{4,5}(-1\pm iv,0),\\ &M_{6,7}(-1\pm iv,-1\pm iv),\quad M_{8,9}\big(-1\pm iv,(1\mp i)r\big).\end{aligned} \tag{3.66}$$

We observe that the singular points M_2 and M_3 could be real (if $p+r<0$) or complex (if $p+r>0$), but they could not coincide due to $p+r\neq 0$.

Assume first $p+r<0$. Then we may use a new parameter w setting $p+r=-w^2<0$. So we have $p=-r-w^2$ and then the above singular points M_2 and M_3 become $M_{2,3}(0,\pm\sqrt{1+v^2}\ w)$.

We observe that all the singularities (3.66) are located at the intersections of the invariant lines, except for $M_{2,3}(0,\pm\sqrt{1+v^2}\ w)$ which lie on the line $x=0$ and are symmetric with respect to the origin of coordinates. Since $w\neq 0$ we deduce that fixing the position of all the invariant straight lines and moving only the singularities $M_{2,3}(0,\pm\sqrt{1+v^2}\ w)$ we could not obtain new configurations.

On the other hand we remark that only three lines are not fixed: the complex lines $L_{2,3}: x=-1\pm iv$ and $L_6: y=-rx$. Moreover four of them (namely L_1,L_4, L_5 and L_6) intersect at the same point $M_1(0,0)$. In addition we point out that according to Notation 1.11, in the triplet of parallel invariant lines L_1 (real) and $L_{2,3}$ (complex) we place both complex invariant lines on one side of the real line.

We also observe that we have two geometrically distinct positions of the invariant line $L_6: y=-rx$: for $r<0$ and for $r>0$, however this does not lead to distinct configurations.

When $p+r>0$ we obtain complex singularities $M_{2,3}(0,\pm\sqrt{-(p+r)(1+v^2)}\ )$ whose location does not change the configuration, i.e. the arguments mentioned above are also valid in this case. So we arrive at the following result.

Lemma 3.11. *The family of systems* (3.64) *with the conditions* (3.65) *possesses the following configurations of invariant lines when the corresponding conditions indicated below are satisfied:*

$$\begin{aligned}&Config.\ 7.46\ \Leftrightarrow\ p+r<0,\quad (p=1,r=-2,v=1);\\ &Config.\ 7.47\ \Leftrightarrow\ p+r>0,\quad (p=3,r=-2,v=1);\end{aligned}$$

1.1.1.2.2. *Possibility* $\gamma = 0$. Taking into account (3.58) we obtain $m = -l(3p-1+2r-r^2)/(p(r-1))$ and then system (3.63) becomes the system

$$\begin{aligned}
\dot{x} &= -\frac{3l+p(1-r)x}{p^3(r-1)^3(p+r)}\Big[\big[p(r-1)(p+r)x-3l(p+r)\big]^2+u^2\Big],\\
\dot{y} &= (y+l/p)\Big[\frac{l^2(p+r)(9p-2+4r-2r^2)+u^2}{p^2(r-1)^2(p+r)}+\frac{l\big[(r-1)^2-6p\big]}{p(r-1)}x\\
&\quad -\frac{l}{p}y+px^2+y^2+(r-1)xy\Big].
\end{aligned} \tag{3.67}$$

Since $up(p+r)(r-1)\neq 0$ we can apply the transformation

$$\begin{gathered}
x_1 = \alpha x - \frac{3l(p+r)}{u}, \quad y_1 = \alpha y + \frac{l(r-1)(p+r)}{u},\\
t_1 = \frac{u^2}{p^2(r-1)^2(p+r)^2}t, \quad \alpha = \frac{p(r-1)(p+r)}{u}
\end{gathered}$$

which brings systems (3.67) to the following family of systems (we keep the old notations for the variables):

$$\dot{x} = (p+r)x(x^2+1), \quad \dot{y} = y\big[p+r+px^2+(r-1)xy+y^2\big]. \tag{3.68}$$

These systems possess the invariant lines

$$L_1: x=0, \quad L_{2,3}: x=\pm i, \quad L_4: y=0, \quad L_5: y=x, \quad L_6: rx+y=0$$

and the following nine finite singularities:

$$M_1(0,0), \quad M_{2,3}(0,\pm\sqrt{-(p+r)}\ ), \quad M_{4,5}(\pm i,0), \quad M_{6,7}(\pm i,\pm i), \quad M_{8,9}(\pm i,\mp ir).$$

We observe that all invariant lines are fixed except for the real line $L_6: rx+y=0$. In addition we point out that according to Notation 1.11 in the triplet of parallel invariant lines L_1 (real) and $L_{2,3}$ (complex) we place the real line between the two complex lines.

As in the previous case, the singularities $M_{2,3}(0,\pm\sqrt{-(p+r)}\ )$ are real if $p+r<0$ and they are complex if $p+r>0$. So applying the same arguments as in the previous case, we arrive at the next result.

Lemma 3.12. *The family of systems* (3.68) *with the condition* (3.65) *(removing the parameter v which is not relevant here) possesses the following configurations of invariant lines when the corresponding conditions indicated below are satisfied:*

$$\begin{aligned}
&Config.\ 7.48 \ \Leftrightarrow\ p+r<0, \quad (p=1, r=-2);\\
&Config.\ 7.49 \ \Leftrightarrow\ p+r>0, \quad (p=3, r=-2).
\end{aligned}$$

1.1.1.3. *Subcase* $\lambda = 0$. Considering (3.55) and solving the equation $\lambda = 0$ with respect to the parameter c, it is clear that we obtain (3.56) for $u=0$. This leads to systems (3.57) with $u=0$, which we denote by $(3.57)_{\{u=0\}}$.

We observe that in this case we obtain $\mu = \gamma^2/(p+r)$ and we again consider two subcases: $\mu\neq 0$ and $\mu = 0$.

1.1.1.3.1. *Possibility* $\mu\neq 0$. Then $\gamma\neq 0$ and via the transformation

$$\begin{gathered}
x_1 = \alpha x - \frac{3l(p+r)-\gamma}{\gamma}, \quad y_1 = \alpha y + \frac{l(r-1)(p+r)}{\gamma},\\
t_1 = \frac{\gamma^2}{p^2(r-1)^2(p+r)^2}t, \quad \alpha = \frac{p(r-1)(p+r)}{\gamma}
\end{gathered}$$

system $(3.57)_{\{u=0\}}$ can be brought to the following family of systems (we keep the old notations for the variables):

$$\begin{aligned} \dot{x} =&(p+r)x^2(x-1), \\ \dot{y} =&y\big[-r+2rx+px^2+(1-r)y+(r-1)xy+y^2\big], \end{aligned} \tag{3.69}$$

for which the condition $p(1+p+2r)(p+2r+r^2)r(r+1)\neq 0$ holds. These systems possess the following five distinct invariant affine straight lines:

$$L_{1,2}: x=0, \quad L_3: x=1, \quad L_4: y=0, \quad L_5: y=x-1, \quad L_6: rx+y=r$$

and six finite singularities

$$M_1(0,0), \quad M_2(0,-1), \quad M_3(0,r), \quad M_4(1,0), \quad M_{5,6}\big(1,\pm\sqrt{-(p+r)}\big). \tag{3.70}$$

We detect that three of the above singularities are double. Indeed considering Lemma 2.7 for system (3.69) we calculate

$$\mu_9=\mu_8=0, \quad \mu_7=r^2(p+r)^3x^4y(px^2+rx^2+y^2)\neq 0$$

i.e. by this lemma $M_1(0,0)$ is a double singular point. Moving M_2 (respectively M_3) to the origin of coordinates we obtain for M_2:

$$\mu_9=\mu_8=0, \quad \mu_7=-(1+r)^2(p+r)^3x^4(x-y)(x^2+px^2+rx^2-2xy+y^2)\neq 0;$$

for M_3:

$$\mu_9=\mu_8=0, \quad \mu_7=r^2(1+r)^2(p+r)^3x^4(rx+y)(px^2+rx^2+r^2x^2+2rxy+y^2)\neq 0.$$

So by Lemma 2.7 each one of these singularities is also double.

We observe that the invariant line $x=0$ is double and only the line L_6 is not fixed (it depends on the parameter r). Moreover four of the invariant lines (namely L_3, L_4, L_5 and L_6) intersect at the same singular point $M_4(1,0)$.

We notice that the singular points M_5 and M_6 could be real (if $p+r<0$) or complex (if $p+r>0$), but they could not coincide due to $p+r\neq 0$.

Assume first $p+r<0$. As before, we may use a new parameter w setting $p+r=-w^2<0$. So we have $p=-r-w^2$ and then the above singular points M_5 and M_6 become $M_{5,6}(1,\pm w)$.

We observe that all the singularities (3.70) are located at the intersections of the invariant lines, except for $M_{5,6}(1,\pm w)$ which lie on the line $x=1$ and are symmetric with to the singular point $M_4(1,0)$. Since $w\neq 0$ we deduce that fixing the position of all the invariant straight lines and moving only the singularities $M_{5,6}$ we could not obtain new configurations. We also observe that we have two geometrically distinct positions of the invariant line $L_6: y=-rx$: for $r<0$ and for $r>0$, however this does not lead to distinct configurations.

In the case $p+r>0$ we obtain complex singularities $M_{5,6}\big(1,\pm\sqrt{-(p+r)}\big)$ whose location does not change the configuration, i.e. the arguments mentioned above, are also valid in this case. So we arrive at the next result.

Lemma 3.13. *The family of systems* (3.69) *with the condition* (3.65) *possesses the following configurations of invariant lines when the corresponding conditions indicated below are satisfied:*

$$\begin{aligned} &Config.\ 7.50 \ \Leftrightarrow\ p+r<0, \quad (p=1, r=-2);\\ &Config.\ 7.51 \ \Leftrightarrow\ p+r>0, \quad (p=3, r=-2). \end{aligned}$$

1.1.1.3.2. *Possibility* $\mu = 0$. So we have $\lambda = \mu = 0$ and this leads to the existence of a triple invariant line among (3.54). This leads to the system

$$\begin{aligned}\dot{x} =&(p+r)\Big[x - \frac{3l}{p(r-1)}\Big]^3,\\ \dot{y} =&(y+l/p)\Big[\frac{l^2(9p-2+4r-2r^2)}{p^2(r-1)^2} - \frac{l(6p-1+2r-r^2)}{p(r-1)}x - \frac{l}{p}y\\ &+px^2+y^2+(r-1)xy\Big]\end{aligned}$$

which via the translation $x_1 = x - \frac{3l}{p(r-1)}$, $y_1 = y + l/p$ can be brought to the cubic homogeneous system

$$\dot{x} = (p+r)x^3, \quad \dot{y} = y\big[px^2 + (r-1)xy + y^2\big]. \tag{3.71}$$

This system has invariant affine lines of total multiplicity six (one triple and three simple):

$$L_{1,2,3} : x = 0, \quad L_4 : y = 0, \quad L_5 : y = x, \quad L_6 : y = -rx$$

and only the last invariant line is not fixed. Since the above homogeneous system has a single singularity $M_1(0,0)$ which is of multiplicity 9, we obtain the unique configuration given by Config. 7.52.

1.1.2. *Case* $p + r = 0$. Hence we have $p = -r$ and this implies

$$\lambda = \big[m(r-1)r + l(1+r+r^2)\big]^2 = \gamma^2, \quad \mu = c(r-1)^2r^2 + 12l\gamma. \tag{3.72}$$

So we observe that $\lambda \geq 0$ and we consider two subcases: $\lambda \neq 0$ and $\lambda = 0$.

1.1.2.1. *Subcase* $\lambda \neq 0$. In this case considering the conditions (3.48), (3.49), (3.50) and (3.53) we arrive at the family of systems

$$\begin{aligned}\dot{x} =&\frac{3l - rx + r^2x}{r^3(r-1)^3}\big[c(r-1)^2r^2 + 6l\gamma - 2(r-1)r\gamma x\big],\\ \dot{y} =&(y - l/r)\Big[\frac{c(r-1)^2r^2 - l^2(2+5r+2r^2) + 6l\gamma}{r^2(r-1)^2} + \frac{lr - l + 2mr}{r}x\\ &+\frac{l}{r}y - rx^2 + (r-1)xy + y^2\Big]\end{aligned} \tag{3.73}$$

and we examine two possibilities: $\mu \neq 0$ and $\mu = 0$.

1.1.2.1.1. *Possibility* $\mu \neq 0$. Since $\gamma \neq 0$ (due to $\lambda \neq 0$) we may apply the transformation

$$\begin{gathered}x_1 = \alpha x + \frac{6l\gamma}{\mu}, \quad y_1 = \alpha y - \frac{2l(r-1)\gamma}{\mu},\\ t_1 = \frac{\mu^2}{4(-1+r)^2r^2\gamma^2}t, \quad \alpha = \frac{2r(r-1)\gamma}{\mu}\end{gathered}$$

bringing systems (3.73) to the 2-parameter family of systems

$$\dot{x} = ax(x-1), \quad \dot{y} = y\big[a(x-1) - rx^2 + (r-1)xy + y^2\big], \tag{3.74}$$

where $a = -4\gamma^2/\mu \neq 0$. For these system we calculate (see the definition of the polynomial $H(X,Y,Z)$ on the page 13, Notation 2.4):

$$H(X,Y,Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = aX(X-Y)Y(rX+Y)(X-Z)Z.$$

So, by Lemma 2.6 the invariant line $Z=0$ at infinity is of multiplicity two and the invariant affine lines are

$$L_1 : x = 0, \quad L_2 : x = 1, \quad L_3 : y = 0, \quad L_4 : y = x, \quad L_5 : y = -rx.$$

We detect that the above system has six finite singularities

$$M_1(0,0),\ M_{2,3}(0,\pm\sqrt{a}), \quad M_4(1,0), \quad M_5(1,1), \quad M_6(1,-r),$$

which are distinct due to $a \neq 0$. We observe that only the line L_5 is not fixed (it depends on the parameter r). Moreover four of the invariant lines (namely L_1, L_3, L_4 and L_5) intersect at the same singular point $M_4(1,0)$.

On the other hand the singular points M_2 and M_3 could be real (if $a>0$) or complex (if $a<0$), but they could not coincide due to $a\neq 0$. These two singularities are located on the invariant line $x=0$ and are symmetric with respect to the origin of coordinates.

So after the examination of the position of the invariant lines we arrive at the next result.

Lemma 3.14. *The family of systems* (3.74) *with the condition* $ar(r-1)(r+1)\neq 0$ *possesses the following configurations of invariant lines when the corresponding conditions indicated below are satisfied:*

$$\begin{gathered} Config.\ 7.53 \ \Leftrightarrow\ a>0, \quad (a=1, r=-2);\\ Config.\ 7.54 \ \Leftrightarrow\ a<0, \quad (a=-1, r=-2). \end{gathered}$$

1.1.2.1.2. *Possibility* $\mu = 0$. Considering (3.72) we obtain

$$\mu = c(r-1)^2r^2 + 12l\gamma = 0 \ \Rightarrow\ c = -12l\gamma/((r-1)^2r^2)$$

and then systems (3.73) with the above value for the parameter c, via the translation

$$x_1 = x + 3l/(r(1-r)), \quad y_1 = u - l/r$$

will be brought to the family of systems:

$$\dot{x} = ax^2, \quad \dot{y} = y\big[ax - rx^2 + (r-1)xy + y^2\big], \tag{3.75}$$

where $a = 2\gamma/(r(1-r)) \neq 0$. For this system we calculate

$$H(X,Y,Z) = -aX^2(X-Y)Y(rX+Y)Z$$

and therefore by Lemma 2.6 the invariant line at infinity, $Z=0$, is of multiplicity two. This system has four distinct invariant affine lines

$$L_{1,2} : x = 0, \quad L_3 : y = 0, \quad L_4 : y = x, \quad L_5 : y = -rx$$

and the unique finite singularity $M_1(0,0)$. Considering Lemma 2.7 for the above system we calculate

$$\begin{gathered} \mu_0 = \mu_1 = \mu_2 = 0, \quad \mu_3 = a^3(x-y)y(rx+y) \neq 0,\\ \mu_4 = \mu_5 = \mu_6 = \mu_7 = \mu_8 = \mu_9 = 0 \end{gathered}$$

and hence by this lemma, three finite singularities coalesced with singularities at infinity (together with one invariant line) and the remaining singularity $M_1(0,0)$ is of multiplicity six.

Since the line $x=0$ is double, it is not too hard to detect that systems (3.75) have the unique configuration of invariant lines given by Config. 7.55.

1.1.2.2. *Subcase* $\lambda = 0$. Considering (3.72) we obtain $\gamma = 0$. So from $(r-1) \neq 0$ we obtain

$$\gamma = m(1-r)r - l(1+r+r^2) = 0 \ \Rightarrow \ m = l(1+r+r^2)/(r(1-r))$$

and then $\mu = c(r-1)^2 r^2 \neq 0$. In this case the right-hand part of the first equation of systems (3.73) becomes linear in x and applying the translation

$$x_1 = x + 3l/(r(1-r)), \quad y_1 = u - l/r$$

we obtain the family of systems:

$$\dot{x} = cx, \quad \dot{y} = y\big[c - rx^2 + (r-1)xy + y^2\big], \tag{3.76}$$

with $cr(r^2-1) \neq 0$. For these systems we calculate

$$H(X,Y,Z) = cX(X-Y)Y(rX+Y)Z^2$$

and therefore by Lemma 2.6 the invariant line at infinity, $Z = 0$, is of multiplicity three. These systems possess the invariant affine lines

$$L_1 : x = 0, \quad L_2 : y = 0, \quad L_3 : y = x, \quad L_4 : y = -rx$$

and three finite singularities

$$M_1(0,0), \quad M_{2,3}(0, \pm\sqrt{-c}).$$

So examining in the same manner as further above the position of the invariant lines as well as the position of the singularities $M_{2,3}(0, \pm\sqrt{-c}\,)$ (which could be real or complex) we obtain the following two distinct configurations: Config. 7.56 if $c < 0$ and Config. 7.57 if $c > 0$.

1.2. *Possibility* $r = 1$. Then considering (3.51) we obtain $G_1 = -3l(3+p) = 0$. On the other hand the condition $1 + p + 2r \neq 0$ (see the paragraph 3.3.2) for $r = 1$ gives $p + 3 \neq 0$. Therefore we obtain $l = 0$ and considering (3.48), (3.49) and (3.50) we have the conditions:

$$\begin{gathered}
k = d = h = e = b = l = 0,\\
f = \frac{c(3+p)^2 - (g-2m)(3g+2mp)}{(3+p)^2},\\
a = \frac{c(3+p)^2 - 2(g-2m)(g+m+mp)}{(3+p)^3}.
\end{gathered}$$

In this case the cubic system (3.44) takes the form

$$\begin{aligned}
\dot{x} =& \big[x + \frac{g-2m}{3+p}\big]\Big[c - \frac{2(g-2m)(g+m+mp)}{(3+p)^2} - \frac{2(g+m+mp)x}{3+p}x + (1+p)x^2\Big],\\
\dot{y} =& y\Big[\frac{(2m-g)(3g+2mp)}{(3+p)^2} + c + 2mx + px^2 + y^2\Big].
\end{aligned} \tag{3.77}$$

These systems possess the invariant lines

$$\begin{gathered}
\tilde{L}_1 = g - 2m + (3+p)x, \ \tilde{L}_{2,3} = c(3+p)^2 - 2(g-2m)(g+m+mp)\\
+ 2(3+p)(g+m+mp)x + (1+p)(3+p)^2x^2,\\
\tilde{L}_4 = y, \quad \tilde{L}_5 = g - 2m + (3+p)(x-y), \quad \tilde{L}_6 = g - 2m + (3+p)(x+y).,
\end{gathered}$$

To determine if the invariant lines $\tilde{L}_{2,3} = 0$ are real or complex as well as if the invariant line $\tilde{L}_1 = 0$ coincides with one of the lines $\tilde{L}_{2,3} = 0$ we calculate:

$$\mathrm{Discrim}[\tilde{L}_{2,3}, x] = 4(3+p)^2\lambda'(c,g,m,p), \quad \mathrm{Res}_x(\tilde{L}_1, \tilde{L}_{2,3}) = (3+p)^2\mu'(c,g,m,p),$$

where

$$\begin{gathered} \lambda' = (3g-3m+2gp-3mp)(g+m+mp) - c(1+p)(3+p)^2, \\ \mu' = c(3+p)^2 + (g-2m)(-3g-6m+gp-6mp). \end{gathered} \tag{3.78}$$

We observe that

$$\mathrm{sign}\big(\mathrm{Discrim}[\tilde{L}_{2,3}, x]\big) = \mathrm{sign}(\lambda'),$$

i.e. the invariant lines $\tilde{L}_{2,3} = 0$ are real (respectively complex; coinciding) if $\lambda' > 0$ (respectively $\lambda' < 0$; $\lambda' = 0$). On the other hand the invariant line $\tilde{L}_1 = 0$ coincides with one of the lines $\tilde{L}_{2,3} = 0$ if and only if $\mu' = 0$.

We observe that the equation $\lambda' = 0$ is linear with respect to the parameter c with the coefficient $(1+p)(3+p)^2$ and since $(3+p) \neq 0$ we examine two cases: $p+1 \neq 0$ and $p = -1$.

1.2.1. *Case* $p+1 \neq 0$. In what follows we examine the possibilities provided by the polynomials λ' and μ'.

1.2.1.1. *Subcase* $\lambda' > 0$. Then we use a new parameter u setting $\lambda' = u^2$ and we obtain

$$c = \frac{1}{(1+p)(3+p)^2}\big[(3g-3m+2gp-3mp)(g+m+mp) - u^2\big]. \tag{3.79}$$

This leads to the system

$$\begin{aligned} \dot{x} =& \frac{1}{1+p}\Big[x + \frac{g-2m}{3+p}\Big]\Big[\frac{g+m+mp-u}{3+p} + (1+p)x\Big] \\ &\times \Big[\frac{g+m+mp+u}{3+p} + (1+p)x\Big], \\ \dot{y} =& y\Big[\frac{m(1+p)(6g-3m+mp) - g^2p - u^2}{(1+p)(3+p)^2} + 2mx + px^2 + y^2\Big]. \end{aligned} \tag{3.80}$$

On the other hand for the value of c given in (3.79) we calculate

$$\mu' = \frac{1}{1+p}(\gamma'^2 - u^2), \quad \gamma' = gp - 3m(1+p) \tag{3.81}$$

and since the condition $\mu' = 0$ leads to the merging of two invariant lines of the triplet, we examine two subcases: $\mu' \neq 0$ and $\mu' = 0$.

1.2.1.1.1. *Possibility* $\mu' \neq 0$. Then $(\gamma' - u)(\gamma' + u) \neq 0$ and via the transformation

$$\begin{gathered} x_1 = \alpha x + \frac{(g-2m)(1+p)}{\gamma' + u}, \quad y_1 = \alpha y, \\ t_1 = \frac{(\gamma'+u)^2}{(1+p)^2(3+p)^2}t, \quad \alpha = \frac{(1+p)(3+p)}{\gamma' + u} \end{gathered}$$

system (3.57) can be brought to the following family of systems (we keep the old notations for the variables):

$$\dot{x} = (p+1)x(x-1)(x-v), \quad \dot{y} = y\big[(p+1)v - (p+1)(1+v)x + px^2 + y^2\big].$$

We observe that this family of systems is a subfamily of the family (3.59), defined by the condition $r = 1$. According to Lemma 3.10 for systems (3.59) the condition (3.60) has to be satisfied, including $r \neq 1$. However we detect that for this family the hypersurface $r = 1$ is not part of the bifurcation set of configurations as it can be seen from Lemma 3.10. So no new configurations can be obtained.

1.2.1.1.2. *Possibility* $\mu' = 0$. Then $(\gamma' - u)(\gamma' + u) = 0$ and we may assume $\gamma' + u = 0$ because of the change $u \to -u$. So setting in system (3.80) $u = -\gamma' \neq 0$ (since $\lambda' = \gamma'^2 > 0$) and applying the transformation

$$x_1 = \alpha x + \frac{(g-2m)(1+p)}{2\gamma'}, \quad y_1 = \alpha y,$$
$$t_1 = \frac{4\gamma'^2}{(1+p)^2(3+p)^2} t, \quad \alpha = \frac{(1+p)(3+p)}{2\gamma'}$$

we obtain the 1-parameter family of systems:

$$\dot{x} = (p+1)x^2(x-1), \quad \dot{y} = y\big[-(p+1)x + px^2 + y^2\big]. \tag{3.82}$$

We observe that (3.82) is a subfamily of systems (3.62) defined by the condition $r = 1$. So we deduce that we do not have new configurations.

1.2.1.2. *Subcase* $\lambda' < 0$. This means that two invariant lines of the triplet in the direction $x = 0$ are complex. We set $\lambda' = -u^2 < 0$ and we obtain

$$c = \frac{1}{(1+p)(3+p)^2}\big[(3g - 3m + 2gp - 3mp)(g + m + mp) + u^2\big]. \tag{3.83}$$

This leads to the systems

$$\begin{aligned}
\dot{x} =& \frac{1}{(1+p)(3+p)^2}\Big[x + \frac{g-2m}{3+p}\Big] \\
&\times \Big[\big(g + m + mp + 3x + 4px + p^2x\big)^2 + u^2\Big], \\
\dot{y} =& y\Big[\frac{m(1+p)(6g - 3m + mp) - g^2p + u^2}{(1+p)(3+p)^2} + 2mx + px^2 + y^2\Big]
\end{aligned} \tag{3.84}$$

We consider two possibilities: $\gamma' \neq 0$ and $\gamma' = 0$.

1.2.1.2.1. *Possibility* $\gamma' \neq 0$. Then via the transformation

$$x_1 = \alpha x - \frac{(g-2m)(1+p)}{\gamma'}, \quad y_1 = \alpha y,$$
$$t_1 = \frac{\gamma'^2}{(1+p)^2(3+p)^2} t, \quad \alpha = -\frac{(1+p)(3+p)}{\gamma'}$$

systems (3.84) can be brought to the following family of systems (we keep the old notations for the variables):

$$\begin{aligned}
\dot{x} =& (p+1)x\big[(x+1)^2 + v^2\big], \\
\dot{y} =& y\big[(p+1)(1+v^2) + 2(p+1)x + px^2 + y^2\big],
\end{aligned} \tag{3.85}$$

where $v = u/\gamma' \neq 0$. It could be easily observed that (3.85) is a subfamily of systems (3.64) defined by the condition $r = 1$. This family was investigated earlier (see Lemma 3.11) and since for $r = 1$ we do not have bifurcation points for the family (3.64) we deduce that there are no new configurations.

1.2.1.2.2. *Possibility* $\gamma'=0$. Taking into account (3.81) we obtain $m = gp/(3(1+p))$ and then system (3.84) becomes system

$$\begin{aligned}\dot{x} =&\big[(1+p)x+g/3\big]\Big[\Big(x+\frac{g}{3(1+p)}\Big)^2+\frac{u^2}{(1+p)^2(3+p)^2}\Big],\\ \dot{y} =&y\Big[\frac{g^2p(3+p)^2+9(1+p)u^2}{9(1+p)^2(3+p)^2}+\frac{2gp}{3(1+p)}x+px^2+y^2\Big].\end{aligned}\tag{3.86}$$

Since $u(1+p)(3+p)\neq 0$ we can apply the transformation

$$x_1=\alpha x-\frac{g(3+p)}{3u},\quad y_1=\alpha y,\quad t_1=\frac{u^2}{(1+p)^2(3+p)^2}t,\quad \alpha=\frac{(1+p)(3+p)}{u}$$

which brings systems (3.86) to the systems

$$\dot{x}=(p+1)x(x^2+1),\quad \dot{y}=y\big[p+1+px^2+y^2\big].$$

We observe that this family of systems is a subfamily of (3.68) defined by the condition $r=1$. So we again do not have new configurations.

1.2.1.3. *Subcase* $\lambda'=0$. Considering (3.78) and solving the equation $\lambda'=0$ with respect to the parameter c, it is clear that we obtain (3.83) for $u=0$. This leads to the system (3.84) with $u=0$, which we denote by $(3.84)_{\{u=0\}}$.

We observe that in this case we obtain $\mu'=\gamma'$ and we again consider two possibilities: $\mu'\neq 0$ and $\mu'=0$.

1.2.1.3.1. *Possibility* $\mu'\neq 0$. Then $\gamma'\neq 0$ and via the transformation

$$x_1=\alpha x-\frac{g+m+mp}{\gamma'},\quad y_1=\alpha y,\quad t_1=\frac{\gamma'^2}{(1+p)^2(3+p)^2}t,\quad \alpha=-\frac{(1+p)(3+p)}{\gamma'}$$

system $(3.84)_{\{u=0\}}$ could be brought to the system

$$\dot{x}=(p+1)x^2(x-1),\quad \dot{y}=y\big[-y+2xy+px^2y+y^3\big].$$

It remains to observe that we obtain a subfamily of (3.69), defined by the condition $r=1$.

1.2.1.3.2. *Possibility* $\mu'=0$. In this case we have $\lambda'=\mu'=\gamma'=0$ (i.e. $m=gp/(3(1+p))$) and we obtain a triple line in the direction $x=0$. Therefore system $(3.84)_{\{u=0\}}$ leads to the system

$$\dot{x}=(1+p)\Big[x+\frac{gp}{3(1+p)}\Big],\quad \dot{y}=y\Big[p\Big(x+\frac{gp}{3(1+p)}\Big)+y^2\Big]$$

and evidently making the translation $x_1=x+\frac{gp}{3(1+p)}$, $y_1=y$ we arrive at the cubic homogeneous systems

$$\dot{x}=(p+1)x^3,\quad \dot{y}=y\big[px^2+y^2\big]$$

which form a subfamily of systems (3.71) for $r=1$.

1.2.2. *Case* $p=-1$. In this case system (3.77) becomes

$$\begin{aligned}\dot{x} =&\big[x+(g-2m)/2\big]\big[c-g(g-2m)/2+gx)\big],\\ \dot{y} =&y\big[c-(g-2m)(3g-2m)/4+2mx-x^2+y^2\big]\end{aligned}\tag{3.87}$$

for which

$$\lambda'=g^2,\quad \mu'=4(c-g^2+2gm).\tag{3.88}$$

So we observe that $\lambda'\geq 0$ and we consider two subcases: $\lambda'\neq 0$ and $\lambda'=0$.

1.2.2.1. *Subcase* $\lambda' \neq 0$. Then $g \neq 0$ and we have to consider two possibilities: $\mu' \neq 0$ and $\mu' = 0$.

1.2.2.1.1. *Possibility* $\mu' \neq 0$. In this case systems (3.87) can be brought via the transformation

$$x_1 = \alpha x + \frac{2g(g-2m)}{\mu'}, \quad y_1 = \alpha y, \quad t_1 = \frac{\mu'^2}{16g^2}t, \quad \alpha = -\frac{4g}{\mu'}$$

to the systems

$$\dot{x} = ax(x-1), \quad \dot{y} = y\big[a(x-1) - x^2 + y^2\big],$$

where $a = -4/\mu'$. We observe that we obtain a subfamily of systems (3.74) defined by $r = 1$.

1.2.2.1.2. *Possibility* $\mu' = 0$. Considering (3.88) we obtain $c = g(g-2m)$ and then system (3.87) takes the form

$$\dot{x} = g(g-2m+2x)^2/4, \quad \dot{y} = y\big[(g^2 - 4m^2 + 8mx - 4x^2 + 4y^2)\big]/4.$$

In this case we apply the transformation $x_1 = x/g + (g-2m)/(2g)$, $y_1 = y/g$, $t_1 = g^2 t$ which brings the above system to the system

$$\dot{x} = x^2, \quad y(x - x^2 + y^2).$$

Evidently this system belongs to the family (3.75) for $a = r = 1$, already studied.

1.2.2.2. *Subcase* $\lambda' = 0$. By (3.88) we obtain $g = 0$ and system (3.87) becomes

$$\dot{x} = c(x-m), \quad y\big[c - (x-m)^2 + y^2\big].$$

Clearly applying the translation $x \to x + m$ we obtain the system

$$\dot{x} = cx, \quad y\big[c - x^2 + y^2\big]$$

which belongs to the family of systems (3.76) for $r = 1$, already studied.

2. *Subcase* $G_1 \neq 0$, $G_2 = G_3 = 0$. Then considering (3.52), since $p(1+p+2r)(p+2r+r^2) \neq 0$, the condition $G_3 = 0$ yields $c = -\Psi(l,g,m,p,r)$. On the other hand we observe that the polynomial G_2 from (3.52) is linear with respect to the parameter m with the coefficient $2p(p+r)w$, where $w = 1 + 3p + 7r + r^2$. Since $p \neq 0$ it remains to consider three possibilities: (a) $(p+r)w \neq 0$, (b) $p + r \neq 0$ and $w = 0$, and (c) $p + r = 0$.

2.1. *Possibility* $(p+r)w \neq 0$. Then the condition $G_2 = 0$ gives

$$m = \frac{1}{2p(p+r)w}\big[l(r-1)(2+r)(p+r)(1+2r) + gp(2p^2 - r + 6pr + 2r^2 - r^3)\big].$$

It can be checked directly that this condition together with (3.48), (3.49), (3.50) and $c = -\Psi(l,g,m,p,r)$ applied to systems (3.44) lead to the family of systems:

$$\begin{aligned}
\dot{x} &= (p+r)\Big[x + \frac{-3l(2+r)(p+r) + gp(1+p+2r)}{p(p+r)w}\Big]\\
&\quad\times\Big[x + \frac{-3l(-1+r)(p+r) + gp(p+3r)}{p(p+r)w}\Big]\\
&\quad\times\ \Big[x + \frac{3l(p+r)(1+2r) + gp(p+2r+r^2)}{p(p+r)w}\Big]\\
&\equiv (p+r)(x+\varphi_1)(x+\varphi_2)(x+\varphi_3),\\
\dot{y} &= -(y+l/p)\Big[\frac{1}{p^2(p+r)^2w^2}\big[(3lg\,p(r-1)(p+r)(2p^2 - r + 3pr - r^2 - r^3)
\end{aligned}$$

$$
\begin{aligned}
&- g^2p^2(p^3+6p^2r-r^2+9pr^2+2r^3-r^4)+l^2(p+r)^2(2+3p+18p^2+r\\
&+21pr-6r^2+3pr^2+r^3+2r^4)\big]-\frac{1}{p(p+r)w}\big[-l(-1+r)(p+r)(-1+3p\\
&+2r-r^2)+gp(2p^2-r+6pr+2r^2-r^3)\big]x+ly/p-px^2+(1-r)xy-y^2\Big].
\end{aligned}
$$

We calculate

$$
\varphi_2-\varphi_1=\frac{gp(r-1)+9l(p+r)}{p(p+r)w},\quad \varphi_3-\varphi_1=\frac{(1+r)\big[gp(r-1)+9l(p+r)\big]}{p(p+r)w}
$$

and hence setting $\delta=gp(r-1)+9l(p+r)$ we deduce that if $\delta=0$ then all the parallel invariant lines in the direction $x=0$ coincide.

On the other hand for the above mentioned values of the parameters c and d we obtain

$$
G_1=-\frac{(1+p+2r)(p+2r+r^2)\delta}{p(p+r)w}
$$

and therefore the condition $G_1\neq 0$ implies $\delta\neq 0$. In this case we can apply the transformation

$$
\begin{gathered}
x_1=\alpha x+\frac{3l(2+r)(p+r)-gp(1+p+2r)}{\delta},\quad y_1=\alpha y-\frac{l(p+r)w)}{\delta},\\
t_1=\frac{\delta^2}{p^2(p+r)^2w^2}t,\quad \alpha=-\frac{p(p+r)w}{\delta}
\end{gathered}
$$

and we arrive at the family of systems

$$
\begin{aligned}
&\dot{x}=(p+r)x(x-1)(x-1-r),\\
&\dot{y}=y\big[(1+r)(p+r)+(r^2-2p-r)x+px^2+(1+p+2r)y\\
&\quad +(r-1)xy+y^2\big],
\end{aligned}
\tag{3.89}
$$

for which the following condition is satisfied

$$
r(r+1)p(1+p+2r)(p+2r+r^2)(p+r)(1+3p+7r+r^2)\neq 0. \tag{3.90}
$$

These systems possess six real invariant affine straight lines

$$
\begin{gathered}
L_1: x=0,\quad L_2: x=1,\quad L_3: x=r+1,\\
L_4: y=0,\quad L_5: y=x-1-r,\quad L_6: rx+y=0
\end{gathered}
$$

which are all distinct because $r(r+1)\neq 0$. The above systems have the following nine real finite singularities:

$$
\begin{gathered}
M_1(0,0),\quad M_2(0,-1-r),\quad M_3(0,-p-r),\quad M_4(1,0),\quad M_5(1+r,0),\\
M_6(1,-r),\quad M_7(1,-p-2r),\quad M_8(1+r,-p-r),\quad M_9\big(1+r,-r(1+r)\big).
\end{gathered}
$$

We observe that because of the condition $r(r+1)(p+r)\neq 0$ all these singularities are distinct except for the case of the singularity $M_7(1,-p-2r)$ which coalesces with the singularity M_4 if $p+2r=0$.

It is easy to determine that 6 of these singularities are located at the intersections of the above invariant lines, more precisely these are the singular points M_i for $i\in\{1,2,4,5,6,9\}$. The singular point M_3 (respectively M_7; M_8) is located on the invariant line L_1 (respectively L_2; L_3). Moreover we have three singular points each one located at the intersection of three invariant lines: L_1, L_2 and L_6 intersect at the point M_1; L_3, L_4 and L_5 intersect at the point M_5; and L_2, L_5 and L_6 intersect at the point M_6.

To determine all possible configurations for system (3.89) we have to examine the positions of the invariant lines as well as of the singularities M_6, M_8 and M_9 depending on the parameters r and u.

Let us first examine the position of the invariant lines. We observe that three of the lines are fixed, namely L_1, L_2 and L_4. Other three invariant lines depend on the parameter r. More exactly the positions of L_1 and L_5 depend on the sign of the expression $r+1$, whereas the position of the invariant line L_6 depends on the sign of the parameter r. So it is clear that we have to examine three cases: (i) $r<-1$; (ii) $-1<r<0$ and (iii) $r>0$.

Next we consider the position of the finite singularity $M_3(0,y_3)$ with $y_3=-(p+r)$ (respectively $M_7(1,y_7)$ with $y_7=-(p+2r)$; $M_8(1+r,y_8)$ with $y_8=-r(1+r)$) on the invariant line $x=0$ (respectively $x=1$; $x=1+r$) with respect to the singular points $M_1(0,0)$ and $M_2(0,-(1+r)$ (respectively $M_4(1,0)$ and $M_6(1,-r)$; $M_5(1+r,0)$ and $M_9(1+r,-r(1+r))\big)$. It is clear that the positions of these singularities in three different cases $(i)-(iii)$ enumerated above could be distinct and therefore we examine each one of these three cases.

Case (i) $r<-1$**.** Then considering Notation 3.6 we have

$$0<-(r{+}1)\ \Rightarrow\ M_0\prec M_2;\quad 0<-r\ \Rightarrow\ M_4\prec M_6;\quad -r(r{+}1)<0\ \Rightarrow\ M_9\prec M_5,$$

and considering the coordinates $y_3=-(p+r)$, $y_7=-(p+2r)$ and $y_8=-r(1+r)$ we have the next implications.

(I) For the singular point M_3:

$$y_3<0\ \Rightarrow\ M_3\prec M_0\prec M_2;\quad 0<y_3<-(r+1)\ \Rightarrow\ M_0\prec M_3\prec M_2;$$
$$y_3>-(r+1)\ \Rightarrow\ M_0\prec M_2\prec M_3.$$

(II) For the singular point M_7:

$$y_7\le 0\ \Rightarrow\ M_7\preceq M_4\prec M_6;\quad 0<y_7<-r\ \Rightarrow\ M_4\prec M_7\prec M_5;$$
$$y_7>-r\ \Rightarrow\ M_4\prec M_5\prec M_7.$$

(III) For the singular point M_8:

$$y_8<-r(r+1)\ \Rightarrow\ M_8\prec M_9\prec M_5;\quad -r(r+1)<y_8<0\ \Rightarrow\ M_9\prec M_8\prec M_5;$$
$$y_8>0\ \Rightarrow\ M_9\prec M_5\prec M_8.$$

Case (ii) $-1<r<0$**.** In this case we have

$$-(r+1)<0\Rightarrow M_2\prec M_0;\quad 0<-r\Rightarrow M_4\prec M_6;\quad 0<-r(r+1)\Rightarrow M_5\prec M_9,$$

and this leads to the next implications.

(I) For the singular point M_3:

$$y_3<-(r+1)\ \Rightarrow\ M_3\prec M_2\prec M_0;\quad -(r+1)<y_3<0\ \Rightarrow\ M_2\prec M_3\prec M_0;$$
$$y_3>0\ \Rightarrow\ M_2\prec M_0\prec M_3.$$

(II) For the singular point M_7:

$$y_7\le 0\ \Rightarrow\ M_7\preceq M_4\prec M_6;\quad 0<y_7<-r\ \Rightarrow\ M_4\prec M_7\prec M_6;$$
$$y_7>-r\ \Rightarrow\ M_4\prec M_6\prec M_7.$$

(III) For the singular point M_8:

$$y_8<0\ \Rightarrow\ M_8\prec M_5\prec M_9;\quad 0<y_8<-r(r+1)\ \Rightarrow\ M_5\prec M_8\prec M_9;$$

$$y_8 > -r(r+1) \Rightarrow M_5 \prec M_9 \prec M_8.$$

Case (iii) $r > 0$**.** Then we have

$$-(r+1) < 0 \Rightarrow M_2 \prec M_0; \quad -r < 0 \Rightarrow M_6 \prec M_4; \quad -r(r+1) < 0 \Rightarrow M_9 \prec M_5,$$

and this leads to the next implications.

(I) For singular point M_3:

$$y_3 < -(r+1) \Rightarrow M_3 \prec M_2 \prec M_0; \quad -(r+1) < y_3 < 0 \Rightarrow M_2 \prec M_3 \prec M_0;$$
$$y_3 > 0 \Rightarrow M_2 \prec M_0 \prec M_3.$$

(II) For singular point M_7:

$$y_7 < -r \Rightarrow M_7 \prec M_6 \prec M_4; \quad -3 < y_7 < 0 \Rightarrow M_6 \prec M_7 \prec M_4;$$
$$y_7 \geq 0 \Rightarrow M_6 \prec M_4 \preceq M_7.$$

(III) For singular point M_8:

$$y_8 < -r(r+1) \Rightarrow M_8 \prec M_9 \prec M_5; \quad -r(r+1) < y_8 < 0 \Rightarrow M_9 \prec M_8 \prec M_5;$$
$$y_8 > 0 \Rightarrow M_9 \prec M_5 \prec M_8.$$

Since we only have two parameters (p and r), clearly not all of the possibilities described above could be realizable. So examining the compatibilities of the conditions it is not too hard to convince ourselves (using for example, the tools "FindInstance" or "Reduce" of computer algebra system Mathematica) that the following lemma is valid.

Lemma 3.15. *The family of systems* (3.89) *with the condition* (3.90) *possesses the following configurations of invariant lines when the corresponding conditions indicated below are satisfied:*

$$Config.\ 7.58 \Leftrightarrow p > 1, p > -2r, p > r^2, \quad (p = 5, r = -2);$$

$$Config.\ 7.59 \Leftrightarrow \begin{cases} r < -1, -2r < p < r^2, & (p = 17/2, r = -3); \\ or\ r < -1, r^2 < p < -2r, & (p = 65/24, r = -3/2); \\ or\ -1 < r < 0, -2r < p < 1, & (p = 3/4, r = -1/4); \\ or\ -1 < r < 0, 1 < p < -2r, & (p = 131/96, r = -3/4); \\ or\ r > 0, 1 < p < r^2, & (p = 7/2, r = 2); \\ or\ r > 0, r^2 < p < 1, & (p = 5/8, r = 1/2); \end{cases}$$

$$Config.\ 7.61 \Leftrightarrow; -2 < r < -1/2, p = -2r, \quad (p = 3, r = -3/2);$$

$$Config.\ 7.62 \Leftrightarrow \begin{cases} r < -2, p = -2r, & (p = 6, r = -3); \\ or\ -1/2 < r < 0, p = -2r, & (p = 3/4, r = -3/8); \end{cases}$$

$$Config.\ 7.64 \Leftrightarrow \begin{cases} r < -1, -r < p < -2r, p < r^2, & (p = 5/2, r = -2); \\ or\ -1 < r < 0, -r < p < -2r, p < 1, & (p = 7/8, r = -1/2); \\ or\ r > 0, -r < p < r^2, p < 1, & (p = -1/2, r = 1); \end{cases}$$

$$Config.\ 7.65 \Leftrightarrow \begin{cases} r < -1, 1 < p < -r, & (p = 3/2, r = -2); \\ or\ -1 < r < 0, r^2 < p < -r, & (p = 3/8, r = -1/2); \\ or\ r > 0, -2r < p < -r, & (p = -1, r = 3/4); \end{cases}$$

$$
\begin{gathered}
Config.\ 7.66 \Leftrightarrow \begin{cases} r<-1, p<1, & (p=-1/2, r=-2); \\ or\ -1<r<0, p<r^2, & (p=-1/2, r=-1/2); \\ or\ r>0, p<-2r, & (p=-7/2, r=1); \end{cases} \\
Config.\ 7.67 \Leftrightarrow r>0, p=-2r, \quad (p=-2, r=1).
\end{gathered}
$$

2.2. *Possibility* $(p+r)\neq 0$, $w=0$. Then the condition $w=1+3p+7r+r^2=0$ gives $p=-(1+7r+r^2)/3\neq 0$ (since $p\neq 0$) and we denote $w'=1+7r+r^2\neq 0$. Then we calculate

$$
\begin{gathered}
G_2 = (r-1)(2+r)(1+2r)\big[g(r-1)w'+9l(1+4r+r^2)\big]/27, \\
p(1+p+2r)(p+2r+r^2)=(r-1)^2(2+r)(1+2r)w'/9\neq 0.
\end{gathered}
$$

Hence the condition $G_2=0$ gives

$$
g=-\frac{9l(1+4r+r^2)}{(r-1)w'}
$$

and this leads to the family of systems

$$
\begin{aligned}
\dot{x} =& -\frac{1+4r+r^2}{3}\Big[x-\frac{3\big[2m(1-r)w'-3l(2+r)(1+5r)\big]}{(r-1)^2(2+r)w'}\Big] \\
&\times\Big[x-\frac{3(3l(5+r)(1+2r)+2m(r-1)w')}{(r-1)^2(1+2r)w'}\Big] \\
&\times\Big[x-\frac{3(3l(2+r)(1+2r)+2m(r-1)w')}{(r-1)(2+r)(1+2r)w'}\Big] \\
&\equiv (p+r)(x-\varphi_1')(x-\varphi_2')(x-\varphi_3'), \\
\dot{y} =&(y-3l/w')\Big[\frac{-3}{(r-1)^4(2+r)^2(1+2r)^2w'}\big[4m^2(r-1)^2w'(1+5r+15r^2 \\
&+5r^3+r^4)+3l^2(2+r)^2(5+r)(1+2r)^2(1+5r)+6lm(r-1)(2+r)(1+2r) \\
&\times(2+19r+66r^2+19r^3+2r^4)\big]+\frac{3l(r-1)+2mw'}{w'}x+3ly/w'-w'x^2/3 \\
&+(r-1)xy+y^2\Big].
\end{aligned}
$$

We calculate

$$
\varphi_2'-\varphi_1'=\frac{18(1+r)\delta'}{(r-1)^2(2+r)(1+2r)w'},\quad \varphi_3'-\varphi_1'=\frac{18r\delta'}{(r-1)^2(2+r)(1+2r)w'}
$$

where $\delta'=3l(2+r)(1+2r)+m(r-1)w'$. Since $r(r+1)\neq 0$ it is clear that the condition $\delta'=0$ is equivalent to the existence of a triple invariant line in the direction $x=0$.

On the other hand for the above mentioned values of the parameters p and g we obtain $G_1=-2\delta'/3\neq 0$. Therefore we can apply the transformation

$$
\begin{gathered}
x_1=\alpha x-\frac{(r-1)\big[m(r-1)w'+\delta'\big]}{6\delta'},\quad y_1=\alpha y-\frac{l(r-1)^2(2+r)(1+2r))}{6\delta'}, \\
t_1=\frac{324\delta'^2}{(r-1)^4(2+r)^2(1+2r)^2w'^2}t,\quad \alpha=\frac{(r-1)^2(2+r)(1+2r)w'}{18\delta'}
\end{gathered}
$$

and we arrive at the family of systems

$$
\dot{x}=-(1+4r+r^2)x(x-1)(x+r)/3,
$$

$$\begin{aligned}\dot{y} =& y\big[-r(1+r+r^2)-3(r-1)rx-(1+7r+r^2)x^2+(r-1)^2y\\ &+3(r-1)xy+3y^2\big]/3.\end{aligned}$$

We observe that these system via the transformation $(x,y,t)\mapsto(1-x,-y,t)$ could be brought to the system (3.89) with $p=-(1+7r+r^2)/3\neq 0$ and hence no new configurations can be obtained.

2.3. *Possibility* $p+r=0$. Considering (3.52) for $p=-r$ we obtain

$$c=g(l-2lr+gr-2mr)/r^2,\quad G_2=-gr^2(1+r)^2$$

and because of $r(r+1)\neq 0$ the condition $G_2=0$ gives $g=0$. However in this case we obtain $c=0$ and this leads to degenerate systems

$$\dot{x}=0,\quad \dot{y}=(y-l/r)\Big[\frac{2mr-l}{r(1+r)}+rx+y\Big]\Big[\frac{l+2m}{1+r}-x+y\Big].$$

So the case $p(1+p+2r)(p+2r+r^2)\neq 0$ is completely examined.
Case $p(1+p+2r)(p+2r+r^2)=0$. We examine two subcases: $p=0$ or $p\neq 0$ and $(1+p+2r)(p+2r+r^2)=0$.

1. *Subcase* $p=0$. Considering (3.45) the condition $Eq'_5=0$ gives $l=0$ and then we calculate

$$Eq'_5=0,\quad Eq'_8=e-2mW+(r-1)W^2,\quad Eq'_{10}=b-fW-W^3. \tag{3.91}$$

Now taking into consideration the conditions $k=d=h=p=l=0$ we consider other two directions. From (3.46) and (3.47) we obtain respectively:

$$\begin{gathered}Eq''_6=-g+2m-(1+2r)W,\quad Eq''_9=-c+e+f-gW+(1-r)W^2,\\ Eq''_{10}=-a+b+(e-c)W-gW^2-rW^3.\end{gathered} \tag{3.92}$$

and

$$\begin{gathered}Eq'''_6=2m-g+(2+r)W,\quad Eq'''_9=f-c-e/r+gW/r+(r-1)W^2/r,\\ Eq'''_{10}=b+ar-(e+cr)W/r+gW^2/r-W^3/r.\end{gathered} \tag{3.93}$$

As we observe the equation $Eq''_6=0$ (respectively $Eq'''_6=0$) is linear in W if and only if $1+2r\neq 0$ (respectively $2+r\neq 0$). So in what follows we consider two possibilities: $(1+2r)(2+r)\neq 0$ and $(1+2r)(2+r)=0$.

1.1. *Possibility* $(1+2r)(2+r)\neq 0$. Then the equation $Eq''_6=0$ gives $W=(2m-g)/(1+2r)\equiv W''_0$ and then the equation $Eq''_9|_{\{W=W''_0\}}=0$ yields

$$f=c-e-\frac{(g-2m)(2g-2m+gr+2mr)}{(1+2r)^2}. \tag{3.94}$$

Considering this value of the parameter f and (3.92) the equation $Eq''_{10}|_{\{W=W''_0\}}=0$ gives

$$a=b+\frac{(c-e)(g-2m)}{1+2r}-\frac{(g-2m)^2(g+gr+2mr))}{(1+2r)^3}. \tag{3.95}$$

We examine now the equations corresponding to the direction $y=-rx$. Considering (3.93) and the conditions (3.94) and (3.95) we detect that the equation $Eq'''_6=0$ gives $W=(g-2m)/(2+r)\equiv W'''_0$ and then the equation $Eq'''_9|_{\{W=W'''_0\}}=0$ yields

$$e=-\frac{(g-2m)(r-1)(g+2m-2gr+14mr+gr^2+2mr^2)}{(2+r)^2(1+2r)^2}. \tag{3.96}$$

Considering this value of the parameter e and (3.93) the equation $Eq_{10}'''|_{\{W=W_0'''\}} = 0$ implies

$$\begin{aligned} b = -\frac{(g-2m)(r-1)}{(2+r)^3(1+2r)^3}\big[&c(2+r)^2(1+2r)^2-(g-2m)(8g-4m+11gr \\ &+8mr+8gr^2-4mr^2)\big]. \end{aligned} \tag{3.97}$$

Now we return to the direction $y=0$. Taking into account (3.91) and the conditions (3.94), (3.95), (3.96) and (3.97) we obtain

$$\begin{aligned} Eq_8' &= -\frac{\Psi(W)}{(2+r)^2(1+2r)^2}\big[(g+2m-2gr+14mr+gr^2+2mr^2 \\ &\quad -(r-1)(2+r)(1+2r)W\big], \\ Eq_{10}' &= \frac{\Psi(W)}{(2+r)^3(1+2r)^3}\big[-c(2+r)^2(1+2r)^2+(g-2m)(8g-4m+11gr \\ &\quad +8mr+8gr^2-4mr^2)+(g-2m)(r-1)(2+r)(1+2r)W \\ &\quad -(2+r)^2(1+2r)^2W^2\big], \end{aligned}$$

where $\Psi(W)=(g-2m)(r-1)+(2+r)(1+2r)W$. So Eq_8' and Eq_{10}' have a common factor which is linear in W. This means that the detected conditions are sufficient for the existence of six invariant affine lines for system (3.44). As a result these systems become

$$\begin{aligned} \dot{x} &= \Big[x+\frac{3(g-2m)}{(2+r)(1+2r)}\Big]\Big[c-\frac{6(g-2m)(g+gr+3mr+gr^2)}{(2+r)^2(1+2r)^2} \\ &\quad +\frac{2(g+gr+3mr+gr^2)}{(2+r)(1+2r)}x+rx^2\Big] \\ &\equiv \tilde{L}_1(x)\tilde{L}_{2,3}(x), \\ \dot{y} &= \Big[y-\frac{(g-2m)(r-1)}{(2+r)(1+2r)}\Big]\Big[c-\frac{(g-2m)(8g-4m+11gr+8mr+8gr^2-4mr^2)}{(2+r)^2(1+2r)^2} \\ &\quad +\frac{(g+2m-2gr+14mr+gr^2+2mr^2)}{(2+r)(1+2r)}x+\frac{(g-2m)(r-1)}{(2+r)(1+2r)}y \\ &\quad -(1-r)xy+y^2\Big]. \end{aligned}$$

We need to detect if the two lines defined by the equation $\tilde{L}_{2,3}=0$ are real or complex and in the case when they are real, if one of them coincides with the invariant line $\tilde{L}_1=0$. So we calculate

$$\operatorname{Discrim}[\tilde{L}_{2,3},x]=\frac{\tilde{\lambda}(c,g,m,r)}{(2+r)^2(1+2r)^2},\quad \operatorname{Res}_x(\tilde{L}_1,\tilde{L}_{2,3})=\frac{\tilde{\mu}(c,g,m,r)}{(2+r)^2(1+2r)^2},$$

where

$$\begin{gathered} \tilde{\lambda}=(g+7gr-9mr+gr^2)(g+gr+3mr+gr^2)-cr(2+r)^2(1+2r)^2, \\ \tilde{\mu}=c(2+r)^2(1+2r)^2-3(g-2m)(4g+gr+18mr+4gr^2). \end{gathered} \tag{3.98}$$

We observe that

$$\operatorname{sign}\big(\operatorname{Discrim}[\tilde{L}_{2,3},x]\big)=\operatorname{sign}(\tilde{\lambda}),$$

i.e. the invariant lines $\tilde{L}_{2,3} = 0$ are real (respectively complex; coinciding) if $\tilde{\lambda} > 0$ (respectively $\tilde{\lambda} < 0$; $\tilde{\lambda} = 0$). And the invariant line $\tilde{L}_1 = 0$ coincides with one of the lines $\tilde{L}_{2,3} = 0$ if and only if $\tilde{\mu} = 0$.

1.1.1. *Case* $\tilde{\lambda} > 0$. Then we use a new parameter u setting $\tilde{\lambda} = u^2$ and since $r(2+r)(1+2r) \neq 0$ we obtain

$$c = \frac{1}{r(2+r)^2(1+2r)^2}\big[(g+7gr-9mr+gr^2)(g+gr+3mr+gr^2)-u^2\big]. \tag{3.99}$$

This leads to the system

$$\begin{aligned}
\dot{x} =& r\Big[x+\frac{3(g-2m)}{(2+r)(1+2r)}\Big]\Big[x+\frac{g+gr+3mr+gr^2-u}{r(2+r)(1+2r)}\Big]\\
&\times\Big[x+\frac{g+gr+3mr+gr^2+u}{r(2+r)(1+2r)}\Big],\\
\dot{y} =& \Big[y-\frac{(g-2m)(r-1)}{(2+r)(1+2r)}\Big]\Big[\frac{g^2(r-1)^2(r+1)^2}{r(r+2)^2(2r+1)^2}+\frac{2gm(7r^2+13r+7)}{(r+2)^2(2r+1)^2}\\
&-\frac{m^2r(8r^2+11r+8)+u^2}{r(r+2)^2(2r+1)^2}+\frac{g(r-1)^2+2m(1+7r+r^2)}{(2+r)(1+2r)}x\\
&+\frac{(g-2m)(r-1)}{(2+r)(1+2r)}y+(r-1)xy+y^2\Big].
\end{aligned} \tag{3.100}$$

On the other hand for the value of c given in (3.99) we calculate

$$\tilde{\mu} = \frac{1}{r}(\tilde{\gamma}^2-u^2), \quad \tilde{\gamma} = g(r-1)^2+9mr \tag{3.101}$$

and since the condition $\tilde{\mu} = 0$ leads to the coincidence of two invariant lines of the triplet, we examine two subcases: $\tilde{\mu} \neq 0$ and $\tilde{\mu} = 0$.

1.1.1.1. *Subcase* $\tilde{\mu} \neq 0$. Then $(\tilde{\gamma}-u)(\tilde{\gamma}+u) \neq 0$ and via the transformation

$$\begin{gathered}
x_1 = \alpha x - \frac{3(g-2m)r}{\tilde{\gamma}-u}, \quad y_1 = \alpha y + \frac{(g-2m)(r-1)r}{\tilde{\gamma}-u},\\
t_1 = \frac{(\tilde{\gamma}-u)^2}{r^2(2+r)^2(1+2r)^2}t, \quad \alpha = -\frac{r(2+r)(1+2r)}{\tilde{\gamma}-u}
\end{gathered}$$

system (3.100) could be brought to the following family of systems (we keep the old notations for the variables):

$$\dot{x} = rx(x-1)(x-v), \quad \dot{y} = y\big[rv-r(1+v)x+(r-1)xy+y^2\big]. \tag{3.102}$$

It remains to note that this family of systems is a subfamily of (3.59) defined by the condition $p = 0$ and hence it is already examined.

1.1.1.2. *Subcase* $\tilde{\mu} = 0$. Then $(\tilde{\gamma}-u)(\tilde{\gamma}+u) = 0$ and we may assume $\tilde{\gamma}+u = 0$ because of the change $u \to -u$. So setting $u = -\tilde{\gamma} \neq 0$ in system (3.100) and applying the transformation

$$\begin{gathered}
x_1 = \alpha x - \frac{3(g-2m)r}{2\tilde{\gamma}}, \quad y_1 = \alpha y + \frac{(g-2m)(r-1)r}{2\tilde{\gamma}},\\
t_1 = \frac{4\tilde{\gamma}^2}{p^2(r-1)^2(p+r)^2}t, \quad \alpha = -\frac{r(2+r)(1+2r)}{2\tilde{\gamma}}
\end{gathered}$$

we obtain the family of systems:

$$\dot{x} = rx^2(x-1), \quad \dot{y} = y\big[-rx+(r-1)xy+y^2\big].$$

We observe that this family of systems is a subfamily of (3.62) defined by the condition $p = 0$, i.e. no new configurations could be obtained.

1.1.2. *Case* $\tilde{\lambda} < 0$. In this case we set $\tilde{\lambda} = -u^2$ and since $r(2+r)(1+2r) \neq 0$ we obtain

$$c = \frac{1}{r(2+r)^2(1+2r)^2}\ \big[(g+7gr-9mr+gr^2)(g+gr+3mr+gr^2)+u^2\big]. \tag{3.103}$$

This leads to the system

$$\begin{aligned}
\dot{x} =& r\Big[x+\frac{3(g-2m)}{(2+r)(1+2r)}\Big]\Big[\Big(x+\frac{g+gr+3mr+gr^2}{r(2+r)(1+2r)}\Big)^2\\
&+\frac{u^2}{r^2(2+r)^2(1+2r)^2}\Big],\\
\dot{y} =& \Big[y-\frac{(g-2m)(r-1)}{(2+r)(1+2r)}\Big]\Big[\frac{g^2(r-1)^2(r+1)^2}{r(r+2)^2(2r+1)^2}+\frac{2gm(7r^2+13r+7)}{(r+2)^2(2r+1)^2}\\
&-\frac{m^2r(8r^2+11r+8)-u^2}{r(r+2)^2(2r+1)^2}+\frac{g(r-1)^2+2m(1+7r+r^2)}{(2+r)(1+2r)}x\\
&+\frac{(g-2m)(r-1)}{(2+r)(1+2r)}\,y+(r-1)xy+y^2\Big]
\end{aligned} \tag{3.104}$$

and we consider two subcases: $\tilde{\gamma} \neq 0$ and $\tilde{\gamma} = 0$ ($\tilde{\gamma}$ is given in (3.101)).

1.1.2.1. *Subcase* $\tilde{\gamma} \neq 0$. Then via the transformation

$$\begin{gathered}
x_1 = \alpha x + \frac{3(g-2m)r}{\tilde{\gamma}}, \quad y_1 = \alpha y - \frac{(g-2m)(r-1)r}{\tilde{\gamma}},\\
t_1 = \frac{\tilde{\gamma}^2}{r^2(2+r)^2(1+2r)^2}t, \quad \alpha = \frac{r(2+r)(1+2r)}{\tilde{\gamma}}
\end{gathered}$$

the above system can be brought to the canonical form

$$\dot{x} = rx\big[(x+1)^2+v^2\big], \quad \dot{y} = y\big[r(1+v^2)+2rx+(r-1)xy+y^2\big], \tag{3.105}$$

where $v = u/\tilde{\gamma} \neq 0$. We observe that these systems belong to the family (3.64) for $p = 0$ and this family is already examined.

1.1.2.2. *Subcase* $\tilde{\gamma} = 0$. Considering (3.101) we obtain $m = -g(r-1)^2/(9r)$ and then system (3.104) becomes system

$$\begin{aligned}
\dot{x} =& r\Big[x+\frac{g)}{3r}\Big]\Big[\Big(x+\frac{g)}{3r}\Big)^2+\frac{u^2}{r^2(2+r)^2(1+2r)^2}\Big],\\
\dot{y} =& -\Big[y-\frac{g(r-1)}{9r}\Big]\Big[\frac{2g^2(-1+r)^2(2+r)^2(1+2r)^2-81ru^2}{81r^2(2+r)^2(1+2r)^2}\\
&+\frac{g(r-1)^2}{9r}x-\frac{g(r-1)}{9r}\,y+(1-r)xy-y^2\Big].
\end{aligned}$$

Since $ur(2+r)(1+2r) \neq 0$ we can apply the transformation

$$\begin{gathered}
x_1 = \alpha x + \frac{g(2+r)(1+2r)}{3u}, \quad y_1 = \alpha y - \frac{g(r-1)(2+r)(1+2r)}{9u},\\
t_1 = \frac{u^2}{r^2(2+r)^2(1+2r)^2}t, \quad \alpha = \frac{r(2+r)(1+2r)}{u}
\end{gathered}$$

and we arrive at the family of systems

$$\dot{x} = rx(x^2+1), \quad \dot{y} = y\big[r + (r-1)xy + y^2\big]. \tag{3.106}$$

It remains to observe that this family of systems is a subfamily of (3.68) defined by $p = 0$, i.e. no new configurations could be obtained.

1.1.3. *Case* $\tilde{\lambda} = 0$. Considering (3.98) and solving the equation $\tilde{\lambda} = 0$ with respect to the parameter c it is clear that we obtain (3.103) for $u = 0$. This leads to system (3.104) with $u = 0$, which we denote by $(3.104)_{\{u=0\}}$.

We observe that in this case we obtain $\tilde{\mu} = \tilde{\gamma}^2/r$ and we again consider two subcases: $\tilde{\mu} \neq 0$ and $\tilde{\mu} = 0$.

1.1.3.1. *Subcase* $\tilde{\mu} \neq 0$. Then $\tilde{\gamma} \neq 0$ and via the transformation

$$x_1 = \alpha x - \frac{g + gr + 3mr + gr^2}{\tilde{\gamma}}, \quad y_1 = \alpha y + \frac{(g-2m)(r-1)r}{\tilde{\gamma}},$$
$$t_1 = \frac{\tilde{\gamma}^2}{r^2(2+r)^2(1+2r)^2}t, \quad \alpha = \frac{r(2+r)(1+2r)}{\tilde{\gamma}}$$

system $(3.104)_{\{u=0\}}$ can be brought to the family of systems

$$\dot{x} = rx^2(x-1), \quad \dot{y} = y\big[-r + 2rx + (1-r)y + (r-1)xy + y^2\big]. \tag{3.107}$$

We notice that the above systems belong to the family (3.69) for which $p = 0$ and this family is already examined.

1.1.3.2. *Subcase* $\tilde{\mu} = 0$. In this case we have $\tilde{\lambda} = \tilde{\mu} = 0$ and this leads to the existence a triple invariant line in the direction $x = 0$. Since $\tilde{\mu} = \tilde{\gamma}^2/r$ we obtain $\tilde{\gamma} = 0$ and considering (3.101) we obtain $m = -g(r-1)^2/(9r)$. Therefore taking into account (3.103) for $u = 0$ we have $c = g^2/(3r)$ and then for these values of the parameters m, u and c system (3.104) becomes

$$\dot{x} = \frac{(g+3rx)^3}{27r^2}, \quad \dot{y} = \frac{(gr - g - 9ry)^2)}{729r^3}\big[2g(-1+r) + 9(-1+r)rx + 9ry\big].$$

However these systems possess invariant lines of total multiplicity 8, because for these system we have (see Notation 2.4):

$$\begin{aligned} H(X,Y,Z) = {}& \frac{\alpha}{r^{11}}(3rX + gZ)^3(-9rY - gZ + grZ)^2(9rX - 9rY + 2gZ + grZ) \\ & \times (9r^2X + 9rY + gZ + 2grZ), \end{aligned}$$

where $\alpha \in \mathbb{R}$. So we are out of the class of system studied in this article.

1.2. *Possibility* $(1+2r)(2+r) = 0$. Considering Remark 3.9 we may assume $2r + 1 = 0$ because of the rescaling $(x, y, t) \mapsto (x, -ry, t/r^2)$ in system (3.44). So $r = -1/2$ and then the equations (3.91) and (3.92) become, respectively:

$$Eq_5' = 0, \quad Eq_8' = e - 2mW - 3W^2/2, \quad Eq_{10}' = b - fW - W^3,$$

and

$$Eq_6'' = -g + 2m, \quad Eq_9'' = -c + e + f - gW + 3W^2/2,$$
$$Eq_{10}'' = -a + b + (e-c)W - gW^2 + W^3/2.$$

So the condition $Eq_6'' = 0$ gives $g = 2m$ and therefore equation (3.93) becomes

$$Eq_6''' = 3W/2, \quad Eq_9''' = -c + 2e + f - 4mW + 3W^2,$$
$$Eq_{10}''' = b - a/2 + (2e-c)W - 4mW^2 + 2W^3.$$

We observe that the equation $Eq_6''' = 0$ gives $W = 0$ and hence we obtain the conditions:

$$Eq_9''' = -c + 2e + f = 0, \quad Eq_{10}''' = b - a/2 = 0.$$

Thus we obtain $f = c - 2e$ and $a = 2b$ and then we calculate $\operatorname{Res}_W(Eq_9'', Eq_{10}'') = -\operatorname{Res}_W(Eq_8', Eq_{10}') = \widetilde{\Psi}(b, c, e, m)/8$, where

$$\widetilde{\Psi}(b, c, e, m) = 27b^2 - 18c^2e - 32e^3 + 64e^2m^2 + 64bm^3 + 4c(12e^2 + 9bm - 8em^2).$$

So in order to have invariant lines of total multiplicity seven the condition $\widetilde{\Psi} = 0$ is necessary. This equation is quadratic in b and we calculate

$$\operatorname{Discrim}[\widetilde{\Psi}, b] = 8(3e + 2m^2)(9c - 12e + 16m^2)^2$$

and clearly we could have a real solution of the equation $\widetilde{\Psi} = 0$ with respect to the parameter b only if either $(3e + 2m^2) \geq 0$ or $9c - 12e + 16m^2 = 0$. We consider both cases.

1.2.1. *Case* $(3e + 2m^2) \geq 0$. Then setting $3e + 2m^2 = 2u^2 \geq 0$ we obtain $e = 2(u^2 - m^2)$ and we have

$$\begin{aligned}\widetilde{\Psi} &= (27b + 18cm + 32m^3 + 18cu + 48m^2u - 16u^3)\\ &\quad\times (27b + 18cm + 32m^3 - 18cu - 48m^2u + 16u^3)/27 = 0.\end{aligned}$$

Because of the change $u \to -u$ we assume without loss of generality that the first factor vanishes and we have the condition

$$b = -2(m + u)(9c + 16m^2 + 8mu - 8u^2)/27.$$

Then considering also the conditions

$$k = d = h = p = l = 0, \quad r = -1/2, \quad g = 2m, \quad f = c - 2e, \quad a = 2b \tag{3.108}$$

we detect that system (3.44) has the form

$$\begin{aligned}\dot{x} &= -\frac{(4m + 4u - 3x)}{54}\big[2(9c + 16m^2 + 8mu - 8u^2) + 12(2m - u)x - 9x^2\big])\\ &\equiv\ -L_1'(x)L_{2,3}'(x)/54,\\ \dot{y} &= -\frac{(2m + 2u - 3y)}{54}\big[2(9c + 16m^2 + 8mu - 8u^2) + 18(m - u)x\\ &\quad + 12(m + u)y - 27xy + 18y^2\big].\end{aligned} \tag{3.109}$$

Following the same algorithm as before we calculate

$$\begin{gathered}\operatorname{Discrim}[L_{2,3}', x] = 216(3c + 8m^2 - 2u^2) \equiv 216\lambda'(c, m, u),\\ \operatorname{Res}_x(L_1', L_{2,3}') = 54(3c + 8m^2 - 8u^2) \equiv 54\mu'(c, m, u)\end{gathered} \tag{3.110}$$

and hence the invariant lines $L_{2,3}' = 0$ are real (respectively complex) if $\lambda' > 0$ (respectively $\lambda' < 0$) and they coincide if $\lambda' = 0$. On the other hand the third line ($L_1' = 0$) of this triplet coalesces with another invariant line if and only if $\mu' = 0$. So we examine the possibilities given by these conditions.

1.2.1.1. *Subcase* $\lambda' > 0$. Setting $\lambda' = 6w^2 > 0$ we obtain $c = -2(4m^2 - u^2 - 3w^2)/3$ and then system (3.109) becomes

$$\begin{aligned}\dot{x} =&(4m+4u-3x)(4m-2u-6w-3x)(4m-2u+6w-3x)/54,\\ \dot{y} =&(2m+2u-3y)\big[4(2m-u-3w)(2m-u+3w)-18(m-u)x\\ &-12(m+u)y+27xy-18y^2\big]/54.\end{aligned}\tag{3.111}$$

For the above value of the parameter c we calculate $\mu' = -6(u-w)(u+w)$ and we consider two possibilities: $\mu' \neq 0$ and $\mu' = 0$.

1.2.1.1.1. *Possibility* $\mu' \neq 0$. Then applying the transformation

$$x_1 = \frac{x}{2(w-u)} + \frac{2(m+u)}{3(u-w)}, \quad y_1 = \frac{y}{2(w-u)} + \frac{m+u}{3(u-w)}, \quad t_1 = 4(w-u)^2 t$$

we obtain the 1-parameter family of systems:

$$\dot{x} = x(x-1)(v-x)/2, \quad \dot{y} = y\big[-v+(1+v)x-3xy+2y^2\big]/2.$$

We observe that this family is a subfamily of (3.102) defined by $r = -1/2$ and hence no new configurations could be obtained.

1.2.1.1.2. *Possibility* $\mu' = 0$. In this case we may assume $w = u$ (due to the change $w \to -w$) and system (3.111) becomes

$$\begin{aligned}\dot{x} =&(4m-8u-3x)(4m+4u-3x)^2/54,\\ \dot{y} =&(2m+2u-3y)\big[16(m-2u)(m+u)-18(m-u)x-12(m+u)y\\ &+27xy-18y^2\big]/54.\end{aligned}$$

Since in this case we have $\lambda' = 6u^2 > 0$ (i.e. $u \neq 0$) applying the transformation

$$x_1 = -\frac{x}{4u} + \frac{m+u}{3u}, \quad y_1 = -\frac{y}{4u} + \frac{m+u}{6u}, \quad t_1 = 16u^2 t$$

we obtain the system

$$\dot{x} = x^2(1-x)/2, \quad \dot{y} = y(x-3xy+2y^2)/2.$$

It remains to observe that this system is contained in the family (3.62) for $p = 0$ and $r = -1/2$, so this family is already examined.

1.2.1.2. *Subcase* $\lambda' < 0$. Setting $\lambda' = -6w^2 < 0$ we obtain $c = -2(4m^2 - u^2 + 3w^2)/3$ and then system (3.109) becomes

$$\begin{aligned}\dot{x} =&(4m+4u-3x)\big[(4m-2u-3x)^2+36w^2\big]/54,\\ \dot{y} =& (2m+2u-3y)\big[4(4m^2-4mu+u^2+9w^2)-18(m-u)x\\ &-12(m+u)y+27xy-18y^2\big]/54.\end{aligned}\tag{3.112}$$

For the above value of the parameter c we calculate $\mu' = -6(u^2+w^2) \neq 0$ since $\lambda' = -6w^2 < 0$. So we consider two possibilities: $u \neq 0$ and $u = 0$.

1.2.1.2.1. *Possibility* $u \neq 0$. Then we apply the transformation

$$x_1 = \frac{x}{2u} - \frac{2(m+u)}{3u}, \quad y_1 = \frac{y}{2u} - \frac{m+u}{3u}, \quad t_1 = 4u^2 t$$

obtaining the systems

$$\dot{x} = -x\big[(x+1)^2+v^2\big]/2, \quad \dot{y} = y(-1-v^2-2x-3xy+2y^2)/2,$$

where $v = u/w \neq 0$. We observe that the above systems form a subfamily of (3.105) defined by $r = -1/2$ and hence no new configurations could be obtained.

1.2.1.2.2. *Possibility* $u = 0$. In this case doing the transformation

$$x_1 = \frac{x}{2w} - \frac{2m}{3w}, \quad y_1 = \frac{y}{2w} - \frac{m}{3w}, \quad t_1 = 4w^2 t$$

system (3.112) can be brought to the system

$$\dot{x} = -x(x^2+1)/2, \quad \dot{y} = y(-1-3xy+2y^2)/2.$$

It remains to notice that this system is contained in the family (3.106) for $r = -1/2$, i.e. no new configurations could be obtained.

1.2.1.3. *Subcase* $\lambda' = 0$. Considering (3.110) this condition gives $c = -2(2m-u)(2m+u)/3$ and then system (3.109) becomes

$$\begin{aligned} \dot{x} =& (4m+4u-3x)(4m-2u-3x)^2/54,\\ \dot{y} =& (2m+2u-3y)\big[4(2m-u)^2 - 18(m-u)x - 12(m+u)y \\ &+ 27xy - 18y^2\big]/54. \end{aligned} \tag{3.113}$$

For the value of the parameter c, given above, we calculate $\mu' = -6u^2$ and we examine two possibilities: $\mu' \neq 0$ and $\mu' = 0$.

1.2.1.3.1. *Possibility* $\mu' \neq 0$. Then $u \neq 0$ and performing the transformation

$$x_1 = \frac{x}{2u} - \frac{2m-u}{3u}, \quad y_1 = \frac{y}{2u} - \frac{m+u}{3u}, \quad t_1 = 4u^2 t$$

we obtain the system

$$\dot{x} = -x^2(x-1)/2, \quad \dot{y} = y(1-2x+3y-3xy+2y^2)/2.$$

We notice that this system belongs to the family (3.107) for $r = -1/2$, already examined.

1.2.1.3.2. *Possibility* $\mu' = 0$. Therefore $u = 0$ and systems (3.113) become

$$\dot{x} = (4m-3x)^3/54, \quad \dot{y} = (2m-3y)^2(8m-9x+6y)/54.$$

For these systems we calculate (see Notation 2.4)

$$H(X,Y,Z) = 2^{-4}3^{-9}(X-2Y)(3X-4mZ)^3(3X-3Y-2mZ)^2(3Y-2mZ)^2$$

and hence by Lemma 2.6 we have invariant lines of total multiplicity nine, i.e. we are not in the class of systems with invariant lines of total multiplicity exactly seven.

1.2.2. *Case* $9c - 12e + 16m^2 = 0$. Then we obtain $c = 4(3e-4m^2)/9$ and this implies $\widetilde{\Psi} = (9b+8em)^2/3$. Therefore the condition $\widetilde{\Psi} = 0$ yields $b = -8em/9$ and considering also the conditions (3.108) we detect that system (3.44) becomes

$$\begin{aligned} \dot{x} =& -(4m-3x)(8e+8mx-3x^2)/18,\\ \dot{y} =& -(8m-9x+6y)(2e+4my-3y^2)/18. \end{aligned}$$

However for these systems, calculations yield

$$\begin{aligned} H(X,Y,Z) = & -2^{-4}3^{-7}(X-2Y)(3X-4mZ)(3X^2-8mXZ-8eZ^2)(3Y^2\\ &- 4mYZ - 2eZ^2)(3X^2-6XY+3Y^2-4mXZ+4mYZ-2eZ^2) \end{aligned}$$

and according to Lemma 2.6 the above systems have invariant lines of total multiplicity nine, so we are out of the class we examine in this work.

2. *Subcase* $p \neq 0$ *and* $(1+p+2r)(p+2r+r^2) = 0$. We claim that in this case we may assume $1+p+2r = 0$ because the case $p+2r+r^2 = 0$ can be reduced to the first one via a rescaling. Indeed, considering Remark 3.9 we observe that by the rescaling $(x, y, t) \mapsto (x, -ry, t/r^2)$ in system (3.44) (which replaces the direction $y = -rx$ with $y = x$) the cubic homogeneous system associated to these system becomes

$$\dot{x}_1 = (p_1 + r_1)x_1^3, \quad \dot{y}_1 = p_1 x^2 y_1 + (r_1 - 1)x_1 y_1^2 + y_1^3,$$

where $p_1 = p/r^2$ and $r_1 = 1/r$. Therefore,

$$p_1 + 2r_1 + r_1^2 = \frac{p}{r^2} + \frac{2}{r} + \frac{1}{r^2} = \frac{1}{r^2}(p + 2r + 1) = 0$$

and this completes the proof of our claim.

So $p = -(2r + 1) \neq 0$ and considering the condition $k = d = h = 0$ (for the existence of a triplet in the direction $x = 0$) and (3.46) we detect that the condition $Eq_6'' = 0$ gives $l = g - 2m$. Now taking into consideration the conditions $k = d = h = 0$, $p = -(2r + 1)$, $l = g - 2m$ as well as the equations (3.45), (3.46) and (3.47) for the remaining three directions, we calculate:

(a) for the direction $y = 0$:

$$\begin{gathered} Eq_5' = g - 2m + (1 + 2r)W, \quad Eq_8' = e - 2mW + (r - 1)W^2, \\ Eq_{10}' = b - fW - W^3; \end{gathered} \tag{3.114}$$

(b) for the direction $y = x$:

$$\begin{aligned} Eq_6'' = 0, \quad Eq_9'' &= -c + e + f - 2mW + (2 + r)W^2, \\ Eq_{10}'' &= -a + b + (e - c)W - 2mW^2 + (1 + r)W^3; \end{aligned} \tag{3.115}$$

(c) for the direction $y = -rx$:

$$\begin{aligned} Eq_6''' &= \frac{r + 1}{r}\big[2m - g + (r - 1)W\big], \\ Eq_9''' &= f - c - e/r + \frac{g - 2m + gr}{r^2}W + \frac{1 + r + r^2}{r^2}W^2, \\ Eq_{10}''' &= b + ar - \frac{e + cr}{r}W + \frac{g - 2m + gr}{r^2}W^2 + \frac{1 + r}{r^2}W^3. \end{aligned} \tag{3.116}$$

Since $1 + 2r \neq 0$ the equation $Eq_5' = 0$ is linear in W and hence we obtain $W = (2m - g)/(1 + 2r)$. Therefore we obtain the equations

$$\begin{aligned} Eq_8' &= \frac{e(1 + 2r)^2 + (g - 2m)(-g + 4m + gr + 2mr)}{(1 + 2r)^2} = 0, \\ Eq_{10}' &= \frac{b(1 + 2r)^3 + (g - 2m)\big[(g - 2m)^2 + f(1 + 2r)^2\big]}{(1 + 2r)^3} = 0, \end{aligned}$$

which lead to the two conditions:

$$e = -\frac{(g - 2m)(4m - g + gr + 2mr)}{(1 + 2r)^2}, \quad b = -\frac{(g - 2m)\big[(g - 2m)^2 + f(1 + 2r)^2\big]}{(1 + 2r)^3}.$$

We observe that the equation $Eq_6''' = 0$ from (3.116) is linear in W if and only if $r - 1 \neq 0$. So in what follows we consider two possibilities: $r - 1 \neq 0$ and $r - 1 = 0$.

2.1. *Possibility* $r - 1 \neq 0$. Then the equation $Eq_6''' = 0$ gives $W = (g - 2m)/(r - 1)$ and considering the above conditions the equations $Eq_9''' = 0$ and $Eq_{10}''' = 0$ give us the following two new conditions:

$$
\begin{aligned}
a &= \frac{3(g-2m)\big[f(r-1)^2(1+2r)^2+3(g-2m)^2(1+r+r^2)\big]}{(r-1)^3(1+2r)^3},\\
c &= \frac{f(r-1)^2(1+2r)^2+3(g-2m)\big[-2m(2+r)^2+3g(1+r+r^2)\big]}{(r-1)^2(1+2r)^2}.
\end{aligned}
$$

So considering all the above conditions we could convince ourselves that $\mathrm{Res}_W(Eq_9'', Eq_{10}'') = 0$ which means that the obtained conditions guarantee the existence of at least one invariant line in the direction $y = 0$.

Thus setting all these conditions in system (3.44) we arrive at the following family of systems:

$$
\begin{aligned}
\dot{x} =& \Big[x+\frac{3(g-2m)}{(r-1)(1+2r)}\Big]\Big[f+\frac{3(g-2m)^2(1+r+r^2)}{(r-1)^2(1+2r)^2}\\
&+\frac{2(g-3m+gr-3mr+gr^2)}{(r-1)(1+2r)}x-(r+1)x^2\Big] \;\equiv\; L_1'(x)L_{2,3}'(x),\\
\dot{y} =& \Big[y-\frac{(g-2m)}{(1+2r)}\Big]\Big[f+\frac{(g-2m)^2}{(1+2r)^2}+\frac{4m-g+gr+2mr}{(1+2r)}x+\frac{(g-2m)}{(1+2r)}\,y\\
&-(1+2r)x^2+(r-1)xy+y^2\Big].
\end{aligned}
$$

We again need to detect if the two lines defined by the equation $L_{2,3}' = 0$ are real or complex and in the case when they are real, if one of them coincides with the invariant line $L_1' = 0$. So we calculate

$$
\begin{aligned}
\mathrm{Discrim}[L_{2,3}', x] &= 4(r-1)^2(1+2r)^2\lambda_1(f,g,m,r),\\
\mathrm{Res}_x(L_1', L_{2,3}') &= (r-1)^2(1+2r)^2\mu^{(1)}(c,g,m,r)
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_1 &= f(r-1)^2(1+r)(1+2r)^2-18gm(1+r)(1+r+r^2)\\
&\quad+g^2(2+r)^2(1+r+r^2)+3m^2(1+r)(7+7r+4r^2),\\
\mu^{(1)} &= c(2+r)^2(1+2r)^2-3(g-2m)(4g+gr+18mr+4gr^2).
\end{aligned}
\tag{3.117}
$$

We observe that

$$
\mathrm{sign}\big(\mathrm{Discrim}[L_{2,3}', x]\big) = \mathrm{sign}(\lambda_1),
$$

i.e. the invariant lines $L_{2,3}' = 0$ are real (respectively complex; coinciding) if $\lambda_1 > 0$ (respectively $\lambda_1 < 0$; $\lambda_1 = 0$). The invariant line $L_1' = 0$ coincides with one of the lines $L_{2,3}' = 0$ if and only if $\mu^{(1)} = 0$.

2.1.1. *Case* $\lambda_1 > 0$. Then we may use a new parameter u setting $\lambda_1 = u^2$ and since $(r-1)(1+r)(1+2r) \neq 0$ we obtain

$$
\begin{aligned}
f =& \frac{1}{(r-1)^2(1+r)(1+2r)^2}\big[18gm(1+r)(1+r+r^2)\\
&-g^2(2+r)^2(1+r+r^2)-3m^2(1+r)(7+7r+4r^2)+u^2\big].
\end{aligned}
\tag{3.118}
$$

This leads to the family of systems

$$
\begin{aligned}
\dot{x} &= -(r+1)\Big[x+\frac{3(g-2m)}{(r-1)(1+2r)}\Big]\Big[x-\frac{g-3m+gr-3mr+gr^2-u}{(r-1)(1+r)(1+2r)}\Big]\\
&\quad\times\Big[x-\frac{g-3m+gr-3mr+gr^2+u}{(r-1)(1+r)(1+2r)}\Big],\\
\dot{y} &= -\Big[y-\frac{(g-2m)}{1+2r}\Big]\Big[\frac{g^2(r^4+4r^3+10r^2+9r+3)}{(r-1)^2(r+1)(2r+1)^2}\\
&\quad-\frac{2gm(7r^2+13r+7)}{(r-1)^2(2r+1)^2}+\frac{m^2(r+1)(8r^2+29r+17)-u^2}{(r-1)^2(r+1)(2r+1)^2}-y^2\\
&\quad+\frac{4m-g+gr+2mr}{1+2r}x-\frac{(g-2m)}{1+2r}y+(1+2r)x^2+(1-r)xy\Big].
\end{aligned}
\tag{3.119}
$$

On the other hand for the value of f given in (3.118) we calculate

$$
\mu^{(1)} = -\frac{1}{r+1}(\gamma_1^2-u^2), \quad \gamma_1 = g(2+r)^2-9m(r+1) \tag{3.120}
$$

and since the condition $\mu^{(1)}=0$ leads to the coalescence of two invariant lines of the triplet, we examine two possibilities: $\mu^{(1)}\neq 0$ and $\mu^{(1)}=0$.

2.1.1.1. *Subcase* $\mu^{(1)}\neq 0$. Then $(\gamma_1-u)(\gamma_1+u)\neq 0$ and via the transformation

$$
\begin{gathered}
x_1 = \alpha x-\frac{3(g-2m)(1+r)}{\gamma_1+u}, \quad y_1 = \alpha y-\frac{(g-2m)(r-1)(1+r)}{\gamma_1+u},\\
t_1 = \frac{(\gamma_1+u)^2}{(r-1)^2(1+r)^2(1+2r)^2}t, \quad \alpha = \frac{(r-1)(1+r)(1+2r)}{\gamma_1+u}
\end{gathered}
$$

system (3.100) can be brought to the following family of systems (we keep the old notations for the variables):

$$
\begin{aligned}
\dot{x} &= -(r+1)x(x-1)(x-v),\\
\dot{y} &= y\big[-(1+r)v+(1+r)(1+v)x-(1+2r)x^2+(r-1)xy+y^2\big],
\end{aligned}
\tag{3.121}
$$

where $v=(\gamma_1-u)/(\gamma_1+u)\neq 0$. It remains to observe that this family of systems is a subfamily of (3.59) defined by the condition $p=-(2r+1)$ which was already examined.

2.1.1.2. *Subcase* $\mu^{(1)}=0$. Then $(\gamma_1-u)(\gamma_1+u)=0$ and we may assume $\gamma_1+u=0$ due to the change $u\to -u$. So setting $u=-\gamma_1\neq 0$ in system (3.119) and applying the transformation

$$
\begin{gathered}
x_1 = \alpha x+\frac{3(g-2m)(1+r)}{2\gamma_1}, \quad y_1 = \alpha y-\frac{(g-2m)(r-1)(1+r)}{2\gamma_1},\\
t_1 = \frac{4\gamma_1^2}{(r-1)^2(1+r)^2(1+2r)^2}t, \quad \alpha = \frac{(r-1)(1+r)(1+2r)}{2\gamma_1}
\end{gathered}
$$

we obtain the family of systems

$$
\dot{x} = -(r+1)x^2(x-1), \quad \dot{y} = y\big[(1+r)x-(1+2r)x^2+(r-1)xy+y^2\big]. \tag{3.122}
$$

We observe that this family of systems is a subfamily of (3.62) defined by the condition $p=-(2r+1)$, i.e. no new configurations could be obtained.

2.1.2. *Case* $\lambda_1 < 0$. Then we set $\lambda_1 = -u^2$ and since $r(2+r)(1+2r) \neq 0$ we obtain

$$\begin{aligned} f = &\frac{1}{(r-1)^2(1+r)(1+2r)^2}\big[18gm(1+r)(1+r+r^2) \\ &- g^2(2+r)^2(1+r+r^2) - 3m^2(1+r)(7+7r+4r^2) - u^2\big]. \end{aligned} \tag{3.123}$$

This leads to the system

$$\begin{aligned} \dot{x} =& -(r+1)\Big[x + \frac{3(g-2m)}{(r-1)(1+2r)}\Big]\Big[\Big(x - \frac{g-3m+gr-3mr+gr^2}{(r-1)(1+r)(1+2r)}\Big)^2 \\ &+ \frac{u^2}{(r-1)^2(1+r)^2(1+2r)^2}\Big], \\ \dot{y} =& -\Big[y - \frac{(g-2m)}{1+2r}\Big]\Big[\frac{g^2(r^4+4r^3+10r^2+9r+3)}{(r-1)^2(r+1)(2r+1)^2} \\ &- \frac{2gm(7r^2+13r+7)}{(r-1)^2(2r+1)^2} + \frac{m^2(r+1)(8r^2+29r+17)-u^2}{(r-1)^2(r+1)(2r+1)^2} - y^2 \\ &+ \frac{4m-g+gr+2mr}{1+2r}x - \frac{(g-2m)}{1+2r}y + (1+2r)x^2 + (1-r)xy\Big] \end{aligned} \tag{3.124}$$

and we examine two subcases: $\gamma_1 \neq 0$ and $\gamma_1 = 0$ (γ_1 is given in (3.120)).

2.1.2.1. *Subcase* $\gamma_1 \neq 0$. Then via the transformation

$$\begin{gathered} x_1 = \alpha x - \frac{3(g-2m)(1+r)}{\gamma_1}, \quad y_1 = \alpha y + \frac{(g-2m)(r-1)(1+r)}{\gamma_1}, \\ t_1 = \frac{(\gamma_1)^2}{(r-1)^2(1+r)^2(1+2r)^2}t, \quad \alpha = -\frac{(r-1)(1+r)(1+2r)}{\gamma_1} \end{gathered}$$

the above system can be brought to the canonical form

$$\begin{aligned} \dot{x} =& -(r+1)x\big[(x+1)^2+v^2\big], \\ \dot{y} =& y\big[-(1+r)(1+v^2) - 2(1+r)x - (1+2r)x^2 + (r-1)xy + y^2\big], \end{aligned} \tag{3.125}$$

where $v = u/\gamma_1 \neq 0$. We observe that these systems belong to the family (3.64) for which $p = -(2r+1)$ and hence no new configurations could be obtained.

2.1.2.2. *Subcase* $\gamma_1 = 0$. Considering (3.120) we obtain $m = g(2+r)^2/(9(1+r))$ and then systems (3.124) become

$$\begin{aligned} \dot{x} =& -(1+r)\Big[x - \frac{g)}{3(1+r)}\Big]\Big[\Big(x - \frac{g)}{3(1+r)}\Big)^2 + \frac{u^2}{(-1+r)^2(1+r)^2(1+2r)^2}\Big], \\ \dot{y} =& -\Big[y + \frac{g(r-1)}{9(1+r)}\Big]\Big[\frac{g^2(r-1)^2(1+2r)^2(11+14r+2r^2)+81(1+r)u^2}{81(r-1)^2(1+r)^2(1+2r)^2} \\ &- \frac{g(7+10r+r^2)}{9(1+r)}x + \frac{g(r-1)}{9(1+r)}y + (1+2r)x^2 + (1-r)xy - y^2\Big]. \end{aligned}$$

Since $ur(1+r)(r-1)(1+2r) \neq 0$ we can apply the transformation

$$\begin{gathered} x_1 = \alpha x - \frac{g(r-1)(1+2r)}{3u}, \quad y_1 = \alpha y + \frac{g(r-1)^2(1+2r)}{9u}, \\ t_1 = \frac{u^2}{(r-1)^2(1+r)^2(1+2r)^2}t, \quad \alpha = \frac{(r-1)(1+r)(1+2r)}{u} \end{gathered}$$

and we arrive at the family of systems

$$\dot{x} = -(1+r)x(x^2+1), \quad \dot{y} = y\big[-1-r-(1+2r)x^2+(r-1)xy+y^2\big]. \quad (3.126)$$

It remains to observe that this is a subfamily of (3.68) defined by $p=-(2r+1)$. So no new configurations could be obtained.

2.1.3. *Case* $\lambda_1 = 0$. Considering (3.117) and solving the equation $\lambda_1 = 0$ with respect to the parameter f it is clear that we obtain (3.123) for $u=0$. This leads to system (3.124) with $u=0$, which we denote by $(3.124)_{\{u=0\}}$. We observe that in this case we obtain $\mu^{(1)} = -\gamma_1^2/(1+r)$ and we again consider two subcases: $\mu^{(1)} \neq 0$ and $\mu^{(1)} = 0$.

2.1.3.1. *Subcase* $\mu^{(1)} \neq 0$. Then $\gamma_1 \neq 0$ and via the transformation

$$x_1 = \alpha x + \frac{g-3m+gr-3mr+gr^2}{\gamma_1}, \quad y_1 = \alpha y + \frac{(g-2m)(r-1)(1+r)}{\gamma_1},$$
$$t_1 = \frac{\gamma_1^2}{(r-1)^2(1+r)^2(1+2r)^2}t, \quad \alpha = -\frac{(r-1)(1+r)(1+2r))}{\gamma_1}$$

systems (3.124) with $\{u=0\}$ can be brought to the family of systems

$$\begin{aligned} \dot{x} =& -(1+r)x^2(x-1), \\ \dot{y} =& y\big[-r+2rx-(1+2r)x^2+(1-r)y+(r-1)xy+y^2\big]. \end{aligned} \quad (3.127)$$

We notice that the above system belongs to the family (3.69) for $p=-(2r+1)$ and this family was already examined.

2.1.3.2. *Possibility* $\mu^{(1)} = 0$. In this case we have $\lambda_1 = \gamma_1 = 0$ and this leads to the existence of a triple invariant line in the direction $x=0$. Considering (3.120) the condition $\gamma_1 = 0$ gives $m = g(2+r)^2/(9(1+r))$. Therefore taking into account the condition $u=0$ system (3.124) becomes

$$\begin{aligned} \dot{x} =& -(r+1)\Big[x - \frac{g}{3(1+r)}\Big]^3, \\ \dot{y} =& \Big[y + \frac{g(r-1)}{9(1+r)}\Big]\Big[-\frac{g^2(11+14r+2r^2)}{81(1+r)^2} + \frac{g(7+10r+r^2)}{9(1+r)}x - \frac{g(r-1)}{9(1+r)}y \\ & -(1+2r)x^2+(r-1)xy+y^2\Big]. \end{aligned}$$

However for these system we have (see Notation 2.4):

$$\begin{aligned} H(X,Y,Z) =& \frac{\alpha}{(1+r)^{11}}(3X+3rX-gZ)^3(9rX+9r^2X+9Y+9rY-gZ-2grZ) \\ & \times(9X+9rX-9Y-9rY-2gZ-grZ)^2(9Y+9rY-gZ+grZ), \end{aligned}$$

where $\alpha \in \mathbb{R}$. Therefore by Lemma 2.6 the above system have invariant lines of total multiplicity 8, i.e. we are out of the family we study here.

2.2. *Possibility* $r-1=0$. So $r=1$ and considering (3.116) we obtain $Eq_6''' = (r+1)(2m-g)/r = 0$ which implies $g=2m$. Then from (3.114) we obtain that $W=0$ must be a common solution of the equations $Eq_5' = Eq_8' = Eq_{10}' = 0$. This implies $e=b=0$ and we obtain the following conditions on the parameters

$$k=d=h=l=e=b=0, \quad p=-3, \quad g=2m, \quad r=1, \qquad (3.128)$$

which guarantees the existence of a triplet of parallel invariant lines in the direction $x=0$ and one invariant line in the direction $y=0$. For the other two directions $y=x$ and $y=-rx$ considering (3.115) and (3.116) we obtain respectively:

$$Eq_9'' = -c+f-2mW+3W^2, \quad Eq_{10}'' = -a-cW-2mW^2+2W^3$$

and

$$Eq_9''' = f-c+2mW+3W^2, \quad Eq_{10}''' = a-cW+2mW^2+2W^3.$$

We calculate $\mathrm{Res}_W(Eq_9'', Eq_{10}'') = \mathrm{Res}_W(Eq_9''', Eq_{10}''') = \Psi'(a,c,f,m)$, where

$$\Psi'(a,c,f,m) = 27a^2 + 2am(9c+4m^2) - (c-f)(c^2+4cf+4f^2+4fm^2). \tag{3.129}$$

So in order to have invariant lines of total multiplicity seven the condition $\Psi'=0$ is necessary. This equation is quadratic in a and we calculate

$$\mathrm{Discrim}[\Psi', a] = 4(3c-3f+m^2)(3c+6f+4m^2)^2$$

and clearly we could have a real solution of the equation $\Psi'=0$ with respect to the parameter a only if either $(3c-3f+m^2)\geq 0$ or $(3c+6f+4m^2)=0$. We consider each one of these cases.

2.2.1. *Case* $(3c-3f+m^2)\geq 0$. Then setting $3c-3f+m^2=u^2\geq 0$ we obtain $f=(3c+m^2-u^2)/3$ and we obtain

$$\begin{aligned}\Psi' &= (27a+9cm+4m^3+9cu+6m^2u-2u^3)\\ &\quad\times\big(27a+9cm+4m^3-9cu-6m^2u+2u^3\big)/27 = 0.\end{aligned}$$

From the change $u\to -u$ we may assume without loss of generality that the first factor vanishes and we have the condition

$$a = -(m+u)(9c+4m^2+2mu-2u^2)/27.$$

So considering also the conditions (3.128) we detect that system (3.44) has the form

$$\begin{aligned}\dot{x} &= -\frac{(m+u-3x)}{27}\big[9c+4m^2+2mu-2u^2+6(2m-u)x-18x^2\big])\\ &\equiv\ -L_1''(x)L_{2,3}''(x)/27,\\ \dot{y} &= y(3c+m^2-u^2+6mx-9x^2+3y^2)/3.\end{aligned} \tag{3.130}$$

So we need again to detect if the two lines defined by the equation $L_{2,3}''=0$ are real or complex and in the case when they are real, if one of them coincides with the invariant line $L_1''=0$. So we calculate

$$\begin{aligned}\mathrm{Discrim}[L_{2,3}'', x] &= 108(6c+4m^2-u^2)\equiv 108\lambda_2(c,m,u),\\ \mathrm{Res}_x(L_1'', L_{2,3}'') &= 27(3c+2m^2-2u^2)\equiv 27\mu^{(2)}(c,m,u).\end{aligned} \tag{3.131}$$

Therefore $\mathrm{sign}\big(\mathrm{Discrim}[L_{2,3}'', x]\big) = \mathrm{sign}(\lambda_2)$ and hence the invariant lines $L_{2,3}''=0$ are real (respectively complex; coinciding) if $\lambda_2>0$ (respectively $\lambda_2<0$; $\lambda_2=0$). The invariant line $L_1''=0$ coincides with one of the lines $L_{2,3}''=0$ if and only if $\mu^{(2)}=0$.

2.2.1.1. *Subcase* $\lambda_2>0$. Then we may use a new parameter w setting $\lambda_1=3w^2$ and we obtain $c=(u^2+3w^2-4m^2)/6$. This leads to the system

$$\begin{aligned}\dot{x} =&(2m-u-3w-6x)(2m-u+3w-6x)(m+u-3x)/54,\\ \dot{y} =&-y(2m^2+u^2-3w^2-12mx+18x^2-6y^2)/6.\end{aligned} \tag{3.132}$$

On the other hand for the value of c given above, we calculate $\mu^{(2)} = -3(u-w)(u+w)/2$ and since the condition $\mu^{(2)} = 0$ leads to the coalescence of two invariant lines of the triplet, we examine two possibilities: $\mu^{(2)} \neq 0$ and $\mu^{(2)} = 0$.

2.2.1.1.1. *Possibility* $\mu^{(2)} \neq 0$. Then $(u-w)(u+w) \neq 0$ and via the transformation

$$x_1 = \frac{2x}{w-u} + \frac{2(m+u)}{3(u-w)}, \quad y_1 = \frac{2y}{w-u}, \quad t_1 = (u-w)^2 t/4$$

system (3.132) can be brought to the following family of systems (we keep the old notations for the variables):

$$\dot{x} = -2x(x-1)(x-v), \quad \dot{y} = y\big[-2v + 2(1+v)x - 3x^2 + y^2\big],$$

where $v = (u+w)/(u-w) \neq 0$. It remains to observe that this family of systems is a subfamily of (3.121) defined by the condition $r = 1$ and this family was already examined.

2.2.1.1.2. *Possibility* $\mu^{(2)} = 0$. In this case $(u-w)(u+w) = 0$ and we may assume $w = u$ (because of the change $w \to -w$). Then system (3.132) becomes

$$\dot{x} = 2(m-2u-3x)(m+u-3x)^2/27, \quad \dot{y} = -y(m^2 - u^2 - 6mx + 9x^2 - 3y^2)/3.$$

Since in this case we have $\lambda_2 = 3u^2 > 0$ (i.e. $u \neq 0$) via the transformation

$$x_1 = -\frac{x}{u} + \frac{m+u}{3u}, \quad y_1 = -\frac{y}{u}, \quad t_1 = u^2 t$$

we obtain the system

$$\dot{x} = -2x^2(1-x), \quad \dot{y} = y(2x - 3x^2 + y^2).$$

We notice that this system is contained in the family (3.122) for $r = 1$, i.e. no new configurations could be obtained.

2.2.1.2. *Subcase* $\lambda_2 < 0$. Setting $\lambda_2 = -3w^2 < 0$ we obtain $c = (u^2 - 3w^2 - 4m^2)/6$ and then system (3.130) becomes

$$\begin{aligned} \dot{x} =& (m+u-3x)\big[(2m-u-6x)^2 + 9w^2\big]/54, \\ \dot{y} =& -y(2m^2 + u^2 + 3w^2 - 12mx + 18x^2 - 6y^2)/6. \end{aligned} \tag{3.133}$$

For the above value of the parameter c we calculate $\mu^{(2)} = -3(u^2+w^2)/2 \neq 0$ since $\lambda_2 = -3w^2 < 0$. We consider two possibilities: $u \neq 0$ and $u = 0$.

2.2.1.2.1. *Possibility* $u \neq 0$. Then we may apply the transformation

$$x_1 = \frac{2x}{u} - \frac{2(m+u)}{2u}, \quad y_1 = \frac{2y}{u}, \quad t_1 = u^2 t/4$$

getting the system

$$\dot{x} = -2x\big[(x+1)^2 + v^2\big], \quad \dot{y} = -y(2 + 2v^2 + 4x + 3x^2 - y^2),$$

where $v = w/u \neq 0$. We observe that the above systems form a subfamily of the family (3.125) defined by $r = 1$ and this family was already examined.

2.2.1.2.2. *Possibility* $u = 0$. In this case via the transformation

$$x_1 = \frac{x}{2w} - \frac{2m}{3w}, \quad y_1 = \frac{y}{2w} - \frac{m}{3w}, \quad t_1 = 4w^2 t$$

system (3.133) will be brought to the system

$$\dot{x} = -2x(x^2+1), \quad \dot{y} = y(-2 - 3x^2 + y^2).$$

It remains to notice that this system is contained in the family (3.126) for $r = 1$, i.e. no new configurations could be obtained.

2.2.1.3. *Subcase* $\lambda_2 = 0$. Considering (3.131) this condition gives $c = (u^2 - 4m^2)/6$ and then system (3.130) becomes

$$\begin{aligned} \dot{x} =& (2m - u - 6x)^2(m + u - 3x)/54,\\ \dot{y} =& -y(2m^2 + u^2 - 12mx + 18x^2 - 6y^2)/6. \end{aligned} \tag{3.134}$$

For the above value of the parameter c we calculate $\mu^{(2)} = -3u^2/2$ and we examine two possibilities: $\mu^{(2)} \neq 0$ and $\mu^{(2)} = 0$.

2.2.1.3.1. *Possibility* $\mu^{(2)} \neq 0$. Then $u \neq 0$ and via the transformation

$$x_1 = \frac{2x}{u} - \frac{2m - u}{3u}, \quad y_1 = \frac{2y}{u}, \quad t_1 = u^2t/4$$

we obtain the system

$$\dot{x} = -2(-1 + x)x^2, \quad \dot{y} = y(-1 + 2x - 3x^2 + y^2).$$

We notice that this system belongs to the family (3.127) for $r = 1$, i.e. no new configurations could be obtained.

2.2.1.3.2. *Possibility* $\mu^{(2)} = 0$. In this case $u = 0$ and system (3.134) becomes

$$\dot{x} = 2(m - 3x)^3/29, \quad \dot{y} = -y(m^2 - 6mx + 9x^2 - 3y^2)/3.$$

For these systems we calculate (see Notation 2.4)

$$H(X, Y, Z) = 23^{-9}Y(3X - mZ)^3(3X + 3Y - mZ)^2(-3X + 3Y + mZ)^2$$

and hence, by Lemma 2.6 we have invariant lines of total multiplicity nine, i.e. we are not in the class of systems with invariant lines of total multiplicity exactly seven.

2.2.2. *Case* $(3c + 6f + 4m^2) = 0$. Then we obtain $c = -2(3f + 2m^2)/3$ and this implies (see (3.129)) $\Psi' = (27a - 18fm - 8m^3)^2/27$. Therefore the condition $\Psi' = 0$ yields $a = 2(9fm + 4m^3)/27$ and considering also the conditions (3.128) we detect that systems (3.44) have the form

$$\dot{x} = 2(m - 3x)(9f + 4m^2 - 6mx + 9x^2)/27, \quad \dot{y} = y(f + 2mx - 3x^2 + y^2).$$

However for these systems calculations yield

$$\begin{aligned} H(X, Y, Z) = & -2 \cdot 3^{-8}Y(3X - mZ)(9X^2 - 6mXZ + 9fZ^2 + 4m^2Z^2)\\ & \times (9X^2 + 18XY + 9Y^2 - 6mXZ - 6mYZ + 9fZ^2 + 4m^2Z^2)\\ & \times (9X^2 - 18XY + 9Y^2 - 6mXZ + 6mYZ + 9fZ^2 + 4m^2Z^2) \end{aligned}$$

and according to Lemma 2.6 the above system has invariant lines of total multiplicity nine.

Existence of a triplet in a direction different from $x = 0$. Considering Remark 3.9 it is sufficient to consider two cases: when either the triplet exists in the direction $y = 0$, or $y = x$.

We claim that in each one of these cases the corresponding systems (3.44) could be brought via an affine transformation to systems, which form a subfamily of systems (3.44) possessing a triplet in the direction $x = 0$. To prove this claim we consider each one of the possibilities.

1. *A triplet in the direction* $y = 0$. Considering (3.45) it is clear that for the existence of a triplet in this direction it is necessary and sufficient $l = p = e = m = 0$ and $r = 1$. In this case system (3.44) becomes

$$\dot{x} = a + cx + dy + gx^2 + 2hxy + ky^2 + x^3, \quad \dot{y} = b + fy + y^3.$$

So by the transformation $x_1 = y$, $y_1 = x$ this system becomes

$$\dot{x}_1 = a_1 + c_1x_1 + x_1^3, \quad \dot{y}_1 = b_1 + e_1x_1 + f_1y_1 + l_1x_1^2 + 2m_1x_1y_1 + n_1y_1^2 + y_1^3$$

where

$$a_1 = b, \quad b_1 = a, \quad c_1 = f, \quad e_1 = d, \quad f_1 = c, \quad l_1 = k, \quad m_1 = h, \quad n_1 = g$$

are free parameters. Then we may consider $n_1 = 0$ due to a translation and going back to the old notations we obtain the family of systems

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + ex + fy + lx^2 + 2mxy + y^3$$

which possess a triplet of invariant lines in the direction $x = 0$. Evidently this family of systems is a subfamily of (3.44) defined by the conditions $p = d = h = k = g = 0$ and $r = 1$ and hence our claim is proved in this case.

2. *Triplet in the direction* $y = x$. In this case considering (3.46) we must impose the equations $Eq_6'' = 0$ and $Eq_9'' = 0$ to vanish identically. So by (3.46) we arrive at the conditions

$$c = -d + e + f, \quad k = -h + m, \quad l = g + h - m, \quad p = 3, \quad r = -2$$

and this leads to the family of systems

$$\begin{aligned}
&\dot{x} = a + (e - d + f)x + dy + gx^2 + 2hxy + (m - h)y^2x^3, \\
&\dot{y} = b + ex + fy + (g + h - m)x^2 + 2mxy + 3x^2y - 3xy^2 + y^3.
\end{aligned}$$

Applying the transformation $x_2 = x - y$ and $y_2 = -y$ these systems can be brought to the form

$$\begin{aligned}
&\dot{x}_2 = a_2 + c_2x_2 + g_2x_2^2 + x_2^3, \\
&\dot{y}_2 = b_2 + e_2x_2 + f_2y_2 + l_2x_2^2 + 2m_2x_2y_2 + n_2y_2^2 - 3xy^2 + y^3,
\end{aligned}$$

where

$$\begin{gathered}
a_2 = a - b, \quad b_2 = -b, \quad c_2 = f - d, \quad e_2 = -e, \quad f_2 = e + f, \quad g_2 = m - h, \\
l_2 = m - g - h, \quad m_2 = g + h, \quad n_2 = -(g + h + m)
\end{gathered}$$

are free parameters. We may consider $n_2 = 0$ via a translation and returning to the old notations we obtain the family of systems

$$\dot{x} = a + cx + gx^2 + x^3, \quad \dot{y} = b + ex + fy + lx^2 + 2mxy + ny^2 - 3xy^2 + y^3$$

which evidently possess a triplet of invariant lines in the direction $x = 0$. It remains to observe that this family of systems is a subfamily of (3.44) defined by the conditions $p = d = h = k = 0$, $p = 3$ and $r = -2$ and this completes the proof of our claim.

3.4. **Systems with configuration type $\mathfrak{T} = (2,2,2)$.** In this subsection we construct a cubic system with four real infinite singular points which has six invariant affine straight lines with the configuration of the type $\mathfrak{T} = (2,2,2)$, having total multiplicity seven, as always the invariant straight line at infinity is included. According to Theorem 2.10, in this case the condition $\mathcal{V}_3 = 0$ is necessary.

In [6, Subsection 3.4.1] it was proved that in this case via a linear transformation and time rescaling the associated cubic homogeneous system can be brought to the form

$$\dot{x} = rx^3 + (2+r)x^2y, \quad \dot{y} = (1+2r)xy^2 + y^3. \tag{3.135}$$

Consider generic cubic systems with cubic homogeneities (3.135). Since $r \neq 0$ via a translation we may assume $n = 0$ in system (2.4), i.e. a system possessing invariant lines in the configuration $(2,2,2)$ necessarily belongs to the following family:

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + rx^3 + (2+r)x^2y, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + (1+2r)xy^2 + y^3, \quad r(r+1) \neq 0. \end{aligned} \tag{3.136}$$

In what follows we shall determine necessary and sufficient conditions for a systems (3.136) to have invariant affine straight lines with the configuration of the type $\mathfrak{T} = (2,2,2)$.

Considering Remark 2.13 for the homogeneous systems (3.135), corresponding to system (3.136), we calculate

$$H(X,Y,Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = X^2(X-Y)Y^2(rX+Y)^2.$$

So each one of the invariant lines $x = 0$, $y = 0$ and $rx+y = 0$ of system (3.135) is of multiplicity two and the line $y = x$ is of multiplicity one. However for some values of the parameter r the common divisor $\gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ could contain additional factors (see Notation 2.4 and Lemma 2.6). Since the factor $(X - Y)$ depends on X as well as on Y, in order to increase its multiplicity it is necessary

$$\mathrm{Res}_X(\mathcal{G}_2/H,\, \mathcal{G}_1/H) = \mathrm{Res}_Y(\mathcal{G}_2/H, \mathcal{G}_1/H) = 0.$$

We calculate:

$$\begin{aligned} \mathrm{Res}_X(\mathcal{G}_2/H, \mathcal{G}_1/H) &= 24(r-1)(1+r)^2(2+r)(1+2r)Y^3 = 0, \\ \mathrm{Res}_Y(\mathcal{G}_2/H, \mathcal{G}_1/H) &= -24(r-1)(1+r)^2(2+r)(1+2r)X^3 = 0, \end{aligned}$$

and since $r(r+1) \neq 0$ the condition $(r-1)(2+r)(1+2r) = 0$ must hold. However for systems (3.135) with the conditions $(r-1)(2+r)(1+2r) = 0$ calculations yield

$$H(X,Y,Z) = \begin{cases} 3X^2(X-Y)Y^2(X+Y)^3 & \text{if } r = 1; \\ -6X^3(X-Y)(2X-Y)^2Y^2 & \text{if } r = -2; \\ 3X^2(X-2Y)^2(X-Y)Y^3/8 & \text{if } r = -1/2, \end{cases}$$

and hence, by Remark 2.13 we deduce that systems (3.136) could not possess a couple of parallel invariant line in the direction $y = x$. So systems (3.136) could possess three couples of invariant straight lines only in the directions $x = 0$, $y = 0$ and $rx + y = 0$. In [6, Subsection 3.4.2] these directions were examined and the following statement was proved:

Lemma 3.16 ([6]). *Systems* (3.136) *have one couple of parallel invariant lines in the direction $x=0$ and one such couple in the direction $y=0$ if and only if the following conditions hold:*

$$\begin{gathered}(r+2)(2r+1)\neq 0,\quad k=l=h=0,\quad g=2m,\quad d=c(2+r)/r,\\ a=2cm/r,\quad e=f(1+2r)+4m^2/(1+2r),\\ b=-2\big[4m^3+fm(1+2r)^2\big]/(1+2r)^3.\end{gathered}\tag{3.137}$$

We examine the third direction $y=-rx$ in which we could also have a couple of parallel invariant straight lines. Considering equation (2.5), Remark 2.12 and the conditions (3.137) we obtain

$$\begin{gathered}Eq_8=(1+r)(f-cr)+\frac{4m^2}{1+2r}-2mW-(r-1)W^2=0,\\ Eq_{10}=2cm-\frac{8m^3}{(1+2r)^3}-\frac{2fm}{1+2r}-(2c+f+cr)W-W^3=0,\\ R_W^{(1)}(Eq_8,Eq_{10})=2c(r-1)-2fr(r-1)-\frac{12m^2r}{1+2r}=0.\end{gathered}$$

It is clear that in order to have two common solutions of the equations $Eq_8=0$ and $Eq_{10}=0$ the condition $r-1\neq 0$ is necessary. Therefore we obtain $c=fr+\frac{6m^2r}{(r-1)(1+2r)}$ and we calculate

$$\begin{aligned}R_W^{(0)}(Eq_8,Eq_{10})&=-\frac{144m^2r^2(1+r)^2\big[f(r-1)^2(1+2r)^2+3m^2(1-2r+4r^2)\big]^2}{(r-1)^3(1+2r)^6}\\&=0\end{aligned}$$

and we have either $m=0$ or $m\neq 0$ and

$$f=-\frac{3m^2(1-2r+4r^2)}{(-1+r)^2(1+2r)^2}.$$

We claim, that in the case $m=0$ we arrive at systems possessing invariant lines of total multiplicity 8. Indeed, suppose $m=0$. Then considering (3.137) and the above expression for the parameter c we obtain the conditions

$$k=l=h=g=m=a=b=0,\quad c=fr,\quad d=f(2+r),\ e=f(1+2r).$$

Thus we obtain the family of systems

$$\dot x=(f+x^2)(rx+2y+ry),\quad \dot y=(f+y^2)(x+2rx+y)$$

with the condition $r(r^2-1)(r+2)(2r+1)\neq 0$. This system possesses the following seven invariant affine straight lines:

$$x^2+f=0,\quad y^2+f=0,\quad y-x=0,\quad (rx+y)^2+f(1+r)^2=0$$

and this proves our claim.

In what follows we assume $m\neq 0$ and we examine the second possibility: $f=-\frac{3m^2(1-2r+4r^2)}{(-1+r)^2(1+2r)^2}$. Then we obtain $c=-\frac{9m^2r}{(-1+r)^2(1+2r)^2}$ and taking into account the conditions (3.137) this leads to the 2-parameter family of systems:

$$\begin{aligned}\dot x&=\Big[x-\frac{3m}{(r-1)(1+2r)}\Big]\Big[x+\frac{3m}{(r-1)(1+2r)}\Big]\big[rx+(2+r)y+2m\big],\\ \dot y&=\Big[y-\frac{m}{(r-1)}\Big]\Big[y+\frac{m(4r-1)}{(r-1)(1+2r)}\Big]\Big[(1+2r)x+y-\frac{2m}{1+2r}\Big].\end{aligned}$$

As it was mentioned earlier, for these systems the condition $mr(r+1)(r+2)(2r+1)(r-1)\neq 0$ must hold. Then via the transformation

$$(x,y,t)\mapsto\Big(\frac{3m(1-2x)}{(r-1)(1+2r)},\ \frac{m(1+2r+6ry)}{(r-1)(1+2r)},\ \frac{(r-1)^2(1+2r)^2}{36m^2r}t\Big)$$

we arrive at the 1-parameter family of systems

$$\dot x = x(x-1)\big[x-(r+2)y-1-r\big],\quad \dot y = y(y+1)\big[ry-(1+2r)x+1+r\big]. \tag{3.138}$$

These systems possess six distinct invariant affine straight lines

$$L_1: x=0,\quad L_2: x=1,\quad L_3: y=0,\quad L_4: y=-1,\quad L_5: y=x,\ L_6: y=x-1$$

and the following nine finite singularities:

$$M_1(0,0),\quad M_2(0,-1),\quad M_3(0,-1-1/r),\quad M_4(1,0),\quad M_5(1,1),$$
$$M_6(1,-1),\quad M_7(-1,-1),\quad M_8(1+r,0),\quad M_9(1/(1+r),-1+1/(1+r)).$$

It is easy to determine that 6 of these singularities are located at the intersections of the above invariant lines, more precisely these are the singular points M_i for $i\in\{1,2,4,5,6,7\}$. The singular point M_3 (respectively M_8; M_9) is located on the invariant line L_1 (respectively L_3; L_6). Moreover we have three singular points located at the intersections of three invariant lines. More exactly L_1, L_3 and L_5 intersect at the point M_1; L_1, L_4 and L_6 intersect at the point M_2 and L_2, L_3 and L_6 intersect at the point M_4.

To determine all the possible configurations for system (3.138) we have to examine the positions of the singularities M_3 (located on the invariant line $x=0$), M_8 (located on the invariant line $y=0$) and M_9 (located on the invariant line $x-y=1$) depending on the value of the parameter r.

Considering Notation 3.6 and the coordinates $x_8=1+r$, $y_3=-(1+r)/r$ and $x_9=1/(1+r)$ it is not too hard to detect the following implications:

(i) $x_8<0 \Rightarrow -1<y_3<0,\ x_9<0 \Rightarrow M_8\prec M_1\prec M_4,\ M_2\prec M_3\prec M_1,\ M_9\prec M_2\prec M_4$;

(ii) $0<x_8<1 \Rightarrow y_3>0,\ x_9>1 \Rightarrow M_1\prec M_8\prec M_4,\ M_2\prec M_1\prec M_3,\ M_2\prec M_2\prec M_9$;

(iii) $x_8>1 \Rightarrow y_3<-1,\ 0<x_9<1 \Rightarrow M_1\prec M_4\prec M_8,\ M_3\prec M_2\prec M_1,\ M_2\prec M_9\prec M_4$.

We observe that in the case (i) (respectively, case (ii); (iii)) we arrive at the configuration given by Figure 2(a) (respectively, Figure 2(b); Figure 2(c)). However it is not too difficult to detect that each one of these three configurations is equivalent (see Definition 1.3 to Config. 7.66).

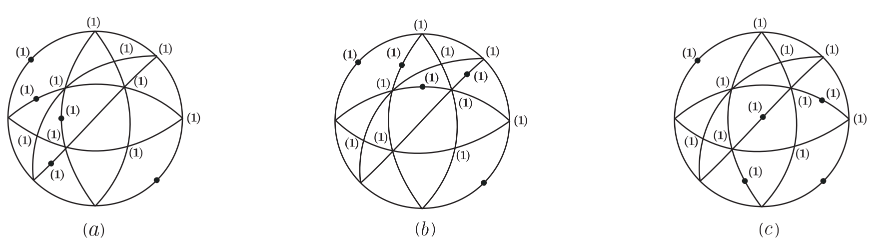


FIGURE 2. Configurations of invariant lines of type (2,2,2)

3.5. **Systems with configuration type $\mathfrak{T} = (2,2,1,1)$.** In this subsection we construct a cubic system with four real infinite singular points which has six invariant affine straight lines with the configuration of the type $\mathfrak{T} = (2,2,1,1)$, having total multiplicity seven, as always the invariant straight line at infinity is included. According to Theorem 2.10 in this case necessarily the condition $\mathcal{V}_5 = 0$ holds.

3.5.1. *Construction of the associated homogeneous cubic system.* As a first step we construct the associated to the system (3.1) homogeneous cubic system for which the condition $\mathcal{V}_5 = 0$ is fulfilled. Since we have 4 real infinite distinct singularities, according to Lemma 2.14 we consider the family of systems

$$\begin{aligned}\dot{x} =&(p+r)x^3 + (s+v)x^2y + qxy^2,\\ \dot{y} =&px^2y + (r+v)xy^2 + (q+s)y^3, \quad rs(r+s) \neq 0,\end{aligned} \tag{3.139}$$

and we shall force the condition $\mathcal{V}_5 = 0$ to be satisfied.

We observe that the invariant polynomial $\mathcal{V}_5$ is a homogeneous polynomial of degree four in x and y. So we shall use the following notations:

$$\mathcal{V}_5 = \sum_{j=0}^{4} \mathcal{V}_{5j} x^{4-j} y^j.$$

Calculating the value of the polynomial $\mathcal{V}_5$ for system (3.38) we obtain

$\mathcal{V}_{51} = 128pq(pr + 2r^2 - ps - rs + rv)/9, \quad \mathcal{V}_{53} = -128pq(qr - qs + rs - 2s^2 - sv)/9$

and we consider two cases: $pq \neq 0$ and $pq = 0$. **(1)** *Case $pq \neq 0$.* Then $\mathcal{V}_{51} = \mathcal{V}_{53} = 0$ give us

$$p(r-s) + r(2r - s + v) = 0 = q(r-s) + s(r - 2s - v).$$

(a) *Subcase $r - s \neq 0$.* In this case we obtain

$$p = (-2r^2 + rs - rv)/(r-s), \quad q = (-rs + 2s^2 + sv)/(r-s)$$

and this implies $\mathcal{V}_5 = 0$. Then after a time rescaling we obtain the family of systems

$$\begin{aligned}\dot{x} &= r(r+v)x^3 - (r-s)(s+v)x^2y + s(r-2s-v)xy^2,\\ \dot{y} &= r(2r-s+v)x^2y - (r-s)(r+v)xy^2 - s(s+v)y^3\end{aligned} \tag{3.140}$$

with $rs(r-s)(r+s)(2r-s+v)(r-2s-v) \neq 0$ for which we calculate

$$H(X,Y,Z) = (r-s)X(X-Y)^2Y(rX+sY)^2.$$

So we observe that these system could have the two couple of parallel invariant affine straight lines in the directions $x - y = 0$ and $rx + sy = 0$. However it is more convenient to have such lines in the directions $x = 0$ and $y = 0$. So applying the change

$$(x,y,t) \mapsto \Big(\frac{rx+sy)}{r},\ x-y,\ \frac{r}{(r+s)(r-s)}t\Big)$$

systems (3.140) can be brought to the form

$$\dot{x} = x^2\Big[rx + \frac{2rs + rv + sv}{r-s}\,y\Big], \quad \dot{y} = y^2\Big[\frac{(r^2 + s^2 + rv + sv)}{r-s}\,y + sy\Big].$$

Finally, since $r - s \neq 0$ setting a new parameter $u = (r+s)(s+v)/(r-s)$ we obtain the 3-parameter family of homogeneous systems:

$$\dot{x} = x^2\big[rx + (s+u)y\big], \quad \dot{y} = y^2\big[(r+u)x + sy\big] \tag{3.141}$$

with $rs(r+s)(r-s) \neq 0$.

(b) *Subcase* $r-s=0$. Setting $s=r$ we calculate $\mathcal{V}_{51} = 128pqr(r+v)/9$ and since $pqr \neq 0$ we obtain $v=-r$. Then we obtain $\mathcal{V}_{52} = 64pqr(p+q+3r)/3 = 0$ which implies $p=-(q+3r)$. In this case we obtain $\mathcal{V}_5 = 0$ and applying the time rescaling $t \to -t$ we arrive at the following family of systems

$$\dot{x} = x\big[(q+2r)x^2 - qy^2\big], \quad \dot{y} = y\big[(q+3r)x^2 - (q+r)y^2\big].$$

with $qr(q+3r) \neq 0$. In this case we apply the transformation $(x,y,t) \mapsto (x+y,\ y-x,\ t/(2r))$ and we obtain the systems

$$\dot{x} = x^2\big[x + (2q+3r)y/r\big], \quad \dot{y} = y^2\big[(2q+3r)x/r + y\big].$$

Setting a new parameter $u_1 = (2q+3r)/r$ we arrive at the systems

$$\dot{x} = x^2(x+u_1y), \quad \dot{y} = y^2(u_1x+y), \quad u_1 \neq 1.$$

We observe that these systems could be a subfamily of systems (3.141) if we allow $r=s$. Indeed, setting $s=r\neq 0$ in systems (3.141) we may assume $r=1$ (due to the time rescaling $t \to t/r$) and then for $u=u_1-1$ we obtain the above systems.

(2) *Case* $pq=0$. Then without loss of generality we may assume $p=0$ due to the change $(x,y,p,q,r,s,v) \mapsto (y,x,q,p,s,r,v)$ which conserves systems (3.38). For these systems for $p=0$ we calculate

$$\mathcal{V}_{50} = \mathcal{V}_{51} = \mathcal{V}_{52} = \mathcal{V}_{53} = 0, \quad \mathcal{V}_{54} = -32q(q+2r+2s+v)(qr+rs+s^2-sv)/9 = 0.$$

So we consider three subcases: (i) $q=0$; (ii) $q\neq 0$ and $v=-(q+2r+2s)$ and (iii) $q(q+2r+2s+v)\neq 0$ and $v=(qr+rs+s^2)/s$.

(a) *Subcase* $q=0$. In this case we obtain the family of systems

$$\dot{x} = x^2\big[rx+(s+v)y\big], \quad \dot{y} = y^2\big[(r+v)x+sy\big]$$

with $rs(r+s)\neq 0$ which coincides with the family (3.141) (removing the restriction $r-s\neq 0$).

(b) *Subcase* $q\neq 0$ *and* $v=-(q+2r+2s)$. This leads to the systems

$$\dot{x} = x\big[rx^2-(q+2r+s)xy+qy^2\big], \quad \dot{y} = y^2\big[-(q+r+2s)x+(q+s)y\big]$$

with $qrs(r+s)\neq 0$. Applying the transformation $(x,y,t)\mapsto(x-y,-y,-t)$ and setting three new parameters $s_1=r+s$, $r_1=-r$ and $u_1=-(q+2s)$ (i.e. $r=-r_1, s=r_1+s_1, q=-2r_1-2s_1-u_1$) we arrive at the family of systems

$$\dot{x} = x^2\big[r_1x+(s_1+u_1)y\big], \quad \dot{y} = y^2\big[(r_1+u_1)x+s_1y\big]$$

which coincides with the family (3.141).

(c) *Subcase* $q(q+2r+2s+v)\neq 0$ *and* $v=(qr+rs+s^2)/s$. Then we obtain the systems

$$\dot{x} = x\big[rx^2+(qr/s+r+2s)xy+qy^2\big], \quad \dot{y} = y^2\big[(qr/s+2r+s)x+(q+s)y\big]$$

with $qrs(r+s)\neq 0$. Applying the transformation $(x,y,t)\mapsto((x-y)/r,\ y/s,\ -rst)$ we arrive at the family of systems

$$\dot{x} = x^2\big[-sx+(-qr-rs+s^2)y/s\big], \quad \dot{y} = y^2\big[-(qr+2rs+s^2)x/s+(r+s)y\big].$$

Then setting three new parameters

$$r_2=-s, \quad s_2=r+s, \quad u_2=-\frac{3(q+2s)}{s}$$

$$\Rightarrow\ r = r_2 + s_2, \quad s = -r_2, \quad q = \frac{r_2(2r_2 + 2s_2 + u_2)}{r_2 + s_2}$$

we obtain the family of systems

$$\dot{x} = x^2\big[r_2x + (s_2 + u_2)y\big], \quad \dot{y} = y^2\big[(r_2 + u_2)x + s_2y\big]$$

which again coincides with the family (3.141).

Thus for further examination it remains only the family (3.141) with the condition $rs(r + s) \neq 0$. Since $rs \neq 0$ in the generic systems with the associated homogeneous cubic systems (3.141), we may consider $g = n = 0$ in system (2.4) (due to a translation) and $s = 1$ (due to the time rescaling $t \to t/s$). So we obtain the following family of cubic systems

$$\begin{aligned} &\dot{x} = a + cx + dy + 2hxy + ky^2 + rx^3 + (1 + u)x^2y,\\ &\dot{y} = b + ex + fy + lx^2 + 2mxy + (r + u)xy^2 + y^3, \quad r(r + 1) \neq 0 \end{aligned} \tag{3.142}$$

which will be considered below.

3.5.2. *Construction of a cubic system possessing invariant lines with configuration type* $\mathfrak{T} = (2, 2, 1, 1)$. In what follows we shall determine necessary and sufficient conditions for a system (3.142) to have invariant lines with the configuration of the type $\mathfrak{T} = (2, 2, 1, 1)$.

Considering Remark 2.13 for the homogeneous systems (3.141), associated to system (3.142) we calculate

$$H(X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = X^2Y^2(X - Y)(rX + Y).$$

So each one of the invariant lines $x = 0$ and $y = 0$ (respectively, $y = x$ and $y = -rx$) of systems (3.141) is of multiplicity two (respectively, one). However for some values of the parameters r and u the common divisor $\gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ could contain additional factors (see Notation 2.4 and Lemma 2.6). We prove the next lemma.

Lemma 3.17. *For the existence of a couple of parallel invariant line in the direction $y = x$ (respectively, $y = -rx$) of a system* (3.142) *the condition* $2(r + 1) + u = 0$ *(respectively $r + 1 - u = 0$) is necessary.*

Proof. We examine each one of the directions mentioned in the statement of the lemma. **(1)** *Direction $y = x$.* Considering the equation (2.5) and Remark 2.12 for system (3.142) we obtain

$$\begin{aligned} Eq_6 &= l - 2h - k + 2m - (2r + 2 + u)W,\\ Eq_9 &= e + f - c - d + (l + k)W + (1 - r)W^2,\\ Eq_{10} &= b - a + (e - c)W + lW^2 - rW^3. \end{aligned} \tag{3.143}$$

We observe that the equation $Eq_6 = 0$ is of degree one with respect to W. So it is clear that for the existence of two distinct solutions of the above equations (with respect to W), the condition $2(r + 1) + u = 0$ is necessary. This proves the validity of the lemma concerning the direction $y = x$.

(2) *Direction $y=-rx$.* In a similar way considering Remark 2.12 for system (3.142) we calculate:

$$\begin{aligned} Eq_6 &= 2hr+2m-(kr^3+l)/r+(r+1-u)W,\\ Eq_9 &= dr-c+f-e/r-\frac{(kr^3-l)}{r^2}W+\frac{r-1}{r}W^2,\\ Eq_{10} &= ar+b+\big[r(cr+e)W+lW^2-rW^3\big]/r^2. \end{aligned} \tag{3.144}$$

We again observe that the equation $Eq_6=0$ is of degree one with respect to W. So for the existence of two distinct solutions of the above equations (with respect to W), the condition $r+1-u=0$ is necessary. This completes the proof. □

Lemma 3.18. *A system* (3.142) *possesses one couple of parallel invariant lines in the direction $x=0$ if and only if the following conditions hold*

$$\begin{gathered} 1+u\neq 0, \quad k=0,\\ d=\frac{4h^2r+c(1+u)^2}{r(1+u)}, \quad a=-\frac{2h\big[4h^2r+c(1+u)^2\big]}{(1+u)^3}. \end{gathered} \tag{3.145}$$

Systems (3.142) *possesses one couple of parallel invariant lines in the direction $y=0$ if and only if*

$$r+u\neq 0, \quad l=0, \quad e=\frac{4m^2+f(r+u)^2}{r+u}, \quad b=-\frac{2m\big[4m^2+f(r+u)^2\big]}{(r+u)^3}. \tag{3.146}$$

Proof. We consider each one of the directions $x=0$ and $y=0$ and force the existence of a couple of parallel invariant lines in each one of these directions.

(1) *Direction $x=0$.* Considering the equations (2.5) and Remark 2.12 for system (3.142) we obtain

$$Eq_7=k, \quad Eq_9=d-2hW+(1+u)W^2, \quad Eq_{10}=a-cW-rW^3.$$

Therefore the condition $Eq_7=0$ gives us $k=0$ and the equation $Eq_9=0$ could have two solutions only if $1+u\neq 0$. On the other hand since $r(1+u)\neq 0$, by Lemma 2.8 for the existence of two common solutions of the equations $Eq_9=0$ and $Eq_{10}=0$ the following conditions are necessary and sufficient:

$$R_W^{(0)}(Eq_9,Eq_{10})=R_W^{(1)}(Eq_9,Eq_{10})=0.$$

Calculations yield

$$R_W^{(1)}(Eq_9,Eq_{10})=-c(1+u)^2-r(4h^2-d-du)=0$$

and this gives $c=\frac{r(d+du-4h^2)}{(1+u)^2}$. Then we calculate

$$R_W^{(0)}(Eq_9,Eq_{10})=\big[a(1+u)^2+2dhr\big]^2/(1+u)=0$$

and we obtain $a=-2dhr/(1+u)^2$. For these values of the parameters c and a we obtain

$$Eq_9=d-2hW+(1+u)W^2, \quad Eq_{10}=-\frac{r(2h+W+uW)}{(1+u)^2}\big[d-2hW+(1+u)W^2\big],$$

i.e. we have two common solutions, which could be real or complex, distinct or coinciding.

(2) *Direction* $y = 0$. In this case considering Remark 2.12 for system (3.142) calculations yield

$$Eq'_5 = l, \quad Eq'_8 = e - 2mW + (r+u)W^2, \quad Eq'_{10} = b - fW - W^3. \tag{3.147}$$

Hence the condition $Eq'_5 = 0$ gives us $l = 0$ and the equation $Eq'_8 = 0$ could have two solutions only if $r + u \neq 0$. Then by Lemma 2.8 for the existence of two common solutions of the equations $Eq'_8 = 0$ and $Eq'_{10} = 0$ the following conditions are necessary and sufficient

$$R_W^{(0)}(Eq'_8, Eq'_{10}) = R_W^{(1)}(Eq'_8, Eq'_{10}) = 0.$$

We calculate

$$R_W^{(1)}(Eq'_8, Eq'_{10}) = -4m^2 + e(r+u) - f(r+u)^2 = 0$$

and this gives $f = \frac{er+eu-4m^2}{(r+u)^2}$. Then calculations yield

$$R_W^{(0)}(Eq'_8, Eq'_{10}) = \big[b(r+u)^2 + 2em\big]^2/(r+u) = 0$$

and we obtain $b = -2em/(r+u)^2$. For these values of the parameters f and b we obtain

$$Eq'_8 = e - 2mW + (r+u)W^2,$$
$$Eq'_{10} = -\frac{2m + rW + uW}{(r+u)^2}\big[e - 2mW + (r+u)W^2\big],$$

i.e. we have two common solutions, which could be real or complex, distinct or coinciding. This completes the proof of the lemma. □

Next we examine the remaining two directions. Taking into account Lemma 3.17 we consider two cases: $(2r+2+u)(r+1-u) \neq 0$ and $(2r+2+u)(r+1-u) = 0$. Case $(2r+2+u)(r+1-u) \neq 0$. By Lemma 3.17 system (3.142) could possess a couple of parallel invariant lines, neither in the direction $y = x$, nor in the direction $y = -rx$. Therefore we could have two couples of such invariant lines: one in the direction $x = 0$ and one in the direction $y = 0$. So assuming that the conditions (3.145) and (3.146) are fulfilled (this guarantees the existence of the two couples of parallel invariant lines), we have to force the existence of two invariant lines: one in the direction $y = x$ and the other in the direction $y = -rx$.

(a) *Direction* $y = x$. In this case we consider (3.143). Since $(2+2r+u) \neq 0$ the equation $Eq_6 = 0$ gives $W = \frac{l-2h-k+2m}{2+2r+u} \equiv W_0$ and we calculate

$$Eq_9|_{\{W=W_0\}} = \frac{H_1(c,d,e,f,l,k,m,r,u)}{(2+2r+u)^2},$$
$$Eq_{10}|_{\{W=W_0\}} = \frac{H_2(c,d,e,f,l,k,m,r,u)}{(2+2r+u)^3},$$

where H_1 and H_2 are the polynomials of degree 3 and 4 in the indicated parameters. It is clear that we could have an invariant straight line in the direction $y = x$ if and only if $H_1 = H_2 = 0$.

(b) *Direction* $y = -rx$. We consider now the equations (3.144) and analogously as above we detect that since $(r+1-u) \neq 0$ the equation $Eq_6 = 0$ gives $W =$

$-\frac{l-2mr-2hr^2+kr^3}{r(1+r-u)} \equiv W_0'$. Herein considering (3.144) we calculate

$$
\begin{aligned}
Eq_9|_{\{W=W_0'\}} &= \frac{H_1'(c,d,e,f,l,k,m,r,u)}{r^3(1+r-u)^2(1+u)(r+u)},\\
Eq_{10}|_{\{W=W_0'\}} &= \frac{H_2'(c,d,e,f,l,k,m,r,u)}{r(1+r-u)^3(1+u)^3(r+u)^3}.
\end{aligned}
$$

with some polynomials H_1' and H_2'. So we deduce that the condition $H_1' = H_2' = 0$ is necessary and sufficient for the existence of exactly one invariant line in the direction $y = -rx$.

Thus we conclude that to have invariant lines with the configuration of the type $\mathfrak{T} = (2,2,1,1)$, we need to join the conditions (3.145), (3.146) and $H_1 = H_2 = H_1' = H_2' = 0$. We stress that in the considered case for system (3.142) the additional condition

$$
r(r+1)(u+1)(r+u)(2+2r+u)(r+1-u) \neq 0 \tag{3.148}
$$

must hold.

Assume that the conditions (3.145) and (3.146) are satisfied. For these values of the parameters a, b, d, e, k and l we evaluate the polynomials H_1 and H_1' which turn out to be linear with respect to the parameters c and f. We obtain

$$
\begin{aligned}
H_1 &= \frac{1}{r}(1+r+u)(2+2r+u)^2(-c+fr) + \frac{4\Phi_1(h,m,r,u)}{(1+u)(r+u)},\\
H_1' &= r^2u(1+r-u)^2(cr-f)\frac{4r^2\Phi_1'(h,m,r,u)}{(1+u)(r+u)},
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_1 =& 2hm(1-r)(1+u)(r+u) - m^2(1+u)(4+9r+3r^2+5u+3ru+u^2)\\
&+ h^2(r+u)(3+9r+4r^2+3u+5ru+u^2),\\
\Phi_1' =& 2hm(-1+r)r(1+u)(r+u) + m^2(1+u)(-1-3r+u+3ru-u^2)\\
&+ h^2r^2(r+u)(3r+r^2-3u-ru+u^2).
\end{aligned}
$$

We calculate

$$
\begin{aligned}
&\text{Coefficient}[H_1,c] \times \text{Coefficient}[H_1',f] - \text{Coefficient}[H_1,f] \times \text{Coefficient}[H_1',c]\\
&= ru(1-r)(1+r+u)(1+r)(1+r-u)^2(2+2r+u)^2 \equiv \Delta_{cf}.
\end{aligned}
$$

Clearly the equations $H_1 = 0$ and $H_1' = 0$ have a unique solution with respect to the parameters c and f if and only if the condition $\Delta_{cf} \neq 0$. Considering the condition (3.148) we deduce that this condition is equivalent to $u(1-r)(1+r+u) \neq 0$. So in what follows we examine two possibilities: $u(1-r)(1+r+u) \neq 0$ and $u(1-r)(1+r+u) = 0$.

1. *Possibility* $u(1-r)(1+r+u) \neq 0$. Then solving the equations $H_1 = 0$ and $H_1' = 0$ with respect to parameters c and f we obtain

$$
\begin{aligned}
c &= \frac{-4r}{(1+u)(r+u)\Delta_{cf}}\big[u(1+r-u)^2\Phi_1 - (1+r+u)(2+2r+u)^2\Phi_1'\big],\\
f &= \frac{-4r^2}{(1+u)(r+u)\Delta_{cf}}\big[r^2(1+r-u)^2u\Phi_1 - (1+r+u)(2+2r+u)^2\Phi_1'\big].
\end{aligned} \tag{3.149}
$$

So considering these conditions as well as the conditions (3.145) and (3.146) we calculate

$$H_2=\frac{8r(1+r)(2+2r+u)^3}{(1+u)^3(r+u)^3\Delta_{cf}}\,V_1V_2V_3,\quad H_2'=-\frac{8r^6(1+r)(1+r-u)^3}{(1+u)^3(r+u)^3\Delta_{cf}}\,V_1V_2V_4,$$

where

$$\begin{aligned}
V_1=&\ h(2r-u)(r+u)+m(u-2)(1+u),\\
V_2=&\ hr(r+u)(3+r+u)+m(1+u)(1+3r+u),\\
V_3=&\ hr(r+u)(2r+4r^2+2r^3-u+2ru+3r^2u-8u^2+2r^2u^2-4u^3\\
&+3ru^3+u^4)+m(1+u)(r^3u-2r-4r^2-2r^3-3ru-2r^2u-2u^2\\
&+8r^2u^2-3u^3+4ru^3-u^4),\\
V_4=&\ h(r+u)(2r^3+2r^4-2r-2r^2+u-7ru-9r^2u-r^3u+u^2\\
&-15ru^2-10r^2u^2-u^3-8ru^3-u^4)+m(1+u)(2+2r-2r^2-2r^3\\
&-u-9ru-7r^2u+r^3u-10u^2-15ru^2+r^2u^2-8u^3-ru^3-u^4).
\end{aligned}\tag{3.150}$$

Therefore the equations $H_2=0$ and $H_2'=0$ have a common solution if and only if either $V_1=0$ or $V_2=0$ or $V_3=V_4=0$. We examine each one of these cases.

1.1. *Case* $V_1=0$. In this case we consider two subcases: $2r-u\neq 0$ and $2r-u=0$.

1.1.1. *Subcase* $2r-u\neq 0$. Since $r+u\neq 0$ (see the condition (3.148)) the equation $V_1=0$ implies $h=\frac{m(2-u)(1+u)}{(2r-u)(r+u)}$ and we arrive at the family of systems:

$$\begin{aligned}
\dot x=&\Big[x-\frac{2m(2u+ru-1-5r)}{(1+r-u)(2r-u)(r+u)}\Big]\Big[x+\frac{2m(3r-1+u-u^2)}{(1+r-u)(2r-u)(r+u)}\Big]\\
&\times\Big[rx+(1+u)y+\frac{2mr(u-2)}{(2r-u)(r+u)}\Big],\\
\dot y=&\Big[y+\frac{2m(5r+r^2-u-2ru)}{(1+r-u)(2r-u)(r+u)}\Big]\Big[y+\frac{2m(r^2-3r-ru+u^2)}{(1+r-u)(2r-u)(r+u)}\Big]\\
&\times\Big[(r+u)x+y-\frac{2m}{r+u}\Big].
\end{aligned}\tag{3.151}$$

To simplify these system we need to use a transformation which depends on two possibilities: $m(u^2+u+ru-8r)\neq 0$ and $m(u^2+u+ru-8r)=0$.

1.1.1.1. *Possibility* $m(u^2+u+ru-8r)\neq 0$. In this case, because of the condition $(1+r-u)(2r-u)(r+u)\neq 0$ we may apply the transformation

$$\begin{gathered}
x_1=\alpha x-\frac{1+5r-2u-ru}{u^2+u+ru-8r},\quad y_1=\alpha y+\frac{5r+r^2-u-2ru}{u^2+u+ru-8r},\\
t_1=\frac{4m^2(u^2+u+ru-8r)^2}{(1+r-u)^2(2r-u)^2(r+u)^2}t,\quad \alpha=-\frac{(1+r-u)(2r-u)(r+u)}{2m(u^2+u+ru-8r)}.
\end{gathered}$$

This leads to the 2-parameter family of systems (we keep the old notations for the variables)

$$\begin{aligned}
\dot x&=x(x-1)\big[rx+(1+u)y-r-1\big],\\
\dot y&=y(y-1)\big[(r+u)x+y-r-1\big],
\end{aligned}\tag{3.152}$$

for which the condition

$$ru(r^2-1)(u+1)(r+u)(2+2r+u)(2r-u)(u^2+u+ru-8r)\big[(r+1)^2-u^2\big]\neq 0\tag{3.153}$$

holds. These systems possess six distinct real invariant affine straight lines

$$L_1 : x = 0, \quad L_2 : x = 1, \quad L_3 : y = 0,$$
$$L_4 : y = 1, \quad L_5 : y = x, \quad L_6 : rx + y = r + 1$$

and the following nine finite singularities:

$$\begin{gathered} M_1(0,0), \quad M_2(0,1), \quad M_3(0,1+r), \quad M_4(1,0), \quad M_5(1,1), \quad M_6(1,1-u), \\ M_7\Big(\frac{r+1}{r},0\Big), \quad M_8\Big(\frac{r-u}{r},1\Big), \quad M_9\Big(\frac{r+1}{1+r+u},\frac{r+1}{1+r+u}\Big). \end{gathered} \tag{3.154}$$

We determine that due to the condition (3.153) all these singularities are distinct if and only if $(u-1)(u-r) \neq 0$. In the case $u = 1$ (respectively $u = r$) the singular point M_6 (respectively M_8) coalesces with M_4 (respectively M_2).

It is easy to determine that six of these singularities are located at the intersections of the above invariant lines, more precisely, these are the singular points M_i for $i \in \{1,2,3,4,5,7\}$. The singular point M_6 (respectively M_8; M_9) is located on the invariant line L_2 (respectively L_4; L_5). Moreover we have one singular point located at the intersection of four invariant lines and one singularity is located at the intersection of three invariant lines. More exactly L_2, L_4, L_5 and L_6 intersect at the point M_5 whereas L_1, L_3 and L_5 intersect at the point M_1.

To determine all the possible configurations for system (3.152) we have to examine the positions of the invariant lines as well as of the singularities M_6, M_8 and M_9 depending on the parameters r and u.

Let us first examine the position of the invariant lines. We observe that five of the lines are fixed and only the position of the invariant line L_6 depends on the parameter r. Since this line is defined by $y = -rx + r + 1$, comparing with other three invariant lines (L_2, L_4 and L_5) which intersect L_6 at the same point $M_5(1,1)$ we detect that we could have geometrically distinct situations in three cases: (i) $r < -1$; (ii) $-1 < r < 0$ and (iii) $r > 0$.

Next we consider the position of the finite singularity $M_6(1, y_6)$ with $y_6 = 1-u$ (respectively $M_8(x_8, 1)$ with $x_8 = (r-u)/r$; $M_9(x_9, y_9)$ with $x_9 = y_9 = (r+1)/(1+r+u)$) on the invariant line $x = 1$ (respectively $y = 1$; $y = x$) with respect to the singular points $M_4(1,0)$ and $M_5(1,1)$ (respectively $M_2(0,1)$ and $M_5(1,1)$; $M_1(0,0)$ and $M_5(1,1)$). Considering Notation 3.6 and the coordinates $y_6 = 1-u$, $x_8 = (r-u)/r$ and $x_9 = (r+1)/(1+r+u)$ we have the next implications.

(I) For the singular point M_6,

$$\begin{gathered} y_6 \le 0 \ \Rightarrow\ M_6 \preceq M_4 \prec M_5; \quad 0 < y_6 < 1 \ \Rightarrow\ M_4 \prec M_6 \prec M_5; \\ y_6 > 1 \ \Rightarrow\ M_4 \prec M_5 \prec M_6. \end{gathered}$$

(II) For the singular point M_8,

$$\begin{gathered} x_8 \le 0 \ \Rightarrow\ M_8 \preceq M_2 \prec M_5; \quad 0 < x_8 < 1 \ \Rightarrow\ M_2 \prec M_8 \prec M_5; \\ x_8 > 1 \ \Rightarrow\ M_2 \prec M_5 \prec M_8. \end{gathered}$$

(III) For the singular point M_9,

$$\begin{gathered} x_9 < 0 \ \Rightarrow\ M_9 \prec M_1 \prec M_5; \quad 0 < x_9 < 1 \ \Rightarrow\ M_1 \prec M_9 \prec M_5; \\ x_9 > 1 \ \Rightarrow\ M_1 \prec M_5 \prec M_9. \end{gathered}$$

So it is clear that the three possibilities mentioned before, for the position of the invariant line L_6, defined by the parameter r (i.e. $r < -1$, $-1 < r < 0$ and

$r>0$), we have to confront with the possibilities for the three singularities M_6 M_8 and M_9 mentioned above. Since we only have two parameters it is evident that not all of the above possibilities are realizable. So examining the compatibilities of the conditions it is not too hard to convince ourselves (using, for example, the tools "FindInstance" or "Reduce" of computer algebra system Mathematica) that the following lemma is valid.

Lemma 3.19. *The family of systems* (3.152) *with the condition* (3.153) *possesses the following configurations of invariant lines when the corresponding conditions indicated below are satisfied:*

$$Config.\ 7.67 \Leftrightarrow \begin{cases} r<-1, u>1, r>-1-u, & (r=-2, u=5/4);\\ or\ -1<r<0, u<r<-1-u, & (r=-1/8, u=-3/4);\\ Config.\ 7.68 \Leftrightarrow r<-1-u,\ u>1, & (r=-4, u=2);\\ or\ u<r<0, r>-1-u, & (r=-7/8, u=1/4);\end{cases}$$

$$Config.\ 7.69 \Leftrightarrow \begin{cases} -2<r<-1, u=1, & (r=-5/4, u=1);\\ or\ -1<r<-1/2, u=r, & (r=-5/6, u=-5/6);\end{cases}$$

$$Config.\ 7.70 \Leftrightarrow \begin{cases} r<-2, u=1, & (r=-5/2, u=1);\\ or\ -1/2<r<0, u=r, & (r=-1/4, u=-1/4);\end{cases}$$

$$Config.\ 7.71 \Leftrightarrow \begin{cases} -1-u<r<-1, 0<u<1, & (r=-11/8, u=1/2);\\ or\ -1<r<u, u<-1-u, & (r=-5/6, u=-7/12);\end{cases}$$

$$Config.\ 7.72 \Leftrightarrow \begin{cases} r<-1-u, 0<u<1, & (r=-2, u=1/2);\\ or\ -1-u<r<0, u<0, & (r=-5/8, u=-1/8);\end{cases}$$

$$Config.\ 7.73 \Leftrightarrow \begin{cases} u<r<-1, & (r=-2, u=-3);\\ or\ -1<r<0, u>1, & (r=-3/4, u=2);\end{cases}$$

$$Config.\ 7.74 \Leftrightarrow \begin{cases} r<-1, r<u<0, & (r=-4, u=-2);\\ or\ -1<r<0, 0<u<1, & (r=-7/8, u=1/4);\end{cases}$$

$$Config.\ 7.75 \Leftrightarrow \begin{cases} u=r<-1, & (r=-2, u=-2);\\ or\ -1<r<0, u=1, & (r=-1/2, u=1);\end{cases}$$

$$Config.\ 7.7 \Leftrightarrow 0<r<u,\ u>1, \quad (r=1/8, u=5/4);$$

$$Config.\ 7.77 \Leftrightarrow \begin{cases} 1<u<r, & (r=2, u=5/4);\\ or\ 0<r<u, u<1, & (r=1/16, u=1/2);\end{cases}$$

$$Config.\ 7.78 \Leftrightarrow \begin{cases} 0<r<1, u=1, & (r=1/7, u=1);\\ or\ r=u>1, & (r=2, u=2);\end{cases}$$

$$Config.\ 7.79 \Leftrightarrow \begin{cases} r>1, u=1, & (r=2, u=1);\\ or\ 0<r=u<1, & (r=1/2, u=1/2);\end{cases}$$

$$Config.\ 7.80 \Leftrightarrow r>u, 0<u<1, \quad (r=7/8, u=1/2);$$

$$Config.\ 7.81 \Leftrightarrow 0<r<-1-u, \quad (r=1/2, u=-5);$$

$$Config.\ 7.82 \Leftrightarrow r>0, r>-1-u, u<0, \quad (r=1/2, u=-5).$$

1.1.1.2. *Possibility* $m(u^2+u+ru-8r)=0$.

1.1.1.2.1. *Case* $m=0$. Then (3.151) become cubic homogeneous

$$\dot{x}=x^2(rx+y+uy), \quad \dot{y}=y^2(rx+ux+y), \quad r(r+1)\neq 0 \tag{3.155}$$

possessing four distinct invariant affine lines: $x=0$ (double), $y=0$ (double), $y=x$ and $y=-rx$. We observe that the unique singularity $(0,0)$ of these systems is of multiplicity 9. Since the invariant lines $x=0$ and $y=0$ are double and other two are simple we deduce that we could have two geometrically distinct possibilities: when the simple invariant lines are adjacent (if $r<0$) and when they are not adjacent (if $r>0$). Therefore we obtain the configuration of invariant lines given by Config. 7.83 if $r<0$ and by Config. 7.84 if $r>0$.

1.1.1.2.2. *Case* $m\neq 0$ *and* $(u^2+u+ru-8r)=0$. So we have $r(u-8)+u(1+u)=0$ and clearly $u\neq 8$ (due to $u(1+u)\neq 0$). Therefore we obtain $r=u(1+u)/(8-u)$ and system (3.151) becomes

$$\begin{aligned}\dot{x}=&\frac{(1+u)\big[m(u-8)^2-27u^2x\big]^2}{19683(8-u)u^5}\big[2m(u-8)^2+27u^2x-27(u-8)uy\big],\\ \dot{y}=&\frac{1}{729(u-8)u^3}(mu-8m-9uy)^2\big[2m(u-8)^2-81u^2x+9(u-8)uy\big].\end{aligned}\tag{3.156}$$

Since $u\neq 0$ applying the transformation

$$x_1=x-\frac{m(u-8)^2}{27u^2}, \quad y_1=y-\frac{m(u-8)}{9u}, \quad t_1=\frac{t}{8-u}$$

we arrive at the 1-parameter family of systems (we keep the old notations for variables)

$$\dot{x}=(1+u)x^2\big[ux+(8-u)y\big], \quad \dot{y}=y^2\big[9ux+(8-u)y\big]. \tag{3.157}$$

On the other hand, setting in systems (3.155) the condition $r=u(1+u)/(8-u)$ we obtain the 1-parameter family of systems

$$\dot{x}=(1+u)x^2\big[ux+(8-u)y\big]/(8-u), \quad \dot{y}=y^2\big[9ux+(8-u)y\big]/(8-u).$$

Evidently applying the time rescaling $t\to(8-u)t$ we obtain systems (3.157) and hence we could not have new configurations.

1.1.2. *Subcase* $2r-u=0$. Then $u=2r$ and considering (3.150) the condition $V_1=0$ becomes $V_1=2m(r-1)(1+2r)=0$. Since by the condition (3.148) we have $(u+1)(r+1-u)=(2r+1)(1-r)\neq 0$ the condition $V_1=0$ implies $m=0$. This leads to the family of systems

$$\begin{aligned}\dot{x}=&\Big(x-\frac{h}{r-1}\Big)\Big[x+\frac{h(-1+4r)}{(r-1)(1+2r)}\Big]\Big[rx+(1+2r)y-\frac{2hr}{1+2r}\Big],\\ \dot{y}=&\Big[y-\frac{3hr}{(r-1)(1+2r)}\Big]\Big[y+\frac{3hr}{(r-1)(1+2r)}\Big](3rx+y).\end{aligned}\tag{3.158}$$

1.1.2.1. *Possibility* $h\neq 0$. Then we can apply the transformation

$$x_1=\frac{(1-r)(1+2r)}{6hr}x+\frac{1+2r}{6r}, \quad y_1=\frac{(1-r)(1+2r)}{6hr}y+1/2,$$
$$t_1=\frac{36h^2r^2t}{(r-1)^2(1+2r)^2}.$$

and this leads to the systems (we keep the old notations for variables)

$$\dot{x} = x(x-1)\big[rx + (1+2r)y - r - 1\big], \quad \dot{y} = y(y-1)\big[3rx + y - r - 1\big].$$

We observe that after the translation $y \to y + r$ these systems become a subfamily of the family (3.152) defined by the condition $u = 2r$ and hence the corresponding configurations are already determined.

1.1.2.2. *Possibility* $h = 0$. In this case system (3.158) becomes homogeneous system (3.155) with $u = 2r$, i.e. we obtain Config. 7.83 if $r < 0$ and by Config. 7.84 if $r > 0$.

1.2. *Case* $V_1 \neq 0$ *and* $V_2 = 0$. Considering (3.150) we obtain the condition

$$V_2 = hr(r+u)(3+r+u) + m(1+u)(1+3r+u) = 0$$

and as $r(r+u) \neq 0$ (see the condition (3.148)) we examine two subcases: $3+r+u \neq 0$ and $3 + r + u = 0$.

1.2.1. *Subcase* $3 + r + u \neq 0$. In this case the condition $V_2 = 0$ gives

$$h = -\frac{m(1+u)(1+3r+u)}{r(r+u)(3+r+u)}$$

and considering also the conditions (3.149), (3.145) and (3.146) we arrive at the system

$$\begin{aligned}
\dot{x} =& \Big[x - \frac{2m(1+8r+3r^2+u+2ru)}{r(r+u)(3+r+u)(2+2r+u)}\Big] \\
& \times \Big[x + \frac{2m(1+3r^2+2u+3ru+u^2)}{r(r+u)(3+r+u)(2+2r+u)}\Big] \\
& \times \Big[rx + (1+u)y + \frac{2m(1+3r+u)}{(r+u)(3+r+u)}\Big], \\
\dot{y} =& \Big[y + \frac{2m(3+8r+r^2+2u+ru)}{(r+u)(3+r+u)(2+2r+u)}\Big]\Big[(r+u)x + y - \frac{2m}{r+u}\Big] \\
& \times \Big[y + \frac{2m(3+r^2+3u+2ru+u^2)}{(r+u)(3+r+u)(2+2r+u)}\Big].
\end{aligned} \tag{3.159}$$

We again have to examine two possibilities: $m(u^2 + u + ru - 8r) \neq 0$ and $m(u^2 + u + ru - 8r) = 0$.

1.2.1.1. *Possibility* $m(u^2 + u + ru - 8r) \neq 0$. In this case due to the condition $r(r + u)(3 + r + u)(2 + 2r + u) \neq 0$ we apply the transformation

$$x_1 = \alpha x - \frac{1+8r+3r^2+u+2ru}{u^2+u+ru-8r}, \quad y_1 = \alpha y + \frac{r(3+8r+r^2+2u+ru)}{u^2+u+ru-8r},$$
$$t_1 = \frac{4m^2(u^2+u+ru-8r)^2 t}{r^2(r+u)^2(3+r+u)^2(2+2r+u)^2}, \quad \alpha = \frac{r(r+u)(3+r+u)(2+2r+u)}{2m(u^2+u+ru-8r)}.$$

This leads to the 2-parameter family of systems

$$\begin{aligned}
\dot{x} &= x(x-1)\big[r_1x + (1+u_1)y - r_1(1+r_1)\big], \\
\dot{y} &= y(y+r_1)\big[(r_1+u_1)x + y + 1 + r_1\big].
\end{aligned}$$

However we detect, that setting $r_1 = 1/r$, $u_1 = -(1 + r + u)/r$ and applying the rescaling $(x, y, t) \mapsto (x, -y/r, r^2t)$ we arrive at systems (3.152) for which all possible configurations of invariant lines are provided by Lemma 3.19.

1.2.1.2. *Possibility* $m(u^2+u+ru-8r)=0$. The straightforward calculations give us that for $m=0$ system (3.159) becomes the cubic homogeneous system (3.155), whereas for $r=u(1+u)/(8-u)$ system (3.159) becomes exactly the system (3.156), i.e. we do not have new configurations.

1.2.2. *Subcase* $3+r+u=0$. Then $u=-(3+r)$ and considering (3.150) the condition $V_2=0$ becomes $V_2=2m(1-r)(2+r)=0$. Since by the condition (3.148) we have $(u+1)(2+2r+u)=(1-r)(2+r)\neq 0$ we deduce that the condition $V_2=0$ implies $m=0$. This leads to the family of systems

$$\begin{aligned}\dot x &=\Big(x-\frac{h}{r-1}\Big)\Big[x-\frac{h(r-4)}{(r-1)(2+r)}\Big]\Big[rx-(2+r)y+\frac{2hr}{2+r}\Big],\\ \dot y &=\Big[y-\frac{3hr}{(r-1)(2+r)}\Big]\Big[y+\frac{3hr}{(r-1)(2+r)}\Big]\big(y-3x\big).\end{aligned} \tag{3.160}$$

1.2.2.1. *Possibility* $h\neq 0$. Then we can apply the transformation

$$x_1=\frac{(1-r)(2+r)}{6h}x+\frac{2+r}{6},\quad y_1=\frac{(1-r)(2+r)}{6h}\,y+\frac{r}{2},\quad t_1=\frac{36h^2t}{(r-1)^2(2+r)^2}.$$

and this leads to the systems (we keep the old notations for the variables)

$$\dot x=x(x-1)\big[rx-(2+r)y+r\big],\quad \dot y=y(y-r)\big[y-3x+1\big]$$

We observe that applying the translation $y\to y+r$ this system becomes a subfamily of the family (3.152) defined by the condition $u=-(3+r)$ and hence their corresponding configurations are already determined.

1.2.2.2. *Possibility* $h=0$. Setting this condition in (3.160) we arrive at the homogeneous system (3.155) with $u=-(3+r)$, i.e. we could not have new configurations.

1.3. *Case* $V_3=V_4=0$. Considering (3.150) we observe that both polynomials V_3 and V_4 are linear and homogeneous with respect to the parameters h and m. Moreover calculations yield

$$\begin{aligned}&\mathrm{Coefficient}[V_3,h]\times\mathrm{Coefficient}[V_4,m]-\mathrm{Coefficient}[V_3,m]\times\mathrm{Coefficient}[V_4,h]\\ &=(1+r)(1+r-u)u(1+u)^2(r+u)^2(1+r+u)(2+2r+u)(u^2+u+ru-8r)\\ &\equiv\Delta_{hm}.\end{aligned}$$

We observe that $\Delta_{hm}\neq 0$ due to the condition (3.148) and $\Delta_{cf}\neq 0$. Hence we detect that the unique solution of the equations $V_3=0$ and $V_4=0$ is $h=m=0$.

Therefore considering the conditions (3.149), (3.145) and (3.146), systems (3.142) become homogeneous systems (3.155), possessing two configurations given by Config. 7.83 or Config. 7.84.

2. *Possibility* $u(1-r)(1+r+u)=0$. In this case $\Delta_{cf}=0$ and we have to return and impose the conditions $H_1=H_2=H_1'=H_2'=0$ to be satisfied. We examine each one of three conditions: (i) $u=0$; (ii) $u\neq 0$ and $r=1$ and (iii) $u(1-r)\neq 0$ and $(1+r+u)=0$.

2.1. *Case* $u=0$. Then we obtain

$$H_1'=-4r(m-hr^2)\big[hr^2(3+r)+m(1+3r)\big]=0 \tag{3.161}$$

and since $r\neq 0$ we have to examine two subcases: $m-hr^2=0$ and $hr^2(3+r)+m(1+3r)=0$. We claim that the first condition leads to degenerate systems.

Indeed, assume $m = hr^2$. Then calculations yield

$$H_1' = H_2' = 0, \quad H_1 = \ 4(1+r)^3(-c+fr-3h^2r+3h^2r^3)/r,$$
$$H_2 = \ 8h(r-3)(1+r)^3(-c+fr-3h^2r+3h^2r^3).$$

Since by condition (3.148) we have $r(r+1) \neq 0$, the equation $H_1 = 0$ gives $c = r(f-3h^2+3h^2r^2)$. Considering the conditions $u=0$ and $m = hr^2$ and this value of the parameter c we arrive at the systems

$$\begin{aligned}\dot{x} &= -(2hr-rx-y)(f+h^2+3h^2r^2+2hx+x^2),\\ \dot{y} &= -(2hr-rx-y)(f+4h^2r^2+2hry+y^2)\end{aligned}$$

which evidently are degenerate. So our claim is proved and we need to examine the condition $hr^2(3+r)+m(1+3r) = 0$, considering two subcases: $1+3r \neq 0$ and $1+3r = 0$.

2.1.1. *Subcase* $1+3r \neq 0$. Then $m = -hr^2(3+r)/(1+3r)$ and we calculate

$$H_1 = \frac{4(1+r)^2}{r}\Big[(1+r)(fr-c)+\frac{h^2r(r-1)}{(1+3r)^2}\,(3+24r+58r^2+24r^3+3r^4)\Big].$$

So the condition $H_1 = 0$ gives

$$c = fr + \frac{h^2r(r-1)}{(1+r)(1+3r)^2}\,\big(3+24r+58r^2+24r^3+3r^4\big)$$

and therefore,

$$H_2 = \frac{64hr(1+r)^2}{(1+3r)^3}\big[f(1+r)^2(1+3r)^2+h^2r^2(27+72r+82r^2+24r^3+3r^4)\big] = 0.$$

Since $r(r+1) \neq 0$ and $h \neq 0$ (otherwise we obtain $m = 0$ and this leads to degenerate systems) the condition $H_2 = 0$ yields

$$f = -\frac{h^2r^2(27+72r+82r^2+24r^3+3r^4)}{(1+r)^2(1+3r)^2}.$$

This implies $H_2' = 0$ and we obtain the following family of systems

$$\begin{aligned}\dot{x} &= \Big[x+\frac{h(1+3r^2)}{(1+r)(1+3r)}\Big]\Big[x+\frac{h(1+8r+3r^2)}{(1+r)(1+3r)}\Big]\big(rx+y-2hr\big),\\ \dot{y} &= \Big[y-\frac{hr(3+r^2)}{(1+r)(1+3r)}\Big]\Big[y-\frac{hr(3+8r+r^2)}{(1+r)(1+3r)}\Big]\Big[rx+y+\frac{2hr(3+r)}{1+3r}\Big].\end{aligned}$$

Then setting $r = 1/r_1$ and applying the transformation

$$x_1 = \frac{(1+r_1)(3+r_1)}{8hr_1}x - \frac{3+8r_1+r_1^2}{8r_1},$$
$$y_1 = -\frac{(1+r_1)(3+r_1)}{8h}\,y + \frac{1+8r_1+3r_1^2}{8r_1}, \quad t_1 = \frac{64h^2\,t}{(3+4r_1+r_1^2)^2}$$

we arrive at the 1-parameter family of systems (we keep the old notations)

$$\begin{aligned}\dot{x} &= x(x-1)(rx-ry-1-r),\\ \dot{y} &= y(y-1)(y-x-1-r), \quad r(r+1)(3r+1) \neq 0.\end{aligned} \tag{3.162}$$

We observe that this family is a subfamily of systems (3.152) defined by the condition $u = -(1+r)$, i.e. it corresponds to the case $r+u+1 = 0$. On the other hand for system (3.152) all possible configurations of invariant lines were constructed provided that the condition (3.153) is satisfied. This condition includes in particular the condition $r+u+1 \neq 0$. So considering the singularities (3.154) of system (3.152) we deduce that the condition $u = -(1+r)$ forces the finite singularity M_9 to coalesce with the infinite singularity $N[1:1:0]$.

We observe that the invariant lines of systems (3.162) coincide with those of systems (3.152) (since they do not depend on parameter u):

$$L_1 : x = 0, \quad L_2 : x = 1, \quad L_3 : y = 0,$$
$$L_4 : y = 1, \quad L_5 : y = x, \quad L_6 : rx + y = r + 1,$$

but system (3.162) has only eight finite singularities:

$$M_1(0,0), \quad M_2(0,1), \quad M_3(0,1+r), \quad M_4(1,0), \quad M_5(1,1),$$
$$M_6(1,2+r), \quad M_7\Big(\frac{r+1}{r},0\Big), \quad M_8\Big(\frac{2r+1}{r},1\Big).$$

We observe that because of the condition $r(r+1) \neq 0$ all these singularities are distinct if and only if $(r+2)(2r+1) \neq 0$. In the case $r=-2$ (respectively $r=-1/2$) the singular point M_6 (respectively M_8) coalesces with M_4 (respectively M_2).

We determine again that 6 of these singularities are located at the intersections of the above invariant lines, more precisely, these are the singular points M_i for $i \in \{1,2,3,4,5,7\}$. The singular point M_6 (respectively M_8) is located on the invariant line L_2 (respectively L_4). Moreover we have one singular point located at the intersection of four invariant lines and one singularity is located at the intersection of three invariant lines. More exactly L_2, L_4, L_5 and L_6 intersect at the point M_5 whereas L_1, L_3 and L_5 intersect at the point M_1.

To determine all possible configurations of invariant lines for system (3.162) we examine the positions of the invariant lines as well as of the singularities M_6 and M_8 depending on the parameter r.

Since the invariant lines of systems (3.162) coincide exactly with those of system (3.152), as it was mentioned, for the invariant lines of system (3.152) (see page 83), in order to detect the position of the invariant line L_6 (which is the only line which depends on the parameter r) it is necessary to examine three cases: (i) $r<-1$; (ii) $-1<r<0$ and (iii) $r>0$.

In addition we need to examine the position of the finite singularity $M_6(1,y_6)$ with $y_6 = 2+r$ (respectively $M_8(x_8,1)$ with $x_8 = (2r+1)/r$) on the invariant line $x=1$ (respectively $y=1$) with respect to the singular points $M_4(1,0)$ and $M_5(1,1)$ (respectively $M_2(0,1)$ and $M_5(1,1)$). Considering Notation 3.6 a we have the following implications.

(I) For the singular point M_6:

$$y_6 \le 0 \;\Rightarrow\; M_6 \preceq M_4 \prec M_5; \quad 0<y_6<1 \;\Rightarrow\; M_4 \prec M_6 \prec M_5;$$
$$y_6 > 1 \;\Rightarrow\; M_4 \prec M_5 \prec M_6.$$

(II) For the singular point M_8:

$$x_8 \le 0 \;\Rightarrow\; M_8 \preceq M_2 \prec M_5; \quad 0<x_8<1 \;\Rightarrow\; M_2 \prec M_8 \prec M_5;$$
$$x_8 > 1 \;\Rightarrow\; M_2 \prec M_5 \prec M_8.$$

Considering the values of the coordinates y_6 and x_8 we obtain

$$y_6 \le 0 \;\Leftrightarrow\; r \le -2; \quad 0<y_6<1 \;\Leftrightarrow\; -2<r<-1; \quad y_6>1 \;\Leftrightarrow\; r>-1;$$
$$x_8 \le 0 \;\Leftrightarrow\; -1/2 \le r < 0; \quad 0<x_8<1 \;\Leftrightarrow\; -1<r<-1/2;$$

$$x_8 > 1 \;\Leftrightarrow\; r < -1 \text{ or } r > 0.$$

Taking into account the three possibilities mentioned before for the position of the invariant line L_6 defined by the parameter r (i.e. $r < -1$, $-1 < r < 0$ and $r > 0$) it is not too difficult to convince ourselves that the following lemma is valid.

Lemma 3.20. *The family of systems* (3.162) *with the condition* $r(r+1)(3r+1) \neq 0$ *possesses the following configurations of invariant lines if and only if the corresponding conditions indicated below are satisfied:*

$$\begin{gathered} Config.\ 7.85 \;\Leftrightarrow\; r < -2 \text{ or } -1/2 < r < 0, r \neq -1/3; \\ Config.\ 7.86 \;\Leftrightarrow\; r \in \{-2, -1/2\}; \\ Config.\ 7.87 \;\Leftrightarrow\; -2 < r < -1/2,\; r \neq -1; \\ Config.\ 7.88 \;\Leftrightarrow\; r > 0. \end{gathered}$$

2.1.2. *Subcase* $1+3r=0$. Then $r=-1/3$ and the condition $hr^2(3+r)+m(1+3r)=0$ (see (3.161)) gives $h=0$. Then we calculate

$$H_1' = 0, \quad H_1 = 16(6c+2f-27m^2) = 0 \;\Rightarrow\; f = -3(2c-9m^2)/2$$

and this implies

$$H_2 = -8m(16c-243m^2)/3 = 0, \quad H_2' = -4(16c-243m^2)/729 = 0.$$

Since $m \neq 0$ (otherwise we obtain $m=h=0$ and this leads to degenerate system), the condition $c = 243m^2/16$ must hold. In this case we arrive at the family of systems

$$\begin{aligned} \dot{x} &= (27m-4x)(27m+4x)(x-3y)/48, \\ \dot{y} &= (3m-4y)(21m-4y)(18m-x+3y)/48 \end{aligned}$$

with $m \neq 0$. Then applying the transformation

$$x_1 = \frac{2}{27m}x + 1/2, \quad y_1 = \frac{2}{9m}\,y - 1/6, \quad t_1 = 81m^2t/4$$

we obtain the system (we keep the old notations)

$$\dot{x} = x(x-1)(3y-3x+2), \quad \dot{y} = y(y-1)(y-x+2).$$

We observe that this system belongs to the family (3.162) for $r=-1/3$ and by Lemma 3.20 its configuration corresponds to Config. 7.85. So we can remove from the lemma the condition $r \neq -1/3$, which corresponds to this configuration.

2.2. *Case* $u \neq 0,\ r=1$. We detect that in this case for the system (3.142) the following condition must hold:

$$r(r+1)(u+1)(r+u)(2r+2+u)(r+1-u) \neq 0 \;\Rightarrow\; (u+1)(u+4)(u-2) \neq 0 \quad (3.163)$$

For $r=1$ we calculate

$$H_1' = (u-2)^2\big[u(c-f) + 4(h^2-m^2)\big]/(1+u)$$

and as $u(u-2) \neq 0$ the condition $H_1' = 0$ gives $f = \frac{4h^2-4m^2+cu+cu^2}{u(1+u)}$. In this case calculations yield

$$H_1 = 8(h-m)(h+m)(4+u)^2/(u(1+u)) = 0.$$

Considering condition (3.163) we obtain $(h-m)(h+m)=0$.

2.2.1. *Subcase $h = m$.* In this case we have $H_1 = H_1' = H_2 = 0$ and

$$H_2' = \frac{4mu(4+u)}{(1+u)^3}\big[c(u-2)^2(1+u)^2 + 4m^2(7-u+u^2)\big] = 0 \tag{3.164}$$

and we consider two possibilities: $m = 0$ and $m \neq 0$.

2.2.1.1. *Possibility $m = 0$.* Then $H_2' = 0$ and we arrive at the following family of systems:

$$\dot{x} = (c+x^2)(x+y+uy), \quad \dot{y} = (c+y^2)(x+ux+y). \tag{3.165}$$

We observe that for $c = 0$ these systems become homogeneous and belong to the family (3.155) for $r = 1$.

Assume that condition $c \neq 0$ holds. Since the invariant lines $c + x^2 = 0$ and $c + y^2 = 0$ could be real or complex depending on the sign of the parameter c we examine both cases.

2.2.1.1.1. *Case $c < 0$.* Then setting $c = -v^2$ and applying the transformation $(x, y, t) \mapsto \big(vx, vy, t/v^2\big)$ we arrive at the 1-parameter family of systems

$$\dot{x} = (x^2-1)\big[x + (1+u)y\big], \quad \dot{y} = (y^2-1)\big[(1+u)x + y\big] \tag{3.166}$$

with condition (3.163). These systems possess six distinct fixed invariant affine straight lines

$$L_1 : x = 1, \quad L_2 : x = -1, \quad L_3 : y = 1,\ L_4 : y = -1, \quad L_5 : y = x, \quad L_6 : y = -x$$

and the following nine finite singularities:

$$\begin{gathered} M_1(0,0), \quad M_2(-1,-1), \quad M_3(-1,1), \quad M_4(1,-1), \quad M_5(1,1), \\ M_6(-1,1+u), \quad M_7(1+u,-1), \quad M_8(1,-1-u), \quad M_9(-1-u,1). \end{gathered} \tag{3.167}$$

We observe that the above systems are symmetric with respect to the origin of coordinates, because of the change $(x, y) \to (-x, -y)$. The systems are also symmetric with respect to the invariant line $y = x$, because the change $(x, y) \to (y, x)$ also conserves these system. As a consequence we have two pairs of finite singularities depending on the parameter u which are symmetric with respect to $(0, 0)$ (M_7 with M_9 and M_6 with M_8), and two pairs of singularities symmetric with respect to the line $y = x$ (M_6 with M_7 and M_8 with M_9).

Thus, because of the symmetry, it is sufficient to determine the position of the singularity M_6 with respect to the singular points M_2 and M_3 located on the same invariant line $x = -1$. Moreover we observe that M_6 could coalesce with M_2 for $u = -2$ and due to the symmetry, M_7 could also coalesce with M_2 getting a triple singular point $M_2 \equiv M_6 \equiv M_7$. Clearly due to the symmetry we obtain simultaneously a triple singular point $M_5 \equiv M_8 \equiv M_9$.

Thus for the singular point $M_6(-1, 1+u)$ we have (see Notation 3.6):

$$\begin{gathered} u \leq -2 \ \Rightarrow\ M_6 \preceq M_2 \prec M_3; \quad -2 < u < 0 \ \Rightarrow\ M_2 \prec M_6 \prec M_3; \\ u > 0 \ \Rightarrow\ M_2 \prec M_3 \prec M_6. \end{gathered}$$

Examining the positions of all the singularities (3.167) corresponding to these conditions we conclude that the following lemma is valid.

Lemma 3.21. *The family of systems* (3.166) *with the condition* $u(u+1) \neq 0$ *possesses the following configurations of invariant lines when the corresponding conditions indicated below are satisfied:*

$$Config.\ 7.89 \ \Leftrightarrow\ u < -2 \ or\ u > 0;$$

$$Config.\ 7.90 \ \Leftrightarrow\ u=-2;$$
$$Config.\ 7.91 \ \Leftrightarrow\ -2<u<0;$$

2.2.1.1.2. *Case $c>0$.* Then setting $c=v^2$ and applying the same transformation $(x,y,t)\mapsto \big(vx,vy,t/v^2\big)$ we obtain the 1-parameter family of systems

$$\begin{aligned}\dot x=&(x^2+1)\big[x+(1+u)y\big],\\ \dot y=&(y^2+1)\big[(1+u)x+y\big],\quad u(u+1)(u+4)(u-2)\neq 0.\end{aligned}$$

These systems possess two real and four complex invariant lines

$$y=\pm x,\quad x=\pm i,\quad y=\pm i$$

as well as nine distinct finite singularities among which only one is real, namely $(0,0)$. As a result we obtain the configuration Config. 7.92.

2.2.1.2. *Possibility $m\neq 0$.* In this case considering (3.164) the condition $H_2'=0$ gives

$$c=-\frac{4m^2(7-u+u^2)}{(u-2)^2(1+u)^2}$$

and therefore we arrive at the family of systems

$$\begin{aligned}\dot x=&\Big[x-\frac{6m}{(u-2)(1+u)}\Big]\Big[x+\frac{2m}{u-2}\Big]\Big[x+(1+u)y-\frac{2m}{u+1}\Big],\\ \dot y=&\Big[y-\frac{6m}{(u-2)(1+u)}\Big]\Big[y+\frac{2m}{u-2}\Big]\Big[(1+u)x+y-\frac{2m}{u+1}\Big].\end{aligned}$$

As $m\neq 0$ we can apply the transformation

$$x_1=\frac{(u-2)(1+u)}{2m(4+u)}x+\frac{u+1}{u+4},$$
$$y_1=\frac{(u-2)(1+u)}{2m(4+u)}\,y+\frac{u+1)}{u+4},\quad t_1=\frac{4m^2(4+u)^2\,t}{(u-2)^2(1+u)^2}$$

and we arrive at the 1-parameter family of systems (we keep the old notation)

$$\begin{aligned}\dot x=&x(x-1)\big[x+(1+u)y-u\big],\\ \dot y=&y(y-1)\big[(1+u)x+y-u\big],\quad u(u+1)(u+4)(u-2)\neq 0.\end{aligned}\tag{3.168}$$

These systems possess six distinct fixed invariant affine straight lines

$$L_1:x=0,\quad L_2:x=1,\quad L_3:y=0,\quad L_4:y=1,\quad L_5:y=x,\quad L_6:y=-x$$

and the following nine finite singularities:

$$\begin{gathered}M_1(0,0),\quad M_2(0,1),\quad M_3(0,u),\quad M_4(1,0),\quad M_5(1,1),\\ M_6(1,-1),\quad M_7(-1,1),\quad M_8(u,0),\quad M_9\big(u/(2+u),u/(2+u)\big).\end{gathered}\tag{3.169}$$

It is clear that for $u=1$ the singularity M_3 (respectively M_8) coalesces with M_2 (respectively M_4) producing two double finite singularities. Moreover if $u=-2$ the singularity M_9 coalesces with the singularity at infinity $N[1:1:0]$.

We observe that the above systems are symmetric with respect to the invariant line $y=x$, because the change $(x,y)\to(y,x)$ conserves these system. As a consequence we have one pair of finite singularities depending on the parameter u which are symmetric with respect the line $y=x$: M_3 and M_8.

On the other hand due to the symmetry it is sufficient to determine the position of the singularity M_3 with respect to the singular points M_1 and M_2 located on the same invariant line $x = 0$ as well as the position of the singularity M_9 with respect to the singular points M_1 and M_5 located on the same invariant line $y = x$.

So examining the positions of the singular points $M_3(0, u)$ and $M_9\big(u/(2+u), u/(2+u)\big)$ we obtain, respectively

$$u < 0 \Rightarrow M_3 \prec M_1 \prec M_2; \quad 0 < u < 1 \Rightarrow M_1 \prec M_3 \prec M_2;$$
$$u \geq 1 \Rightarrow M_1 \prec M_2 \preceq M_3,$$

and

$$u = -2 \Rightarrow M_9 \equiv N[1:1:0]; \quad -2 < u < 0 \Rightarrow M_9 \prec M_1 \prec M_5;$$
$$u > 0 \Rightarrow M_1 \prec M_9 \prec M_5; \quad u < -2 \Rightarrow M_1 \prec M_5 \prec M_9.$$

So considering the positions of all the singularities (3.169) corresponding to these conditions we conclude that the following lemma is valid.

Lemma 3.22. *The family of systems* (3.168) *with the condition* $u(u+1)(u+4)(u-2) \neq 0$ *possesses the following configurations of invariant lines when the corresponding conditions indicated below are satisfied:*

$$Config.\ 7.76 \ \Leftrightarrow\ u > 1;$$
$$Config.\ 7.80 \ \Leftrightarrow\ 0 < u < 1;$$
$$Config.\ 7.81 \ \Leftrightarrow\ u < -2;$$
$$Config.\ 7.82 \ \Leftrightarrow\ -2 < u < 0;$$
$$Config.\ 7.88 \ \Leftrightarrow\ u = -2;$$
$$Config.\ 7.93 \ \Leftrightarrow\ u = 1.$$

2.2.2. *Subcase $h = -m$.* In this case we have

$$H_1 = H_1' = H_2' = 0, \quad H_2 = \frac{4m(u^2-4)}{(1+u)^3}\big[c(1+u)^2(4+u)^2 + 4m^2(13+5u+u^2)\big] = 0.$$

If $m = 0$ we obtain $h = 0$ and this case was considered above and leads to systems (3.165) and therefore we assume $m \neq 0$. On the other hand by the condition (3.163) we have $(1+u)(4+u)(u-2) \neq 0$. So the condition $H_2 = 0$ gives

$$c = -\frac{4m^2(13+5u+u^2)}{(1+u)^2(4+u)^2}$$

and we arrive at the family of systems

$$\begin{aligned} \dot{x} &= \Big[x - \frac{2m}{4+u}\Big]\Big[x - \frac{6m}{(1+u)(4+u)}\Big]\Big[x + (1+u)y + \frac{2m}{u+1}\Big], \\ \dot{y} &= \Big[y + \frac{2m}{4+u}\Big]\Big[y + \frac{6m}{(1+u)(4+u)}\Big]\Big[(1+u)x + y - \frac{2m}{u+1}\Big]. \end{aligned} \tag{3.170}$$

As $m \neq 0$ we can apply the transformation

$$x_1 = -\frac{(1+u)(4+u)}{2m(u-2)}x + \frac{u+1}{u-2},$$
$$y_1 = -\frac{(1+u)(4+u)}{2m(u-2)}\,y - \frac{u+1}{u-2}, \quad t_1 = \frac{4m^2(u-2)^2\,t}{(1+u)^2(4+u)^2}$$

and this leads to the 1-parameter family of systems (we keep the old notation)

$$\begin{aligned}\dot{x} =& x(x-1)\big[x+(1+u)y+u+2\big],\\ \dot{y} =& y(y+1)\big[(1+u)x+y-u-2\big], \quad u(u+1)(u+4)(u-2)\neq 0.\end{aligned} \tag{3.171}$$

We observe that via the change $(x,y,t,u)\mapsto(x,-y,t,-u-2)$ the above systems can be brought to the systems (3.168) and hence no new configurations could be obtained.

2.3. *Case* $u(r-1)\neq 0$ *and* $1+r+u=0$. In this case the following condition must hold for system (3.142):

$$r(r+1)(u+1)(r+u)(2r+2+u)(r+1-u)u(r-1)\neq 0 \;\Rightarrow\; u(u+1)(u+2)\neq 0.$$

For $r=-u-1$ calculations yield

$$H_1=4(h-m-mu)(2h-2m+3hu-mu+mu^2)/(1+u)=0$$

and we consider two subcases: $h-m-mu=0$ and $h-m-mu\neq 0$.

2.3.1. *Subcase* $h-m-mu=0$. Then $h=m(1+u)$ and we calculate

$$H_1=H_2=0,\quad H_1'=4u^3(1+u)^2\big[-f-c(1+u)+6m^2u+3m^2u^2\big]=0.$$

Therefore from $u(u+1)\neq 0$ we obtain $f=-c(1+u)+6m^2u+3m^2u^2$. This implies $H_2'=0$ and we arrive at the family of degenerate system,

$$\begin{aligned}\dot{x} =& (2m-x+y)\big[-c+4m^2+4m^2u+2m(1+u)x+(1+u)x^2\big],\\ \dot{y} =& (2m-x+y)\big[-c+4m^2-cu+6m^2u+3m^2u^2-2my+y^2\big].\end{aligned}$$

2.3.2. *Subcase* $h-m-mu\neq 0$. Then the condition $H_1=0$ yields

$$h(2+3u)+m(u-2)(1+u)=0. \tag{3.172}$$

2.3.2.1 *Possibility* $2+3u=0$. Then $u=-2/3$ and this implies $m=0$. In this case we have

$$H_1=0,\quad H_1'=16(2c+6f+9h^2)/729=0$$

and this gives $f=-(2c+9h^2)/6$. Then calculation yields

$$H_1=H_1'=0,\quad H_2=4h(16c-171h^2)/27=0,\quad H_2'=8h(16c-171h^2))/2187=0.$$

If $h=0$ we obtain the degenerate systems

$$\dot{x}=(3c-x^2)(x-y)/3,\quad \dot{y}=(x-y)(c-3y^2)/3.$$

So $h\neq 0$ and we must have $c=171h^2/16$. In this case we arrive at the family of systems

$$\dot{x}=(3h+4x)(21h+4x)(6h-x+y)/48,\quad \dot{y}=(9h-4y)(x-y)(9h+4y)/16$$

which via the transformation

$$x_1=-2x/(9h)-1/6,\quad y_1=-2y/(9h)-1/2,\quad t_1=-27h^2t/4$$

can be brought to the system

$$\dot{x}=x(x-1)(1+x-y),\quad \dot{y}=y(y+1)(3x-3y-1).$$

This system possesses six distinct fixed invariant affine straight lines

$$\begin{gathered}L_1: x=0,\quad L_2: x=1,\quad L_3: y=0,\\ L_4: y=-1,\quad L_5: y=x-1,\quad L_6: x-3y=1\end{gathered}$$

and eight finite singularities:

$$M_1(0,0), \quad M_2(0,-1), \quad M_3(0,-1/3), \quad M_4(1,0),$$
$$M_5(1,-1), \quad M_6(1,2/3), \quad M_7(-1,0), \quad M_8(-2,-1).$$

Considering Lemma 2.7 we calculate $\mu_0 = 0$ and $\mu_1 = -36(x-y) \neq 0$ and we deduce that one finite singular point has coalesced with $N[1:1:0]$. As a result we arrive at the configuration of invariant lines equivalent to Config. 7.85.

2.3.2.2. *Possibility* $2+3u \neq 0$. Then from the condition (3.172) we obtain $h = -m(-2+u)(1+u)/(2+3u)$ and calculations yield

$$H_1 = 0, \quad H_2 = -\frac{8mu^3}{(2+3u)^3}\Big[(2+3u)^2(c+f+fu) + 24m^2u(1+u)(2+u)\Big] = 0.$$

Since $u \neq 0$ we have to consider two cases: $m = 0$ and $m \neq 0$.

2.3.2.2.1. *Case* $m = 0$. This implies $h = 0$ and we obtain

$$H_1 = H_2 = H_2' = 0, \quad H_1' = -4u^3(1+u)^2(c+f+cu) = 0.$$

Therefore from $u(u+1) \neq 0$ we obtain $f = -c(1+u)$ and this leads to the degenerate system

$$\dot{x} = (c - x^2 - ux^2)(x-y), \quad \dot{y} = (x-y)(c + cu - y^2).$$

2.3.2.2.2. *Case* $m \neq 0$. Since $u \neq 0$ the condition $H_2 = 0$ implies

$$f = -\frac{c(2+3u)^2 + 24m^2u(1+u)(2+u)}{(1+u)(2+3u)^2}$$

and then we have $H_1 = H_2 = 0$ and

$$H_1' = \frac{4u^2(1+u)(2+u)}{(2+3u)^2}\Big[-cu^2(2+3u)^2 + m^2(1+u)(16+32u+28u^2-12u^3+3u^4)\Big] = 0.$$

Because $u(1+u)(2+u) \neq 0$ and $2+3u \neq 0$, the condition $H_1' = 0$ gives

$$c = \frac{m^2(1+u)(16+32u+28u^2-12u^3+3u^4)}{u^2(2+3u)^2}.$$

This implies $H_2' = 0$ and we arrive at the family of systems

$$\begin{aligned}
\dot{x} &= -(1+u)\Big[x - \frac{m(u^2-6u-4)}{u(2+3u)}\Big]\Big[x - \frac{m(4+2u+u^2)}{u(2+3u)}\Big]\Big[x - y + \frac{2m(u-2)}{2+3u}\Big], \\
\dot{y} &= \Big[y - \frac{m(3u^2-2u-4)}{u(2+3u)}\Big]\Big[y - \frac{m(4+6u+3u^2)}{u(2+3u)}\Big](2m - x + y).
\end{aligned}$$

Since $mu(1+u)(2+3u) \neq 0$ we can apply the transformation

$$x_1 = \frac{u(2+3u)}{8m(1+u)}x + \frac{4+6u-u^2}{8(1+u)},$$
$$y_1 = \frac{u(2+3u)}{8m(1+u)}y + \frac{4+2u-3u^2}{8(1+u)}, \quad t_1 = \frac{64m^2(1+u)^2t}{u^2(2+3u)^2}$$

and we obtain the system

$$\dot{x} = x(x-1)\big[(1+u)y - (1+u)x + u\big], \quad \dot{y} = y(y-1)(u - x + y).$$

It remains to observe that replacing in system (3.162) $r = -1-u$ we obtain exactly the above system, i.e. no new configurations could be obtained.

Case $(2r+2+u)(r+1-u)=0$. We observe that the condition $(2r+2+u)^2+(r+1-u)^2\neq 0$ has to be fulfilled, otherwise we obtain a contradiction: $r+1=0$. So we examine both subcases given by this relation.

1. *Subcase* $2r+2+u=0$. Then $r=-(u+2)/2\neq 0$ and we have $r(r+1)=u(2+u)/4\neq 0$.

According to Lemma 3.17 in this case we could not have a couple of parallel invariant lines in the direction $y=-rx$. Hence we deduce that the possible directions in which systems (3.142) could possess two couples of parallel invariant lines could be either (i) $x=0$ and $y=0$, or (ii) $x=0$ and $y=x$, or (iii) $y=0$ and $y=x$. We examine each one of these possibilities.

1.1. *Directions $x=0$ and $y=0$.* The existence of two couples of the parallel invariant straight lines in these directions was examined earlier (see Lemma 3.18) in the generic case and we have determined the conditions (3.145) (for direction $x=0$) and (3.146) (for direction $y=0$). These conditions were detected without imposing the condition $2r+2+u\neq 0$. The only restriction was $(1+u)(r+u)\neq 0$, which now becomes $(1+u)(u-2)\neq 0$. So the conditions (3.145) and (3.146) remain valid also in the case $2r+2+u=0$ and setting $r=-(u+2)/2$ we arrive at the conditions

$$\begin{gathered}
d=-\frac{2\big[c(1+u)^2-2h^2(2+u)\big]}{(1+u)(2+u)},\quad a=\frac{2h\big[2h^2(2+u)-c(1+u)^2\big]}{(1+u)^3},\\
k=l=0,\quad e=\frac{16m^2+f(u-2)^2}{2(u-2)},\\
r=-(u+2)/2\quad b=-\frac{4m\big[16m^2+f(u-2)^2\big]}{(u-2)^3}.
\end{gathered}\tag{3.173}$$

We point out that these conditions guarantee the existence of two couples of parallel invariant lines in the directions $x=0$ and $y=0$. We recall that in this case the parameter u must satisfy the following condition

$$u(u+1)(u-2)(u+2)\neq 0,\tag{3.174}$$

Now we consider the other two directions provided that the conditions (3.173) are satisfied.

(a) *Direction $y=x$.* Considering (3.143) it remains to examine the equations $Eq_6=0$, $Eq_9=0$ and $Eq_{10}=0$. When the conditions (3.173) are satisfied we obtain $Eq_6=-2(h-m)=0$ and this yields $h=m$. By Lemma 2.8 in order to have a common solution of the equations $Eq_9=0$ nd $Eq_{10}=0$ with respect to W the condition $R_W^{(0)}(Eq_9,Eq_{10})=0$ is necessary. We calculate

$$R_W^{(0)}(Eq_9,Eq_{10})=\frac{uU_1U_2}{2(u-2)^6(1+u)^6(2+u)}$$

where

$$\begin{aligned}
U_1=&2c(u-2)^2(1+u)^2+f(u-2)^2(1+u)^2(2+u)\\
&+9m^2u(2+u)(4+u),\\
U_2=&(u-2)^4(1+u)^4(2c+2f+cu)^2-4m^2(u-2)^2(1+u)^2(4+u)\\
&\times\big[c(2+u)(16+6u+u^3)-f(32+36u+12u^2-u^3)\big]\\
&+4m^4(4+u)^2(4+2u+u^2)(112+112u+24u^2-2u^3+u^4)
\end{aligned}\tag{3.175}$$

and because $u \neq 0$ we necessarily must have $U_1U_2 = 0$. We will later examine this condition together with the conditions of the existence of one invariant line in the direction $y = -rx$.

(b) *The direction* $y = -rx$. In this case, considering the equations (3.144) and the conditions (3.173) and $h = m$, we obtain $Eq_6 = -u(2m + 3W)/2 = 0$ and due to $u \neq 0$ this yields $W = -2m/3$. Clearly that this value of W must be a common solution of the equations $Eq_9 = 0$ nd $Eq_{10} = 0$ from (3.144). For this value of W we obtain

$$Eq_9 = \frac{U_1'}{9(u^2-4)(1+u)}, \quad Eq_{10} = \frac{U_2'}{27(u-2)^3(1+u)^3(2+u)}$$

where

$$\begin{aligned} U_1' &= 9u(u-2)(1+u)(2c+2f+cu) - -2m^2(4+u)(-32-16u+7u^2), \\ U_2' &= m\big[9c(u-2)^3(8+5u) - 18f(u-2)^2(1+u)^3(16+2u+u^2) \\ &\quad - 2m^2(4+u)(128+1744u+1880u^2+452u^3-20u^4+35u^5)\big]. \end{aligned}$$

It is clear that in order to have one invariant line in the direction $y = -rx$ the condition $U_1' = U_2' = 0$ must hold. In addition we must also have an invariant line in the direction $y = x$ and hence the following condition must be satisfied

$$U_1U_2 = U_1' = U_2' = 0,$$

where U_1 and U_2 are given in (3.175).

Since by condition (3.174) we have $u(u+1)(u-2) \neq 0$, condition $U_1' = U_2' = 0$ gives

$$\begin{aligned} c &= -\frac{2m^2(2+u)(64+64u+51u^2-23u^3+7u^4)}{9u^2(u-2)^2(1+u)^2}, \\ f &= -\frac{m^2(256+256u+204u^2+232u^3+109u^4)}{9u^2(u-2)^2(1+u)^2}. \end{aligned}$$

This implies $U_1 = 0$ and we arrive at the family of systems

$$\begin{aligned} \dot x &= \Big[x + \frac{2m(u-8)}{3u(u-2)}\Big]\Big[x + \frac{2m(8+u+2u^2)}{3u(u-2)(1+u)}\Big]\Big[(1+u)y - (2+u)x/2 \\ &\quad + \frac{m(2+u)}{1+u}\Big], \\ \dot y &= \Big[y + \frac{m(8+5u)}{3u(1+u)}\Big]\Big[y + \frac{m(16+14u+7u^2)}{3u(u-2)(1+u)}\Big]\Big[(u-2)x/2 + y - \frac{4m}{u-2}\Big]. \end{aligned} \tag{3.176}$$

To simplify these systems we need to use a transformation which depends on two cases: $m(4+u) \neq 0$ and $m(4+u) = 0$.

1.1.1. *Case* $m(4+u) \neq 0$. Then we can apply the transformation

$$\begin{gathered} x_1 = -\frac{3u(u-2)(1+u)}{2m(4+u)^2}x - \frac{(u-8)(1+u)}{(4+u)^2}, \quad t_1 = \frac{4m^2(4+u)^4\, t}{9u^2(u-2)^2(1+u)^2} \\ y_1 = -\frac{3u(u-2)(1+u)}{2m(4+u)^2}y - \frac{(u-2)(8+5u)}{2(4+u)^2} \end{gathered}$$

and this leads to the 1-parameter family of systems

$$\begin{aligned} \dot x &= x(x-1)\big[2(1+u)y - (2+u)x + u\big]/2, \\ \dot y &= y(y-1)\big[(u-2)x + 2y + u\big]/2. \end{aligned} \tag{3.177}$$

It remains to observe that this family of systems is a subfamily of (3.151) defined by the condition $r = -(u+2)/2$. So we could not obtain new configurations.

1.1.2. *Case* $m(4+u) = 0$. If $m = 0$ then system (3.176) becomes cubic homogeneous

$$\dot{x} = -x^2(2x + ux - 2y - 2uy)/2, \quad \dot{y} = y^2(-2x + ux + 2y) \tag{3.178}$$

for which we calculate (see the definition of the polynomial $H(X, Y, Z)$ on the page 13):

$$H(X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = X^2(2X + uX - 2Y)(X - Y)^2Y^2/16.$$

So by Lemmas 2.5 and 2.6 we conclude that the above system has invariant lines of total multiplicity 8 (including the line at infinity).

Assume now $m \neq 0$ and $u = -4$. Then system (3.176) becomes

$$\dot{x} = (m - 3x)^2(2m + 3x - 9y)/27, \quad \dot{y} = (m - 3y)^2(2m - 9x + 3y)/27$$

for which we also calculate

$$H(X, Y, Z) = -3^{-8}(X - Y)^3(3X + 3Y - 2mZ)(3X - mZ)^2(3Y - mZ)^2.$$

So in this case we obtain a system possessing invariant lines of total multiplicity 9.

1.2. *Directions* $x = 0$ *and* $y = x$. According to Lemma 3.18 for the existence of a couple of parallel invariant straight lines in the direction $x = 0$ the conditions (3.145) are necessary and sufficient. For $r = -(u+2)/2$ these conditions become

$$\begin{gathered} 1 + u \neq 0, \quad k = 0, \quad d = \frac{2h^2(2+u) - c(1+u)^2}{(1+u)(2+u)}, \\ a = \frac{2h^2(2+u) - c(1+u)^2\big]}{(1+u)^3}, \quad r = -(2+u)/2. \end{gathered} \tag{3.179}$$

We consider now the direction $y = x$ and ask for the existence of a couple of parallel invariant straight lines in this direction. Considering (3.143) for $r = -(u+2)/2$ we obtain $Eq_6 = l - 2h - k + 2m = 0$, i.e. $l = 2h + k - 2m$. Then taking into account also (3.179) we calculate

$$\begin{aligned} Eq_9 =& \frac{(1+u)(2f + cu + fu) + (2+u)(e - 4h^2 + eu)}{2(1+u)(2+u)} + 2(h - m)W \\ &+ (4+u)W^2/2, \\ Eq_{10} =& \frac{(1+u)^2(b + 2ch + bu) - 4h^3(2+u)}{(1+u)^3} + (e - c)W + 2(h - m)W^2 \\ &+ (2+u)W^3/2. \end{aligned}$$

We observe that Eq_9 is of degree two with respect to the parameter W if and only if $u + 4 \neq 0$. Therefore in order to have two common solutions of the equations $Eq_9 = 0$ and $Eq_{10} = 0$ (i.e. to have a couple of parallel invariant lines, which could coincide, in the direction $y = x$) the condition $u + 4 \neq 0$ must be fulfilled.

On the other hand since by (3.174) the condition $2+u \neq 0$ holds, then by Lemma 2.8 for the existence of two common solutions in W of the equations $Eq_9 = 0$ and $Eq_{10} = 0$ the following conditions are necessary and sufficient:

$$R_W^{(0)}(Eq_9, Eq_{10}) = R_W^{(1)}(Eq_9, Eq_{10}) = 0.$$

Calculations yield

$$R_W^{(1)}(Eq_9,Eq_{10})=\frac{(4+u)}{4}\big[2e-(2c+f)(2+u)\big]+h^2+8hm-4m^2+h^2u+\frac{3h^2}{1+u}=0$$

and from $(u+1)(u+4)\neq 0$ we obtain

$$e=\frac{(2c+f)(2+u)}{2}+\frac{2h^2(2+u)}{1+u}-\frac{8(h-m)^2}{4+u}. \tag{3.180}$$

Then condition $R_W^{(0)}(Eq_9,Eq_{10})=0$ gives

$$\begin{aligned} b=&\frac{4f(h-m)}{4+u}+\frac{2c\big[h(3u^2+2u-4)-4m(1+u)^2\big]}{(1+u)(2+u)(4+u)}\\ &-\frac{4h^3(2+u)}{(1+u)^3}+\frac{16h^2(h-m)}{(1+u)(4+u)}-\frac{64(h^3-3h^2m+3hm^2-m^3)}{(4+u)^3} \end{aligned} \tag{3.181}$$

and therefore,

$$\begin{gathered} Eq_9=\Psi(W),\quad Eq_{10}=\frac{\big[8(h-m)+(2+u)(4+u)W\big]}{(4+u)^2}\Psi(W),\\ \Psi(W)=\frac{(4+u)(2c+2f+2cu+fu)}{2(2+u)}-\frac{h^2(u^2+4u+12)}{(u+1)(u+4)}+\frac{4m(2h-m)}{u+4}\\ +2(h-m)W+(4+u)W^2/2. \end{gathered}$$

Therefore the equations $Eq_9=0$ and $Eq_{10}=0$ have two common solutions in W given by $\Psi(W)=0$, real or complex, distinct or coinciding.

Thus, considering the conditions (3.179), (3.180) and (3.181) we ask for the existence of two invariant line in each one of the directions $y=0$ and $y=-rx$.

(a) *Direction $y=0$.* Taking into account (3.147) we obtain $Eq_5=2(h-m)=0$ and hence we obtain $h=m$. Then we calculate

$$\begin{aligned} Eq_8&=\frac{(2+u)(2c+f-4m^2+2cu+fu)}{2(1+u)}-2mW+(u-2)W^2/2,\\ Eq_{10}&=-\frac{2m\big[c(1+u)^2-2m^2(2+u)\big]}{(1+u)^3}-fW-W^3. \end{aligned}$$

Since by (3.174) the condition $u-2\neq 0$ holds, according to Lemma 2.8, for the existence of a common solution in W of the equations $Eq_8=0$ and $Eq_{10}=0$, the following conditions are necessary and sufficient:

$$\begin{aligned} R_W^{(0)}(Eq_8,Eq_{10})&=\frac{V_1V_2}{2(1+u)^6}=0,\quad V_1=2c(1+u)^2+f(1+u)^2-m^2(7+4u),\\ V_2&=c^2(1+u)^4(2+u)^3+4c((1+u)^2)\big[f(1+u)^2(2+u)^2\\ &\quad-m^2u(4+u)(2+4u+u^2)\big]+4(2+u)\big[f^2(1+u)^4\\ &\quad-fm^2u(1+u)^2(4+u)+m^4u^2(4+u)^2\big]. \end{aligned}$$

Hence we have to examine the condition $V_1V_2=0$. We will later examine this condition together with conditions of the existence of one invariant line in the direction $y=-rx$.

(b) *Direction $y=-rx$.* In this case from (3.144) considering the conditions (3.179), (3.180) and (3.181) we obtain $Eq_6=-u(2m+3W)/2=0$ which gives

$W=-2m/3$. Substituting in Eq_9 and Eq_{10} from (3.144) we obtain

$$Eq_9=\frac{V_1'}{9(1+u)(2+u)},\quad Eq_{10}=\frac{V_2'}{27(1+u)^3(2+u)}.$$

where

$$V_1'=9(u-2)u(1+u)(2c+2f+cu)-2m^2(4+u)(16+7u),$$
$$V_2'=9(1+u)^2(2+u)\big[c(u-2)-2f(1+u)\big]+2m^2(4+u)^3$$

So considering (3.174) the condition $V_1'=V_2'=0$ gives

$$c=\frac{2m^2(31+29u+7u^2)}{9(1+u)^2(2+u)},\quad f=\frac{m^2(2+19u+8u^2)}{9(1+u)^2(2+u)}$$

and this implies $V_1=0$. So we arrive at the family of systems

$$\begin{aligned}\dot x=&\Big[x+\frac{2m}{3(2+u)}\Big]\Big[x+\frac{2m(5+2u)}{3(1+u)(2+u)}\Big]\Big[(1+u)y-(2+u)x/2+\frac{m(2+u)}{1+u}\Big],\\ \dot y=&\Big[y+\frac{m}{1+u}\Big]\Big[\frac{4m^2(5+2u)}{9(1+u)(2+u)}+\frac{3m(2+u)}{2(1+u)}x-\frac{m}{1+u}\,y+\frac{u-2}{2}xy+y^2\Big].\end{aligned}$$

If $m=0$ then the above systems become exactly systems (3.178) which have invariant lines of total multiplicity 9.

In the case $m\neq 0$ via the transformation

$$x_1=-\frac{(1+u)}{2m(4+u)}\,\big[3(2+u)x+2m\big],\quad t_1=\frac{4m^2t(4+u)^2}{9(1+u)^2(2+u)^2},$$
$$y_1=-\frac{3(1+u)(2+u)}{2m(4+u)}\,(x-y)+1/2$$

we arrive at the family of systems

$$\dot x=x(x-1)\big[ux-2(1+u)y-2-u\big]/2,\quad \dot y=-y(y-1)\big[(4+u)x-2y+2+u\big]/2.$$

It remains to observe that these systems become systems (3.177) by the change of the parameter: $u\to-(u+2)$.

1.3. *Directions $y=0$ and $y=x$.* We claim, that this case could be brought to the case of the directions $x=0$ and $y=x$ which we investigated above (beginning from the page 96).

Recall that we are in the subcase $2r+2+u=0$. This condition depends only on the parameters r and u of homogeneous cubic parts of systems (3.142) with $s=1$ (we denote this family (3.142) with $\{s=1\}$). So we consider the associated homogeneous cubic systems

$$\dot x=rx^3+(1+u)x^2y,\quad \dot y=(r+u)xy^2+y^3,\quad r(r+1)\neq 0 \tag{3.182}$$

and apply the change $(x,y,t,r,u)\mapsto(y,x,r_1t_1,1/r_1,u_1/r_1)$. Then we obtain the systems

$$\dot x=r_1x^3+(1+u_1)x^2y,\quad \dot y=(r_1+u_1)xy^2+y^3$$

with $r_1=1/r$ and $u_1=u/r$. For these systems we have

$$r_1(r_1+1)=(1+r)/r^2\neq 0,\quad 2r_1+2+u_1=(2r+2+u)/r=0.$$

Therefore we have passed from the homogeneous systems (3.182) with the condition $2r+2+u=0$ to the above systems with the condition $2r_1+2+u_1=0$, interchanging the directions $y=0$, $x=0$.

It remains to observe that all the coefficients of non-cubic parts of the whole system (3.142) are free parameters, except for the coefficient in front of x^2 (respectively y^2) in the first (respectively second) equation, which vanishes. However the change $x \leftrightarrow y$ does not affect this property.

2. *Subcase* $r+1-u=0$. According to Lemma 3.17 in this case we could have a couple of parallel invariant straight lines in the direction $y=-rx$. We claim that via a rescaling (which replaces the direction $y=-rx$ with $y=x$) system $(3.142)_{\{s=1\}}$ with the condition $r+1-u=0$ could be brought to system of the same form but with the condition $2r+2+u=0$.

Indeed assume that for system (3.142) the condition $r+1-u=0$ is satisfied. Then applying the rescaling $(x,y,t) \mapsto (-x/r, y, t)$ to system $(3.142)_{\{s=1\}}$ we obtain the system

$$\begin{aligned}
\dot{x} &= a_1 + c_1x + d_1y + 2h_1xy + k_1y^2 + r_1x^3 + (1+u_1)x^2y,\\
\dot{y} &= b_1 + e_1x + f_1y + l_1x^2 + 2m_1xy + (r_1+u_1)xy^2 + y^3, \quad r(r+1) \neq 0
\end{aligned}$$

where $a_1, b_1, \dots, m_1$ are new free parameters and $r_1 = 1/r$ and $u_1 = -(r+1+u)/r$. Then $r = 1/r_1$, $u = -(r_1+1+u_1)/r1$ and for these system we have

$$r(r+1) = (1+r_1)/r_1^2 \neq 0, \quad r+1-u = (2r_1+2+u_1)/r_1 = 0.$$

This completes the proof of our claim and this means that it is not necessary to investigate the case $r+1-u=0$ because no new configurations could be obtained.

Since all the possible cases were investigated we conclude that the class $\mathrm{CSL}_7^{4s\infty}$ of cubic systems possess at most 93 distinct configurations of invariant lines. In the next subsection we prove that all of them are non-equivalent according to Definition 1.3.

3.6. **Geometric invariants and the proof of the non-equivalence of the 93 configurations.** In this subsection we complete the proof of the Main Theorem by showing that all 93 configurations of invariant lines we constructed are non-equivalent according to Definition 1.3.

Notation 3.23. Let us denote

$$\begin{aligned}
CS = \big\{(S) : (S) &\text{ is a system (2.1) with } \gcd(P(x,y), Q(x,y)) = 1 \text{ and}\\
&\max\big(\deg(P(x,y)), \deg(Q(x,y))\big) = 3\big\}
\end{aligned}$$

$$\begin{aligned}
CSL = \big\{(S) \in CS : (S) &\text{ possesses at least one invariant affine line or}\\
&\text{the line at infinity with multiplicity at least two}\big\}.
\end{aligned}$$

Notation 3.24. Let

$$\begin{aligned}
\widetilde{P}(X,Y,Z) &= p_0(\mathbf{a})Z^2 + p_1(\mathbf{a},X,Y)Z + p_2(\mathbf{a},X,Y);\\
\widetilde{Q}(X,Y,Z) &= q_0(\mathbf{a})Z^2 + q_1(\mathbf{a},X,Y)Z + q_2(\mathbf{a},X,Y);\\
\widetilde{C}(X,Y,Z) &= Y\widetilde{P}(X,Y,Z) - X\widetilde{Q}(X,Y,Z);\\
\sigma(P,Q) &= \{w \in \mathbb{R}^2) |\ P(w) = Q(w) = 0\};\\
D_S(\widetilde{P},\widetilde{Q}) &= \sum_{w \in \sigma(\widetilde{P},\widetilde{Q})} I_w(\widetilde{P},\widetilde{Q})w;
\end{aligned}$$

$$D_S(\widetilde{C}, Z) = \sum_{w \in \{Z=0\}} I_w(\widetilde{C}, Z)w \quad \text{if} \quad Z \nmid \widetilde{C}(X, Y, Z);$$
$$D_S(\widetilde{P}, \widetilde{Q}; Z) = \sum_{w \in \{Z=0\}} I_w(\widetilde{P}, \widetilde{Q})w;$$
$$\widehat{D}_S(\widetilde{P}, \widetilde{Q}, Z) = \sum_{w \in \{Z=0\}} \Big(I_w(\widetilde{C}, Z), I_w(\widetilde{P}, \widetilde{Q})\Big)w,$$

where $I_w(F, G)$ is the intersection number (see [14]) of the curves defined by homogeneous polynomials $F, G \in \mathbb{C}[X, Y, Z]$ and $\deg(F), \deg(G) \geq 1$.

A complex projective line $uX + vY + wZ = 0$ is invariant for a system (S) if either it coincides with $Z = 0$ or it is the projective completion of an invariant affine line $ux + vy + w = 0$.

Notation 3.25. Let $(S) \in CSL$. Let us denote

$$IL(S) = \big\{l : l \text{ is a line in } P_2(\mathbb{C}) \text{ that is invariant for } (S)\big\},$$
$$M(l) = \text{the multiplicity of the invariant line } l \text{ of } (S).$$

In defining $M(l)$ we assume, of course, that (S) has a finite number of invariant lines.

Remark 3.26. We note that the line $L_\infty : Z = 0$ is included in $IL(S)$ for any $(S) \in CSL$.

Assume we have a finite number of invariant lines. Let $l_i : f_i(x, y) = ax+by+c = 0$, $i = 1, \dots, k$, be all the distinct invariant affine lines (real or complex) of a system $(S) \in CSL$. Let $L_i : \mathcal{F}_i(X, Y, Z) = aX + bY + cZ = 0$ be the complex projective completion of l_i. Let m_i be the multiplicity of the line L_i and let m be the multiplicity of the line at infinity $Z = 0$.

Notation 3.27. Let

$$\mathcal{G} : \prod_i \mathcal{F}_i(X, Y, Z)^{m_i} Z^m = 0;$$
$$\operatorname{Sing} \mathcal{G} = \{w \in \mathcal{G} : w \text{ is a singular point of } \mathcal{G}\};$$
$$\nu(w) = \text{the multiplicity of the point } w, \text{ as a point of } \mathcal{G}.$$
$$D_{IL}(S) = \sum_{l \in IL(S)} M(l)l, \quad (S) \in CSL.$$

Next we define the geometric invariants which will be used in this work.

$$N_{\mathbb{R}} = \#\{l \in IL(S) : l : aX + bY + cZ = 0, a, b, c \in \mathbb{R}\};$$
$$\mathcal{M}_{IL} = \max\{M(l) : l \in IL(S)\};$$

$\mathcal{N}_{ss}$ =the total number of real singular points of the system which are located on the smooth part of the total curve;

$$M(l_\infty) = \text{the multiplicity of the invariant line } l_\infty : Z = 0 \text{ of } (S);$$

$\mathcal{M}_\sigma$ = the maximum multiplicity of a real affine singular point of a system.

Suppose that a system (2.1) possesses a finite number of invariant lines $L_1, \dots L_k$, including the line at infinity. Sometimes it is convenient to consider in our discussion a number of these invariant lines say $L_{i_1}, \dots L_{i_l}$ of a system (S). We call *marked* system (S) by the lines $L_{i_1}, \dots L_{i_l}$ the object denoted by $(S, L_{i_1}, \dots L_{i_l})$ of the system (S) in which we singled out the lines $L_{i_1}, \dots L_{i_l}$. We shall consider invariants attached to such marked system.

For any non-degenerate cubic polynomial system $(S) \in CSL$ marked with an invariant line L that has a finite number of singularities, we can define the following divisor:

Definition 3.28. Let $D(S, L) = \sum_i \nu(w_i) w_i$ where w_i's are the singular points of the system (S) situated on the line L, and $\nu(w_i)$ is the multiplicity of the point w_i, considered as a point of the curve $\mathcal{G}$.

All singularities of the curve $\mathcal{G}$ are also singular points of the system. Apart from these points, on $\mathcal{G}$ we may also have singular points of the system which are smooth points of the curve $\mathcal{G}$ and these points play a role in constructing geometric invariants for the system which will help distinguishing such system. Clearly for such a point w we have $\nu(w) = 1$.

Definition 3.29. We call s-point of the curve $\mathcal{G}$ a singular point of the system that is a smooth point of the curve $\mathcal{G}$.

Remark 3.30. Looking at the configurations we obtained for the class $\mathrm{CSL}_7^{4s\infty}$ we see that the maximum number of s-points located on a line is two, and that they are never adjacent. Furthermore, for every system $(S) \in \mathrm{CSL}_7^{4s\infty}$ with a simple line L containing an s-point, the number of real singularities of (S) on L is 4. On the other hand the maximum number of distinct invariant affine lines over $\mathbb{R}$ each having an s-point is four. The maximum number of s-points occurring in a system (S) is 4 and this occurs only in three configurations (Config. 7.66, Config. 7.89 and Config. 7.91).

Consider a marked system (S, L) with the property that L is simple and on L we have an s-point w. We attach an invariant to such marked system as follows: Consider the divisor $D(S, L)$ and order the integers $\nu(w_i)$ appearing in this divisor as follows: we start with the minimum of the integers $\nu(w_i)$ which in this case is $1 = \nu(w)$. We have two possibilities: (i) on L w is the only s-point and (ii) on L we have two s-points.

Consider case (i) and look at the two adjacent points on L of the point w and denote them by p and q and consider $\nu(p)$ and $\nu(q)$. Suppose that $\nu(p) \neq \nu(q)$. Take the smaller one of the two, suppose it is $\nu(p)$. This defines an order on L, namely from w to p. Then order the singular points on L considering this order from w to p. This gives us the ordered sequence of integers $S_d(S, L) = (1, \nu(p), \nu(r), \nu(q))$ where r is the fourth singular point on L of the system. Suppose now that $\nu(p) = \nu(q)$. Then since $(1, \nu(p), \nu(r), \nu(q)) = (1, \nu(q), \nu(r), \nu(p))$ we define $S_d(S, L)$ as the common value of these two sequences.

Consider now the case (ii). Suppose we have two s-points on L $(= \mathbb{P}_1(\mathbb{R}) \equiv S^1)$, call them w and p. According to Remark 3.30 these are not adjacent points. We have 4 singular points on L which are w, r, p, q. Suppose that $\nu(r) < \nu(q)$. We define then $S_d(S, L) = (1, \nu(r), 1, \nu(q))$.

Clearly S_d is an invariant of marked system (S, L) corresponding to $D(S, L)$. We use the notation S_d to indicate that this is a *sequence* of integers attached to a *divisor*, namely the divisor $D(S, L)$.

Assume now that we have a number of straight invariant lines L_i over $\mathbb{R}$ on which we have affine s-points. Then for each such line we can form $D(S, L_i)$ and

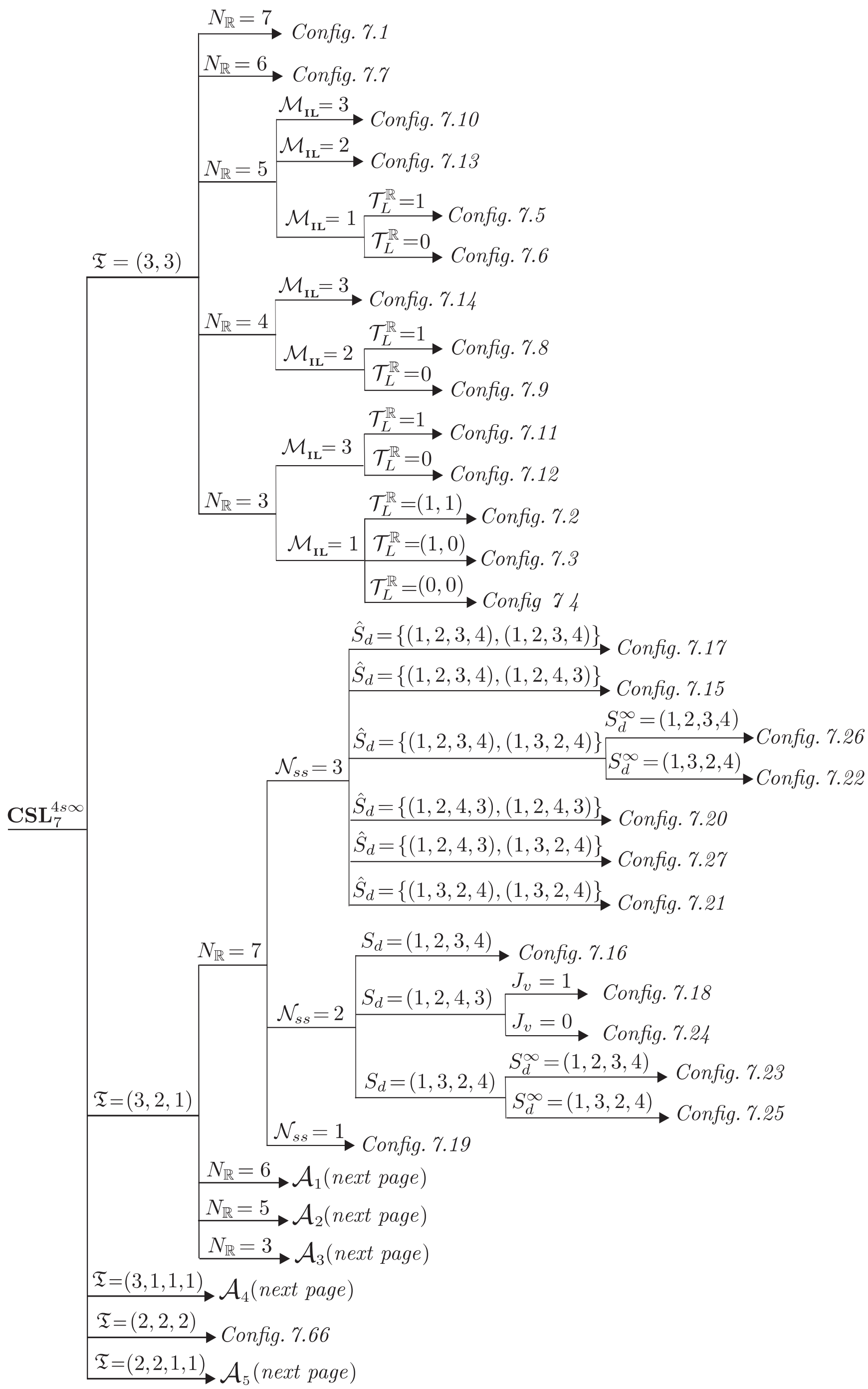


FIGURE 3. Diagram of non-equivalent configurations (to be continued)

$S_d(S, L_i)$. We now consider the marked system $(S, L_i; i \leq 4)$ (see Remark 3.30)

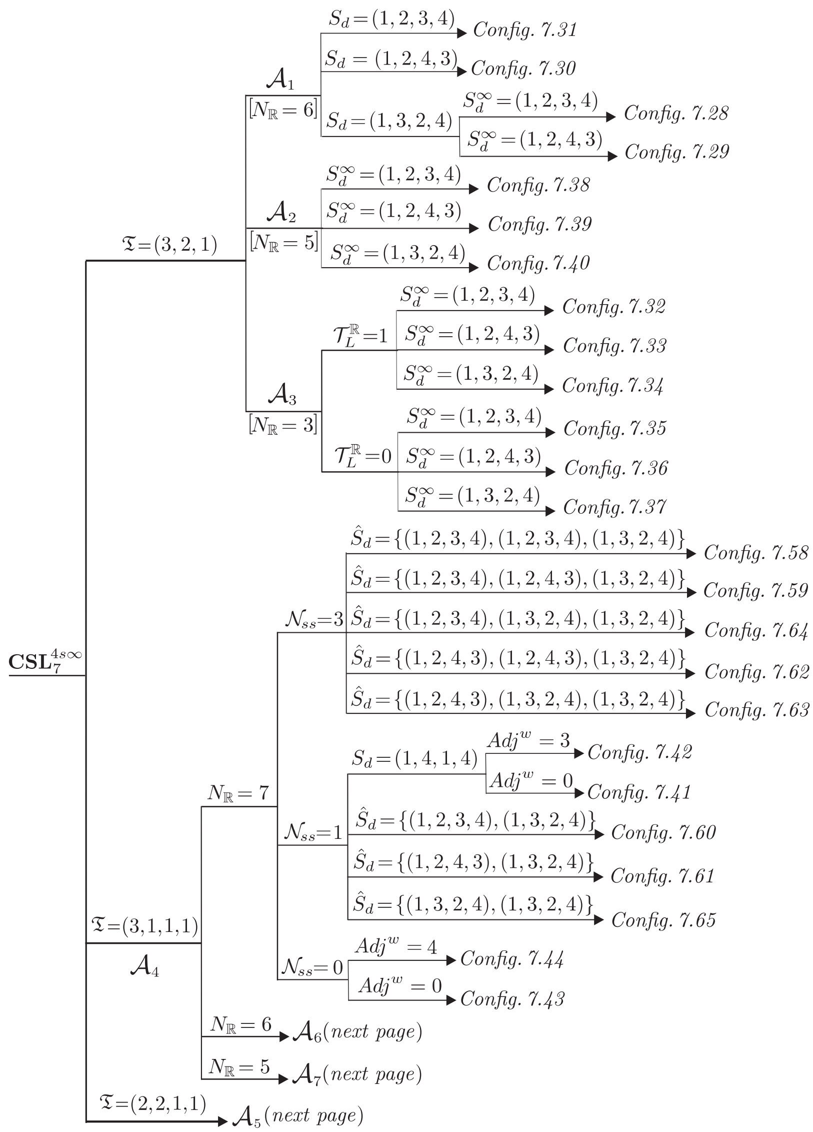


FIGURE 3. (cont.) Diagram of non-equivalent configurations (to be continued)

and to this we associate an invariant $\widehat{S}_d = S_d(S, L_i; i \leq 4)$ defined as follows: We first consider $S_d(S, L_i)$ for each one of the lines L_i. We then define an order on these $S_d(S, L_i)$, all of which start with 1. Consider two such 4-tuples $S_d(S, L_j)$ and $S_d(S, L_k)$ and order these two 4-tuples according to the order of their corresponding first elements where they do not coincide. We now define $\widehat{S}_d = S_d(S, L_i; i \leq 4)$ as this ordered sequence of these 4-tuples.

Let us consider an example: Config. 7.68, with three invariant affine lines with s-points. Then the 4-tuples for the three invariant lines are: $(1, 2, 3, 4)$, $(1, 3, 2, 4)$

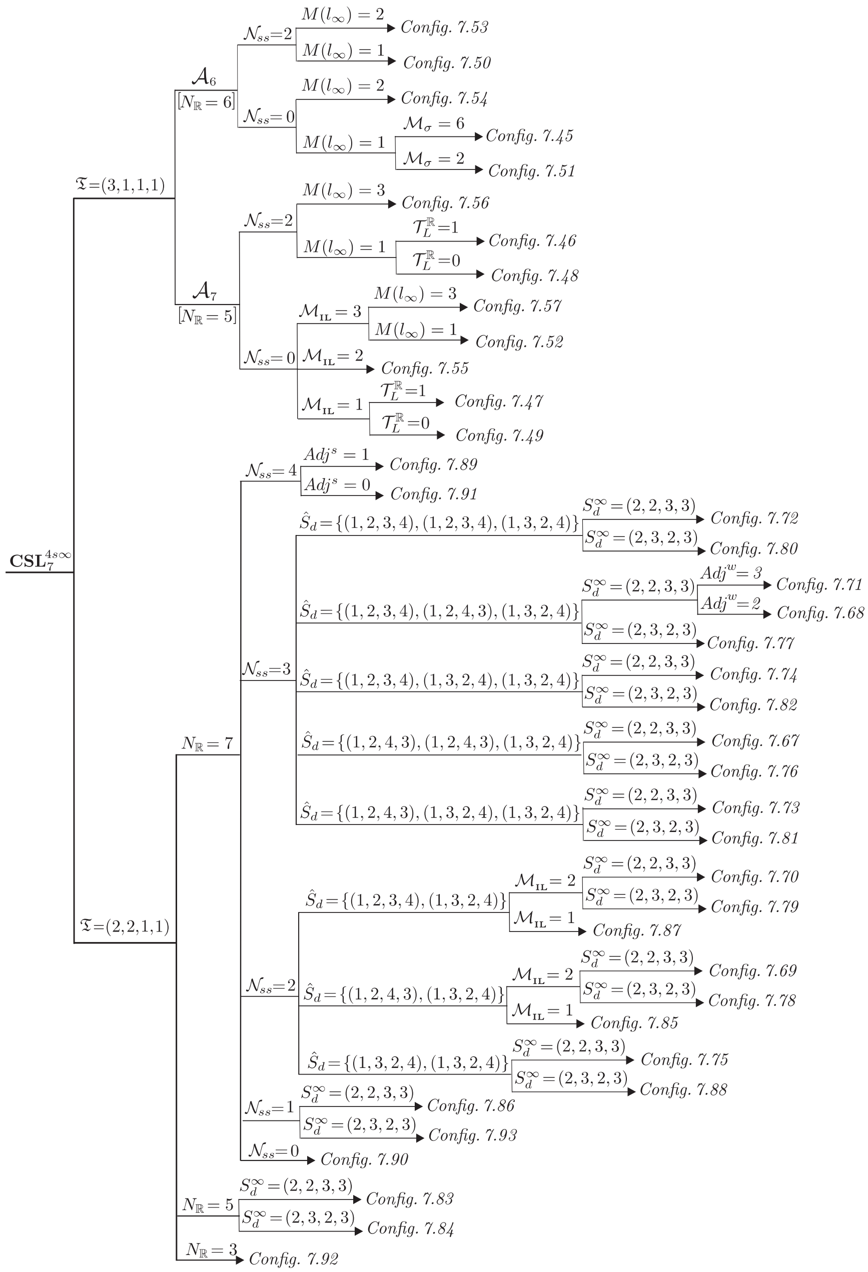


FIGURE 3. (cont.) Diagram of non-equivalent configurations

and $(1,2,4,3)$. We now form the ordered 3-tuple of the 4-tuples: $\widehat{S}_d = \big((1,2,3,4), (1,2,4,3),(1,3,2,4)\big)$.

Suppose the line at infinity $L : Z = 0$ is simple (i.e. of multiplicity one). Consider the divisor $D(S,L)$ and define $S_d^\infty(S,L)$ for the marked system (S,L) as follows:

On the line L we have four real singular points p, q, r, v of the system with corresponding multiplicities $\nu(p), \nu(q), \nu(r), \nu(v)$ of these points considered as points of $\mathcal{G}$. Take the points with minimal multiplicity.

Consider first that we have a single point with minimal multiplicity. Without loss of generality we may assume that this point is p. Consider the two adjacent points of p which without loss of generality we may assume to be q, r and with their multiplicities $\nu(q) \leq \nu(r)$. Consider the ordered 4-tuple $(\nu(p), \nu(q), \nu(v), \nu(r))$ and define $S_d^\infty(S, L)$ as this 4-tuple. For example for the configuration Config. 7.26 we have $S_d^\infty(S, L) = (1, 2, 3, 4)$.

Suppose now that we have two adjacent points p, q with the minimum multiplicity as points of $\mathcal{G}$. Consider the two remaining points r, v and we can assume without loss of generality that $\nu(r) \leq \nu(v)$. We define $S_d^\infty(S, L) = (\nu(p), \nu(q), \nu(r), \nu(v))$.

For example for the configuration $Config.$7.69 we have $S_d^\infty(S, L) = (2, 2, 3, 3)$.

Consider now the case when we have two singular points p, q of (S) with minimum multiplicity i.e. $\nu(p) = \nu(q)$, and which are not adjacent. Let the other two singular points of (S) on L be r, v. Without loss of generality we may assume that $\nu(r) \leq \nu(v)$. Then for each one of the two points p, q we associate an order on L, i.e. the order from p to r or the order from q to r. Consider the first order and form the ordered 4-tuple of the multiplicities of the singular points listed in the first order. We obtain $(\nu(p), \nu(r), \nu(q), \nu(v))$. For the second order we obtain $(\nu(q), \nu(r), \nu(p), \nu(v))$. But these two 4-tuples are equal and we define $S_d^\infty(S, L)$ as the common value of these two 4-tuples.

Consider now the case when we have at least three singular points p, q, r of (S) on L with minimal multiplicity $m \leq \nu(v)$ as points on $\mathcal{G}$. Since these points are on the line at infinity L then we necessarily must have three singular point at infinity, all with multiplicity m, which are consecutive. In this case we define $S_d^\infty(S, L) = (m, m, m, \nu(v))$.

Definition 3.31. Let (S) be a system (2.1) and suppose that at least one of the coefficients j_l in $D_L(S; Z)$ is three and that we actually have three parallel affine lines, one of them L with real coefficients and two others complex: L', L''. Then as indicated in the Notation 1.11 we can bring the cubic system to the form

$$\dot{x} = x[(x+b)^2 + u^2], \quad \dot{y} = q(a, x, y). \tag{3.183}$$

which has the triplet of invariant lines: $x = 0$, $x = -b + iu$, $x = -b - iu$. We have here two possibilities: either $b = 0$ or $b \neq 0$. We define the invariant $\mathcal{T}_L^{\mathbb{R}}$ attached to the marked system (S, L, L', L'') as follows: $\mathcal{T}_L^{\mathbb{R}} = 1$ if and only if $b \neq 0$, and $\mathcal{T}_L^{\mathbb{R}} = 0$ if and only if $b = 0$.

Applying this invariant to configurations Config. 7.5 and Config. 7.6 we then have in the first case $\mathcal{T}_L^{\mathbb{R}}(S) = 1$ and in the second case $\mathcal{T}_L^{\mathbb{R}}(S) = 0$.

The above definition is given for one triplet of invariant lines. In case we have two such triplets (S, L_0, L_0', L_0''), (S, L_1, L_1', L_1'') we shall condense the notation by writing for $\mathcal{T}_L^{\mathbb{R}}$ the two corresponding coordinates which could be either $(1, 1)$ or $(1, 0)$ or $(0, 1)$ as defined in the previous definition. For example for the configuration $Config.$ 7.3 we have: $\mathcal{T}_L^{\mathbb{R}} = (1, 0)$.

Consider a system (S) and define $\mathcal{M}_{\mathcal{G}}^{\mathbb{R}}(S)$ to be the maximum of the numbers $\nu(p)$ where p is a real singular point of the curve $\mathcal{G}$ situated in the affine plane and $\nu(p)$ is its multiplicity. Suppose that $\mathcal{G}$ possesses only one point w such that $\nu(w) = \mathcal{M}_{\mathcal{G}}^{\mathbb{R}}$. Consider all the invariant lines which pass trough w and consider the points at infinity corresponding to all these lines. We now define an invariant attached to such system (S): Define $Adj^w = k$ if and only if w is adjacent to exactly k points at infinity. Then for example consider for the configuration Config. 7.68, $Adj^w = 2$ whereas for the Config. 7.71 we have $Adj^w = 3$.

In a similar manner we introduce the invariant Adj^s. Let (S) be a system of our family. Consider all the invariant affine lines of (S) which have at least one s-point. We define $Adj^s = 1$ if and only if at least one of the s-points of these lines is adjacent to the point at infinity of the line. Define $Adj^s = 0$ if and only if none of the s-points on these lines is adjacent to the point at infinity of the corresponding line. Then for example consider for the configuration Config. 7.89, $Adj^s = 1$ whereas for the Config. 7.91 we have $Adj^s = 0$.

We now introduce another invariant J_v as follows: Suppose a system (S) in our family has a singular point of (S) v with maximum multiplicity $\mathcal{M}_\sigma \geq 2$. We define J_v to be 1 if and only if $\nu(v) = \mathcal{M}_\sigma$ and 0 if and only if $\nu(v) \neq \mathcal{M}_\sigma$.

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