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SYMMETRY ANALYSIS FOR A SECOND-ORDER ORDINARY DIFFERENTIAL EQUATION

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Abstract. In this article, we apply the Lie symmetry analysis to a second-order nonlinear ordinary differential equation, which is a Liénard-type equation with quadratic friction. We find the infinitesimal generators under certain parametric conditions and apply them to construct canonical variables. Also we present some formulas for the first integral for this equation.

1. Introduction

Nonlinear differential equations have a wide array of applications in many scientific fields, and can model a lot of physical and biological phenomena. The study of solutions to nonlinear differential equations has been an important topic in the community of nonlinear sciences. However, it is not always possible to express exact solutions of nonlinear differential equations explicitly in terms of elementary functions. In some cases it is possible to find elementary functions that are constant on solution curves, that is, elementary first integrals. These first integrals allow us to occasionally find some useful properties that an explicit solution may not reveal. Prelle-Singer [16] proposed a method for solving first-order ODEs solutions in terms of elementary functions if such solutions exists. Duarte and his co-authors [8] modified the technique developed by Prelle and Singer and applied it to the second-order ODEs. Their approach was based on the conjecture that if an elementary solution exists for the given second-order ODEs, then there exists as least one elementary first integral $I(x, y, y')$ whose derivatives are all rational functions of x, y, y'. Chandrasekar et al [4, 5] used an extended Prelle-Singer procedure applicable to identify integrable nonlinear oscillator systems and construct integrating factors. Another two powerful techniques for studying explicit solutions and first integrals of various differential equations are the Painlevé test [6, 7, 13], and the Lie symmetry reduction method [2, 3, 11, 12, 14]. The latter method has been applied in a variety of fields in the past decades.

We consider the force-free Duffing-van der Pol equation [15]

$$y'' + (\alpha + \beta y^2)y' - \gamma y + y^3 = 0, \tag{1.1}$$

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where α, β, and γ are arbitrary parameters. It is integrable with the parametric conditions $\alpha = 4/\beta$ and $\gamma = -3/\beta^2$. Under the transformation

$$w = -ye^{(1/\beta)x}, \quad z = e^{-(2/\beta)x}, \tag{1.2}$$

equation (1.1) with restriction $\alpha = 4/\beta$ and $\gamma = -3/\beta^2$ was shown to be transformable into

$$w'' - \frac{\beta^2}{2}w^2w' = 0,$$

which can then be integrated. Equation (1.1) arises in a model describing the propagation of voltage pulses along a neuronal axon and has recently received much attention from many authors. Similar results and more discussions can se found in [10, 15].

In a parallel direction, while performing the invariance analysis of a similar kind of problem, we find that not only (1.1) but also its generalized version

$$y'' + \Big(\frac{4}{\beta} + \beta y^2\Big)y' + \frac{3}{\beta^2}y + y^3 + \delta y^5 = 0, \tag{1.3}$$

where δ is an arbitrary parameter, is invariant under the same set of Lie point symmetries. As a consequence one can use the same transformation (1.2) to trnasform (1.3) into

$$w'' - \frac{\beta^2}{2}w^2w' + \delta w^5 = 0,$$

which is not so simple to integrate. However, we observe that this equation coincides with the second equation in the so-called modified Emden equation (MEE) hierarchy, investigated by Feix et al [9],

$$y'' + y^l y' + gy^{2l+1} = 0, \quad l = 1, 2, \dots, n,$$

where g is an arbitrary parameter. In fact, they have shown that through a direct transformation to a third-order equation, the above equation can be integrated to obtain the general solution for the specific choice of the parameter g, namely, for $g = 1/(l + 2)^2$ [5, 9]. This provides us grounds for expecting that there should be a number of integrable equations which also admits solutions which are both oscillatory and non-oscillatory types in the class

$$y'' + (k_1y^q + k_2)y' + k_3y^{2q+1} + k_4y^{q+1} + \lambda_1 y = 0, \quad q \in \mathbb{R}. \tag{1.4}$$

where $k_i's$, $i = 1, 2, 3, 4$ and λ_1 are arbitrary parameters. In this study, we restrict our attention to this equation for its first integrals by means of the Lie symmetry method. Equation (1.4) is a unified model for several ground-breaking physical systems which includes simple harmonic oscillator, anharmonic oscillator, force-free Helmholtz oscillator, force-free Duffing oscillator, MEE hierarchy, and the generalized DVP hierarchy, see [5, 15]. If $k_3 = 0$, then equation (1.4) is the Duffing-van der Pol-type oscillator. When $q = 1$, equation (1.4) becomes a more general MEE,

$$y'' + (k_1y + k_2)y' + k_3y^3 + k_4y^2 + \lambda_1 y = 0,$$

which provides us the force-free Helmholtz oscillator. When $q = 2$, equation (1.4) reduces to the force-free Duffing-van der Pol oscillator.

Now, let us consider the usage of the Lie symmetry reduction to obtain the first integrals of a second-order nonlinear ODEs. Symmetry is the key to solve differential equations. In this article, we study equation (1.4) to derive its first integrals under certain parametric conditions by applying the Lie point symmetry reduction method. In fact, the exponent q determines the tangent vector, which induces the infinitesimal generator. As a result of this, we can classify the integrable cases by the different values of q. For certain value of parameters, we can find parametric conditions through the determining equations for the infinitesimal generator, which enable us to construct the corresponding canonical coordinates. Through the inverse transformation, we obtain the first integrals of equation (1.4).

2. Preliminaries

Let us briefly recall the Lie symmetry method [11, 14] for the ODEs of the form

$$y^{(n)} = \omega\big(x, y, y', \dots, y^{(n-1)}\big), \quad y^{(k)} \equiv \frac{d^k y}{dx^k}. \tag{2.1}$$

It is assumed that ω is (locally) a smooth function of all its arguments. We first state the symmetry condition. A symmetry condition of (2.1) is a diffeomorphism that maps the set of solutions of the ODE to itself. Any diffeomorphism,

$$\Gamma : (x, y) \mapsto (\hat{x}, \hat{y}),$$

maps smooth planar curves to smooth planar curves. On the plane, the diffeomorphism Γ generates a mapping on the derivatives $y^{(k)}$,

$$\Gamma : \big(x, y, y', \dots, y^{(n)}\big) \mapsto \big(\hat{x}, \hat{y}, \hat{y}', \dots, \hat{y}^{(n)}\big),$$

where

$$\hat{y}^{(k)} \equiv \frac{d^k \hat{y}}{d\hat{x}^k}, \quad k = 1, 2, \dots, n.$$

Using the chain rule, the function $\hat{y}^{(k)}$ can be written as

$$\begin{gathered} \hat{y}^{(k)} = \frac{d\hat{y}^{(k-1)}}{d\hat{x}} = \frac{D_x \hat{y}^{k-1}}{D_x \hat{x}}, \quad k = 1, 2, \dots, n; \\ y^{(0)} \equiv \hat{y}, \end{gathered} \tag{2.2}$$

where $D(x)$ is the total derivative with respect to x:

$$D_x = \partial_x + y' \partial_y + y'' \partial_{y'} + \dots .$$

We obtain the symmetry condition for ODE (2.1),

$$\hat{y}^{(n)} = \omega\big(\hat{x}, \hat{y}, \hat{y}', \dots, \hat{y}^{(n-1)}\big), \tag{2.3}$$

where the function $\hat{y}^{(k)}$ is given by (2.2).

The action of a Lie symmetry maps every point on an orbit to a point on the same orbit. Now consider the orbit through a non-invariant point (x, y). The tangent vector to the orbit at the point $(\hat{x}, \hat{y})$ is $\big(\xi(\hat{x}, \hat{y}), \eta(\hat{x}, \hat{y})\big)$, where

$$\frac{d\hat{x}}{d\varepsilon} = \xi(\hat{x}, \hat{y}), \quad \frac{d\hat{y}}{d\varepsilon} = \eta(\hat{x}, \hat{y}).$$

In particular, the tangent vector at (x, y) is

$$(\xi(x, y), \eta(x, y)) = \Big(\frac{d\hat{x}}{d\varepsilon}\Big|_{\varepsilon=0}, \ \frac{d\hat{y}}{d\varepsilon}\Big|_{\varepsilon=0}\Big).$$

For almost all ODEs, the symmetry condition (2.3) is nonlinear. Lie symmetries are obtained by linearizing (2.3) about $\varepsilon = 0$. It is usually easy to check whether or not a given diffeomorphism is a symmetry of a particular ODE. Since the trivial symmetry condition corresponding to $\varepsilon = 0$ leaves every point unchanged, for ε sufficiently close to 0, the prolonged Lie symmetries are of the form

$$\begin{aligned} \hat{x} &= x + \varepsilon\xi + O(\varepsilon^2), \\ \hat{y} &= y + \varepsilon\eta + O(\varepsilon^2), \\ \hat{y}^{(k)} &= y^{(k)} + \varepsilon\eta^{(k)} + O(\varepsilon^2), \quad k \ge 1. \end{aligned} \tag{2.4}$$

Note that the superscript in $\eta^{(k)}$, $(k = 1, 2, \dots, n)$ is merely an index; it does not denote a derivative of η. Substituting (2.4) into the symmetry condition (2.3); the $O(\varepsilon)$ terms yield the linearized symmetry condition

$$\eta^{(n)} = \xi\omega_x + \eta\omega_y + \eta^{(1)}\omega_{y'} + \dots + \eta^{(n-1)}\omega_{y^{(n-1)}} \tag{2.5}$$

when (2.1) holds. The function $\eta^{(k)}(k = 1, 2, \dots, n)$ can be obtained from (2.2). For $k \ge 1$, we have

$$\begin{aligned} \hat{y}^{(k)} &= \frac{D_x\hat{y}^{(k-1)}}{D_x\hat{x}} = \frac{y^{(k)} + \varepsilon D_x\eta^{(k-1)} + O(\varepsilon^2)}{1 + \varepsilon D_x\xi + O(\varepsilon^2)} \\ &= y^{(k)} + \varepsilon\big(D_x\eta^{(k-1)} - y^{(k)}D_x\xi\big) + O(\varepsilon^2). \end{aligned}$$

From (2.4), we obtain

$$\eta^{(k)}\big(x, y, y', \dots, y^{(k)}\big) = D_x\eta^{(k-1)} - y^{(k)}D_x\xi. \tag{2.6}$$

Now we consider the second-order ODE

$$y'' = \omega\left(x, y, y'\right). \tag{2.7}$$

The diffeomorphism of the form

$$(\hat{x}, \hat{y}) = \big(\hat{x}(x, y), \hat{y}(x, y)\big)$$

is called a point symmetry. To find the Lie point symmetry of a second-order ODE, we need to calculate $\eta^{(1)}$ and $\eta^{(2)}$ first. Since the functions ξ and η depend upon x and y only, (2.6) gives

$$\eta^{(1)} = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2, \tag{2.8}$$

$$\begin{aligned} \eta^{(2)} &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 \\ &\quad + (\eta_y - 2\xi_x - 3\xi_y y')y''. \end{aligned} \tag{2.9}$$

The linearized symmetry condition of (2.7) is obtained by substituting (2.8) and (2.9) into (2.5) and then replacing y'' by $\omega(x, y, y')$. This gives

$$\begin{aligned} &\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 \\ &= (-\eta_y + 2\xi_x + 3\xi_y y')\omega + \xi\omega_x + \eta\omega_y + \{\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2\}\omega_{y'}, \end{aligned} \tag{2.10}$$

which can be solved for many regular cases. Since ξ and η are independent of y', it follows that (2.10) can be decomposed into a system of PDEs, which are the determining equations for the Lie point symmetries. Similarly, for higher-order ODEs, we can also obtain the linearized symmetry condition, which usually looks more complicated.

3. Infinitesimal generator and canonical coordinates

Suppose that a first-order ODE has a one-parameter Lie group of symmetries, whose tangent vector at (x, y) is (ξ, η). Then the partial differential operator

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$$

is called the infinitesimal generator of the Lie group. To deal with the action of Lie symmetries on derivatives of order n or smaller, we introduce the prolonged infinitesimal generator

$$X^{(n)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'} + \cdots + \eta^{(n)}\partial_{y^{(n)}}.$$

The coefficient of $\partial_{y^{(n)}}$ is the $O(\varepsilon)$ term in the expansion of $\hat{y}^{(k)}$, and so $X^{(n)}$ is associated with the tangent vector in the space of variables $(x, y, y', \dots, y^{(n)})$. We can use the prolonged infinitesimal generator to write the linearized symmetry condition (2.5) in a compact form:

$$X^{(n)}\Big(y^{(n)} - \omega(x, y, y', \dots, y^{(n-1)})\Big) = 0$$

when (2.1) holds.

Let $\mathcal{L}$ denote the set of all infinitesimal generators of one-parameter Lie groups of point symmetries of an ODE of order $n \geq 2$. The linearized symmetry condition is linear in ξ and η, and so

$$X_1, X_2 \in \mathcal{L} \Rightarrow c_1X_1 + c_2X_2 \in \mathcal{L} \quad \forall c_1, c_2 \in R.$$

Hence $\mathcal{L}$ is a vector space. The dimension of this vector space is the number of arbitrary constants that appear in the general solution of the linearized symmetry condition.

We know that if an ordinary differential equation admits an infinitesimal generator, then there exists a pair of variables

$$r = r(x, y) \quad \text{and} \quad s = s(x, y),$$

which are called canonical coordinates, with r and s ($s \neq 0$) being arbitrary particular solutions of the first-order linear partial equations

$$\xi(x, y)\frac{\partial r}{\partial x} + \eta(x, y)\frac{\partial r}{\partial y} = 0,$$
$$\xi(x, y)\frac{\partial s}{\partial x} + \eta(x, y)\frac{\partial s}{\partial y} = 1.$$

The change of coordinates should be invertible in some neighbourhood of (x, y), so we impose the nondegeneracy condition

$$r_x s_y - r_y s_x \neq 0.$$

Suppose that $\xi(x, y) \neq 0$. The invariant canonical coordinate $r(x, y)$ is a first integral of

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)}.$$

The coordinate $s(x, y)$ is obtained by the quadrature

$$s(x, y) = \int \frac{dx}{\xi(x, y(r, x))},$$

where the integral is evaluated with r being treated as a constant. Similarly, if $\xi(x,y)=0$ and $\eta(x,y)\neq 0$, then

$$r=x \quad \text{and} \quad s=\int \frac{dy}{\eta(x,y)},$$

are canonical coordinates.

4. Main results

Let us consider (1.4) assuming that q is arbitrary. Chandrasekar et al [5] used the extended Prelle-Singer procedure to identify the first integrals of equation (1.4). Now, let us apply the method of Lie point symmetry [11, 14] to re-consider first integrals of (1.4) under certain parametric conditions. Following the process to determine the symmetries of a differential equation introduced in the preceding section, we can obtain the linearized symmetry condition concerning equation (1.4). Although (2.10) looks complicated, it is not difficult for us to solve $\xi(x,y)$ and $\eta(x,y)$. Since the unknown functions do not depend on the derivative y', after setting the coefficients of the powers $(y')^i$ $(i=0,1,2,3)$ in (2.10) to zero, the linearized symmetry condition (2.10) can be decomposed into the determining system as follows

$$[y']^3 : \xi_{yy}=0, \tag{4.1}$$

$$[y']^2 : \eta_{yy}-2\xi_{xy}=-2\xi_y k_1 y^q-2\xi_y k_2, \tag{4.2}$$

$$\begin{aligned}[y']^1 : 2\eta_{xy}-\xi_{xx}&=-\xi_x(k_1y^q+k_2)-3k_3\xi_y y^{2q+1}-3k_4\xi_y y^{q+1}\\ &\quad -3\lambda_1\xi_y y-qk_1\eta y^{q-1},\end{aligned} \tag{4.3}$$

$$\begin{aligned}[y']^0 : \eta_{xx}&=-2k_3\xi_x y^{2q+1}-2k_4\xi_x y^{q+1}-2\lambda_1\xi_x y+k_3\eta_y y^{2q+1}\\ &\quad +k_4\eta_y y^{q+1}+\lambda_1\eta_y y-(2q+1)k_3\eta y^{2q}\\ &\quad -(q+1)k_4\eta y^q-\lambda_1\eta-(k_1y^q+k_2)\eta_x.\end{aligned} \tag{4.4}$$

The first equation (4.1) gives

$$\xi=a(x)y+b(x). \tag{4.5}$$

Substituting (4.5) into (4.2), we have

$$\eta=-\frac{2a(x)k_1}{(q+1)(q+2)}y^{q+2}+\{a'(x)-a(x)k_2\}y^2+c(x)y+d(x), \tag{4.6}$$

where $q\neq -1$ and $q\neq -2$. And $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are functions of x to be determined. Plugging $(4.5) and (4.6)$ into (4.3) leads to

$$\begin{gathered} : -3k_3a+\frac{2k_1^2q}{(q+1)(q+2)}a=0,\\ [y^{q+1}] : \frac{(q-1)(q+3)}{q+1}k_1a'+3k_4a-k_1k_2qa=0,\\ [y^q] : k_1b'+k_1qc=0,\\ [y^{q-1}] : k_1qd=0,\\ [y] : a''-k_2a'+\lambda_1a=0,\\ [y^0] : 2c'-b''+k_2b'=0. \end{gathered} \tag{4.7}$$

Similarly, plugging (4.5) and (4.6) into (4.4), we find that

$$
\begin{gathered}
[y^{3q+2}] : k_1k_3a = 0,\\
[y^{2q+2}] : (2q-1)k_2k_3a - k_3(2q+1)a' + \frac{2k_1^2}{(q+1)(q+2)}a' - \frac{2k_1k_4}{(q+1)(q+2)}a = 0,\\
[y^{2q+1}] : k_3b' + qk_3c = 0,\\
[y^{2q}] : k_3(2q+1)d = 0,\\
[y^{q+2}] : \frac{q(q+3)}{(q+1)(q+2)}k_1a'' - (q-1)k_2k_4a + (q+1)k_4a'\\
+ \frac{2k_1\lambda_1}{q+2}a - \frac{q^2+3q+4}{(q+1)(q+2)}k_1k_2a' = 0,\\
[y^{q+1}] : 2k_4b' + qk_4c + k_1c' = 0,\\
[y^{q}] : (q+1)k_4d + k_1d' = 0,\\
[y^2] : a''' + k_2\lambda_1a + \lambda_1a' - k_2^2a' = 0,\\
[y] : c'' + 2\lambda_1b' + k_2c' = 0,\\
[y^0] : d'' + \lambda_1d + k_2d' = 0.
\end{gathered} \tag{4.8}
$$

We assume that $q \neq 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, 1, 2$. Since we can combine the coefficients of y with the same power, under this assumption there is no equations with the same power of y^i. We need to carefully consider several cases.

Case 1: $k_1, k_2, k_3, k_4, \lambda_1$ are arbitrary constants. The first equation in (4.8) gives

$$k_1k_3a = 0,$$

which means $a(x) = 0$. The forth equation in (4.7) gives

$$k_1qd = 0,$$

which means $d(x) = 0$. Then the determining system for $b(x)$ and $c(x)$ can be reduced to

$$2c' - b'' + k_2b' = 0, \tag{4.9}$$

$$qc + b' = 0, \tag{4.10}$$

$$c'' + 2\lambda_1b' + k_2c' = 0, \tag{4.11}$$

$$2k_4b' + qk_4c + k_1c' = 0. \tag{4.12}$$

Substituting (4.10) into (4.9), we obtain

$$c(x) = c_0e^{\frac{k_2q}{q+2}x}, \quad b(x) = -\frac{(q+2)c_0}{k_2}e^{\frac{k_2q}{q+2}x} + c_1,$$

where c_0 and c_1 are constants. Substituting $b(x)$ and $c(x)$ into (4.11) and (4.12), we obtain the parametric conditions

$$\lambda_1 = \frac{q+1}{(q+2)^2}k_2^2, \quad k_4 = \frac{k_1k_2}{q+2}, \tag{4.13}$$

respectively. Hence, the general solution of the linearized symmetry condition is

$$\xi = -\frac{(q+2)c_0}{k_2}e^{\frac{k_2q}{q+2}x} + c_1, \quad \eta = c_0e^{\frac{k_2q}{q+2}x}y.$$

By using ξ and η, every infinitesimal generator is of the form

$$\chi = c_0\chi_0 + c_1\chi_1,$$

where

$$\chi_0 = -\frac{(q+2)}{k_2}e^{\frac{k_2 q}{q+2}x}\partial_x + e^{\frac{k_2 q}{q+2}x}y\partial_y, \quad \chi_1 = \partial_x.$$

For the generator χ_1, it is a homothety operator. Generally, it is hard to use this operator to find the first integrals of complicated second-order nonlinear ODEs. So we choose χ_0, which is a translation operator, as a generator to get canonical coordinates. Note that in the following cases, for simplicity, we assume that $c_0 = 1$, $c_1 = 0$ to obtain the generator χ_0.

The invariant canonical coordinate $r(x, y)$ is a first integral of

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)}.$$

Then we obtain

$$r(x,y) = \frac{q+2}{k_2}e^{\frac{k_2}{q+2}x}y. \tag{4.14}$$

The corresponding coordinate $s(x, y)$ is obtained by the quadrature

$$s(x,y) = \int \frac{dx}{\xi(x, y(r,x))}.$$

So we derive

$$s(x,y) = \frac{1}{q}e^{-\frac{qk_2}{q+2}x}. \tag{4.15}$$

Note that equations (4.14)and(4.15) can be rewritten in the parametric form

$$x = -\frac{q+2}{qk_2}\ln(qs), \quad y = \frac{k_2}{q+2}(qs)^{\frac{1}{q}}r. \tag{4.16}$$

Using the nonlinear transformations (4.16) yields

$$\frac{\partial y}{\partial x} = -\frac{k_2^2}{(q+2)^2}q^{\frac{q+1}{q}}\Big(r_s s^{\frac{q+1}{q}} + \frac{1}{q}rs^{\frac{1}{q}}\Big), \tag{4.17}$$

$$\frac{\partial^2 y}{\partial^2 x} = \frac{k_2^3}{(q+2)^3}q^{\frac{2q+1}{q}}\Big(r_{ss}s^{\frac{2q+1}{q}} + \frac{q+2}{q}r_s s^{\frac{q+1}{q}} + \frac{1}{q^2}rs^{\frac{1}{q}}\Big). \tag{4.18}$$

Substituting (4.17) and (4.18) into (1.4), under parametric condition (4.13), we obtain

$$r_{ss} = \frac{k_1k_2^{q-1}}{(q+2)^{q-1}}r_s r^q - \frac{k_3k_2^{2q-2}}{(q+2)^{2q-2}}r^{2q+1}, \tag{4.19}$$

which is integrated as

$$\begin{aligned}&\frac{2k_1}{\sqrt{k_1^2-4(q+1)k_3}}\tanh^{-1}\Big[\frac{k_1(q+2)k_2^q r^{q+1} - 2k_2(q+1)(q+2)^q r_s}{(q+2)k_2^q r^{q+1}\sqrt{k_1^2-4(q+1)k_3}}\Big]\\ &+\ln\big[k_3k_2^{2q-2}r^{2q+2} - k_1k_2^{q-1}(q+2)^{q-1}r_s + (q+1)(q+2)^{2q-2}r_s^2\big] = I,\end{aligned} \tag{4.20}$$

where I is an arbitrary constant. Using the inverse transformation of (4.16), we have

$$r_s = -\frac{q+2}{k_2}e^{\frac{(q+1)k_2}{q+2}x}y - \frac{(q+2)^2}{k_2^2}e^{\frac{(q+1)k_2}{q+2}x}y', \tag{4.21}$$

where $k_2y + (q+2)y' \neq 0$.

(i) If $k_3 \neq \frac{k_1^2}{4(q+1)}$, substituting (4.21) into (4.20), we obtain the first integral of equation (1.4) as follows

$$\begin{aligned}&\frac{2k_1}{\sqrt{k_1^2-4(q+1)k_3}}\tanh^{-1}\Big[\frac{k_1y^{q+1}+\frac{2k_2(q+1)}{q+2}y+2(q+1)y'}{\sqrt{k_1^2-4(q+1)k_3}y^{q+1}}\Big]+\frac{2q+2}{q+2}k_2x\\&+\ln\Big[(q+2)y^{q+1}[k_3(q+2)y^{q+1}+k_1k_2y+k_1(q+2)y']+(q+1)[k_2y+(q+2)y']^2\Big]\\&=I,\end{aligned}$$

where I is an arbitrary constant.

(ii) If $k_3 = \frac{k_1^2}{4(q+1)}$, equation (4.19) can be integrated as

$$\frac{1}{1-\frac{2k_2(q+2)^q(q+1)r_s}{k_1k_2^q(q+2)r^{q+1}}}+\ln\Big[1-\frac{2k_2(q+2)^q(q+1)r_s}{k_1k_2^q(q+2)r^{q+1}}\Big]+\ln(r^{q+1})=I. \tag{4.22}$$

Substituting (4.21) into (4.22), we obtain the first integral of equation (1.4) as follows

$$\begin{aligned}&\frac{q+1}{q+2}k_2x+\ln\Big[k_1y^{q+1}+2(q+1)y'+\frac{2(q+1)}{q+2}k_2y\Big]\\&+\frac{k_1(q+2)y^{q+1}}{k_1(q+2)y^{q+1}+2(q+1)k_2y+2(q+1)(q+2)y'}=I.\end{aligned}$$

The above two formulas of first integrals for equation (1.4) are in good agreement with the results presented in [5, 15].

Case 2: $k_3 = 0$, k_1, k_2, k_4, λ_1 are arbitrary constants. In this case, (1.4) becomes the Duffing van der Pol-type oscillator. The first integrals of this kind of oscillator can also be found in [15].

By the first equation and the forth equation in (4.7), we obtain $a(x) = 0$ and $d(x) = 0$. Then the determining system for $b(x)$ and $c(x)$ is the same as Case 1. So we obtain

$$c(x)=c_0e^{\frac{k_2q}{q+2}x},\quad b(x)=-\frac{(q+2)c_0}{k_2}e^{\frac{k_2q}{q+2}x}+c_1,$$

where c_0 and c_1 are arbitrary constants under the parametric conditions

$$\lambda_1=\frac{q+1}{(q+2)^2}k_2^2,\quad k_4=\frac{k_1k_2}{q+2}. \tag{4.23}$$

Using

$$\xi=-\frac{q+2}{k_2}e^{\frac{k_2q}{q+2}x},\quad \eta=e^{\frac{k_2q}{q+2}x}y,$$

and combining (4.14) and (4.19), under the parametric condition (4.23), we deduce

$$r_{ss}=\frac{k_1k_2^{q-1}}{(q+2)^{q-1}}r_sr^q,$$

which is integrated as

$$r_s=\frac{k_1k_2^{q-1}}{(q+1)(q+2)^{q-1}}r^{q+1}+I_1, \tag{4.24}$$

where I_1 is an arbitrary constant. Substituting equation (4.21) into (4.24), we obtain the first integral of equation (1.4) as

$$e^{\frac{(q+1)k_2}{q+2}x}\Big(y'+\frac{k_2}{q+2}y+\frac{k_1}{q+1}y^{q+1}\Big)=I_1,$$

where $q\neq -1,-2$.

Case 3: $k_1=0$, $k_3=0$ and k_2,k_4,λ_1 are arbitrary constants. In this case, (1.4) becomes the Duffing-type oscillator. The second equation in (4.7) gives $a(x)=0$. And the seventh equation in (4.8) gives $d(x)=0$. The determining system for $b(x)$ and $c(x)$ is reduced to

$$2c'-b''+k_2b'=0, \tag{4.25}$$

$$qc+2b'=0, \tag{4.26}$$

$$c''+2\lambda_1b'+k_2b'=0. \tag{4.27}$$

Substituting (4.26) into (4.25), we obtain

$$c(x)=c_0e^{\frac{qk_2}{q+4}x},\quad b(x)=-\frac{(q+4)c_0}{2k_2}e^{\frac{qk_2}{q+4}x}+c_1,$$

where c_0 and c_1 are arbitrary constants. In this case, we assume $q\neq -4$. If $q=-4$, then we obtain $c(x)=0$, and the equation (1.4) is partially integrable. Substituting $b(x)$ and $c(x)$ into (4.27), we obtain one parametric condition

$$\lambda_1=\frac{2(q+2)}{(q+4)^2}k_2^2. \tag{4.28}$$

For simplicity, we assume taht $c_0=1$ and $c_1=0$ which yield

$$\xi=-\frac{q+4}{2k_2}e^{\frac{qk_2}{q+4}x},\quad \eta=e^{\frac{qk_2}{q+4}x}y.$$

Using ξ and η, we derive

$$r(x,y)=\frac{q+4}{2k_2}e^{\frac{2k_2}{q+4}x}y, \tag{4.29}$$

$$s(x,y)=\frac{2}{q}e^{-\frac{qk_2}{q+4}x}. \tag{4.30}$$

Formulas (4.29)and(4.30) can be rewritten to the parametric form

$$x=-\frac{q+4}{qk_2}\ln\Big(\frac{qs}{2}\Big),\quad y=\frac{2k_2}{q+4}\Big(\frac{qs}{2}\Big)^{2/q}r. \tag{4.31}$$

Using the nonlinear transformation (4.31) yields

$$\frac{\partial y}{\partial x}=-\frac{2qk_2^2}{(q+4)^2}\Big(\frac{q}{2}\Big)^{2/q}\Big(r_ss^{\frac{q+2}{q}}+\frac{2}{q}rs^{2/q}\Big), \tag{4.32}$$

$$\frac{\partial^2 y}{\partial^2 x}=\frac{2q^2k_2^3}{(q+4)^3}\Big(\frac{q}{2}\Big)^{2/q}\Big[r_{ss}s^{\frac{2q+2}{q}}+(1+\frac{4}{q})r_ss^{\frac{q+2}{q}}+\frac{4}{q^2}rs^{2/q}\Big]. \tag{4.33}$$

Substituting (4.32) and (4.33) into (1.4), under the parametric condition (4.28), we obtain

$$r_{ss}=-k_4\Big(\frac{2k_2}{q+4}\Big)^{q-2}r^{q+1},$$

which is integrated as

$$r_s^2 = -\frac{2k_4}{q+2}\Big(\frac{2k_2}{q+4}\Big)^{q-2} r^{q+2} + I_2, \tag{4.34}$$

where I_2 is an arbitrary constant. Using the inverse transformation of (4.31), we can deduce

$$r_s = -\frac{q+4}{2k_2} e^{\frac{k_2(q+2)}{q+4}x}\Big(y + \frac{q+4}{2k_2}y'\Big), \tag{4.35}$$

where $2k_2 y + (q+4)y' \neq 0$. Substituting equation (4.35) into (4.34), we can reproduce the first integral of equation (1.4) as follows

$$e^{\frac{2(q+2)k_2}{q+4}x}\Big[\frac{y'^2}{2} + \frac{2k_2}{(q+4)}yy' + \frac{2k_2^2}{(q+4)^2}y^2 + \frac{k_4}{q+2}y^{q+2}\Big] = I_2.$$

Case 4: $k_1 = 0$, $k_4 = 0$ and k_2, k_3, λ_1 are arbitrary constants. The first equation in (4.7) and the forth equation in (4.8) give $a(x) = 0$ and $d(x) = 0$. Then we can solve for $b(x)$ and $c(x)$ from the system

$$2c' - b'' + k_2 b' = 0, \tag{4.36}$$

$$qc + b' = 0, \tag{4.37}$$

$$c'' + 2\lambda_1 b' + k_2 c' = 0. \tag{4.38}$$

Substituting (4.37) into (4.36), we obtain

$$c(x) = c_0 e^{\frac{qk_2}{q+2}x}, \quad b(x) = -\frac{(q+2)c_0}{k_2} e^{\frac{qk_2}{q+2}x} + c_1,$$

where c_0 and c_1 are arbitrary constants. Substituting $b(x)$ and $c(x)$ into (4.38), we obtain one parametric condition

$$\lambda_1 = \frac{q+1}{(q+2)^2}k_2^2. \tag{4.39}$$

For simplicity, we assume that $c_0 = 1$ and $c_1 = 0$. Then

$$\xi = -\frac{q+2}{k_2} e^{\frac{qk_2}{q+2}x}, \quad \eta = e^{\frac{qk_2}{(q+2)}x}y.$$

By (4.14) and (4.19), under the parametric condition (4.39) we derive

$$r_{ss} = -\frac{k_3 k_2^{2q-2}}{(q+2)^{2q-2}} r^{2q+1},$$

which is integrated as

$$r_s^2 = -\frac{k_3 k_2^{2q-2}}{(q+1)(q+2)^{2q-2}} r^{2q+2} + I_3, \tag{4.40}$$

where I_3 is an arbitrary constant. Substituting equation $(4.21) into (4.40)$, we obtain the first integral of equation (1.4) as

$$e^{\frac{(2q+2)k_2}{q+2}x}\Big[\frac{k_2^2}{2(q+2)^2}y^2 + \frac{k_2}{q+2}yy' + \frac{y'^2}{2} + \frac{k_3}{2q+2}y^{2q+2}\Big] = I_3.$$

Note that all first integrals described herein agree well with those presented in the literature [5, 8, 15] etc. For other cases of k_1, k_2, k_3, k_4 and λ_1, the original equation is partially integrable or the first integral can be derived directly without applying the Lie point symmetry. But the values of q may effect the first integrals of equation (1.4). For example, if $q = -1$ or -2, the tangent vector $\eta(x, y)$ is undefined. However, we can still classify the first integrals of (1.4) in this case by applying the Lie symmetry method as well as the theory of vector fields. In the subsequent work, we will continue to consider this problem for various values of q specifically and its applications.

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