Electronic Journal of Differential Equations, Vol. 2021 (2021), No. 87, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

SOLUTIONS AND EIGENVALUES OF LAPLACE'S EQUATION ON BOUNDED OPEN SETS

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Abstract. We obtain solutions for Laplace's and Poisson's equations on bounded open subsets of $\mathbb{R}^n$ ($n \geq 2$), via Hammerstein integral operators involving kernels and Green's functions, respectively. The new solutions are different from the previous ones obtained by the well-known Newtonian potential kernel and the Newtonian potential operator. Our results on eigenvalue problems of Laplace's equation are different from the previous results that use the Newtonian potential operator and require $n \geq 3$. As a special case of the eigenvalue problems, we provide a result under an easily verifiable condition on the weight function when $n \geq 3$. This result cannot be obtained by using the Newtonian potential operator.

1. Introduction

The Newtonian potential kernel and Newtonian operator have been used to study the following three problems:

(i) solutions of Laplace's equation $\Delta u(x) = 0$ in $\mathbb{R}^n \setminus \{0\}$,
(ii) solutions of Poisson's equation $-\Delta u(x) = v(x)$ in Ω, and
(iii) eigenvalue problems of Laplace's equation $-\Delta u(x) = \mu g(x)u(x)$ in Ω.

It is shown in [3, p.21-22], [4, p.17], and [15, Lemma 2.1 (P_3)] that a solution $\Psi(\cdot, 0)$ is a harmonic function in $\mathbb{R}^n \setminus \{0\}$ for $n \in \mathbb{N}$ with $n \geq 2$; that is, $\Psi(\cdot, 0)$ is a solution of Laplace's equation and belongs to $C^2(\mathbb{R}^n \setminus \{0\})$, where Ψ is the Newtonian potential kernel, and has singularities. We refer the reader to [12, 13] for a study on the Newtonian potential, and to [2, 6, 14] for a study on fractional differential equations, where the related operators involve singularities.

When Ω is a bounded connected open subset in $\mathbb{R}^n$ with $n \geq 2$ and $v \in C^\mu(\Omega)$, it is showed in [4, Lemma 4.2] that Lv is a solution of the Poisson's equation, where L is the Newtonian potential operator. This result is generalized to the case that Ω is a bounded open subset of $\mathbb{R}^n$ in [15, Theorem 2.3].

Eigenvalue problems of the Laplace's equation can be solved by the result on the weight Newtonian potential operator [15, Theorem 2.4] together with the Krein-Rutman theorem. These eigenvalue results can be used to study the existence of positive classical solutions of nonlinear Poisson's equations, see [15, Section 3]. However, the Newtonian potential kernel alone cannot be applied to the eigenvalue

2020 *Mathematics Subject Classification.* 35J05, 31A05, 31B05, 35J08, 47A75.
Key words and phrases. Eigenvalue; Laplace's equation; Poisson's equation; Green's function; Hammerstein integral operator.

Submitted July 12, 2021. Published October 18, 2021.

problems with $n = 2$ because the Newtonian potential kernel changes sign when $n = 2$.

In this article, we study the above three problems via different approaches and deal with the case $n \geq 2$ in a unified setting. We only assume that Ω is a bounded open subset in $\mathbb{R}^n$, and the connectedness on $\bar{\Omega}$ and the smoothness on the boundary of Ω are not required. Some previous results on existence of classical or weak solutions of some linear or nonlinear elliptic boundary value problems required the connectedness and smoothness, for example, see [1, p.633] and [7, 8, 11, 12, 13].

First, we study the Laplace's equation in $\bar{B}_\rho$ and in $\bar{\Omega}$, and obtain solutions of the Laplace's equations via Hammerstein integral operators $S_{\delta,\rho}$ with newly defined kernels Φ_δ. We give properties of Φ_δ and $S_{\delta,\rho}$ including the domain of Φ_δ and compactness of $S_{\delta,\rho}$.

Next, we study solutions and nonnegative solutions of the Poisson's equation in Ω and give these solutions via Hammerstein integral operators $L_{\delta,\rho}$ with Green's functions $k_\delta := \Psi + \Phi_\delta$ for $v \in C^\mu(\Omega)$. These solutions $L_{\delta,\rho}v$ are obviously different from those given by Lv when $n \geq 3$ and are new when $n = 2$.

Finally, we consider the eigenvalue problems of Laplace's equation in Ω and allow $n = 2$. We prove that the spectral radius of the linear integral operator $\mathscr{L}_g$ with the kernel $k_\delta g$ is the eigenvalue of the Laplace's equation in Ω. As a special case, when $n \geq 3$, we obtain a simple condition on g which ensures that the spectral radius of $\mathscr{L}_g$ is the the eigenvalue of the Laplace's equation.

In Section 2 of this paper, we provide some results on the Newtonian potential kernel and the Newtonian potential operator, some of them are new. These results will be used in Sections 3-5. In Section 3, we introduce the kernel Φ_δ and study its properties. The properties of integral operator with the kernel Φ_δ are given. The integral operator is then used to give solutions of the Laplace's equation, and its compactness will be used to obtain compactness of the integral operators involving the Green's functions k_δ in Sections 4 and 5. In Section 4, we give solutions of the Poisson's equation via integral operators involving the Green's functions k_δ. The result will be useful for studying the existence of nonzero nonnegative solutions of the nonlinear Poisson's equation. In Section 5, we show that the spectral radii of the integral operators involving the weight Green's functions are the eigenvalues of the Laplace's equations when $n \geq 2$.

2. Newtonian potential operator

Let $n \in \mathbb{N}$ with $n \geq 2$ and let $\mathbb{R}^n$ be the Euclidean Banach space with norm $|x| = \sqrt{\sum_{i=1}^n x_i^2}$ and inner product $x \cdot y = \sum_{i=1}^n x_i y_i$. We always assume that Ω is a bounded open subset in $\mathbb{R}^n$. Note that Ω is required neither to be a connected set in $\mathbb{R}^n$ nor to have any smoothness on $\partial\Omega$.

We consider the Newtonian potential operator L defined by

$$(Lv)(x) = \int_\Omega \Psi(x,y)v(y)\,dy \quad \text{for } x \in \mathbb{R}^n, \tag{2.1}$$

where $\Psi : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{R}$ is the Newtonian potential kernel defined by

$$\Psi(x,y) := -\Gamma(|x-y|) = \begin{cases} -\frac{1}{2\pi}\ln|x-y| & \text{if } n = 2, \\ \frac{1}{n(n-2)\omega_n|x-y|^{n-2}} & \text{if } n \geq 3, \end{cases} \tag{2.2}$$

where $\Gamma : (0,\infty) \to \mathbb{R}$ is defined by

$$\Gamma(u) = \begin{cases} \frac{\ln u}{2\omega_2} & \text{if } n = 2, \\ -\frac{1}{n(n-2)\omega_n u^{n-2}} & \text{if } n \geq 3, \end{cases} \tag{2.3}$$

where $\omega_2 = \pi$, $\omega_n = \frac{2\pi^{n/2}}{n\Gamma_0(n/2)}$ is the volume of the unit ball in $\mathbb{R}^n$ for $n \geq 3$, and Γ_0 is the Gamma function $\Gamma_0(u) = \int_0^\infty s^{u-1}e^{-s}\,ds$.

The Newtonian potential operator was studied, for example, in [15], where Ω is only required to be a bounded open subset in $\mathbb{R}^n$, and in [3, 4], where Ω is a domain (i.e. a connected open subset) in $\mathbb{R}^n$ with suitable smoothness on $\partial\Omega$.

Notation. For $\rho > 0$ and $x \in \mathbb{R}^n$, let $B_\rho(x) = \{y \in \mathbb{R}^n : |x-y| < \rho\}$, $\bar{B}_\rho(x) = \{y \in \mathbb{R}^n : |x-y| \leq \rho\}$ and $\partial B_\rho(x) = \{y \in \mathbb{R}^n : |x-y| = \rho\}$. We write

$$B_\rho = B_\rho(0), \quad \bar{B}_\rho = \bar{B}_\rho(0), \quad \partial B_\rho = \partial B_\rho(0).$$

Let $p, q \in [1,\infty]$ be the conjugate indices, that is, they satisfy

$$1/p + 1/q = 1; \tag{2.4}$$

and if $p = \infty$, then $q = 1$; and if $p = 1$, then $q = \infty$. Hence, if $p \in (n/2, \infty]$, then

$$q \in \begin{cases} [1,\infty) & \text{if } n = 2, \\ [1, \frac{n}{n-2}) & \text{if } n \geq 3. \end{cases} \tag{2.5}$$

The following results on the Newtonian potential kernel can be found in [15, Lemma 2.1] and will be used to prove Theorem 3.8 in Section 3.

Lemma 2.1. *The Newtonian potential kernel Ψ has the following properties.*

(1) $\Psi(\cdot, y) \in C^\infty(\mathbb{R}^n \setminus \{y\})$ *for each* $y \in \mathbb{R}^n$.
(2) $\Delta_x\Psi(x,y) := \sum_{i=1}^n \frac{\partial^2\Psi(x,y)}{\partial x_i^2} = 0$ *for* $x, y \in \mathbb{R}^n$ *with* $x \neq y$.
(3) *If q satisfies* (2.5)*, then* $\Psi(x,\cdot) \in L^q(\Omega)$ *for each* $x \in \mathbb{R}^n$.

Let D be a nonempty subset of $\mathbb{R}^n$. We denote by $\mathscr{F}(D)$ the set of all the functions from $D \to \mathbb{R}$. Let $D_1 \subset D$ be a nonempty subset and $f \in \mathscr{F}(D)$. We still use f to denote the restriction of f on D_1. For $\mu \in (0,1)$, we denote by $C^\mu(D)$ the vector space of all locally μ-Hölder continuous functions on D, see [1, p.629]. If D is bounded and closed, then we denote by $C(\bar{D})$ the Banach space of all continuous functions from $\bar{D}$ to $\mathbb{R}$ with the maximum norm $\|\cdot\|$.

We denote by $L^p(\Omega)$ and $L^p_+(\Omega)$ the Banach space of functions for which the pth power of the absolute values are Lebesgue integrable with norm $\|\cdot\|_{L^p(\Omega)}$, and its positive cone of all the nonnegative functions in $L^p(\Omega)$, respectively.

Proposition 2.2. *If $p \in (n/2, \infty]$, then L maps $L^p(\Omega)$ into $\mathscr{F}(\mathbb{R}^n)$.*

Proof. Let $n \geq 2$, $p \in (n/2,\infty]$ and $v \in L^p(\Omega)$. Let q satisfy (2.5). Let $x \in \mathbb{R}^n$ and $v \in L^p(\Omega)$. By Lemma 2.1(3), for each $x \in \bar{\Omega}$, we have

$$|Lv(x)| \leq \int_\Omega \big|\Psi(x,y)\big||v(y)|\,dy \leq \Big(\int_\Omega |\Psi(x,y)|^q\,dy\Big)^{1/q}\|v\|_{L_p(\Omega)} < \infty.$$

Hence, $Lv \in \mathscr{F}(\mathbb{R}^n)$. □

Proposition 2.2 is given in the proof of [15, Theorem 2.1]. By Proposition 2.2, if $p \in \big(n/2, \infty]$, then L maps $L^p(\Omega)$ into $\mathscr{F}(D)$ for each nonempty subset D in $\mathbb{R}^n$. We need the following known results from [15, Theorems 2.1, 2.2 and 2.3].

Lemma 2.3. *The operator L defined in* (2.1) *has the following properties.*

(1) *If $p \in (n/2, \infty]$, then L maps $L^p(\Omega)$ into $C(\bar{\Omega})$.*
(2) *If $p \in (n, \infty]$, then L maps $L^p(\Omega)$ to $C^1(\mathbb{R}^n)$.*
(3) *L maps $C^\mu(\Omega)$ into $C^2(\Omega)$.*
(4) *If $v \in C^\mu(\Omega)$, then $-\Delta(Lv)(x) = v(x)$ for each $x \in \Omega$.*

Remark 2.4. We note that the following question has not been solved yet. Is $L : L^p(\Omega) \to C(\bar{\Omega})$ compact for $p \in (n/2, \infty]$?

We generalize Lemma 2.3(1) (that is, [15, Theorem 2.1]) from $C(\bar{\Omega})$ to $C(\mathbb{R}^n)$.

Theorem 2.5. *If $p \in (n/2, \infty]$, then L maps $L^p(\Omega)$ into $C(\mathbb{R}^n)$.*

Proof. Let D be a nonempty bounded open subset in $\mathbb{R}^n$ satisfying $\Omega \subset D$. It suffices to prove $Lv \in C(\bar{D})$. We define a Hammerstein integral operator

$$(L^*\bar{v})(x) = \int_D \Psi(x,y)\bar{v}(y)\,dy \quad \text{for } x \in \mathbb{R}^n.$$

By Lemma 2.3(1), for $p \in (n/2, \infty]$, L^* maps $L^p(D)$ into $C(\bar{D})$. Let $v \in L^p(\Omega)$. We define a function $\bar{v} : \bar{D} \to \mathbb{R}$ by

$$\bar{v}(y) = \begin{cases} v(y) & \text{if } y \in \Omega, \\ 0 & \text{if } y \in \bar{D} \setminus \Omega. \end{cases}$$

Then

$$\begin{aligned} (L^*\bar{v})(x) &= \int_D \Psi(x,y)\bar{v}(y)\,dy = \int_\Omega \Psi(x,y)\bar{v}(y)\,dy + \int_{D\setminus\Omega} \Psi(x,y)\bar{v}(y)\,dy \\ &= \int_\Omega \Psi(x,y)v(y)\,dy = (Lv)(x) \quad \text{for each } x \in \mathbb{R}^n. \end{aligned}$$

This implies

$$(Lv)(x) = (L^*\bar{v})(x) \quad \text{for each } x \in \bar{D}.$$

This, together with $L^*\bar{v} \in C(\bar{D})$, implies $Lv \in C(\bar{D})$. □

Lemma 2.6 ([15, Lemma 2.4])**.** *Let $p \in (n/2, \infty]$ and $g \in L^p_+(\Omega)$. Then*

$$\lim_{x\to\tau} \int_\Omega \big|\Psi(x,y) - \Psi(\tau,y)\big| g(y)\,dy = 0 \quad \textit{for each } \tau \in \bar{\Omega}.$$

Lemma 2.7. *Let $p \in (n/2, \infty]$ and $g \in L^p_+(\Omega)$. Then the operator L_g defined by*

$$L_g v(x) = \int_\Omega \Psi(x,y)g(y)v(y)\,dy \tag{2.6}$$

is a compact linear operator from $C(\bar{\Omega})$ to $C(\bar{\Omega})$.

Proof. (i) By Lemma 2.1(3), for each $x \in \bar{\Omega}$, we have

$$|L_g v(x)| \le \int_\Omega \big|\Psi(x,y)\big| g(y)|v(y)|\,dy \le \Big(\int_\Omega |\Psi(x,y)|^q\,dy\Big)^{1/q} \|g\|_{L_p(\Omega)} \|v\|_{C(\bar{\Omega})} < \infty.$$

By a proof similar to that of [5, Lemma 2.1] and applying Lemma 2.6, L_g maps $C(\bar{\Omega})$ to $C(\bar{\Omega})$ and is compact. □

Note that the linear operator L_g in Lemma 2.7 is different from [15, Theorem 2.4], where the kernel is $|\Psi(x,y)|$.

3. Solutions of Laplace's equation

In this section, we study solutions of the Laplace's equation

$$\Delta u(x) = 0 \quad \text{for each } x \in \bar{B}_\rho. \tag{3.1}$$

A function $u : \bar{B}_\rho \to \mathbb{R}$ is said to be a (classical) solution of (3.1) if $u \in C(\bar{B}_\rho) \cap C^2(B_\rho)$ and u satisfies (3.1).

If Ω is a bounded open subset in $\mathbb{R}^n$ and $\bar{\Omega} \subset B_\rho$, then a solution of (3.1) is a solution of the Laplace's equation

$$\Delta u(x) = 0 \quad \text{for } x \in \bar{\Omega}. \tag{3.2}$$

Recall that a function $u : \Omega \to \mathbb{R}$ is said to be a harmonic function in Ω if $u \in C^2(\Omega)$ and u satisfies $\Delta u(x) = 0$ for $x \in \Omega$, see [3, p.20] or [4, p.13]. Hence, every solution of (3.2) is a harmonic function in Ω.

It is well known that $\Psi(\cdot, 0)$ is a harmonic function in $\mathbb{R}^n \setminus \{0\}$, see [3, p.21-22], [4, p.17], and [15, Lemma 2.1 (P_3)]. At the end of this section, we shall provide other harmonic functions via a Hammerstein integral operator.

Notation. For each $\delta > 0$, let

$$y_\delta = \delta^2 |y|^{-2} y \quad \text{for } y \in \mathbb{R}^n \setminus \{0\}, \tag{3.3}$$

$$\mathscr{D}_\delta = D_1 \cup (D_2)_\delta, \tag{3.4}$$

where $D_1 = \{(x, 0) \in \mathbb{R}^n \times \mathbb{R}^n : x \in \mathbb{R}^n\}$ and

$$(D_2)_\delta = \big\{(x, y) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) : y \in \mathbb{R}^n \setminus \{0\} \text{ and } x \neq y_\delta\big\},$$

$$(D_0)_\delta = \{(x, x) : x \in \bar{B}_\delta\}, \tag{3.5}$$

$$r = \max\{|x| : x \in \bar{\Omega}\}, \quad \delta > r, \quad \rho \in [\delta, \delta^2/r). \tag{3.6}$$

For $x, y \in \mathbb{R}^n$, we denote by $(\mathscr{D}_\delta)_1(x)$ and $(\mathscr{D}_\delta)_2(y)$ the cross sections of $\mathscr{D}_\delta$ at x and y, respectively. Then

$$(\mathscr{D}_\delta)_1(x) = \{y \in \mathbb{R}^n : (x, y) \in \mathscr{D}_\delta\}, \quad (\mathscr{D}_\delta)_2(y) = \{x \in \mathbb{R}^n : (x, y) \in \mathscr{D}_\delta\}. \tag{3.7}$$

Lemma 3.1. *Both $(\mathscr{D}_\delta)_1(x)$ and $(\mathscr{D}_\delta)_2(y)$ are open subsets in $\mathbb{R}^n$ for $x, y \in \mathbb{R}^n$.*

Proof. It is easy to verify that for each $x \in \mathbb{R}^n$,

$$(\mathscr{D}_\delta)_1(x) = \{0\} \cup \{y \in \mathbb{R}^n \setminus \{0\} : y_\delta \neq x\}, \tag{3.8}$$

$$(\mathscr{D}_\delta)_2(y) = \begin{cases} \mathbb{R}^n & \text{if } y = 0 \in \mathbb{R}^n, \\ \mathbb{R}^n \setminus \{y_\delta\} & \text{if } y \in \mathbb{R}^n \setminus \{0\}. \end{cases} \tag{3.9}$$

It follows from (3.8) that $(\mathscr{D}_\delta)_1(0) = \mathbb{R}^n$ and

$$\mathbb{R}^n \setminus (\mathscr{D}_\delta)_1(x) = \{y \in \mathbb{R}^n \setminus \{0\} : y_\delta = x\} \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}.$$

It is easy to verify that the set on the right-hand side of the above equation is a closed set in $\mathbb{R}^n$. Hence, $(\mathscr{D}_\delta)_1(x)$ is open in $\mathbb{R}^n$ for each $x \in \mathbb{R}^n$. By (3.9), it is obvious that $(\mathscr{D}_\delta)_2(y)$ is open in $\mathbb{R}^n$ for each $y \in \mathbb{R}^n$. $\square$

Let $x \in \mathbb{R}^n$. For each $i \in I_n := \{1, \cdots, n\}$, we write

$$x = (x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) = (x_i, \hat{x_i}),$$

where $\hat{x_i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$. For $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ and $i \in I_n$, we write

$$(x, y) = (x_i, \hat{x_i}, y) = (x, y_i, \hat{y_i}).$$

We denote by $\mathscr{D}_\delta(\hat{x_i}, y)$ and $\mathscr{D}_\delta(x, \hat{y_i})$ the cross sections of $\mathscr{D}_\delta$ at $(\hat{x_i}, y)$ and $(x, \hat{y_i})$, respectively. By Lemma 3.1, we see that the cross sections $\mathscr{D}_\delta(\hat{x_i}, y)$ and $\mathscr{D}_\delta(x, \hat{y_i})$ are open in $\mathbb{R}$. These open cross sections will be implicitly used in some partial derivatives such as in the proof of Theorem 3.8.

The following result provides some useful subsets of $\mathscr{D}_\delta$.

Proposition 3.2. (1) *If $\delta \geq r$, then $(\bar{B}_{\delta^2/r} \times \Omega) \cup (B_{\delta^2/r} \times \bar{\Omega}) \subset \mathscr{D}_\delta$.*
(2) *If $\delta > 0$, then $(\bar{B}_\delta \times \bar{B}_\delta) \setminus (D_0)_\delta \subset \mathscr{D}_\delta$.*

Proof. (1) Let $\delta \geq r$ and $(x, y) \in \bar{B}_{\delta^2/r} \times \Omega$. If $y = 0$, then

$$(x, y) = (x, 0) \in D_1 \subset \mathscr{D}_\delta.$$

If $y \neq 0$, then $|x| \leq \delta^2/r$ and $y \in \Omega$. Since Ω is open, we have $0 < |y| < r$. Hence,

$$\Big|\delta^2|y|^{-2}y\Big| = \delta^2/|y| > \delta^2/r \geq |x|.$$

This implies $x \neq \delta^2|y|^{-2}y = y_\delta$. By (3.4), $(x, y) \in (D_2)_\delta \subset \mathscr{D}_\delta$.

Let $(x, y) \in B_{\delta^2/r} \times \bar{\Omega}$. If $y = 0$, then $(x, y) = (x, 0) \in D_1 \subset \mathscr{D}_\delta$. If $y \neq 0$, then $|x| < \delta^2/r$ and $0 < |y| \leq r$. Hence, we have

$$\big|\delta^2|y|^{-2}y\big| = \delta^2/|y| \geq \delta^2/r > |x|.$$

This and (3.4), imply $(x, y) \in (D_2)_\delta \subset \mathscr{D}_\delta$.

(2) Let $(x, y) \in (\bar{B}_\delta \times \bar{B}_\delta) \setminus (D_0)_\delta$. Then $|x| \leq \delta$, $|y| \leq \delta$ and $x \neq y$. If $y = 0$, then $(x, y) = (x, 0) \in D_1 \subset \mathscr{D}_\delta$. If $y \neq 0$, then $x \neq y_\delta$. In fact, if not, then $x = y_\delta$. Since $x \neq y$, we have $y_\delta \neq y$. Hence, $|y| \neq \delta$. This, together with $|y| \leq \delta$, implies $|y| < \delta$. By $x = y_\delta$ and $|y| < \delta$, we have

$$|x| = |y_\delta| = \delta^2/|y| > \delta^2/\delta = \delta \geq |x|,$$

a contradiction. Hence, $(x, y) \in (D_2)_\delta \subset \mathscr{D}_\delta$. □

Remark 3.3. In the proof of Proposition 3.2, we see that we need the hypothesis that Ω is open to prove $\bar{B}_{\delta^2/r} \times \Omega \subset \mathscr{D}_\delta$, but the inclusion $B_{\delta^2/r} \times \bar{\Omega} \subset \mathscr{D}_\delta$ holds for any bounded subset $\Omega \subset \mathbb{R}^n$.

Corollary 3.4. (1) *If* (3.6) *holds, then*

$$\bar{\Omega} \times \bar{\Omega} \subset \bar{B}_\delta \times \bar{\Omega} \subset \bar{B}_\rho \times \bar{\Omega} \subset B_{\delta^2/r} \times \bar{\Omega} \subset \mathscr{D}_\delta. \tag{3.10}$$

(2) *If $\delta \geq r$, then*

$$(\bar{\Omega} \times \bar{\Omega}) \setminus (D_0)_\delta \subset (\bar{B}_\delta \times \bar{\Omega}) \setminus (D_0)_\delta \subset (\bar{B}_\delta \times \bar{B}_\delta) \setminus (D_0)_\delta \subset \mathscr{D}_\delta.$$

Proof. Since $\delta \geq r$, we have $\bar{\Omega} \subset \bar{B}_\delta$. The results (1) and (2) follow from Proposition 3.2 (1) and (2), respectively. □

For each $\delta > 0$, we define a function $\eta_\delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\eta_\delta(x, y) = \big(\delta^{-1}|x||y|\big)^2 - 2x \cdot y + \delta^2. \tag{3.11}$$

Lemma 3.5. *For $\delta > 0$, the function η_δ in* (3.11) *has the following properties.*

(i) $\eta_\delta \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

(ii) $\eta_\delta(x, y) = \begin{cases} \delta^2 & \text{if } (x, 0) \in \mathbb{R}^n \times \mathbb{R}^n, \\ \delta^{-2}|y|^2|x - y_\delta|^2 & \text{if } (x, y) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}). \end{cases}$

(iii) *$\eta_\delta(x, y) > 0$ for each $(x, y) \in \mathscr{D}_\delta$ and $\eta_\delta(x, y) = 0$ for each $(x, y) \in \mathbb{R}^n \setminus \mathscr{D}_\delta$.*

(iv) *$\eta_\delta(x, y) = \delta^{-2}\big(\delta^2 - |x|^2\big)\big(\delta^2 - |y|^2\big) + |x - y|^2$ for $x, y \in \mathbb{R}^n$.*

Proof. (i) By (3.11), we see that $\eta_\delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous, and

$$\eta_\delta(x,y) = \delta^{-2}\Big[\sum_{i=1}^n y_i^2\Big]\Big[\sum_{i=1}^n x_i^2\Big] - 2\Big[\sum_{i=1}^n x_i y_i\Big] + \delta^2.$$

Differentiating both sides of the above equation implies that for each $i \in I_n$,

$$\frac{\partial \eta_\delta(x,y)}{\partial x_i} = 2\Big[\delta^{-2}|y|^2 x_i - y_i\Big], \quad \frac{\partial^2 \eta_\delta(x,y)}{\partial x_i^2} = 2\delta^{-2}|y|^2$$

and

$$\frac{\partial \eta_\delta(x,y)}{\partial y_i} = 2\Big[\delta^{-2}|x|^2 y_i - x_i\Big], \quad \frac{\partial^2 \eta_\delta(x,y)}{\partial y_i^2} = 2\delta^{-2}|x|^2.$$

Moreover, all other partial derivatives are 0. It is easy to see that all of the partial derivatives are continuous on $\mathbb{R}^n \times \mathbb{R}^n$. Hence, the result (i) holds.

(ii) Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. If $y = 0$, then by (3.11), we have $\eta_\delta(x,y) = \delta^2$. If $y \neq 0$, then by (3.11), we have

$$\eta_\delta(x,y) = \big(\delta^{-1}|x||y|\big)^2 - 2x\cdot y + \delta^2 = \frac{|y|^2}{\delta^2}\big(|x|^2 - 2x\cdot y_\delta + |y_\delta|^2\big) = \frac{|y|^2}{\delta^2}|x - y_\delta|^2$$

and the result (ii) holds.

(iii) The result follows from the result (ii).

(iv) Since

$$|x-y|^2 = |x|^2 - 2x\cdot y + |y|^2 \quad \text{for } x, y \in \mathbb{R}^n,$$

for $x, y \in \mathbb{R}^n$ we have

$$\begin{aligned}\eta_\delta(x,y) &= \big(\delta^{-1}|x||y|\big)^2 + \delta^2 - 2x\cdot y = \big(\delta^{-1}|x||y|\big)^2 + \delta^2 + |x-y|^2 - |x|^2 - |y|^2 \\ &= \delta^{-2}\big(\delta^2 - |x|^2\big)\big(\delta^2 - |y|^2\big) + |x-y|^2\end{aligned}$$

and the result holds. □

From Lemma 3.5, we obtain the following result.

Corollary 3.6. *Let $\delta > 0$ and $y \in \mathbb{R}^n$. Then $(y,y) \in \mathscr{D}_\delta$ if and only if $|y| \neq \delta$.*

Proof. Let $y \in \mathbb{R}^n$. It is easy to verify that if $y \in \mathbb{R}^n \setminus \{0\}$, then

$$y = y_\delta \quad \text{if and only if} \quad |y| = \delta.$$

Assume that $(y,y) \in \mathscr{D}_\delta$. If $y = 0$, then $|y| = 0 \neq \delta$. If $y \neq 0$, then by Lemma 3.5 (ii), $y \neq y_\delta$. It follows that $|y| \neq \delta$. Conversely, if $|y| \neq \delta$, then $y = 0$ or $y \neq 0$ and $y \neq y_\delta$. It follows from Lemma 3.5 (ii) that $(y,y) \in \mathscr{D}_\delta$. □

With $\rho > 0$ and the function η_δ defined in (3.11), we define a kernel function $\Phi_\delta : \mathscr{D}_\delta \to \mathbb{R}$ by

$$\Phi_\delta(x,y) = \Gamma\big(\sqrt{\eta_\delta(x,y)}\big). \tag{3.12}$$

By Lemma 3.5 (iii), $\eta_\delta(x,y) > 0$ only when $(x,y) \in \mathscr{D}_\delta$. This, together with (2.3), implies that $\Gamma(\sqrt{\eta_\delta(x,y)})$ exists only when $(x,y) \in \mathscr{D}_\delta$. Hence, $\mathscr{D}_\delta$ is the natural domain of Φ_δ.

Remark 3.7. By Corollary 3.4, if $\delta > r$, then $\bar{\Omega} \times \bar{\Omega} \subset \mathscr{D}_\delta$. By Proposition 3.2 (1) with $\delta = r$ and $\Omega = B_r$ and Lemma 3.5 (iv) with $\delta = r$, $x = y$ and $|x| = \delta$, we see that

$$B_r \times B_r \subset B_r \times \bar{B}_r \subset \mathscr{D}_r, \quad \bar{B}_r \times \bar{B}_r \not\subset \mathscr{D}_r.$$

Hence, the natural domain of Φ_δ contains $\bar{\Omega}\times\bar{\Omega}$ if $\delta > r$, but the natural domain of Φ_r does not contain $\bar{B}_r \times \bar{B}_r$.

Theorem 3.8. *For $\delta > 0$, the function Φ_δ has the following properties.*

(i) $\Phi_\delta : \mathscr{D}_\delta \to \mathbb{R}$ *is continuous.*
(ii) $\Phi_\delta(\cdot, y) \in C^\infty\big((\mathscr{D}_\delta)_2(y)\big)$ *for each* $y \in \mathbb{R}^n$.
(iii) $\Delta_x \Phi_\delta(x, y) = 0$ *for* $(x, y) \in \mathscr{D}_\delta$.

Proof. (i) By Lemma 3.5 (i), $\eta_\delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous. By (2.3), $\Gamma : (0,\infty) \to \mathbb{R}$ is continuous. These, together with Lemma 3.5 (iii), imply that $\Phi_\delta : \mathscr{D}_\delta \to \mathbb{R}$ is continuous.

(ii) By (2.3), (3.11) and (3.12), we have for $(x, y) \in \mathscr{D}_\delta$,

$$\Phi_\delta(x,y) = \begin{cases} \Gamma(\delta) & \text{if } y = 0, \\ (2\pi)^{-1}\ln(\delta^{-1}|y|) - \Psi(x, y_\delta) & \text{if } y \neq 0,\ n = 2, \\ -(\delta|y|^{-1})^{n-2}\Psi(x, y_\delta) & \text{if } y \neq 0,\ n \geq 3. \end{cases} \tag{3.13}$$

This, together with Lemma 2.1 (1), implies that the result (ii) holds.

(iii) By (3.13), we have for each $i \in I_n$,

$$\frac{\partial^2 \Phi_\delta(x,y)}{\partial x_i^2} = \begin{cases} 0 & \text{if } y = 0, \\ -\frac{\partial^2\Psi(x,y_\delta)}{\partial x_i^2} & \text{if } y \neq 0,\ n = 2, \\ -(\delta|y|^{-1})^{n-2}\frac{\partial^2\Psi(x,y_\delta)}{\partial x_i^2} & \text{if } y \neq 0,\ n \geq 3. \end{cases}$$

It follows that for $(x, y) \in \mathscr{D}_\delta$,

$$\Delta_x\Phi_\delta(x,y) = \begin{cases} 0 & \text{if } y = 0, \\ -\sum_{i=1}^n \frac{\partial^2\Psi(x,y_\delta)}{\partial x_i^2} & \text{if } y \neq 0,\ n = 2, \\ -(\delta|y|^{-1})^{n-2}\sum_{i=1}^n \frac{\partial^2\Psi(x,y_\delta)}{\partial x_i^2} & \text{if } y \neq 0,\ n \geq 3. \end{cases} \tag{3.14}$$

Note that $(x, y) \in \mathscr{D}_\delta$ with $y \neq 0$ implies $x \neq y_\delta$. By Lemma 2.1 (2),

$$\Delta_x \Psi(x, y_\delta) = \sum_{i=1}^n \frac{\partial^2\Psi(x,y_\delta)}{\partial x_i^2} = 0 \quad \text{for } (x,y) \in \mathscr{D}_\delta \text{ with } y \neq 0.$$

This and (3.14) imply $\Delta_x\Phi_\delta(x, y) = 0$ for $(x, y) \in \mathscr{D}_\delta$. □

Corollary 3.9. *If* (3.6) *holds, then the following assertions hold.*

(i) $\Phi_\delta \in C^\infty(\bar{B}_\rho \times \bar{\Omega})$.
(ii) $\Delta_x\Phi_\delta(x, y) = 0$ *for* $(x, y) \in \bar{B}_\rho \times \bar{\Omega}$.

Proof. (i) By Corollary 3.4, we have $\bar{B}_\rho \times \bar{\Omega} \subset \mathscr{D}_\delta$. By Lemma 3.5 (i) and (iii), $\eta_\delta \in C^\infty(\bar{B}_\rho \times \bar{\Omega})$. Since $\Gamma, y \in C^\infty(0,\infty)$, where $y(x) = \sqrt{x}$ for $x \in (0,\infty)$. It follows that $\Phi_\delta\Gamma(y(\eta_\delta)) \in C^\infty(\bar{B}_\rho \times \bar{\Omega})$.
(ii) Since $\bar{B}_\rho \times \bar{\Omega} \subset \mathscr{D}_\delta$, by Theorem 3.8 (ii), the result (ii) holds. □

Let δ and ρ satisfy (3.6). With the kernel Φ_δ given in (3.12), we study the Hammerstein integral operator

$$(S_{\delta,\rho}v)(x) = \int_\Omega \Phi_\delta(x,y)v(y)\,dy \quad \text{for } x \in \bar{B}_\rho, \tag{3.15}$$

where $v : \Omega \to \mathbb{R}$ is a function.

Theorem 3.10. *If* (3.6) *holds, then the following assertions hold.*

(1) *$S_{\delta,\rho}$ maps $L^1(\Omega)$ to $C^\infty(\bar{B}_\rho)$.*
(2) *$S_{\delta,\rho} : L^1(\Omega) \to C(\bar{B}_\rho)$ is compact.*

Proof. (1) By Corollary 3.9 (i), $\Phi_\delta \in C^\infty(\bar{B}_\rho \times \bar{\Omega})$. This, (3.15), and the Leibniz integral rule (see [9, Lemma 2.2, p.226]), imply $S_{\delta,\rho} v \in C^\infty(\bar{B}_\rho)$ for $v \in L^1(\Omega)$.

(2) By Corollary 3.9 (i), $\Phi_\delta : \bar{B}_\rho \times \bar{\Omega} \to \mathbb{R}$ is continuous. The result (2) follows from [5, Lemma 2.1]. □

Theorem 3.11. *If (3.6) holds, then for each $v \in L^1(\Omega)$, the function $u_{\delta,\rho} : \bar{B}_\rho \to \mathbb{R}$ defined by*

$$u_{\delta,\rho}(x) = (S_{\delta,\rho} v)(x) = \int_\Omega \Phi_\delta(x,y) v(y)\, dy \quad \text{for each } x \in \bar{B}_\rho$$

is a solution of (3.1).

Proof. By Theorem 3.10, we have

$$u_{\delta,\rho} = S_{\delta,\rho} v \in C(\bar{B}_\rho) \cap C^2(B_\rho).$$

By Lemma 3.9 (i), $\Phi_\delta \in C^\infty(\bar{B}_\rho \times \bar{\Omega})$. By [9, Lemma 2.2, p.226] and Lemma 3.9 (i), we have for each $x \in \bar{B}_\rho$,

$$\Delta u_{\delta,\rho}(x) = \Delta \int_\Omega \Phi_\delta(x,y) v(y) dy = \int_\Omega \Delta_x \Phi_\delta(x,y) v(y) dy. \tag{3.16}$$

By Corollary 3.9 (ii), we have

$$\Delta_x \Phi_\delta(x,y) = 0 \quad \text{for } (x,y) \in \bar{B}_\rho \times \bar{\Omega}.$$

This and (3.16), imply that $u_{\delta,\rho}$ satisfies (3.1). □

As a special case of Theorem 3.11, we give solutions of the Laplace's equation (3.2).

Corollary 3.12. *For $v \in L^1(\Omega)$ and $\delta > r$, the function $u_\delta : \bar{\Omega} \to \mathbb{R}$ defined by*

$$u_\delta(x) = \int_\Omega \Phi_\delta(x,y) v(y)\, dy \quad \text{for each } x \in \bar{\Omega}$$

is a solution of (3.2).

This corollary provides harmonic functions via functions in $L^1(\Omega)$, where Ω is not necessarily a domain. As mentioned in the Introduction, $\Psi(\cdot, 0)$ is a harmonic function in $\mathbb{R}^n \setminus \{0\}$ given in [3, p.21-22], [4, p.17], and [15, Lemma 2.1 (P_3)].

4. Solutions to Poisson's equation

In this section, we study solutions of the Poisson's equation

$$-\Delta u(x) = v(x) \quad \text{for each } x \in \Omega, \tag{4.1}$$

where Ω is a bounded open subset in $\mathbb{R}^n$, $n \geq 2$ and $v \in C^\mu(\Omega)$.

Definition 4.1. Let $\bar{\Omega} \subset D$. A function $u : D \to \mathbb{R}$ is said to be a (classical) solution of (4.1) if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and u satisfies (4.1). A solution u of (4.1) is said to be nonnegative if $u \in P$, where P is the cone in $C(\bar{\Omega})$ defined by

$$P = \{u \in C(\bar{\Omega}) : u(x) \geq 0 \quad \text{for } x \in \bar{\Omega}\}. \tag{4.2}$$

A classical result [4, Lemma 4.2] shows that if Ω is a bounded connected open in $\mathbb{R}^n$ and $v \in C^\mu(\Omega)$, then Lv is a solution of (4.1). This result was generalized to the case that Ω is a bounded open subset in $\mathbb{R}^n$ in [15, Theorem 2.3 (2)]. In the following, we provide other solutions involving an integral operator with the Green's function in bounded open subsets in $\mathbb{R}^n$. We show that some of solutions u of (4.1) satisfy the Dirichlet boundary condition

$$u(x) = 0 \quad \text{for } x \in \partial B_\delta, \tag{4.3}$$

where δ is the same as in (3.6).

For each $\delta > 0$, we define a function $k_\delta : \mathscr{D}_\delta \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{R}$ by

$$k_\delta(x,y) = \Psi(x,y) + \Phi_\delta(x,y), \tag{4.4}$$

where Ψ and Φ_δ are the same as in (2.2) and (3.12). If Ω is a domain (a connected open subset in $\mathbb{R}^n$), then following [4, p.19], $k_\delta : \bar{\Omega} \times \bar{\Omega} \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{R}$ is called the (Dirichlet) Green's function for Ω. When $\Omega = B_\delta$, the expression of the Green's function k_δ is given and studied in [4, (2.23), p.19], where $G(x,y) = -k_\delta(x,y)$. In the following, we study the Green's function k_δ in (4.4) for a general bounded open subset Ω in $\mathbb{R}^n$.

Lemma 4.2. *The Green's function k_δ in* (4.4) *has the following properties.*

(i) *$k_\delta : \mathscr{D}_\delta \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{R}$ is continuous.*

(ii) *If $\delta > r$, then the following assertions hold.*

 (ii.1) *$k_\delta(x,y) \geq 0$ for $(x,y) \in \bar{B}_\delta \times \bar{B}_\delta \setminus (D_0)_\delta$.*

 (ii.2) *$k_\delta(x,y) > 0$ for $(x,y) \in B_\delta \times B_\delta \setminus (D_0)_\delta$.*

 (ii.3) *$k_\delta(x,y) = 0$ for $(x,y) \in (\partial B_\delta \times \bar{B}_\delta) \cup (\bar{B}_\delta \times \partial B_\delta) \setminus (D_0)_\delta$.*

Proof. (i) By (2.2), Ψ is continuous at $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $x \neq y$. By Theorem 3.8 (i), $\Phi_\delta : \mathscr{D}_\delta \to \mathbb{R}$ is continuous. The result follows.

(ii.1) Let $(x,y) \in \bar{B}_\delta \times \bar{B}_\delta$ with $x \neq y$. Then $|x| \leq \delta$ and $|y| < \delta$. By Lemma 3.5 (ii), we have

$$\eta_\delta(x,y) \geq |x-y|^2, \quad \sqrt{\eta_\delta(x,y)} \geq |x-y|.$$

Because Γ is increasing on $(0,\infty)$, we have

$$k_\delta(x,y) = \Gamma(\sqrt{\eta_\delta(x,y)}) - \Gamma(|x-y|) \geq 0.$$

(ii.2) Let $(x,y) \in B_\delta \times B_\delta$ with $x \neq y$. By Lemma 3.5 (iv), we have

$$\eta_\delta(x,y) > |x-y|^2, \quad \sqrt{\eta_\delta(x,y)} > |x-y|.$$

Because Γ is increasing on $(0,\infty)$, we have

$$k_\delta(x,y) = \Gamma(\sqrt{\eta_\delta(x,y)}) - \Gamma(|x-y|) > 0.$$

(ii.3) Let $(x,y) \in (\partial B_\delta \times \bar{B}_\delta) \cup (\bar{B}_\delta \times \partial B_\delta)$ with $x \neq y$. By Lemma 3.5 (iv), we have $\eta_\delta(x,y) = |x-y|^2$ and $\sqrt{\eta_\delta(x,y)} = |x-y|$. Hence,

$$k_\delta(x,y) = \Gamma(\sqrt{\eta_\delta(x,y)}) - \Gamma(|x-y|) = 0$$

and the result holds. ◻

With (3.6), we define an integral operator $L_{\delta,\rho}$ by

$$(L_{\delta,\rho}v)(x) = (Lv)(x) + (S_{\delta,\rho}v)(x) = \int_\Omega k_\delta(x,y)v(y)\,dy \quad \text{for } x \in \bar{B}_\rho. \tag{4.5}$$

Theorem 4.3. *Assume that* (3.6) *holds. Then the operator* $L_{\delta,\rho}$ *in* (4.5) *has the following properties.*

(1) *If* $p \in (n/2, \infty]$, *then* $L_{\delta,\rho}$ *maps* $L^p(\Omega)$ *into* $C(\bar{B}_\rho)$. *Moreover, for each* $v \in L^p(\Omega)$, $(L_{\delta,\delta}v)(x) = 0$ *for* $x \in \partial B_\delta$.
(2) *If* $p \in (n, \infty]$, *then* $L_{\delta,\rho}$ *maps* $L^p(\Omega)$ *to* $C^1(\bar{B}_\rho)$.
(3) $L_{\delta,\rho}$ *maps* $C^\mu(\Omega)$ *into* $C^2(\Omega)$.

Proof. (1) Since $p \in (n/2, \infty]$, then by Theorem 2.5, L maps $L^p(\Omega)$ into $C(\mathbb{R}^n)$. Since $\delta > r$ and $\rho \in [\delta, \delta^2/r)$, by Theorem 3.10 (1), $S_{\delta,\rho}$ maps $L^1(\Omega)$ to $C^\infty(\bar{B}_\rho)$. It follows from (4.5) that $L_{\delta,\rho}v \in C^\infty(\bar{B}_\rho)$ for $v \in L^p(\Omega)$. By Lemma 4.2 (3),

$$k_\delta(x,y) = 0 \quad \text{for } (x,y) \in (\partial B_\delta \times \bar{B}_\delta) \setminus (D_0)_\delta.$$

Hence, by (4.5) with $\rho = \delta$, for each $v \in L^p(\Omega)$ we have

$$(L_{\delta,\delta}v)(x) = \int_\Omega k_\delta(x,y)v(y)\,dy = 0 \quad \text{for each } x \in \bar{B}_\delta.$$

(2) Since $p \in (n, \infty]$, by Lemma 2.3 (2), L maps $L^p(\Omega)$ to $C^1(\mathbb{R}^n)$. By (4.5), $L_{\delta,\rho}v \in C^1(\bar{B}_\rho)$ for $v \in L^p(\Omega)$.

(3) By Lemma 2.3 (3), L maps $C^\mu(\Omega)$ into $C^2(\Omega)$. By Theorem 3.10 (1), $S_{\delta,\rho}$ maps $L^1(\Omega)$ to $C^\infty(\bar{B}_\rho)$. Since $\Omega \subset \bar{B}_\rho$, by (4.5), $L_{\delta,\rho}v \in C^2(\Omega)$ for $v \in C^\mu(\Omega)$. □

Theorem 4.4. *Assume that* (3.6) *holds. Then the following assertions hold.*

(i) *If* $v \in C^\mu(\Omega)$, *then* $L_{\delta,\rho}v$ *is a solution of* (4.1). *Moreover,* $L_{\delta,\delta}v$ *is a solution of* (4.1) *subject to* (4.3).
(ii) *If* $v \in P \cap C^\mu(\Omega)$, *then* $L_{\delta,\rho}v$ *is a nonnegative solution of* (4.1). *Moreover,* $L_{\delta,\delta}v$ *is a nonnegative solution of* (4.1) *subject to* (4.3).

Proof. (i) By Theorem 4.3 (1) and (3), $L_{\delta,\rho}v \in C^2(\Omega) \cap C(\bar{\Omega})$ for $v \in C^\mu(\Omega)$. Since $\delta > r$ and $\rho \in [\delta, \delta^2/r)$, by Theorem 3.11, we have for $v \in C^\mu(\Omega)$,

$$\Delta(S_{\delta,\rho}v)(x) = 0 \quad \text{for each } x \in \bar{B}_\rho.$$

By Lemma 2.3 (4),

$$-\Delta(Lv)(x) = v(x) \quad \text{for each } x \in \Omega.$$

Hence, we have for each $x \in \Omega$,

$$-\Delta(L_{\delta,\rho}v)(x) = -\Delta(Lv)(x) - \Delta(S_{\delta,\rho}v)(x) = v(x)$$

and $L_{\delta,\rho}v$ is a solution of (4.1). By the last result of Theorem 4.3 (1), the solution $L_{\delta,\delta}v$ of (4.1) satisfies (4.3).

(ii) By Lemma 4.2 (1), we have

$$k_\delta(x,y) \geq 0 \quad \text{for } (x,y) \in \bar{B}_\delta \times \bar{B}_\delta \setminus (D_0)_\delta.$$

Since $v \in P$, it follows that $L_{\delta,\rho}v(x) \geq 0$ for $x \in \bar{\Omega}$. The last result follows from the last result of Theorem 4.3 (1). □

Theorem 4.4 gives solutions $(Lv) + (S_{\delta,\rho}v)$ of (4.1) which are different from those Lv obtained in [4, Lemma 4.2] and [15, Theorem 2.3 (2)].

5. Eigenvalues of Laplace's equations

We study the eigenvalue problem of the Laplace's equation

$$-\Delta u(x) = \mu g(x)u(x) \quad \text{for } x \in \Omega, \tag{5.1}$$

where Ω is a bounded open subset in $\mathbb{R}^n$, $n \geq 2$ and $g : \bar{\Omega} \to \mathbb{R}$ is a function.

The eigenvalue problem is to determine that under what conditions on g, there exist $\mu > 0$ and $u \in P \setminus \{0\}$ such that (5.1) holds, where P is the same as in (4.2).

If $n \geq 3$, $p \in (n/2, \infty]$ and $g \in L^p_+(\Omega)$, the eigenvalue problem can be solved by [15, Theorem 2.4], with the well known Krein-Rutman theorem, where the operator L_g in (2.6) is used. However, the method cannot be applied for $n = 2$ because the Newtonian potential kernel Ψ in (2.2) changes sign.

In the following, we study the eigenvalue problem (5.1) using the linear Hammerstein integral operator

$$(\mathscr{L}_g v)(x) = \int_\Omega k_\delta(x, y)g(y)v(y)\, dy \quad \text{for } x \in \bar{\Omega}. \tag{5.2}$$

Proposition 5.1. *Let $n \geq 2$, $p \in (n/2, \infty]$, $g \in L^p_+(\Omega)$ and $\delta > r$. Then the linear integral operator $\mathscr{L}_g$ defined by* (5.2) *is a compact operator from $C(\bar{\Omega})$ to $C(\bar{\Omega})$ satisfying $\mathscr{L}_g(P) \subset P$.*

Proof. (i) Since $n \geq 2$, $p \in (n/2, \infty]$ and $g \in L^p_+(\Omega)$, by Lemma 2.7, L_g is a compact operator from $C(\bar{\Omega})$ to $C(\bar{\Omega})$. We define an operator $(S_{\delta,\rho})_g$ by

$$(S_{\delta,\rho})_g v(x) = \int_\Omega \Phi_\delta(x, y)g(y)v(y)\, dy \quad \text{for } x \in \bar{\Omega}. \tag{5.3}$$

Since $\delta > r$, by Theorem 3.10 (2) with $\rho = \delta$, $S_{\delta,\rho} : L^1(\Omega) \to C(\bar{B}_\delta)$ is compact. It follows from $\bar{\Omega} \subset \bar{B}_\rho$ that $S_{\delta,\rho} : L^1(\Omega) \to C(\bar{\Omega})$ is compact. It is obvious that the map T defined by

$$(Tv)(x) = g(x)v(x)$$

is continuous from $C(\bar{\Omega})$ to $L^1(\Omega)$. Hence, $(S_{\delta,\rho})_g = S_{\delta,\rho}T$ is compact from $C(\bar{\Omega})$ to $C(\bar{\Omega})$. Noting that $\mathscr{L}_g = L_g + (S_{\delta,\rho})_g$, we see that $\mathscr{L}_g : C(\bar{\Omega}) \to C(\bar{\Omega})$ is compact. By Corollary 3.4 (2) and Lemma 4.2 (1), we have

$$(\bar{\Omega} \times \bar{\Omega}) \setminus (D_0)_\delta \subset (\bar{B}_\delta \times \bar{B}_\delta) \setminus (D_0)_\delta, \quad k_\delta(x, y) \geq 0 \quad \text{for } (x, y) \in \bar{B}_\delta \times \bar{B}_\delta \setminus (D_0)_\delta.$$

This implies

$$k_\delta(x, y) \geq 0 \quad \text{for } (x, y) \in (\bar{\Omega} \times \bar{\Omega}) \setminus (D_0)_\delta. \tag{5.4}$$

Since $g \in L^p_+(\Omega)$, it follows from (5.2) and (5.4) that

$$\mathscr{L}_g v(x) \geq 0 \quad \text{for } v \in P,\ x \in \bar{\Omega},$$

and $\mathscr{L}_g v \in P$ for $v \in P$. □

The following Krein-Rutman theorem can be found in [10].

Lemma 5.2. *Assume that P is a total cone in a real Banach space X and $\mathcal{L} : X \to X$ is a compact linear operator such that $\mathcal{L}(P) \subset P$ and $r(\mathcal{L}) > 0$. Then there exists an eigenvector $u \in P \setminus \{0\}$ such that $r(\mathcal{L})u = \mathcal{L}u$.*

It is well known that the cone P defined in (4.2) is a total cone in $C(\bar{\Omega})$.

Theorem 5.3. *Let $n \ge 2$, $p \in (n/2, \infty]$, $g \in L^p_+(\Omega)$ and $\delta > r$. Assume that there exists a measurable set $\Omega_0 \subset \bar{\Omega}$ with* $\operatorname{meas}(\Omega_0) > 0$ *such that*

$$\gamma := \inf\Big\{\int_{\Omega_0} k_\delta(x,y)g(y)\,dy : x \in \Omega_0\Big\} > 0. \tag{5.5}$$

Then the following assertions hold.

(i) *$r(\mathscr{L}_g) > 0$, where $r(\mathscr{L}_g) = \lim_{m\to\infty} \sqrt[m]{\|\mathscr{L}_g^m\|}$ is the spectral radius of $\mathscr{L}_g$,*

(ii) *There exists an eigenvector $u \in P \setminus \{0\}$ such that*

$$-\Delta u(x) = \frac{1}{r(\mathscr{L}_g)} g(x)u(x) \quad \text{for } x \in \Omega, \tag{5.6}$$

where P is the same as in (4.2).

Proof. (i) The proof is similar to that of [15, Theorem 2.4]. Let $u(x) \equiv 1$ for $x \in \bar{\Omega}$. Then

$$(\mathscr{L}_g u)(x) = \int_\Omega k_\delta(x,y)g(y)u(y)\,dy \ge \int_{\Omega_0} k_\delta(x,y)g(y)\,dy \ge \gamma \quad \text{for } x \in \Omega_0.$$

Since $\mathscr{L}_g u \in P$, for $x \in \Omega_0$ we have

$$\mathscr{L}_g^2 u(x) = \int_\Omega k_\delta(x,y)g(y)[\mathscr{L}_g u(y)]\,dy \ge \int_{\Omega_0} k_\delta(x,y)g(y)[\mathscr{L}_g u(y)]\,dy \ge \gamma^2.$$

Repeating the process implies $\mathscr{L}_g^m u(x) \ge \gamma^m$ and $r(\mathscr{L}_g) \ge \gamma$.

(ii) It is well known that P is a total cone in $C(\bar{\Omega})$. The result follows from Lemma 5.2, Proposition 5.1 and the result (i). $\square$

Theorem 5.3 is different from [15, Theorem 2.4], where k_δ is replaced by $|\Psi|$. The condition (5.5) depends on Green's function k_δ. In the following, we provide a sufficient condition for (5.5) with $n \ge 3$ to hold, which is independent of k_δ and is easily verified. To do that, we first prove the following result.

Lemma 5.4. *Let $n \ge 2$ and $0 < \sigma < \delta < \infty$. Then the following assertions hold.*

(i) *$\Phi_\delta(x,y) \ge \Gamma\big(\delta^{-1}\sqrt{\delta^2 - \sigma^2}\big)$ for $(x,y) \in \bar{B}_\sigma \times \bar{B}_\sigma$.*

(ii) *If $n \ge 3$, then*

$$k_\delta(x,y) \ge \Gamma\big(\delta^{-1}\sqrt{\delta^2 - \sigma^2}\big) \quad \text{for } (x,y) \in \bar{B}_\sigma \times \bar{B}_\sigma \setminus (D_0)_\delta.$$

(iii) *If $n = 2$, then*

$$k_\delta(x,y) \ge \frac{1}{4\pi} \ln\Big[1 + \frac{\delta^{-2}\big(\delta^2 - \sigma^2\big)^2}{|x-y|^2}\Big] \quad \text{for } (x,y) \in \bar{B}_\sigma \times \bar{B}_\sigma \setminus (D_0)_\delta.$$

Proof. By Lemma 3.5 (iv), for $x, y \in \bar{B}_\sigma$ we have

$$\begin{aligned}
\eta_\delta(x,y) &= \delta^{-2}\big(\delta^2 - |x|^2\big)\big(\delta^2 - |y|^2\big) + |x-y|^2 \\
&\ge \delta^{-2}\big(\delta^2 - \sigma^2\big)\big(\delta^2 - \sigma^2\big) + |x-y|^2 \\
&= \delta^{-2}\big(\delta^2 - \sigma^2\big)^2 + |x-y|^2.
\end{aligned} \tag{5.7}$$

(i) By (5.7), we have

$$\eta_\delta(x,y) \ge \delta^{-2}\big(\delta^2 - \sigma^2\big) \quad \text{for } (x,y) \in \bar{B}_\sigma \times \bar{B}_\sigma.$$

Since Γ is increasing on $(0,\infty)$, we have

$$\Phi_\delta(x,y) = \Gamma\big(\sqrt{\eta_\delta(x,y)}\big) \ge \Gamma\big(\delta^{-1}\sqrt{\delta^2 - \sigma^2}\big) \quad \text{for } (x,y) \in \bar{B}_\sigma \times \bar{B}_\sigma.$$

(ii) Since $n \ge 3$, by (2.2) we have

$$\Psi(x,y) = \frac{1}{n(n-2)\omega_n} \frac{1}{|x-y|^{n-2}} \ge 0 \quad \text{for } (x,y) \in \bar{B}_\delta \times \bar{B}_\delta \setminus (D_0)_\delta.$$

This, (4.4) and the result (i), imply

$$k_\delta(x,y) \ge \Phi_\delta(x,y) \ge \Gamma\big(\delta^{-1}\sqrt{\delta^2-\sigma^2}\big) \quad \text{for } x,y \in \bar{B}_\sigma \times \bar{B}_\sigma \setminus (D_0)_\delta.$$

(iii) Since $n = 2$, by (2.2) and (4.4), for $(x,y) \in \mathscr{D}_\delta \setminus (D_0)_\delta$, we have

$$\begin{aligned} k_\delta(x,y) &= \Psi(x,y) + \Phi_\delta(x,y) = -\frac{1}{2\pi} \ln|x-y| + \frac{1}{2\pi} \ln \Gamma\big(\sqrt{\eta_\delta(x,y)}\big) \\ &= \frac{1}{4\pi}\big[\ln \eta_\delta(x,y) - \ln|x-y|^2\big] = \frac{1}{4\pi} \ln \frac{\eta_\delta(x,y)}{|x-y|^2}. \end{aligned}$$

This and (5.7), imply that

$$\begin{aligned} k_\delta(x,y) &= \frac{1}{4\pi} \ln \frac{\eta_\delta(x,y)}{|x-y|^2} \ge \frac{1}{4\pi} \ln \frac{\delta^{-2}\big(\delta^2-\sigma^2\big)^2 + |x-y|^2}{|x-y|^2} \\ &\ge \frac{1}{4\pi} \ln\Big[1 + \frac{\delta^{-2}\big(\delta^2-\sigma^2\big)^2}{|x-y|^2}\Big] \quad \text{for } (x,y) \in \bar{B}_\sigma \times \bar{B}_\sigma \setminus (D_0)_\delta. \end{aligned}$$

and result (iii) holds. □

Corollary 5.5. *Let $n \ge 3$, $p \in (n/2, \infty]$ and $g \in L^p_+(\Omega)$ with $\int_\Omega g(y)\,dy > 0$. Then $r(\mathscr{L}_g) > 0$ and there exists an eigenvector $u \in P \setminus \{0\}$ such that* (5.6) *holds.*

Proof. Let $r < \delta$ and $\sigma \in [r, \delta)$. By Lemma 5.4 (ii), we have

$$k_\delta(x,y) \ge \Gamma\big(\delta^{-1}\sqrt{\delta^2-\sigma^2}\big) \quad \text{for } (x,y) \in \bar{B}_\sigma \times \bar{B}_\sigma \setminus (D_0)_\delta.$$

Since $\sigma \ge r$, we have $\bar{\Omega} \subset \bar{B}_r \subset \bar{B}_\sigma$. Hence,

$$\int_\Omega k_\delta(x,y) g(y)\,dy \ge \Gamma\big(\delta^{-1}\sqrt{\delta^2-\sigma^2}\big) \int_\Omega g(y)\,dy > 0 \quad \text{for } x \in \bar{\Omega}.$$

This implies

$$\gamma = \inf\Big\{\int_\Omega k_\delta(x,y) g(y)\,dy : x \in \bar{\Omega}\Big\} \ge \Gamma\big(\delta^{-1}\sqrt{\delta^2-\sigma^2}\big) \int_\Omega g(y)\,dy > 0.$$

The results follow from Theorem 5.3 (ii). □

By Lemma 5.4 (iii), we see that when $n = 2$, k_δ has no positive lower bound on the set $\bar{B}_\sigma \times \bar{B}_\sigma \setminus (D_0)_\delta$ since $|x-y|^2$ may tend to zero on $\bar{B}_\sigma \times \bar{B}_\sigma \setminus (D_0)_\delta$. Hence, it is not clear whether Corollary 5.5 holds when $n = 2$.

Acknowledgments. The author would like to thank the reviewers and handling editor very much for providing valuable comments.

G. Yang was supported by the Applied Basic Research Project of Sichuan Province (No. 2018JY0169). K. Lan was supported by the Natural Sciences and Engineering Research Council of Canada under grant No. 135752-2018.

References

[1] H. Amann; Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM. Rev.*, **18** (1976), 620-709.

[2] K. Diethelm; *The analysis of fractional differential equations*, Springer-Verlag Berlin Heidelberg 2010.

[3] L. C. Evans; *Partial Differential Equations*, Second Edition, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, 2010.

[4] D. Gilbarg, N. S. Trudinger; *Elliptic Partial Differential Equations of Second Order*, 3rd ed., Springer, Berlin, Heidelberg, 2001.

[5] K. Q. Lan; Multiple positive solutions of Hammerstein integral equations with singularities, *Differential Equations Dynam. Systems* **8** (2000) (2), 175–192.

[6] K. Q. Lan; Equivalence of higher order linear Riemann-Liouville fractional differential and integral equations, *Proc. Amer. Math. Soc.*, **148** (2020) (12), 5225–5234.

[7] K. Q. Lan, W. Lin; Positive solutions of elliptic boundary value problems and applications to population dynamics, *J. Dynam. Differ. Equations*, **32** (2020) (2), 873–894.

[8] K. Q. Lan, Zhitao Zhang; Nonzero positive weak solutions of systems of p-Laplace equations, *J. Math. Anal. Appl.* **394** (2012), 581-591.

[9] S. Lang; *Real functional analysis*, Graduate texts in Mathematics 142, third edition, Springer, New York, 1993.

[10] R. D. Nussbaum; Eigenvectors of nonlinear positive operators and the linear Krein-Rutman theorem, *Fixed point theory* (eds. E. Fadell and G. Fournier; Springer, 1981), Lecture Notes in Math., 886, pp. 309-330.

[11] N. S. Papageorgiou, V. D. Radulescu, D. D. Repovs; Nonlinear nonhomogeneous singular problems, *Calc. Var. Partial Differential Equations*, **59** (2020) (1), Art. 9, 31 pp.

[12] X. Ren, J. Wei; A toroidal tube solution to a problem involving mean curvature and Newtonian potential, *Interfaces Free Bound.*, **13** (2011) (1), 127–154.

[13] X. Ren, J. Wei; Double tori solution to an equation of mean curvature and Newtonian potential, *Calc. Var. Partial Differ. Equ.*, **49** (2014) (3), 987–1018.

[14] J. R. Webb; Initial value problems for Caputo fractional equations with singular nonlinearities, *Electr. J. Differential Equ.*, 2019 (2019), No. 117, 1-32.

[15] G. C. Yang, K. Q. Lan; Newtonian potential and positive solutions of Poisson equations, *Nonlinear Anal.* **196** (2020), 111811, 20 pp.

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