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EXISTENCE AND BLOW UP IN A SYSTEM OF WAVE EQUATIONS WITH NONSTANDARD NONLINEARITIES

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Abstract. In this article, we consider a coupled system of two nonlinear hyperbolic equations, where the exponents in the damping and source terms are variables. First, we prove a theorem of existence and uniqueness of weak solution, by using the Faedo Galerkin approximations and the Banach fixed point theorem. Then, using the energy method, we show that certain solutions with positive initial energy blow up in finite time. We also give some numerical applications to illustrate our theoretical results.

1. Introduction

In this work, we study the following initial-boundary-value problem for the unknowns u and v:

$$\begin{gathered} u_{tt} - \operatorname{div}(A\nabla u) + |u_t|^{m(x)-2}u_t = f_1(x,u,v) \quad \text{in } \Omega\times(0,T), \\ v_{tt} - \operatorname{div}(B\nabla v) + |v_t|^{r(x)-2}v_t = f_2(x,u,v) \quad \text{in } \Omega\times(0,T), \\ u = v = 0 \quad \text{on } \partial\Omega\times(0,T), \\ u(0)=u_0, \quad u_t(0)=u_1 \quad \text{in } \Omega, \\ v(0)=v_0 \text{ and } v_t(0)=v_1 \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $T>0$ and Ω is a bounded domain of $\mathbb{R}^n$ $(n=1,2,3)$ with a smooth boundary $\partial\Omega$, m and r are continuous functions on $\overline{\Omega}$ such that, for all $x\in\overline{\Omega}$,

$$\begin{gathered} 2\le m(x), \quad \text{if } n=1,2, \\ 2\le m^-\le m(x)\le m^+\le 6, \quad \text{if } n=3 \end{gathered} \tag{1.2}$$

and

$$\begin{gathered} 2\le r(x), \quad \text{if } n=1,2, \\ 2\le r^-\le r(x)\le r^+\le 6, \quad \text{if } n=3, \end{gathered} \tag{1.3}$$

where

$$m^- = \inf_{x\in\overline{\Omega}} m(x), \quad m^+ = \sup_{x\in\overline{\Omega}} m(x), \quad r^- = \inf_{x\in\overline{\Omega}} r(x), \quad r^+ = \sup_{x\in\overline{\Omega}} r(x).$$

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The coupling terms f_1 and f_2 are as follows: for all $x \in \overline{\Omega}$ and $(u,v) \in \mathbb{R}^2$,

$$f_1(x,u,v) = \frac{\partial}{\partial u} F(x,u,v), \quad f_2(x,u,v) = \frac{\partial}{\partial v} F(x,u,v), \tag{1.4}$$

with

$$F(x,u,v) = a|u+v|^{p(x)+1} + 2b|uv|^{\frac{p(x)+1}{2}}, \tag{1.5}$$

where $a, b > 0$ two positive constants, p is a given continuous function on $\overline{\Omega}$ such that, for all $x \in \overline{\Omega}$,

$$\begin{gathered} 3 \le p^- \le p(x) \le p^+, \quad \text{if } n = 1,2, \\ p(x) = 3, \quad \text{if } n = 3, \end{gathered} \tag{1.6}$$

with

$$\max\{m^+, r^+\} \le p^- = \inf_{x \in \overline{\Omega}} p(x).$$

A and B are symmetric matrices of class $C^1(\overline{\Omega} \times [0,\infty))$ such that for constants a_0, $b_0 > 0$ and all $\xi \in \mathbb{R}^n$,

$$A\xi \cdot \xi \ge a_0|\xi|^2, \quad B\xi \cdot \xi \ge b_0|\xi|^2, \tag{1.7}$$

$$A'\xi \cdot \xi \le 0, \quad B'\xi.\xi \le 0, \tag{1.8}$$

where $A' = \frac{\partial A}{\partial t}(\cdot, t)$ and $B' = \frac{\partial B}{\partial t}(\cdot, t)$.

The study of system (1.1) is motivated by the description of several models in physical phenomena, such as viscoelastic fluids, filtration processes through a porous media, fluids with temperature dependent viscosity, image processing, or robotics, etc. See for example [7] for an application of such functional spaces in the image recovery. Our system can be regarded as a model for interaction between two fields describing the motion of two nonlinear "smart" materials. For more details, see [1, 7].

A considerable effort has been devoted to the study of single wave equations in the case of constant exponents. The equation

$$u_{tt} - \Delta u + a|u_t|^{m-2}u_t = b|u|^{p-2}u \quad \text{in } \Omega \times (0,T),$$

with initial and Dirichlet boundary conditions, has been studied by many researchers. For example, Ball in [5] showed that if $a = 0$, then the source term $b|u|^{p-2}u$, with $b > 0$, forces the negative-energy solutions to explode in finite time. Haraux and Zuazua [10] proved that in the absence of the source term, the damping term $a|u_t|^{m-2}u_t$, with $a > 0$, assures the global existence for arbitrary initial data. In the presence of both terms, the problem was first considered by Levine [12]. He established the blow up for solutions with negative initial energy, when $m = 2$. Georgiev and Todorava [8] pushed Levine's result to the case $m > 2$, by introducing a different method. Messaoudi [14] proved that any solution with negative initial energy only, blows up in finite time when $m < p$.

For a wave equation with variable-exponent nonlinearity, we mention some works. In [3], Antontsev studied the equation

$$u_{tt} - \operatorname{div}(a|\nabla u|^{p(x,t)-2}\nabla u) - \alpha\Delta u_t - bu|u|^{\sigma(x,t)-2} = f \quad \text{in } \Omega \times (0,T),$$

where $\alpha > 0$ is a constant and a, b, p, σ are given functions. Under specific conditions, he proved the local and global existence of some weak solutions and a blow-up result for certain solutions having arbitrary initial energy. Guo and Gao [9] took

$\sigma(x,t) = r > 2$ and established a finite-time blow-up result. Sun et al. [22] studied the equation

$$u_{tt} - \operatorname{div}(a(x,t)\nabla u) + c(x,t)u_t|u_t|^{q(x,t)-1} = b(x,t)u|u|^{p(x,t)-2} \quad \text{in } \Omega \times (0,T)$$

and established a blow-up result. Also, under some conditions on the initial data, the lower and upper bounds for the blow-up time are obtained. In addition, they provided numerical illustrations for their result. After that, Messaoudi and Talahmeh [17] studied the equation

$$u_{tt} - \operatorname{div}(|\nabla u|^{m(x)-2}\nabla u) + \mu u_t = u|u|^{p(x)-2} \quad \text{in } \Omega \times (0,T),$$

for $\mu \geq 0$ supplemented with Dirichlet-boundary conditions. They proved a blow-up result for certain solutions with arbitrary positive initial energy. In [18], the same authors considered the equation

$$u_{tt} - \operatorname{div}(|\nabla u|^{r(x)-2}\nabla u) + au_t|u_t|^{m(x)-2} = bu|u|^{p(x)-2} \text{ in } \Omega \times (0,T),$$

where $a, b > 0$ are two constants and m, r, p are given functions. They established a finite-time blow-up result for negative-initial-energy solutions and for certain solutions with positive initial energy. Recently, Messaoudi et al. [19] considered the equation

$$u_{tt} - \Delta u + au_t|u_t|^{m(x)-2} = bu|u|^{p(x)-2} \text{ in } \Omega \times (0,T).$$

Using the Faedo-Galerkin method, they established the existence and uniqueness of a weak local solution and proved the finite-time blow up for solutions with negative initial-energy.

Concerning the coupled systems of two nonlinear wave equations with constant exponents, Messaoudi and Said-Houari [16] proved a nonexistence theorem for positive-initial-energy solutions of a system of viscoelastic wave equations. Under some restrictions on the nonlinearity of the damping and source terms and the initial data, Said-Houari et al. [21] proved that the rate of decay of the total energy depends on those of the relaxation functions. Agre and Rammaha [2] obtained several results on the local and global existence, uniqueness, and the finite-time blow up of solutions, under appropriate conditions on the parameters in the system. Very recently, Messaoudi and Hassan [15] established a general decay result for a certain system of viscoelastic wave equations.

In the case of coupled systems of two nonlinear hyperbolic equations with variable exponents, there is only the work of Bouhoufani and Hamchi [6], where they proved the global existence of a weak solution and established decay estimates of the solution energy. However, the existence of a local solution was not discussed. In this work, we push the local existence result of Agre and Rammaha [2], which was established for the case of constant-exponent nonlinearities, to our system which deals with variable-exponent nonlinearities. To the best of our knowledge, this is the first result of this kind and the generalization was not trivial at all. In addition to the local existence, we establish the blow up in finite time for certain solutions with positive initial energy and give some numerical illustrations.

This paper consists of four Sections, in addition to the introduction. In Section 2, we give some preliminary results. The existence and uniqueness of weak solution is discussed in Section 3. Section 4 is devoted to the statement and the proof of the finite-time blow up result. In section 5, we present two numerical examples to illustrate our theoretical findings.

2. Preliminaries

In this section, we define the Lebesgue and Sobolev spaces with variable exponents and present some facts and results related to this important class of spaces. See [4, 11] for more details.

Let $q : \Omega \to [1, \infty)$ be a measurable function. We define the Lebesgue space with a variable exponent by

$$L^{q(\cdot)}(\Omega) = \big\{ f : \Omega \to \mathbb{R} \text{ measurable in } \Omega : \varrho_{q(\cdot)}(\lambda f) < +\infty \text{ for some } \lambda > 0 \big\},$$

where

$$\varrho_{q(\cdot)}(f) = \int_\Omega |f(x)|^{q(x)} dx.$$

$L^{q(\cdot)}(\Omega)$ is a Banach space with respect to the following Luxembourg-type norm

$$\|f\|_{q(\cdot)} := \inf \big\{ \lambda > 0 : \int_\Omega |\frac{f(x)}{\lambda}|^{q(x)} dx \leq 1 \big\}.$$

We also define the variable exponent Sobolev space

$$W^{1,q(\cdot)}(\Omega) = \{ f \in L^{q(\cdot)}(\Omega) : \nabla f \text{ exists and } |\nabla f| \in L^{q(\cdot)}(\Omega) \}.$$

Equipped with the norm

$$\|f\|_{W^{1,q(\cdot)}(\Omega)} = \|f\|_{q(\cdot)} + \|\nabla f\|_{q(\cdot)},$$

$W^{1,q(\cdot)}(\Omega)$ is a Banach space. Furthermore, we denote by $H_0^{1,q(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,q(\cdot)}(\Omega)$ and by $W^{-1,q'(\cdot)}(\Omega)$ the dual space of $W_0^{1,q(\cdot)}(\Omega)$ (in the same way as the usual Sobolev spaces), where

$$W_0^{k,p(\cdot)}(\Omega) = \overline{\{u \in W^{k,p(\cdot)}(\Omega) : u = u\chi_K \text{ for a compact } K \subset \Omega\}}$$

and $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

Lemma 2.1 (Young's Inequality [11]). *Let $p, q, s : \Omega \to [1, \infty)$ be measurable functions, such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad \text{for a.e } y \in \Omega.$$

Then, for all $a, b \geq 0$, we have

$$\frac{(ab)^{s(\cdot)}}{s(\cdot)} \leq \frac{a^{p(\cdot)}}{p(\cdot)} + \frac{b^{q(\cdot)}}{q(\cdot)}.$$

By taking $s = 1$ and $1 < p, q < +\infty$, it follows that for any $\varepsilon > 0$,

$$ab \leq \varepsilon a^p + C_\varepsilon b^q, \quad \text{where } C_\varepsilon = 1/q(\varepsilon p)^{q/p}. \tag{2.1}$$

Lemma 2.2 (Hölder's Inequality [11]). *Let $p, q, s : \Omega \to [1, \infty)$ be measurable functions, such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad \text{for a.e. } y \in \Omega.$$

If $f \in L^{p(\cdot)}(\Omega)$ and $h \in L^{q(\cdot)}(\Omega)$, then $fh \in L^{s(\cdot)}(\Omega)$, with

$$\|fh\|_{s(\cdot)} \leq 2\|f\|_{p(\cdot)}\|h\|_{q(\cdot)}.$$

Lemma 2.3 ([11]). *If $1 < q^- \leq q(x) \leq q^+ < +\infty$ holds, then for any $f \in L^{q(\cdot)}(\Omega)$,*

$$\min\{\|f\|_{q(\cdot)}^{q^-}, \|f\|_{q(\cdot)}^{q^+}\} \leq \varrho_{q(\cdot)}(f) \leq \max\{\|f\|_{q(\cdot)}^{q^-}, \|f\|_{q(\cdot)}^{q^+}\}.$$

Lemma 2.4 ([11]). *If $q^+ < +\infty$, then $C_0^\infty(\Omega)$ is dense in $L^{q(\cdot)}(\Omega)$.*

Lemma 2.5 (Embedding Property [11]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$. If $r, q \in C(\overline{\Omega})$ such that $r, q \geq 1$ and $\inf_{x\in\overline{\Omega}}(r^*(x) - q(x)) > 0$ with*

$$r^*(x) = \begin{cases} \frac{nr(x)}{\sup_{x\in\overline{\Omega}}(n-r(x))}, & \text{if } r^+ < n, \\ \infty, & \text{if } r^+ \geq n. \end{cases}$$

Then, the embedding $W_0^{1,r(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

3. Existence of weak solutions

We begin this section by giving the definition of a weak solution for the system (1.1).

Definition 3.1. Let $(u_0, u_1), (v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Any pair of functions (u, v) such that

$$\begin{gathered} u, v \in L^\infty([0, T), H_0^1(\Omega)), \\ u_t \in L^\infty([0, T), L^2(\Omega)) \cap L^{m(\cdot)}(\Omega \times (0, T)), \\ v_t \in L^\infty([0, T), L^2(\Omega)) \cap L^{r(\cdot)}(\Omega \times (0, T)), \end{gathered}$$

is called a weak solution of (1.1) on $[0, T)$, if

$$\begin{gathered} \frac{d}{dt}[\int_\Omega u_t \Phi dx + \int_\Omega A\nabla u.\nabla \Phi dx] + \int_\Omega |u_t|^{m(x)-2} u_t \Phi dx - \int_\Omega \Phi f_1 dx = 0, \\ \frac{d}{dt}[\int_\Omega v_t \Psi dx + \int_\Omega B\nabla v.\nabla \Psi dx] + \int_\Omega |v_t|^{r(x)-2} u_t \Psi dx - \int_\Omega \Psi f_2 dx = 0, \\ u(0) = u_0, \quad u_t(0) = u_1, \quad v(0) = v_0, \quad v_t(0) = v_1, \end{gathered}$$

for a.e. $t \in (0, T)$ and all test functions $\Phi, \Psi \in H_0^1(\Omega)$.

To prove the existence of a local weak solution of problem (1.1), we first consider, as in [13], the initial-boundary-value problem

$$\begin{gathered} u_{tt} - \operatorname{div}(A\nabla u) + |u_t|^{m(x)-2} u_t = f(x, t) \quad \text{in } \Omega \times (0, T), \\ v_{tt} - \operatorname{div}(B\nabla v) + |v_t|^{r(x)-2} v_t = g(x, t) \quad \text{in } \Omega \times (0, T), \\ u = v = 0 \quad \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0, \quad u_t(0) = u_1 \quad \text{in } \Omega, \\ v(0) = v_0, \quad v_t(0) = v_1 \quad \text{in } \Omega, \end{gathered} \tag{3.1}$$

where $f, g \in L^2(\Omega \times (0, T))$.

Theorem 3.2. *Under the above conditions, on m, r, A and B, and for the initial values $(u_0, u_1), (v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$, problem (3.1) has a unique local weak solution (u, v) on $[0, T)$, in the sense of Definition 3.1.*

Proof. (**Uniqueness.**) Suppose that (3.1) has two weak solutions (u_1, v_1) and (u_2, v_2), in the sense of Definition 3.1. Then, $(u, v) = (u_1 - u_2, v_1 - v_2)$ solves the problem

$$\begin{gathered} u_{tt} - \operatorname{div}(A\nabla u) + |u_{1t}|^{m(x)-2} u_{1t} - |u_{2t}|^{m(x)-2} u_{2t} = 0 \quad \text{in } \Omega \times (0, T), \\ v_{tt} - \operatorname{div}(B\nabla v) + |v_{1t}|^{r(x)-2} v_{1t} - |v_{2t}|^{r(x)-2} v_{2t} = 0 \quad \text{in } \Omega \times (0, T), \end{gathered}$$

$$u = v = 0 \quad \text{on } \partial\Omega \times (0, T),$$
$$u(0) = u_t(0) = 0, v(0) = v_t(0) = 0 \quad \text{in } \Omega,$$

in the sense of Definition 3.1. Taking $\Phi = u_t$, we obtain

$$\begin{aligned} &\frac{d}{dt}\Big[\int_\Omega (u_t^2 + A\nabla u.\nabla u)dx\Big] \\ &+ 2\int_\Omega (|u_{1t}|^{m(x)-2}u_{1t} - |u_{2t}|^{m(x)-2}u_{2t})(u_{1t} - u_{2t})dx \le 0, \end{aligned} \tag{3.2}$$

by (1.8) and

$$\frac{d}{dt}\Big(\int_\Omega A\nabla u \cdot \nabla u dx\Big) = \int_\Omega A'\nabla u \cdot \nabla u \, dx + 2\int_\Omega A\nabla u \cdot \nabla u_t \, dx. \tag{3.3}$$

Since, for all $x \in \Omega, Y, Z \in \mathbb{R}$ and $q(x) \ge 2$, we have

$$(|Y|^{q(x)-2}Y - |Z|^{q(x)-2}Z)(Y - Z) \ge 0, \quad q(x) \ge 2, \tag{3.4}$$

inequality (3.2) leads to

$$\frac{d}{dt}\int_\Omega (u_t^2 + A\nabla u \cdot \nabla u)dx \le 0.$$

Integrating over $(0, t), t \le T$, and using (1.7), we find that

$$\|u_t\|_2^2 + a_0\|\nabla u\|_2^2 = 0.$$

Similarly, we obtain

$$\|v_t\|_2^2 + b_0\|\nabla v\|_2^2 = 0.$$

Therefore, $u_t(\cdot, t) = v_t(\cdot, t) = 0$ and $\nabla u(\cdot, t) = \nabla v(\cdot, t) = 0$ for all$t \in (0, T)$. Which implies $u = v = 0$ on $\Omega \times (0, T)$, since $u = v = 0$ on $\partial\Omega \times (0, T)$. This proves the uniqueness.

Existence. To prove the existence of a weak solution of (3.1), we proceed in four steps:

Step 1. Approximate problem. We consider an orthonormal basis $\{\omega_j\}_{j=1}^\infty$ of $H_0^1(\Omega)$ and define, for all $k \ge 1$, a sequence (u^k, v^k) in the finite-dimensional subspace $V_k = \text{span}\{\omega_1, \omega_2, \dots, \omega_k\}$, as follows

$$u^k(x, t) = \Sigma_{j=1}^k a_j(t)\omega_j(x), \quad v^k(t) = \Sigma_{j=1}^k b_j(t)\omega_j(x),$$

for $x \in \Omega$ and $t \in (0, T)$, satisfying the approximate problems

$$\begin{aligned} &\int_\Omega u_{tt}^k(x, t)\omega_j dx + \int_\Omega A\nabla u^k(x, t) \cdot \nabla\omega_j dx \\ &+ \int_\Omega |u_t^k(x, t)|^{m(x)-2}u_t^k(x, t)\omega_j dx = \int_\Omega f(x, t)\omega_j dx, \\ &\int_\Omega v_{tt}^k(x, t)\omega_j dx + \int_\Omega B\nabla v^k(x, t) \cdot \nabla\omega_j dx \\ &+ \int_\Omega |v_t^k(x, t)|^{r(x)-2}v_t^k(x, t)\omega_j dx = \int_\Omega g(x, t)\omega_j dx, \end{aligned} \tag{3.5}$$

for $j = 1, 2, \dots, k$, and with the initial data

$$u^k(0) = u_0^k = \Sigma_{i=1}^k \langle u_0, \omega_i\rangle\omega_i, \quad u_t^k(0) = u_1^k = \Sigma_{i=1}^k \langle u_1, \omega_i\rangle\omega_i$$
$$v^k(0) = v_0^k = \Sigma_{i=1}^k \langle v_0, \omega_i\rangle\omega_i, \quad v_t^k(0) = v_1^k = \Sigma_{i=1}^k \langle v_1, \omega_i\rangle\omega_i,$$

where (u_0^k) and (v_0^k) are two sequences such that

$$
\begin{gathered}
u_0^k \to u_0, \quad v_0^k \to v_0 \quad \text{in } H_0^1(\Omega), \\
u_1^k \to u_1, \quad v_1^k \to v_1 \quad \text{in } L^2(\Omega).
\end{gathered}
$$

This generates a system of k nonlinear ordinary differential equations, which admits a unique local solution (u^k, v^k) in $[0, T_k), T_k \le T$, by the standard ODE theory.

Step 2. A priori Estimates. Now, we show, by a priory estimates, that $T_k = T$, for all $k \ge 1$. For this, we multiply $(3.5)_1$ and $(3.5)_2$ by $a_j'(t)$ and $b_j'(t)$, respectively. Then sum each result over j, from 1 to k, and integrate each equation over $(0, t)$, with $t \le T_k$. We obtain

$$
\begin{aligned}
&\|u_t^k\|_2^2 - \|u_1^k\|_2^2 + \int_\Omega A\nabla u^k \cdot \nabla u^k dx - \int_\Omega A(x,0)\nabla u_0^k \cdot \nabla u_0^k dx \\
&+ 2\int_0^t \int_\Omega |u_t^k(x,t)|^{m(x)}\,dx\,dt \le 2\int_0^t \int_\Omega f(x,t)u_t^k(x,t)\,dx\,dt
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
&\|v_t^k\|_2^2 - \|v_1^k\|_2^2 + \int_\Omega B\nabla v^k \cdot \nabla v^k dx - \int_\Omega B(x,0)\nabla v_0^k \cdot \nabla v_0^k dx \\
&+ 2\int_0^t \int_\Omega |v_t^k(x,t)|^{r(x)}\,dx\,dt \le 2\int_0^t \int_\Omega g(x,t)v_t^k(x,t)\,dx\,ds,
\end{aligned} \tag{3.7}
$$

by (3.3) and (1.8). Under the assumptions on A and B, the addition of (3.6) and (3.7), Young's inequality (2.1) gives

$$
\begin{aligned}
&\|u_t^k\|_2^2 + \|v_t^k\|_2^2 + a_0\|\nabla u^k\|_2^2 + b_0\|\nabla v^k\|_2^2 \\
&+ 2\int_0^{T_k} \int_\Omega (|u_t^k(x,t)|^{m(x)} + |v_t^k(x,t)|^{r(x)})\,dx\,ds \\
&\le 2\varepsilon \int_0^{T_k} (\|u_t^k\|_2^2 + \|v_t^k\|_2^2) ds + 2C_\varepsilon \int_0^T \int_\Omega (|f(x,t)|^2 + |g(x,t)|^2))\,dx\,ds \\
&\quad + \|u_1^k\|_2^2 + \|v_1^k\|_2^2 + \alpha\|\nabla u_0^k\|_2^2 + \beta\|\nabla v_0^k\|_2^2,
\end{aligned} \tag{3.8}
$$

where

$$
\alpha = \sup_{\Omega\times(0,T)} A(x,t), \quad \beta = \sup_{\Omega\times(0,T)} B(x,t).
$$

Since $f, g \in L^2(\Omega \times (0,T))$ and

$$
\begin{gathered}
u_0^k \to u_0, \quad v_0^k \to v_0 \quad \text{in } H_0^1(\Omega), \\
u_1^k \to u_1, \quad v_1^k \to v_1 \quad \text{in } L^2(\Omega),
\end{gathered}
$$

invoking Gronwall's lemma, estimate (3.8) leads to

$$
\begin{aligned}
&\sup_{(0,T_k)} [\|u_t^k\|_2^2 + \|v_t^k\|_2^2 + \|\nabla u^k\|_2^2 + \|\nabla v^k\|_2^2] \\
&+ \int_0^{T_k} \int_\Omega (|u_t^k(x,t)|^{m(x)} + |v_t^k(x,t)|^{r(x)})\,dx\,ds \le C,
\end{aligned}
$$

where $C > 0$, for all $T_k \le T$ and $k \ge 1$. Therefore, the local solution (u^k, v^k) of system (3.5) can be extended to $(0, T)$, for all $k \ge 1$. Furthermore,

$$
\begin{gathered}
(u^k)_k, (v^k)_k \text{ are bounded in } L^\infty((0,T), H_0^1(\Omega)), \\
(u_t^k)_k \text{ is bounded in } L^\infty((0,T), L^2(\Omega)) \cap L^{m(\cdot)}(\Omega \times (0,T)),
\end{gathered}
$$

$$(v_t^k)_k \text{ is bounded in } L^\infty((0,T),L^2(\Omega)) \cap L^{r(\cdot)}(\Omega\times(0,T)).$$

Consequently, we can extract two subsequences of $(u^k)_k$ and $(v^k)_k$, which we denote by $(u_l)_l$ and $(v_l)_l$, respectively, such that

$$\begin{gathered} u^l \to u \text{ and } v^l \to v \quad \text{weakly * in } L^\infty((0,T),H_0^1(\Omega)),\\ u_t^l \to u_t \text{ weakly * in } L^\infty((0,T),L^2(\Omega)) \text{ and weakly in } L^{m(\cdot)}(\Omega\times(0,T)),\\ v_t^l \to v_t \text{ weakly * in } L^\infty((0,T),L^2(\Omega)) \text{ and weakly in } L^{r(\cdot)}(\Omega\times(0,T)), \end{gathered}$$

as $l\to+\infty$

Step 3. Nonlinear terms. In this step, we show that

$$\begin{gathered} |u_t^l|^{m(\cdot)-2}u_t^l \to |u_t|^{m(\cdot)-2}u_t \quad \text{weakly in } L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega\times(0,T)),\\ |v_t^l|^{r(\cdot)-2}v_t^l \to |v_t|^{r(\cdot)-2}v_t \quad \text{weakly in } L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega\times(0,T)) \end{gathered}$$

and then, we establish that (u,v) satisfies the differential equations (3.1) on $\Omega\times(0,T)$. So, by exploiting Hölder's inequality (Lemma 2.2), one easily deduce that $(|u_t^l|^{m(\cdot)-2}u_t^l)_l$ is bounded in $L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega\times(0,T))$. Then, there exists a subsequence, still denoted by $(|u_t^l|^{m(\cdot)-2}u_t^l)_l$, such that

$$|u_t^l|^{m(\cdot)-2}u_t^l \to \Phi \quad \text{weakly in } L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega\times(0,T)).$$

To prove that $\Phi=|u_t|^{m(\cdot)-2}u_t$, we set $h(z)=|z|^{m(\cdot)-2}z$ and define, as in [13], the sequence

$$S_l=\int_0^T\int_\Omega (h(u_t^l)-h(z))(u_t^l-z),$$

for $z\in L^{m(\cdot)}((0,T),H_0^1(\Omega))$ and $l\ge 1$. Replacing u^k by u^l in (3.6), integrating the result over $(0,T)$, and letting $l\to\infty$, by (3.3), we obtain

$$\begin{aligned} 0\le \limsup_l S_l \le \frac12\Big[&\|u_1\|_2^2-\|u_t(T)\|_2^2+\int_\Omega A(x,0)\nabla u_0\cdot\nabla u_0\Big]\\ &-\frac12\int_\Omega A(x,T)\nabla u(T)\cdot\nabla u(T)]-\int_0^T\int_\Omega \Phi z\\ &-\int_0^T\int_\Omega h(z)(u_t-z)+\int_0^T\int_\Omega f u_t. \end{aligned} \tag{3.9}$$

On the other hand, if we use u^l instead of u^k in $(3.5)_1$ and integrate the result over $(0,t)$, we arrive at

$$\int_\Omega u_t\omega-\int_\Omega u_1\omega+\int_0^t\int_\Omega A\nabla u\cdot\nabla\omega+\int_0^t\int_\Omega \Phi\omega=\int_0^t\int_\Omega f\omega,\ \forall\omega\in H_0^1(\Omega),$$

since $\{\omega_j\}_{j=1}^\infty$ is a basis of $H_0^1(\Omega)$. Therefore,

$$\int_\Omega u_{tt}\omega+\int_\Omega (A\nabla u\cdot\nabla\omega+\Phi\omega)=\int_\Omega f\omega,\quad \forall\omega\in H_0^1(\Omega), \tag{3.10}$$

for all $t \in (0,T)$. Using the denseness of $H_0^1(\Omega)$ in $L^2(\Omega)$, we use u_t instead of ω in (3.10) and integrate the result over $(0,T)$ to obtain

$$\begin{aligned} \int_0^T \int_\Omega f u_t = \frac{1}{2}\Big[\|u_t(T)\|_2^2 - \|u_1\|_2^2 + \int_\Omega A(x,T)\nabla u(T)\cdot\nabla u(T)\Big] \\ - \frac{1}{2}\int_\Omega A(x,0)\nabla u_0.\nabla u_0] + \int_0^T\int_\Omega \Phi u_t. \end{aligned} \tag{3.11}$$

Combining (3.9) and (3.11), we infer that

$$\int_0^T \int_\Omega [\Phi - h(z)](u_t - z) \geq \limsup_l S_l \geq 0, \quad \forall z \in L^{m(\cdot)}(\Omega\times(0,T)), \tag{3.12}$$

since $H_0^1(\Omega)$ is dense in $L^{m(\cdot)}(\Omega)$ (see Lemma 2.4).

Now, let $z = \lambda\omega + u_t,\ \omega \in L^{m(\cdot)}(\Omega\times(0,T))$. Hence, inequality ((3.12) can be rewritten as

$$-\lambda\int_0^T\int_\Omega [\Phi - h(\lambda\omega + u_t)]\omega \geq 0, \quad \forall \lambda \neq 0.$$

The continuity of h with respect to λ yields

$$\int_0^T\int_\Omega (\Phi - h(u_t))\omega = 0, \quad \forall \omega \in L^{m(\cdot)}(\Omega\times(0,T)).$$

Thus, $\Phi = h(u_t) = |u_t|^{m(x)-2}u_t$. Therefore, inequality ((3.10) becomes

$$\int_\Omega u_{tt}\omega + \int_\Omega A\nabla u\cdot\nabla\omega + \int_\Omega |u_t|^{m(x)-2}u_t\omega = \int_\Omega f\omega,\ \forall \omega \in H_0^1(\Omega).$$

Consequently,

$$u_{tt} - \operatorname{div}(A\nabla u) + |u_t|^{m(x)-2}u_t = f \quad \text{in } D'(\Omega\times(0,T)). \tag{3.13}$$

Likewise and since $H_0^1(\Omega)$ is dense in $L^{r(\cdot)}(\Omega)$ (Lemma 2.4), we obtain

$$|v_t^l|^{r(\cdot)-2}v_t^l \to\ |v_t|^{r(\cdot)-2}v_t \quad \text{weakly in } L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega\times(0,T))$$

and

$$v_{tt} - \operatorname{div}(B\nabla v) + |v_t|^{r(x)-2}v_t = g \quad \text{in } D'(\Omega\times(0,T)). \tag{3.14}$$

Step 4. Initial conditions.

First, by Lions' lemma [13, Lemma 1.2, page 7] and since

$$\begin{aligned} u^l &\rightharpoonup u \quad \text{weakly * in } L^\infty((0,T), H_0^1(\Omega)),\\ u_t^l &\rightharpoonup u_t \quad \text{weakly * in } L^\infty((0,T), L^2(\Omega)), \end{aligned}$$

we deduce that $u^l \to u$ in $C([0,T], L^2(\Omega))$. Therefore, $u^l(\cdot,0)$ is defined and

$$u^l(\cdot,0) \to u(\cdot,0) \quad \text{in } L^2(\Omega).$$

But, $u^l(\cdot,0) = u_0^l \to u_0$, in $H_0^1(\Omega)$. Then $u(\cdot,0) = u_0$. Similarly, we obtain $v(\cdot,0) = v_0$.

Second, for any $\phi \in C_0^\infty(0,T)$ and $j \le l$, we obtain from (3.4) that

$$\begin{aligned}
&\int_0^T \int_\Omega u_{tt}^l(x,t)\omega_j(x)\phi(t) + \int_0^T \int_\Omega A\nabla u^l(x,t)\cdot\nabla\omega_j(x)\phi(t) \\
&= -\int_0^T \int_\Omega |u_t^l(x,t)|^{m(x)-2}u_t^l(x,t)\omega_j(x)\phi(t) \\
&\quad + \int_0^T \int_\Omega f(x,t)\omega_j(x)\phi(t).
\end{aligned} \tag{3.15}$$

By routine computations and taking $l \to +\infty$, we find that for all $\omega \in H_0^1(\Omega)$,

$$\begin{aligned}
&\int_0^T \int_\Omega u_{tt}(x,t)\omega(x)\phi(t) \\
&= \int_0^T \int_\Omega [\operatorname{div}(A\nabla u(x,t)) - |u_t(x,t)|^{m(x)-2}u_t(x,t) + f(x,t)]\omega(x)\phi(t),
\end{aligned}$$

which means $u_{tt} \in L^{\frac{m(\cdot)}{m(\cdot)-1}}([0,T), H^{-1}(\Omega))$ and that u solves the equation

$$u_{tt} - \operatorname{div}(A\nabla u) + |u_t|^{m(x)-2}u_t = f, \quad \text{in } D'(\Omega\times(0,T)).$$

So, we have

$$u_t \in L^\infty((0,T), L^2(\Omega)), \quad u_{tt} \in L^{\frac{m(\cdot)}{m(\cdot)-1}}([0,T), H^{-1}(\Omega)).$$

By Lions' lemma [13], $u_t \in C([0,T), H^{-1}(\Omega))$ and consequently, $u_t(\cdot,0)$ has a meaning and, in addition,

$$u_t^l(\cdot,0) \to u_t(\cdot,0), \quad \text{in } H^{-1}(\Omega).$$

Since, $u_t^l(\cdot,0) = u_1^l \to u_1$ in $L^2(\Omega)$, this implies that $u_t(\cdot,0) = u_1$. Similarly, one has $v_t(\cdot,0) = v_1$.

Therefore, (u,v) is the unique local solution of (3.1). $\square$

To state and prove the existence of a solution for problem (1.1), we recall the following elementary inequalities:

$$\big||X|^k - |Y|^k\big| \le C|X-Y|(|X|^{k-1} + |Y|^{k-1}), \tag{3.16}$$

for some constant $C > 0$, all $k \ge 1$ and all $X, Y \in \mathbb{R}$. Also

$$\big||X|^{k'}X - |Y|^{k'}Y\big| \le C|X-Y|(|X|^{k'} + |Y|^{k'}), \tag{3.17}$$

for some constant $C > 0$, all $k' \ge 0$ and all $X, Y \in \mathbb{R}$.

Theorem 3.3. *Let $(u_0,u_1),(v_0,v_1) \in H_0^1(\Omega)\times L^2(\Omega)$ be given. Assume that the conditions on $p(\cdot)$, $r(\cdot)$, $m(\cdot)$, A and B, given in Section 1, hold. Then, problem (1.1) has a unique weak local solution (u,v) on $[0,T)$, for some $T > 0$.*

Proof. (**Existence.**) Recall that the source terms are defined for all $x \in \Omega$ and $(y,z) \in \mathbb{R}^2$ by

$$f_1(x,y,z) = \frac{\partial}{\partial y}F(x,y,z), \quad f_2(x,y,z) = \frac{\partial}{\partial z}F(x,y,z),$$

where

$$F(x,y,z) = a|y+z|^{p(x)+1} + 2b|yz|^{\frac{p(x)+1}{2}}, \quad a,b > 0.$$

So,

$$f_1(x,y,z) = (p(x)+1)[a|y+z|^{p(x)-1}(y+z) + by|y|^{\frac{p(x)-3}{2}}|z|^{\frac{p(x)+1}{2}}],$$

$$f_2(x,y,z) = (p(x)+1)[a|y+z|^{p(x)-1}(y+z) + bz|z|^{\frac{p(x)-3}{2}}|y|^{\frac{p(x)+1}{2}}].$$

Let $y, z \in L^\infty((0,T), H_0^1(\Omega))$. Using Young's inequality (2.1) and the Sobolev embedding (Lemma 2.5), $f_1(y,z)$ and $f_2(y,z)$ are in $L^2(\Omega\times(0,T))$. Indeed, we have

$$\begin{aligned}
&\int_\Omega |f_1(x,y,z)|^2 dx \\
&\le 2\int_\Omega (p(x)+1)^2[a^2|y+z|^{2p(x)} + b^2|y|^{p(x)-1}|z|^{p(x)+1}]dx \\
&\le 2(p^+ +1)^2\Big[a^2\int_\Omega |y+z|^{2p(x)}dx + b^2\int_\Omega |y|^{p(x)-1}|z|^{p(x)+1}dx\Big] \\
&\le C_0\Big[\int_\Omega |y+z|^{2p^+}dx + \int_\Omega |y+z|^{2p^-}dx + \int_\Omega |y|^{3(p^+-1)}dx\Big] \\
&+ C_0\Big[\int_\Omega |y|^{3(p^--1)}dx + \int_\Omega |z|^{\frac{3}{2}(p^++1)}dx + \int_\Omega |z|^{\frac{3}{2}(p^-+1)}dx\Big],
\end{aligned} \tag{3.18}$$

where $C_0 = 2(p^+ +1)^2\max\{a^2, 3b^2\} > 0$. By the embeddings, one can obtain the following results

- If $n = 1, 2$, then

$$1 < \frac{3}{2}(p^- + 1) \le \frac{3}{2}(p^+ + 1) \le 2p^+ \le 3(p^+ - 1) < \infty,$$

since $3 \le p^- \le p(x) \le p^+ < \infty$. Therefore, estimate (3.18) leads to

$$\begin{aligned}
&\int_\Omega |f_1(x,y,z)|^2 dx \\
&\le C_1\big[\|\nabla(y+z)\|_2^{2p^+} + \|\nabla(y+z)\|_2^{2p^-} + \|\nabla y\|_2^{3(p^+-1)} + \|\nabla y\|_2^{3(p^--1)}\big] \\
&\quad + C_1\big[\|\nabla z\|_2^{\frac{3}{2}(p^++1)} + \|\nabla z\|_2^{\frac{3}{2}(p^-+1)}\big] < +\infty, \quad C_1 = C_0C_e
\end{aligned} \tag{3.19}$$

- If $n = 3$, then the Sobolev embeddings used in (3.19) are also satisfied, since $p \equiv 3$ on $\overline{\Omega}$.

Consequently, under assumption (1.6), for all $t \in (0,T)$, we have

$$\int_\Omega |f_1(x,y,z)|^2 dx < \infty$$

and similarly,

$$\int_\Omega |f_2(x,y,z)|^2 dx < \infty.$$

Therefore,

$$f_1(y,z), f_2(y,z) \in L^2(\Omega\times(0,T)).$$

By Theorem 3.2, there exists a unique weak solution (u,v) for the problem

$$\begin{gathered}
u_{tt} - \operatorname{div}(A\nabla u) + |u_t|^{m(x)-2}u_t = f_1(y,z) \quad \text{in } \Omega\times(0,T), \\
v_{tt} - \operatorname{div}(B\nabla v) + |v_t|^{r(x)-2}v_t = f_2(y,z) \quad \text{in } \Omega\times(0,T), \\
u = v = 0 \quad \text{on } \partial\Omega\times(0,T), \\
u(0) = u_0, \quad u_t(0) = u_1 \quad \text{in } \Omega, \\
v(0) = v_0, \quad v_t(0) = v_1 \quad \text{in } \Omega.
\end{gathered} \tag{3.20}$$

Now, let $G : W_T \times W_T :\to W_T \times W_T$ be a map defined by $G(y,z) = (u,v)$, where

$$W_T = \{w \in L^\infty((0,T), H_0^1(\Omega))/w_t \in L^\infty((0,T), L^2(\Omega))\}.$$

Note that W_T is a Banach space with respect to the norm

$$\|w\|_{W_T}^2 = \sup_{(0,T)} \int_\Omega |\nabla w|^2 dx + \sup_{(0,T)} \int_\Omega |w_t|^2 dx.$$

Our task is to prove that G is a contraction mapping from a bounded ball $B(0,d)$ into itself, where

$$B(0,d) = \big\{(y,z) \in W_T \times W_T / \|(y,z)\|_{W_{T_0} \times W_{T_0}} \le d\big\},$$

for $d > 0$ sufficiently large and $T_0 \ge T$ to be fixed later. To do so, let (y,z) be in $B(0,d)$ and (u,v) be the corresponding solution of system (3.20). Taking $(\Phi,\Psi) = (u_t, v_t)$ in Definition 3.1 and by (3.3) and (1.8), and then integrating each identity over $(0,t)$, for all $t \le T$, we obtain

$$\begin{aligned}
&\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\int_\Omega A\nabla u \cdot \nabla u dx - \frac{1}{2}\int_\Omega A(x,0)\nabla u_0 \cdot \nabla u_0 dx \\
&+ \int_0^t \int_\Omega |u_t(x,t)|^{m(x)} \\
&\le \int_0^t \int_\Omega u_t f_1(y,z)\, dx\, ds
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&\frac{1}{2}\|v_t\|_2^2 - \frac{1}{2}\|v_1\|_2^2 + \frac{1}{2}\int_\Omega B\nabla v \cdot \nabla v\, dx - \frac{1}{2}\int_\Omega B(x,0)\nabla v_0^k \cdot \nabla v_0\, dx \\
&+ \int_0^t \int_\Omega |v_t(x,t)|^{r(x)} \\
&\le \int_0^t \int_\Omega v_t f_2(y,z)\, dx\, ds.
\end{aligned} \tag{3.22}$$

Recalling the assumptions on A and B, inequalities (3.21), (3.22) lead to

$$\begin{aligned}
\frac{1}{2}\big(\|u_t\|_2^2 + a_0\|\nabla u\|_2^2\big) &\le \frac{1}{2}\big(\|u_1\|_2^2 + \alpha\|\nabla u_0\|_2^2\big) + \int_0^t \int_\Omega u_t f_1(y,z)\, dx\, ds, \\
\frac{1}{2}\big(\|v_t\|_2^2 + b_0\|\nabla v\|_2^2\big) &\le \frac{1}{2}\big(\|v_1\|_2^2 + \beta\|\nabla v_0\|_2^2\big) + \int_0^t \int_\Omega v_t f_2(y,z)\, dx\, ds.
\end{aligned}$$

Consequently, we arrive at

$$\begin{aligned}
\frac{1}{2}\|u\|_{W_T}^2 &\le \lambda_0 + \frac{1}{\min\{1,a_0\}} \sup_{(0,T)} \int_0^t \int_\Omega u_t f_1(y,z)\, dx\, ds, \\
\frac{1}{2}\|v\|_{W_T}^2 &\le \beta_0 + \frac{1}{\min\{1,b_0\}} \sup_{(0,T)} \int_0^t \int_\Omega v_t f_2(y,z)\, dx\, ds,
\end{aligned}$$

where,

$$\lambda_0 = \frac{\|u_1\|_2^2 + \alpha\|\nabla u_0\|_2^2}{2\min\{1,a_0\}}, \quad \beta_0 = \frac{\|v_1\|_2^2 + \beta\|\nabla v_0\|_2^2}{2\min\{1,b_0\}}.$$

The addition the last two inequalities yield

$$
\begin{aligned}
&\frac{1}{2}\|(u,v)\|^2_{W_T\times W_T}\\
&\le \gamma_0 + C_2 \sup_{(0,T)}\int_0^t \Big(|\int_\Omega u_t f_1(y,z)dx| + |\int_\Omega v_t f_2(y,z)dx|\Big)ds,
\end{aligned}
\tag{3.23}
$$

where

$$
\gamma_0=\lambda_0+\beta_0,\quad C_2=\frac{1}{\min\{1,a_0\}}+\frac{1}{\min\{1,b_0\}}.
$$

Under assumption (1.6), we apply Young's inequality (2.1) and the Sobolev embeddings (Lemma 2.5) to obtain, for a.e. $t\in(0,T)$,

$$
\begin{aligned}
&|\int_\Omega u_t f_1(y,z)dx|\\
&\le (p^+ +1)\Big[a\int_\Omega |u_t||y+z|^{p(x)} + b\int_\Omega |u_t|\,|y|^{\frac{p(x)-1}{2}}|z|^{\frac{p(x)+1}{2}}\Big]\\
&\le (p^+ +1)\Big[\frac{\varepsilon(a+b)}{2}\int_\Omega |u_t|^2 + \frac{2a}{\varepsilon}\int_\Omega |y+z|^{2p(x)} + \frac{2b}{\varepsilon}\int_\Omega |y|^{p(x)-1}|z|^{p(x)+1}\Big]\\
&\le c_1\Big[\frac{\varepsilon}{2}\|u_t\|_2^2 + C_\varepsilon\Big(\int_\Omega |y+z|^{2p^+}dx + \int_\Omega |y+z|^{2p^-}dx\Big)\Big]\\
&\quad + c_1C_\varepsilon\Big(\int_\Omega |y|^{3(p(x)-1)}dx + \int_\Omega |z|^{\frac{3}{2}(p(x)+1)}dx\Big)\\
&\le c_2\Big[\varepsilon\|u_t\|_2^2 + \|\nabla y\|_2^{2p^-} + \|\nabla z\|_2^{2p^-} + \|\nabla y\|_2^{2p^+} + \|\nabla z\|_2^{2p^+}\Big]\\
&\quad + c_2\Big[\|\nabla y\|_2^{3(p^- -1)} + \|\nabla y\|_2^{3(p^+ -1)} + \|\nabla z\|_2^{\frac{3}{2}(p^- +1)} + \|\nabla z\|_2^{\frac{3}{2}(p^+ +1)}\Big],
\end{aligned}
\tag{3.24}
$$

where ε, c_1, c_2 are positive constants. Likewise, we obtain

$$
\begin{aligned}
&|\int_\Omega v_t f_2(y,z)dx|\\
&\le (p^+ +1)\Big[a\int_\Omega |v_t||y+z|^{p(x)} + b\int_\Omega |v_t|.|z|^{\frac{p(x)-1}{2}}|y|^{\frac{p(x)+1}{2}}\Big]\\
&\le c_2\Big[\varepsilon\|v_t\|_2^2 + \|\nabla y\|_2^{2p^-} + \|\nabla z\|_2^{2p^-} + \|\nabla y\|_2^{2p^+} + \|\nabla z\|_2^{2p^+}\Big]\\
&\quad + c_2\Big[\|\nabla z\|_2^{3(p^- -1)} + \|\nabla z\|_2^{3(p^+ -1)} + \|\nabla y\|_2^{\frac{3}{2}(p^- +1)} + \|\nabla y\|_2^{\frac{3}{2}(p^+ +1)}\Big].
\end{aligned}
\tag{3.25}
$$

Combining (3.24) and (3.25), we find

$$
\begin{aligned}
&\sup_{(0,T)}\int_0^t \Big(|\int_\Omega u_t f_1(y,z)dx| + |\int_\Omega v_t f_2(y,z)dx|\Big)ds\\
&\le \varepsilon c_2 T\|(u,v)\|^2_{W_T\times W_T}\\
&\quad + c_2T\Big(\|(y,z)\|^{2p^-}_{W_T\times W_T} + \|(y,z)\|^{2p^+}_{W_T\times W_T} + \|(y,z)\|^{3(p^- -1)}_{W_T\times W_T}\Big)\\
&\quad + c_2T\Big(\|(y,z)\|^{3(p^+ -1)}_{W_T\times W_T} + \|(y,z)\|^{\frac{3}{2}(p^- +1)}_{W_T\times W_T} + \|(y,z)\|^{\frac{3}{2}(p^+ +1)}_{W_T\times W_T}\Big).
\end{aligned}
\tag{3.26}
$$

By substituting (3.26) into (3.23), we obtain, for some $c_3 > 0$,

$$\begin{aligned}
&\frac{1}{2}\|(u,v)\|^2_{W_T\times W_T} \\
&\le \gamma_0 + \varepsilon c_3 T\|(u,v)\|^2_{W_T\times W_T} \\
&\quad + c_3T\Big(\|(y,z)\|^{2p^-}_{W_T\times W_T} + \|(y,z)\|^{2p^+}_{W_T\times W_T} + \|(y,z)\|^{3(p^--1)}_{W_T\times W_T}\Big) \\
&\quad + c_3T\Big(\|(y,z)\|^{3(p^+-1)}_{W_T\times W_T} + \|(y,z)\|^{\frac{3}{2}(p^-+1)}_{W_T\times W_T} + \|(y,z)\|^{\frac{3}{2}(p^++1)}_{W_T\times W_T}\Big).
\end{aligned} \tag{3.27}$$

Choosing ε such that $\varepsilon c_3 T = 1/4$ and recalling that $\|(y,z)\|_{W_T\times W_T} \le d$, for some $d > 1$, inequality (3.27) implies, for some $c_4 > 0$,

$$\begin{aligned}
&\|(u,v)\|^2_{W_T\times W_T} \\
&\le 4\gamma_0 + 4c_4T\Big(\|(y,z)\|^{2p^-}_{W_T\times W_T} + \|(y,z)\|^{2p^+}_{W_T\times W_T} + \|(y,z)\|^{3(p^--1)}_{W_T\times W_T}\Big) \\
&\quad + 4c_4T\Big(\|(y,z)\|^{3(p^+-1)}_{W_T\times W_T} + \|(y,z)\|^{\frac{3}{2}(p^-+1)}_{W_T\times W_T} + \|(y,z)\|^{\frac{3}{2}(p^++1)}_{W_T\times W_T}\Big) \\
&\le 4\gamma_0 + c_4 d^{3(p^+-1)}T \\
&\le 4\gamma_0 + c_4 d^{3(p^+-1)}T_0.
\end{aligned}$$

Hence, if we take (d, T_0) such that $d^2 >> 4\gamma_0$ and $T_0 < \frac{d^2-4\gamma_0}{c_4 d^{3(p^+-1)}}$, we obtain

$$\|(u,v)\|^2_{W_T\times W_T} \le \|(u,v)\|^2_{W_{T_0}\times W_{T_0}} \le d^2,$$

which implies $(u,v) \in B(0,d)$. Thus, $G : B(0,d) \to B(0,d)$.

Next, we show that $G : B(0,d) \to B(0,d)$ is a contraction. For this, let (y_1,z_1) and (y_2,z_2) be in $B(0,d)$ and set $(u_1,v_1) = G(y_1,z_1)$ and $(u_2,v_2) = G(y_2,z_2)$. Then $(u,v) = (u_1-u_2, v_1-v_2)$ is a weak solution of the problem

$$\begin{gathered}
u_{tt} - \operatorname{div}(A\nabla u) + (|u_{1t}|^{m(x)-2}u_{1t} - |u_{2t}|^{m(x)-2}u_{2t}) \\
= f_1(y_1,z_1) - f_1(y_2,z_2) \quad \text{in } \Omega\times(0,T), \\
v_{tt} - \operatorname{div}(B\nabla v) + (|v_{1t}|^{r(x)-2}v_{1t} - |v_{2t}|^{r(x)-2}v_{2t}) \\
= f_2(y_1,z_1) - f_2(y_2,z_2) \quad \text{in } \Omega\times(0,T), \\
u = v = 0 \quad \text{on } \partial\Omega\times(0,T), \\
(u(0),v(0)) = (u_t(0),v_t(0)) = (0,0) \quad \text{in } \Omega,
\end{gathered} \tag{3.28}$$

in the sense of Definition 3.1. Therefore, by taking $\Phi = u_t$ (in Definition 3.1) and integrating the result over $(0,t)$, we obtain, for a.e. $t \le T$,

$$\begin{aligned}
&\frac{d}{dt}\Big[\|u_t\|_2^2 + \int_\Omega A\nabla u\cdot\nabla u\Big] \\
&- \int_\Omega A'\nabla u\cdot\nabla u + 2\int_\Omega u_t\Big(|u_{1t}|^{m(x)-2}u_{1t} - |u_{2t}|^{m(x)-2}u_{2t}\Big) \\
&= 2\int_\Omega u_t(f_1(y_1,z_1) - f_1(y_2,z_2))dx.
\end{aligned} \tag{3.29}$$

Integrating over $(0,t)$ and using the initial conditions, the assumptions (1.7), (1.8) and inequality (3.3), we arrive at

$$\|u_t\|_2^2 + a_0\|\nabla u\|_2^2 \le 2\int_0^t\int_\Omega u_t(f_1(y_1,z_1) - f_1(y_2,z_2))\,dx\,ds,$$

for all $t \in (0,T)$. Consequently,

$$\|u\|_{W_T}^2 \le C \sup_{(0,T)} \int_0^t \int_\Omega |u_t|\, |f_1(y_1,z_1) - f_1(y_2,z_2)|\, dx\, ds, \tag{3.30}$$

where $C = 2/\min\{1, a_0\}$. By repeating the same computations with $\Psi = v_t$, in the second equation in Definition 3.1, we obtain

$$\|v\|_{W_T}^2 \le \mathrm{C} \sup_{(0,T)} \int_0^t \int_\Omega |v_t|\, |f_2(y_1,z_1) - f_2(y_2,z_2)|\, dx\, ds, \tag{3.31}$$

where $C = 2/\min\{1, b_0\}$. Exploiting Young's inequality (2.1), estimates (3.30) and (3.31), we arrive at

$$\begin{aligned}
\|u\|_{W_T}^2 &\le \varepsilon CT \|u\|_{W_T}^2 + C_\varepsilon \sup_{(0,T)} \int_0^t \int_\Omega |f_1(y_1,z_1) - f_1(y_2,z_2)|^2\, dx\, ds,\\
\|v\|_{W_T}^2 &\le \varepsilon CT \|v\|_{W_T}^2 + C_\varepsilon \sup_{(0,T)} \int_0^t \int_\Omega |f_2(y_1,z_1) - f_2(y_2,z_2)|^2\, dx\, ds.
\end{aligned}$$

By addition and choosing ε small enough, we obtain

$$\begin{aligned}
\|(u,v)\|_{W_T \times W_T}^2 \le C_\varepsilon \sup_{(0,T)} \int_0^t \int_\Omega &[|f_1(y_1,z_1) - f_1(y_2,z_2)|^2 \\
&+ |f_2(y_1,z_1) - f_2(y_2,z_2)|^2]\, dx\, ds.
\end{aligned} \tag{3.32}$$

Now, we set $Y = y_1 - y_2$, $Z = z_1 - z_2$ and estimate

$$\int_\Omega |f_1(y_1,z_1) - f_1(y_2,z_2)|^2 dx \quad \text{and} \quad \int_\Omega |f_2(y_1,z_1) - f_2(y_2,z_2)|^2 dx.$$

For this purpose, we recall inequalities (3.16) and (3.17) to obtain the following estimates satisfied by f_1 and f_2, respectively (as in [2]).

$$\begin{aligned}
|f_1(y_1,z_1) - f_1(y_2,z_2)| \le\ & C_4(|y_1 - y_2| + |z_1 - z_2|)\Big(|y_1|^{p(x)-1} + |z_1|^{p(x)-1} \\
& + |y_2|^{p(x)-1} + |z_2|^{p(x)-1}\Big) \\
& + C_5 |z_1 - z_2| . |y_1|^{\frac{p(x)-1}{2}} \Big(|z_1|^{\frac{p(x)-1}{2}} + |z_2|^{\frac{p(x)-1}{2}}\Big) \\
& + C_5 |y_1 - y_2| . |z_2|^{\frac{p(x)+1}{2}} \Big(|y_1|^{\frac{p(x)-3}{2}} + |y_2|^{\frac{p(x)-3}{2}}\Big),
\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}
|f_2(y_1,z_1) - f_2(y_2,z_2)| \le\ & C_4(|y_1 - y_2| + |z_1 - z_2|)\Big(|y_1|^{p(x)-1} + |z_1|^{p(x)-1} \\
& + |y_2|^{p(x)-1} + |z_2|^{p(x)-1}\Big) \\
& + C_5 |y_1 - y_2| . |z_1|^{\frac{p(x)-1}{2}} \Big(|y_1|^{\frac{p(x)-1}{2}} + |y_2|^{\frac{p(x)-1}{2}}\Big) \\
& + C_5 |z_1 - z_2| . |y_2|^{\frac{p(x)+1}{2}} \Big(|z_1|^{\frac{p(x)-3}{2}} + |z_2|^{\frac{p(x)-3}{2}}\Big),
\end{aligned} \tag{3.34}$$

for some constants C_4, $C_5 > 0$ and for almost all $x \in \Omega$ and $t \in (0,T)$. So,

$$\int_\Omega |f_1(y_1,z_1) - f_1(y_2,z_2)|^2 dx \le I_1 + I_2 + I_3 + I_4, \tag{3.35}$$

where

$$\begin{aligned}
I_1 &= C_4 \int_\Omega |y_1 - y_2|^2 (|y_1|^{2(p(x)-1)} + |z_1|^{2(p(x)-1)}) dx \\
&\quad + C_4 \int_\Omega |y_1 - y_2|^2 (|y_2|^{2(p(x)-1)} + |z_2|^{2(p(x)-1)}) dx, \\
I_2 &= C_4 \int_\Omega |z_1 - z_2|^2 (|y_1|^{2(p(x)-1)} + |z_1|^{2(p(x)-1)}) dx \\
&\quad + C_4 \int_\Omega |z_1 - z_2|^2 (|y_2|^{2(p(x)-1)} + |z_2|^{2(p(x)-1)}) dx, \\
I_3 &= C_5 \int_\Omega |z_1 - z_2|^2 |y_1|^{p(x)-1} \Big(|z_1|^{p(x)-1} + |z_2|^{p(x)-1}\Big) dx, \\
I_4 &= C_5 \int_\Omega |y_1 - y_2|^2 |z_2|^{p(x)+1} \Big(|y_1|^{p(x)-3} + |y_2|^{p(x)-3}\Big) dx.
\end{aligned}$$

By using Hölder's and Young's inequalities (Lemmas 2.1 and 2.2) and the Sobolev embeddings (Lemma 2.5), we obtain the following estimate for a typical term in I_1 and I_2,

$$\begin{aligned}
&\int_\Omega |y_1 - y_2|^2 |y_1|^{2(p(x)-1)} dx \\
&\le 2\Big(\int_\Omega |y_1 - y_2|^6 dx\Big)^{1/3} \Big(\int_\Omega |y_1|^{3(p(x)-1)}\Big)^{2/3} \\
&\le C\|y_1 - y_2\|_6^2 \Big[\Big(\int_\Omega |y_1|^{3(p^+-1)} dx\Big)^{2/3} + \Big(\int_\Omega |y_1|^{3(p^--1)} dx\Big)^{2/3}\Big] \\
&\le C\|\nabla(y_1 - y_2)\|_2^2 \big(\|y_1\|_{3(p^+-1)}^{2(p^+-1)} + \|y_1\|_{3(p^--1)}^{2(p^--1)}\big) \\
&\le C\|\nabla Y\|_2^2 \big(\|\nabla y_1\|_2^{2(p^+-1)} + \|\nabla y_1\|_2^{2(p^--1)}\big) \\
&\le C\|\nabla Y\|_2^2 \big(\|(y_1, z_1)\|_{W_T \times W_T}^{2(p^+-1)} + \|(y_1, z_1)\|_{W_T \times W_T}^{2(p^--1)}\big),
\end{aligned} \tag{3.36}$$

since $1 < 3(p^- - 1) \le 3(p^+ - 1) < \infty$, when $n = 1, 2$; and $1 < 3(p^- - 1) = 3(p^+ - 1) = 6 = \frac{2n}{n-2}$, when $n = 3$.

Similarly, we obtain

$$\begin{aligned}
&\int_\Omega |z_1 - z_2|^2 |y_2|^{2(p(x)-1)} dx \\
&\le C\|\nabla Z\|_2^2 \big(\|(y_2, z_2)\|_{W_T \times W_T}^{2(p^+-1)} + \|(y_2, z_2)\|_{W_T \times W_T}^{2(p^--1)}\big).
\end{aligned} \tag{3.37}$$

Since $(y_1, z_1), (y_2, z_2) \in B(0, d)$ and $d > 1$, estimates (3.36) and (3.37) lead to

$$I_1 \le Cd^{2(p^+-1)} \|\nabla Y\|_2^2 \quad \text{and} \quad I_2 \le Cd^{2(p^+-1)} \|\nabla Z\|_2^2.$$

Hence,

$$I_1 + I_2 \le Cd^{2(p^+-1)} (\|\nabla Y\|_2^2 + \|\nabla Z\|_2^2). \tag{3.38}$$

Similarly, a typical term in I_3 can be handled as follows:

$$\begin{aligned}
&\int_\Omega |z_1 - z_2|^2 |y_1|^{p(x)-1} |z_1|^{p(x)-1} dx \\
&\le 2\Big(\int_\Omega |z_1 - z_2|^6 dx\Big)^{1/3} \Big(\int_\Omega |y_1|^{\frac{3}{2}(p(x)-1)} |z_1|^{\frac{3}{2}(p(x)-1)}\Big)^{2/3}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|z_1-z_2\|_6^2\Big[\Big(\int_\Omega |y_1|^{3(p(x)-1)}dx\Big)^{2/3}+\Big(\int_\Omega |z_1|^{3(p(x)-1)}dx\Big)^{2/3}\Big]\\
&\le C\|\nabla(z_1-z_2)\|_2^2\Big(\|y_1\|_{3(p^+-1)}^{2(p^+-1)}+\|y_1\|_{3(p^--1)}^{2(p^--1)}+\|z_1\|_{3(p^+-1)}^{2(p^+-1)}+\|z_1\|_{3(p^--1)}^{2(p^--1)}\Big)\\
&\le C\|\nabla(z_1-z_2)\|_2^2\big(\|\nabla y_1\|_2^{2(p^+-1)}+\|\nabla y_1\|_2^{2(p^--1)}\Big)\\
&+C\|\nabla(z_1-z_2)\|_2^2\big(\|\nabla z_1\|_2^{2(p^+-1)}+\|\nabla z_1\|_2^{2(p^--1)}\Big)\\
&\le 2C\|\nabla Z\|_2^2\Big(\|(y_1,z_1)\|_{W_T\times W_T}^{2(p^+-1)}+\|(y_1,z_1)\|_{W_T\times W_T}^{2(p^--1)}\Big).
\end{aligned}
$$

since $1<3(p^--1)\le 3(p^+-1)<\infty$, when $n=1,2$; and $1<3(p^--1)=3(p^+-1)=6=\frac{2n}{n-2}$, when $n=3$. Therefore,

$$
I_3\le Cd^{2(p^+-1)}\|\nabla Z\|_2^2. \tag{3.39}
$$

Using the same arguments, we estimate a typical term in I_4, as follows:

Case 1. If $n=1,2$, we have $3\le p^-\le p^+<\infty$. So

$$
\begin{aligned}
&\int_\Omega |y_1-y_2|^2|z_2|^{p(x)+1}|y_1|^{p(x)-3}dx\\
&\le 2\Big(\int_\Omega |y_1-y_2|^3dx\Big)^{2/3}\Big(\int_\Omega |z_2|^{3(p(x)+1)}|y_1|^{3(p(x)-3)}\Big)^{1/3}\\
&\le C\|y_1-y_2\|_3^2\Big[\Big(\int_\Omega |z_2|^{6(p(x)+1)}dx\Big)^{1/3}+\Big(\int_\Omega |y_1|^{6(p(x)-3)}dx\Big)^{1/3}\Big]\\
&\le C\|\nabla Y\|_2^2\Big(\|\nabla z_2\|_2^{2(p^++1)}+\|\nabla z_2\|_2^{2(p^-+1)}+\|\nabla y_1\|_2^{2(p^+-3)}+\|\nabla y_1\|_2^{2(p^--3)}\Big)\\
&\le 4Cd^{2(p^++1)}\|\nabla Y\|_2^2,
\end{aligned}
$$

since $(y_1,z_1),(y_2,z_2)\in B(0,d)$ and $d>1$.

Case 2. If $n=3$, then $p\equiv 3$ on $\overline{\Omega}$ and, hence,

$$
\begin{aligned}
\int_\Omega |y_1-y_2|^2|z_2|^{p(x)+1}|y_1|^{p(x)-3}dx&=\int_\Omega |y_1-y_2|^2|z_2|^4dx\\
&\le C\Big(\int_\Omega |y_1-y_2|^6dx\Big)^{1/3}\Big(\int_\Omega |z_2|^6dx\Big)^{2/3}\\
&\le C\|y_1-y_2\|_6^2.\|z_2\|_6^4\\
&\le C\|\nabla Y\|_2^2.\|(y_2,z_2)\|_{W_T\times W_T}^4.
\end{aligned}
$$

We deduce that

$$
I_4\le Cd^{2(p^++1)}\|\nabla Y\|_2^2. \tag{3.40}
$$

Finally, by substituting (3.40), (3.39) and ((3.38) in (3.35), we arrive at

$$
\int_\Omega |f_1(y_1,z_1)-f_1(y_2,z_2)|^2dx\le Cd^{2(p^++1)}(\|\nabla Y\|_2^2+\|\nabla Z\|_2^2), \tag{3.41}
$$

for all $t\in(0,T)$. Similarly, we obtain

$$
\int_\Omega |f_2(y_1,z_1)-f_2(y_2,z_2)|^2dx\le Cd^{2(p^++1)}(\|\nabla Y\|_2^2+\|\nabla Z\|_2^2). \tag{3.42}
$$

Now, we replace (3.41) and (3.42) in (3.32) to obtain

$$\begin{aligned}\|(u,v)\|^2_{W_T\times W_T} &\le C_\varepsilon d^{2(p^++1)} \sup_{(0,T)}\int_0^t \big(\|\nabla Y(s)\|_2^2+\|\nabla Z(s)\|_2^2\big)ds\\ &\le C_\varepsilon d^{2(p^++1)}T\|(Y,Z)\|^2_{W_T\times W_T}\\ &\le \gamma T_0\|(Y,Z)\|^2_{W_T\times W_T},\end{aligned}$$

where $\gamma = C_\varepsilon d^{2(p^++1)}$. So, if we take T_0 small enough, we obtain, for $0<k<1$,

$$\|(u,v)\|^2_{W_T\times W_T}\le k\|(Y,Z)\|^2_{W_T\times W_T}.$$

Thus,

$$\|G(y_1,z_1)-G(y_2,z_2)\|^2_{W_T\times W_T}\le k\|(y_1,z_1)-(y_2,z_2)\|^2_{W_T\times W_T}.$$

This proves that $G: B(0,d)\to B(0,d)$ is a contraction. The Banach-fixed-point theorem guarantees the existence of a unique $(u,v)\in B(0,d)$, such that $G(u,v)=(u,v)$, which is obviously a local weak solution of (1.1).

Uniqueness. Suppose that (1.1) has two solutions (u_1,v_1) and (u_2,v_2), in the sense of Definition 3.1. Therefore, $(u,v)=(u_1-u_2,v_1-v_2)$ satisfies the problem

$$\begin{gathered}u_{tt}-\operatorname{div}(A\nabla u)+(|u_{1t}|^{m(x)-2}u_{1t}-|u_{2t}|^{m(x)-2}u_{2t})\\ =f_1(u_1,v_1)-f_1(u_2,v_2)\quad \text{in } \Omega\times(0,T),\\ v_{tt}-\operatorname{div}(B\nabla v)+(|v_{1t}|^{r(x)-2}v_{1t}-|v_{2t}|^{r(x)-2}v_{2t})\\ =f_2(u_1,v_1)-f_2(u_2,v_2)\quad \text{in } \Omega\times(0,T),\\ u=v=0\quad \text{on } \partial\Omega\times(0,T),\\ (u(0),v(0))=(u_t(0),v_t(0))=(0,0)\quad \text{in } \Omega.\end{gathered}$$

By taking $(\Phi,\Psi)=(u_t,v_t)$ in this definition, integrating each equation over $(0,t)$ $(t\le T)$ and adding the two results, we obtain (as in (3.30) and (3.31)) the following

$$\begin{aligned}\|(u_t,v_t)\|_2^2+\|(\nabla u,\nabla v)\|_2^2 &\le C\int_0^t\int_\Omega |u_t||f_1(u_1,v_1)-f_1(u_2,v_2)|\,dx\,dt\\ &\quad + C\int_0^t\int_\Omega |v_t||f_2(u_1,v_1)-f_2(u_2,v_2)|\,dx\,dt.\end{aligned}$$

Under assumption (1.6) and applying similar arguments as in above, we arrive at

$$\|(u_t,v_t)\|_2^2+\|(\nabla u,\nabla v)\|_2^2\le C_\varepsilon\int_0^t (\|(u_t(s),v_t(s))\|_2^2+\|(\nabla u(s),\nabla v(s))\|_2^2)ds,$$

for all $t\in(0,T)$. Gronwall's lemma leads to

$$\|(u_t,v_t)\|_2^2+\|(\nabla u,\nabla v)\|_2^2=0,\quad \text{for all } t\in(0,T).$$

Thanks to the boundary conditions, we obtain $u=v=0$ on $\Omega\times(0,T)$. This proves the uniqueness of the solution of (1.1). □

Theorem 3.3 is a generalization of the local existence of Agre and Rammaha [2], which dealt with constant exponents, to the situation of variable exponents.

4. Blow up result

To prove our main blow-up result, we give some Lemmas. First we derive the energy functional associated with the problem.

$$\begin{aligned} E(t) = \frac{1}{2}(\|u_t\|_2^2 + \|v_t\|_2^2) + \frac{1}{2}\int_\Omega (A\nabla u\cdot\nabla u + B\nabla v\cdot\nabla v)dx \\ - \int_\Omega F(x,u,v)dx, \end{aligned} \tag{4.1}$$

for all $t \in [0,T)$, and

$$\alpha_1 = (k(p^- + 1))^{\frac{1}{1-p^-}}, \quad E_1 = (\frac{1}{2} - \frac{1}{p^- + 1})\alpha_1^2, \tag{4.2}$$

where

$$k = \big(a2^{\frac{p^-+1}{2}} + 2b\big)\Big(\frac{\tilde{B}^2}{c_0}\Big)^{\frac{p^-+1}{2}}, \quad c_0 = \min\{a_0, b_0\} > 0$$

and $\tilde{B}$ is the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)+1}(\Omega)$.

4.1. **Lemmas.**

Lemma 4.1 ([6]). *The energy functional E is decreasing function and, for a.e. $t \in (0,T)$, we have*

$$\begin{aligned} &E'(t) \\ &= -\int_\Omega |u_t|^{m(x)}dx - \int_\Omega |v_t|^{r(x)}dx + \frac{1}{2}\int_\Omega (A'\nabla u\cdot\nabla u + B'\nabla v\cdot\nabla v)dx. \end{aligned} \tag{4.3}$$

Lemma 4.2. [16] *(1) There exist $C_1, C_2 > 0$ such that, for all $x \in \overline{\Omega}$ and $(u,v) \in \mathbb{R}^2$, we have*

$$C_1\big(|u|^{p(x)+1} + |v|^{p(x)+1}\big) \le F(x,u,v) \le C_2(|u|^{p(x)+1} + |v|^{p(x)+1}). \tag{4.4}$$

(2) For all $x \in \overline{\Omega}$ and $(u,v) \in \mathbb{R}^2$, we have

$$u\ f_1(x,u,v) + vf_2(x,u,v) = (p(x)+1)F(x,u,v). \tag{4.5}$$

Lemma 4.3. *For any solution (u,v) of the system* (1.1)*, with initial energy*

$$E(0) < E_1 \tag{4.6}$$

and

$$\alpha_1 < \Big(\int_\Omega (A\nabla u_0\cdot\nabla u_0 + B\nabla v_0\cdot\nabla v_0)dx\Big)^{1/2} \le (\frac{c_0}{2\tilde{B}^2})^{1/2},$$

there exists $\alpha_2 > \alpha_1$ such that

$$\alpha_2 \le \Big(\int_\Omega (A\nabla u\cdot\nabla u + B\nabla v\cdot\nabla v)dx\Big)^{1/2}, \quad \forall t \in [0,T). \tag{4.7}$$

Proof. From the definition of the energy, it results that

$$E(t) \ge \frac{1}{2}\int_\Omega (A\nabla u\cdot\nabla u + B\nabla v\cdot\nabla v)dx - \int_\Omega F(x,u,v)dx.$$

If we set

$$\alpha = \Big(\int_\Omega (A\nabla u\cdot\nabla u + B\nabla v\cdot\nabla v)dx\Big)^{1/2} \tag{4.8}$$

then

$$E(t) \geq \frac{1}{2}\alpha^2 - \int_\Omega F(x,u,v)dx. \tag{4.9}$$

From (1.7), we have

$$\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \leq \frac{1}{c_0}\int_\Omega (A\nabla u\cdot\nabla u + B\nabla v\cdot\nabla v)dx.$$

So

$$\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \leq \frac{\alpha^2}{c_0}. \tag{4.10}$$

On the other hand, by the definition of F, we have

$$\int_\Omega F(x,u,v) = a\int_\Omega |u+v|^{p(x)+1}dx + 2b\int_\Omega |uv|^{\frac{p(x)+1}{2}}dx.$$

Invoking Lemma 2.3, this leads to

$$\begin{aligned}\int_\Omega F(x,u,v) &\leq a\max\big\{\|u+v\|_{p(\cdot)+1}^{p^-+1}, \|u+v\|_{p(\cdot)+1}^{p^++1}\big\}\\ &\quad + 2b\max\big\{\|uv\|_{\frac{p(\cdot)+1}{2}}^{\frac{p^-+1}{2}}, \|uv\|_{\frac{p(\cdot)+1}{2}}^{\frac{p^++1}{2}}\big\}.\end{aligned} \tag{4.11}$$

First, by the embedding (Lemma 2.5), we have

$$\|u+v\|_{p(\cdot)+1} \leq \tilde{B}\|\nabla(u+v)\|_2 \leq \tilde{B}[(\|\nabla u\|_2 + \|\nabla v\|_2)^2]^{1/2}.$$

Since

$$(X+Y)^\delta \leq 2^{\delta-1}(X^\delta + Y^\delta), \quad \text{for all } X,Y\geq 0 \text{ and } \delta\geq 1, \tag{4.12}$$

it follows that

$$\|u+v\|_{p(\cdot)+1} \leq \tilde{B}[2(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)]^{1/2}.$$

By (4.10),

$$\|u+v\|_{p(\cdot)+1}\Big(\frac{2\tilde{B}^2\alpha^2}{c_0}\Big)^{1/2}.$$

Hence,

$$\|u+v\|_{p(\cdot)+1}^{p^-+1} \leq \Big(\frac{2\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^-+1}{2}}, \quad \|u+v\|_{p(\cdot)+1}^{p^++1} \leq (\frac{2\tilde{B}^2\alpha^2}{c_0})^{\frac{p^++1}{2}}.$$

Therefore,

$$\begin{aligned}&\max\big\{\|u+v\|_{p(\cdot)+1}^{p^-+1}, \|u+v\|_{p(\cdot)+1}^{p^++1}\big\}\\ &\leq \max\Big\{\Big(\frac{2\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^-+1}{2}}, \Big(\frac{2\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^++1}{2}}\Big\}.\end{aligned} \tag{4.13}$$

Likewise, Hölder's and Young's inequalities (Lemmas 2.1 and 2.2) give

$$\|uv\|_{\frac{p(\cdot)+1}{2}} \leq 2\|u\|_{p(\cdot)+1}\|v\|_{p(\cdot)+1} \leq \|u\|_{p(\cdot)+1}^2 + \|v\|_{p(\cdot)+1}^2 \leq \tilde{B}^2(\|\nabla u\|_2^2 + \|\nabla v\|_2^2).$$

Again, by (4.10), we find that

$$\|uv\|_{\frac{p(\cdot)+1}{2}} \leq \frac{\tilde{B}^2\alpha^2}{c_0}.$$

So, we have

$$\|uv\|_{\frac{p(\cdot)+1}{2}}^{\frac{p^-+1}{2}} \leq \Big(\frac{\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^-+1}{2}}, \quad \|uv\|_{\frac{p(\cdot)+1}{2}}^{\frac{p^++1}{2}} \leq \Big(\frac{\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^++1}{2}}.$$

Therefore,

$$\max\Big\{\|uv\|_{\frac{p(\cdot)+1}{2}}^{\frac{p^-+1}{2}}, \|uv\|_{\frac{p(\cdot)+1}{2}}^{\frac{p^++1}{2}}\Big\} \le \max\Big\{\Big(\frac{\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^-+1}{2}}, \Big(\frac{\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^++1}{2}}\Big\}. \tag{4.14}$$

Substituting (4.13) and (4.14) in (4.11), we infer that

$$\begin{aligned} \int_\Omega F(x,u,v) &\le a\max\Big\{\Big(\frac{2\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^-+1}{2}}, \Big(\frac{2\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^++1}{2}}\Big\} \\ &\quad + 2b\max\Big\{\Big(\frac{\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^-+1}{2}}, \Big(\frac{\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^++1}{2}}\Big\}. \end{aligned} \tag{4.15}$$

By inserting (4.15) into (4.9), we obtain

$$E(t) \ge h(\alpha), \quad \text{for all } \alpha \ge 0, \tag{4.16}$$

where

$$\begin{aligned} h(\alpha) := \frac{1}{2}\alpha^2 &- a\max\Big\{\Big(\frac{2\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^-+1}{2}}, \Big(\frac{2\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^++1}{2}}\Big\} \\ &- 2b\max\Big\{\Big(\frac{\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^-+1}{2}}, \Big(\frac{\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^++1}{2}}\Big\}. \end{aligned}$$

For α in $[0, (\frac{c_0}{2\tilde{B}^2})^{1/2}]$, one can easily check that

$$\frac{\tilde{B}^2\alpha^2}{c_0} \le \frac{2\tilde{B}^2\alpha^2}{c_0} \le 1.$$

Consequently,

$$\Big(\frac{2\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^-+1}{2}} \ge \Big(\frac{2\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^++1}{2}}, \quad \Big(\frac{\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^-+1}{2}} \ge \Big(\frac{\tilde{B}^2\alpha^2}{c_0}\Big)^{\frac{p^++1}{2}}.$$

Thus, (4.16) leads to

$$E(t) \ge \frac{1}{2}\alpha^2 - \big(a2^{\frac{p^-+1}{2}} + 2b\big)\Big(\frac{\tilde{B}^2}{c_0}\Big)^{\frac{p^-+1}{2}}\alpha^{p^-+1}.$$

That is,

$$E(t) \ge g(\alpha), \quad \text{for all } \alpha \in \Big[0, \Big(\frac{c_0}{2\tilde{B}^2}\Big)^{1/2}\Big], \tag{4.17}$$

where

$$g(\alpha) = \frac{1}{2}\alpha^2 - k\alpha^{p^-+1}.$$

It is easy to verify that g is strictly increasing on $[0,\alpha_1)$ and strictly decreasing on $[\alpha_1,+\infty)$. Since

$$E(0) < E_1 \quad \text{and} \quad E_1 = g(\alpha_1),$$

then, we can find $\alpha_2 > \alpha_1$ such that $g(\alpha_2) = E(0)$. But

$$\alpha_0 = \Big(\int_\Omega (A\nabla u_0 \cdot \nabla u_0 + B\nabla v_0 \cdot \nabla v_0)dx\Big)^{1/2} \in \Big[\alpha_1, \Big(\frac{c_0}{2\tilde{B}^2}\Big)^{1/2}\Big],$$

therefore, by (4.17), we obtain $g(\alpha_2) = E(0) \ge g(\alpha_0)$.

This implies $\alpha_0 \ge \alpha_2$. Consequently $\alpha_2 \in (\alpha_1, (\frac{c_0}{2\tilde{B}^2})^{1/2}]$.

To establish (4.7), we suppose on the contrary that

$$\Big(\int_\Omega (A\nabla u(.,t^*).\nabla u(.,t^*) + B\nabla v(.,t^*).\nabla v(.,t^*))dx\Big)^{1/2} < \alpha_2,$$

for some $t^* \in [0,T)$. By the continuity of $\left(\int_\Omega A\nabla u \cdot \nabla u + B\nabla v \cdot \nabla v dx\right)^{1/2}$ and since $\alpha_2 > \alpha_1$, we can choose t^* such that

$$\Big[\int_\Omega (A\nabla u(.,t^*).\nabla u(.,t^*) + B\nabla v(.,t^*).\nabla v(.,t^*))dx\Big]^{1/2} > \alpha_1.$$

The g being decreasing of on $[\alpha_1, (\frac{c_0}{2B^2})^{1/2}]$ and (4.17) imply that

$$\begin{aligned} E(t^*) &\geq g\Big(\Big[\int_\Omega (A\nabla u(.,t^*).\nabla u(.,t^*) + B\nabla v(.,t^*).\nabla v(.,t^*))dx\Big]^{1/2}\Big) \\ &> g(\alpha_2) = E(0). \end{aligned}$$

This is impossible since $E(t) \leq E(0)$, for all $t \in [0.T)$. Thus, (4.17) is established. □

Now, we set

$$H(t) = E_1 - E(t), \quad \text{for all } t \in [0,T). \tag{4.18}$$

Lemma 4.4. *We have*

$$0 < H(0) \leq H(t) \leq \int_\Omega F(x,u,v)dx, \quad \textit{for all } t \in [0,T), \tag{4.19}$$

$$\int_\Omega F(x,u,v)dx \geq k\alpha_2^{p^-+1}. \tag{4.20}$$

Proof. From Lemma 4.1 and assumption (4.6), we have

$$0 < E_1 - E(0) = H(0) \leq H(t) \tag{4.21}$$

and by (4.9), we obtain

$$H(t) \leq E_1 - \frac{1}{2}\alpha^2 + \int_\Omega F(x,u,v)dx.$$

Since $E_1 = g(\alpha_1)$ and $\alpha \geq \alpha_2 > \alpha_1$, we obtain

$$\begin{aligned} H(t) &\leq \Big(g(\alpha_1) - \frac{1}{2}\alpha_1^2\Big) + \int_\Omega F(x,u,v)dx \\ &\leq -k\alpha_1^{p^-+1} + \int_\Omega F(x,u,v)dx \\ &\leq \int_\Omega F(x,u,v)dx. \end{aligned}$$

Thus, (4.19) is established. To prove (4.20), we note that E is nonincreasing. Hence,

$$E(0) \geq E(t) \geq \frac{1}{2}\alpha^2 - \int_\Omega F(x,u,v)dx.$$

Consequently,

$$\int_\Omega F(x,u,v)dx \geq \frac{1}{2}\alpha^2 - E(0).$$

But $E(0) = g(\alpha_2)$ and $\alpha \geq \alpha_2$, so

$$\int_\Omega F(x,u,v)dx > \frac{1}{2}\alpha_2^2 - g(\alpha_2) = k\alpha_2^{p^-+1}.$$

□

In what follows and for simplicity, we denote

$$\rho(u)=\int_\Omega |u\ |^{p(x)+1}dx,\quad \rho(v)=\int_\Omega |v\ |^{p(x)+1}dx$$

and we define

$$\Omega_+=\{x\in\Omega:|u(x,t)|\ge 1\},\quad \Omega_-=\{x\in\Omega:|u(x,t)|<1\}.$$

Lemma 4.5. *There exists $C_3>0$ such that any solution of* (1.1) *satisfies*

$$\|u\|_{p^-+1}^{p^-+1}+\|v\|_{p^-+1}^{p^-+1}\le C_3(\rho(u)+\rho(v)). \tag{4.22}$$

Proof. Since $p^-\le p(\cdot)\le p^+$, one easily sees that

$$\begin{aligned}
\rho(u)&=\int_{\Omega_+}|u|^{p(x)+1}dx\ +\int_{\Omega_-}|u|^{p(x)+1}dx\\
&\ge\int_{\Omega_+}|u|^{p^-+1}dx\ +\int_{\Omega_-}|u|^{p^++1}dx\\
&\ge\int_{\Omega_+}|u|^{p^-+1}dx\ +c_1(\int_{\Omega_-}|u|^{p^-+1}dx)^{\frac{p^++1}{p^-+1}},
\end{aligned}$$

for some $c_1>0$. Thus,

$$\rho(u)\ge\int_{\Omega_+}|u|^{p^-+1}dx\quad\text{and}\quad\Big(\frac{\rho(u)}{c_1}\Big)^{\frac{p^-+1}{p^++1}}\ge\int_{\Omega_-}|u|^{p^-+1}dx.$$

By addition, for some $c_2>0$, we obtain

$$\begin{aligned}
\|u\|_{p^-+1}^{p^-+1}&\le\rho(u)+c_2(\rho(u))^{\frac{p^-+1}{p^++1}}\\
&\le\rho(u)+\rho(v)+c_2(\rho(u)+\rho(v))^{\frac{p^-+1}{p^++1}}\\
&=(\rho(u)+\rho(v))\big[1+c_2(\rho(u)+\rho(v))^{\frac{p^--p^+}{p^++1}}\big].
\end{aligned}$$

Recalling (4.19) and (4.4), we infer that

$$0<H(0)\le H(t)\le C_2(\rho(u)+\rho(v)), \tag{4.23}$$

then $\rho(u)+\rho(v)\ge H(0)/C_2$. Therefore,

$$\|u\|_{p^-+1}^{p^-+1}\le(\rho(u)+\rho(v))\big[1+c_2(H(0)/C_2)^{\frac{p^--p^+}{p^++1}}\big].$$

Hence

$$\|u\|_{p^-+1}^{p^-+1}\le c_3(\rho(u)+\rho(v)),$$

where $c_3=1+c_2(H(0)/C_2)^{\frac{p^--p^+}{p^++1}}>0$. Similarly, we arrive at

$$\|v\|_{p^-+1}^{p^-+1}\le c_3(\rho(u)+\rho(v)).$$

Therefore, (4.22) is satisfied with $C_3=2c_3$. □

Corollary 4.6. *There exist constants $C_4,C_5>0$ such that*

$$\int_\Omega|u|^{m(x)}dx\le C_4\big[(\rho(u)+\rho(v))^{\frac{m^+}{p^-+1}}+(\rho(u)+\rho(v))^{\frac{m^-}{p^-+1}}\big], \tag{4.24}$$

$$\int_\Omega|v|^{r(x)}dx\le C_5\big[(\rho(u)+\rho(v))^{\frac{r^+}{p^-+1}}+(\rho(u)+\rho(v))^{\frac{r^-}{p^-+1}}\big]. \tag{4.25}$$

Proof. Since $p^- \geq \max\{m^+, r^+\}$, it follows that

$$\begin{aligned}
\int_\Omega |u|^{m(x)}dx &\leq \int_{\Omega_+} |u|^{m^+}dx \ + \int_{\Omega_-} |u|^{m^-}dx \\
&\leq c_1\Big(\int_{\Omega_+} |u|^{p^-+1}dx\Big)^{\frac{m^+}{p^-+1}} + c_1\Big(\int_{\Omega_-} |u|^{p^-+1}dx\Big)^{\frac{m^-}{p^-+1}} \\
&\leq c_1\Big(\|u\|_{p^-+1}^{m^+} + \|u\|_{p^-+1}^{m^-}\Big), \quad c_1 > 0.
\end{aligned}$$

Recalling Lemma 4.5, we obtain, for a constant $C_4 > 0$,

$$\int_\Omega |u|^{m(x)}dx \leq C_4\Big[(\rho(u)+\rho(v))^{\frac{m^+}{p^-+1}} + (\rho(u)+\rho(v))^{\frac{m^-}{p^-+1}}\Big].$$

Similarly, we obtain, for some $C_5 > 0$,

$$\int_\Omega |v|^{r(x)}dx \leq C_5\Big[(\rho(u)+\rho(v))^{\frac{r^+}{p^-+1}} + (\rho(u)+\rho(v))^{\frac{r^-}{p^-+1}}\Big].$$

Thus, (4.23) and (4.24) are proved. □

4.2. **Main result.** Now, we state and prove our main blow-up result.

Theorem 4.7. *Let the assumptions given in Subsection 4.1 hold. Then any solution of the system* (1.1) *blows up in finite time.*

Proof. We assume that the solution exists for any $t > 0$ and reach to a contradiction. This will be established in 4 steps.

Step 1. For small $\varepsilon > 0$ to be fixed later, we define the auxiliary functional

$$G(t) = H^{1-\sigma}(t) + \varepsilon\int_\Omega (uu_t + vv_t)dx, t > 0,$$

where

$$0 < \sigma \leq \min\Big\{\frac{p^- - m^+ + 1}{(p^-+1)(m^+-1)}, \frac{p^- - r^+ + 1}{(p^-+1)(r^+-1)}, \frac{p^- - 1}{2(p^-+1)}\Big\}. \tag{4.26}$$

Our goal is to show that G satisfies a differential inequality which leads to a blow up in finite time. Now, we have

$$\begin{aligned}
G'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2) \\
&\quad + \varepsilon\int_\Omega (uf_1(x,u,v) + vf_2(x,u,v))dx \\
&\quad - \varepsilon\int_\Omega (A\nabla u\cdot\nabla u + B\nabla v\cdot\nabla v)dx \\
&\quad - \varepsilon\int_\Omega (|u_t|^{m(x)-2}u_tu + |v_t|^{r(x)-2}v_tv)dx.
\end{aligned} \tag{4.27}$$

From Lemma 4.2, it follows that

$$\begin{aligned}
\int_\Omega (uf_1(x,u,v) + vf_2(x,u,v))dx &= \int_\Omega (p(x)+1)F(x,u,v)dx \\
&\geq (p^-+1)\int_\Omega F(x,u,v)dx.
\end{aligned} \tag{4.28}$$

By the definition of H and E, we obtain

$$\begin{aligned} &\int_\Omega (A\nabla u\cdot\nabla u + B\nabla v\cdot\nabla v)dx \\ &= 2\int_\Omega F(x,u,v)dx - \|u_t\|_2^2 - \|v_t\|_2^2 + 2E_1 - 2H(t). \end{aligned} \tag{4.29}$$

If we insert (4.28) and (4.29) into (4.27), we then obtain

$$\begin{aligned} G'(t) &\ge (1-\sigma)H^{-\sigma}(t)H'(t) + 2\varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2) + 2\varepsilon H(t) \\ &\quad - 2\varepsilon E_1 + \varepsilon(p^- - 1)\int_\Omega F(x,u,v)dx \\ &\quad - \varepsilon\int_\Omega (|u||u_t|^{m(x)-1} + |v||v_t|^{r(x)-1})dx. \end{aligned} \tag{4.30}$$

Using (4.20), we have

$$E_1 \le (k\alpha_2^{p^-+1})^{-1}E_1\int_\Omega F(x,u,v)dx.$$

Hence, (4.30) becomes

$$\begin{aligned} &G'(t) \\ &\ge (1-\sigma)H^{-\sigma}(t)H'(t) + 2\varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2) + \varepsilon c_1\int_\Omega F(x,u,v)dx \\ &\quad + 2\varepsilon H(t) - \varepsilon\int_\Omega (|u||u_t|^{m(x)-1} + |v|\,|v_t|^{r(x)-1})dx, \end{aligned} \tag{4.31}$$

where $c_1 = p^- - 1 - 2(k\alpha_2^{p^-+1})^{-1}E_1 > 0$, since $\alpha_2 > \alpha_1$.

Step 2. In this step, we estimate the last two terms in the right-hand side of (4.31). We set

$$I_1 := \int_\Omega |u||u_t|^{m(x)-1}dx, \quad I_2 := \int_\Omega |v||v_t|^{r(x)-1}dx$$

and apply the Young inequality

$$XY \le \frac{\delta^\lambda}{\lambda}X^\lambda + \frac{\delta^{-\beta}}{\beta}Y^\beta, \quad \text{for all } X,\ Y \ge 0, \delta > 0 \text{ and } \frac{1}{\lambda} + \frac{1}{\beta} = 1,$$

with

$$X = |u|, \quad Y = |u_t|^{m(x)-1}, \quad \lambda = m(x), \quad \beta = \frac{m(x)}{m(x)-1}, \quad \delta > 0,$$

to obtain

$$I_1 \le \int_\Omega \frac{\delta^{m(x)}}{m(x)}|u|^{m(x)}dx + \int_\Omega \frac{m(x)-1}{m(x)}\delta^{-m(x)/(m(x)-1)}|u_t|^{m(x)}dx. \tag{4.32}$$

By taking

$$\delta = [KH^{-\sigma}(t)]^{\frac{1-m(x)}{m(x)}},$$

where K is a large constant to be chosen later, we obtain

$$\begin{aligned} I_1 &\le \frac{K^{1-m^-}}{m^-}\int_\Omega [H(t)]^{\sigma(m(x)-1)}|u|^{m(x)}dx \\ &\quad + \frac{m^+ - 1}{m^-}KH^{-\sigma}(t)\int_\Omega |u_t|^{m(x)}dx. \end{aligned} \tag{4.33}$$

From Lemma 4.1, we have

$$H'(t) = \int_\Omega |u_t|^{m(x)}dx + \int_\Omega |v_t|^{r(x)}dx - \frac{1}{2}\int_\Omega (A'\nabla u\cdot\nabla u + B'\nabla v\cdot\nabla v)dx$$

and by (1.8)), we obtain

$$\int_\Omega |u_t|^{m(x)}dx \le H'(t). \tag{4.34}$$

On the other hand, since $m(x) \le m^+$ and $H(t) \ge H(0) > 0$, one has

$$\begin{aligned}\int_\Omega [H(t)]^{\sigma(m(x)-1)}|u|^{m(x)}dx &= \int_\Omega \big[\frac{H(t)}{H(0)}\big]^{\sigma(m(x)-1)}[H(0)]^{\sigma(m(x)-1)}|u|^{m(x)}dx\\ &\le c_2[H(t)]^{\sigma(m^+-1)}\int_\Omega [H(0)]^{\sigma(m(x)-1)}|u|^{m(x)}dx,\end{aligned}$$

where $c_2 = 1/[H(0)]^{\sigma(m^+-1)}$. But $[H(0)]^{\sigma(m(x)-1)} \le c_3$ for all $x \in \Omega$, where $c_3 > 0$. So, for a constant $c_4 > 0$, we obtain

$$\int_\Omega [H(t)]^{\sigma(m(x)-1)}|u|^{m(x)}dx \le c_4[H(t)]^{\sigma(m^+-1)}\int_\Omega |u|^{m(x)}dx. \tag{4.35}$$

Replacing (4.35) and (4.34)) in (4.33), we infer that

$$\begin{aligned} I_1 \le& \frac{K^{1-m^-}}{m^-}c_4[H(t)]^{\sigma(m^+-1)}\int_\Omega |u|^{m(x)}dx\\ &+ \frac{m^+-1}{m^-}KH^{-\sigma}(t)H'(t).\end{aligned} \tag{4.36}$$

Likewise, we obtain, for some $c_5 > 0$,

$$I_2 \le \frac{K^{1-r^-}}{r^-}c_5[H(t)]^{\sigma(r^+-1)}\int_\Omega |v|^{r(x)}dx + \frac{r^+-1}{r^-}KH^{-\sigma}(t)H'(t). \tag{4.37}$$

Also, from (4.19), we have, for some $c_6 > 0$,

$$[H(t)]^{\sigma(m^+-1)} \le c_6(\rho(u)+\rho(v))^{\sigma(m^+-1)}.$$

This inequality and estimate (4.24) imply that for some $c_7 > 0$,

$$\begin{aligned}&[H(t)]^{\sigma(m^+-1)}\int_\Omega |u|^{m(x)}dx\\ &\le c_7(\rho(u)+\rho(v))^{\sigma(m^+-1)+\frac{m^+}{p^-+1}} + c_7(\rho(u)+\rho(v))^{\sigma(m^+-1)+\frac{m^-}{p^-+1}}.\end{aligned} \tag{4.38}$$

Now, if we use (4.26) and the algebraic inequality

$$z^\tau \le z+1 \le (1+\frac{1}{a})(z+a), \quad \text{for all } z \ge 0,\ 0<\tau\le 1 \text{ and } a>0, \tag{4.39}$$

with

$$z = \rho(u)+\rho(v), \quad a = H(0), \quad \tau = \sigma(m^+-1) + \frac{m^+}{p^-+1}$$

and then with $\tau = \sigma(m^+-1) + \frac{m^-}{p^-+1}$, respectively, we obtain

$$\begin{aligned}(\rho(u)+\rho(v))^{\sigma(m^+-1)+\frac{m^+}{p^-+1}} &\le [1+\frac{1}{H(0)}](\rho(u)+\rho(v)+H(0))\\ &\le \gamma(\rho(u)+\rho(v)+H(t))\end{aligned} \tag{4.40}$$

and

$$(\rho(u)+\rho(v))^{\sigma(m^+-1)+\frac{m^-}{p^-+1}} \le \gamma(\rho(u)+\rho(v)+H(t)) \tag{4.41}$$

where $\gamma = 1+\frac{1}{H(0)}$. Combining (4.41) and (4.40) with (4.38), we obtain that for some $c_8>0$,

$$[H(t)]^{\sigma(m^+-1)} \int_\Omega |u|^{m(x)}dx \le c_8(\rho(u)+\rho(v)+H(t)). \tag{4.42}$$

Similarly, we have for some $c_9>0$,

$$[H(t)]^{\sigma(r^+-1)} \int_\Omega |v|^{r(x)}dx \le c_9(\rho(u)+\rho(v)+H(t)). \tag{4.43}$$

Substituting (4.42) into (4.36), we find that

$$I_1 \le \frac{K^{1-m^-}}{m^-}c_{10}(\rho(u)+\rho(v)+H(t)) + \frac{m^+-1}{m^-}KH^{-\sigma}(t)H'(t), \tag{4.44}$$

and substituting (4.43) into (4.37), we obtain

$$I_2 \le \frac{K^{1-r^-}}{r^-}c_{11}(\rho(u)+\rho(v)+H(t)) + \frac{r^+-1}{r^-}KH^{-\sigma}(t)H'(t), \tag{4.45}$$

where c_{10} and c_{11} are two positive constants.

Step 3. Now, we estimate G'. By inserting (4.44) and (4.45) into (4.31), we arrive at

$$\begin{aligned} G'(t) &\ge (1-\sigma-\varepsilon M)H^{-\sigma}(t)H'(t) + 2\varepsilon(\|u_t\|_2^2+\|v_t\|_2^2) + 2\varepsilon H(t) \\ &\quad + c_{12}(\rho(u)+\rho(v)) - \varepsilon\frac{K^{1-m^-}}{m^-}\ c_{10}(\rho(u)+\rho(v)+H(t)) \\ &\quad - \varepsilon\frac{K^{1-r^-}}{r^-}c_{11}(\rho(v)+\rho(u)+H(t)), \end{aligned}$$

where $c_{12}>0$ and $M = K(\frac{m^+-1}{m^-}+\frac{r^+-1}{r^-})$. Therefore,

$$\begin{aligned} G'(t) &\ge (1-\sigma-\varepsilon M)H^{-\sigma}(t)H'(t) + 2\varepsilon(\|u_t\|_2^2+\|v_t\|_2^2) \\ &\quad + \varepsilon\Big(2 - \frac{K^{1-m^-}}{m^-}c_{10} - \frac{K^{1-r^-}}{r^-}c_{11}\Big)H(t) \\ &\quad + \varepsilon\Big(c_{12} - \frac{K^{1-m^-}}{m^-}c_{10} - \frac{K^{1-r^-}}{r^-}c_{11}\Big)(\rho(u)+\rho(v)). \end{aligned}$$

For a large value of K, we can find $c_{13}>0$ such that

$$\begin{aligned} G'(t) &\ge (1-\sigma-\varepsilon M)H^{-\sigma}(t)H'(t) \\ &\quad + \varepsilon c_{13}(\|u_t\|_2^2+\|v_t\|_2^2+H(t)+\rho(u)+\rho(v)). \end{aligned}$$

Once K is fixed, we select ε small enough so that

$$1-\sigma-\varepsilon M \ge 0 \text{ and } G(0) = H^{1-\sigma}(0) + \varepsilon\int_\Omega (u_0u_1+v_0v_1)dx > 0.$$

From Lemma 4.1, we have $H'(t)\ge 0$. Therefore, there exists $h>0$ satisfying

$$G'(t) \ge \varepsilon h(H(t)+\|u_t\|_2^2+\|v_t\|_2^2+\rho(u)+\rho(v)). \tag{4.46}$$

Consequently,

$$G(t) \ge G(0) > 0, \quad \text{for } t>0.$$

Step 4. Finally, we complete the proof of the blow up result. By the definition of G and using (4.12), it follows that

$$\begin{aligned} G^{1/(1-\sigma)}(t) &\le \Big(H^{1-\sigma}(t)+\varepsilon\int_\Omega |uu_t+vv_t|dx\Big)^{1/(1-\sigma)} \\ &\le 2^{\sigma/(1-\sigma)}\Big(H(t)+(\varepsilon\int_\Omega(|uu_t|+|vv_t|)dx)^{1/(1-\sigma)}\Big) \\ &\le c_{14}\Big(H(t)+(\int_\Omega(|u||u_t|+|v||v_t|)dx)^{1/(1-\sigma)}\Big), \end{aligned}$$

where $c_{14}=2^{\sigma/(1-\sigma)}\max\{1,\varepsilon^{1/(1-\sigma)}\}$. Also, we have

$$\begin{aligned} &\Big(\int_\Omega(|u||u_t|+|v||v_t|)dx\Big)^{1/(1-\sigma)} \\ &\le 2^{\sigma/(1-\sigma)}\Big(\int_\Omega|u\|u_t|dx\Big)^{1/(1-\sigma)}+2^{\sigma/(1-\sigma)}\Big(\int_\Omega|v||v_t|dx\Big)^{1/(1-\sigma)}. \end{aligned} \tag{4.47}$$

Since $p^-\ge 2$, Hölder's and Young's inequalities yield, for $c_{15},c_{16}>0$,

$$\begin{aligned} \Big(\int_\Omega|u\|u_t|dx\Big)^{1/(1-\sigma)} &\le \|u\|_2^{1/(1-\sigma)}\|u_t\|_2^{1/(1-\sigma)} \\ &\le c_{15}\|u\|_{p^-+1}^{1/(1-\sigma)}\|u_t\|_2^{1/(1-\sigma)} \\ &\le c_{16}(\|u\|_{p^-+1}^{\mu/(1-\sigma)}+\|u_t\|_2^{\beta/(1-\sigma)}), \end{aligned}$$

where $\frac{1}{\mu}+\frac{1}{\beta}=1$. If we set $\beta=2(1-\sigma)$, we obtain $\mu/(1-\sigma)=2/(1-2\sigma)$. Hence,

$$\Big(\int_\Omega|u\|u_t|dx\Big)^{1/(1-\sigma)}\le c_{16}(\|u\|_{p^-+1}^{2/(1-2\sigma)}+\|u_t\|_2^2). \tag{4.48}$$

By Lemma 4.4, estimate (4.48) becomes

$$\Big(\int_\Omega|u\|u_t|dx\Big)^{1/(1-\sigma)}\le c_{17}((\rho(u)+\rho(v))^\tau+\|u_t\|_2^2),$$

where $c_{17}>0$ and $\tau=2/(p^-+1)(1-2\sigma)$. Again, by (4.26), (4.39) and since $\tau\le 1$, we obtain, for some $c_{18}>0$,

$$\Big(\int_\Omega|u||u_t|dx\Big)^{1/(1-\sigma)}\le c_{18}(\rho(u)+\rho(v)+H(t)+\|u_t\|_2^2). \tag{4.49}$$

Similar computations lead to

$$(\int_\Omega|v||v_t|dx\Big)^{1/(1-\sigma)}\le c_{18}(\rho(u)+\rho(v)+H(t)+\|v_t\|_2^2). \tag{4.50}$$

By adding (4.49) and (4.50), estimate (4.47) yields, for some $c_{19}>0$,

$$\Big(\int_\Omega(|u||u_t|+|v||v_t|)dx\Big)^{1/(1-\sigma)}\le c_{19}\Big(\rho(u)+\rho(v)+\|u_t\|_2^2+\|v_t\|_2^2+H(t)\Big).$$

Therefore, for some $c_{20}>0$, we arrive at

$$G^{1/(1-\sigma)}(t)\le c_{20}(\rho(u)+\rho(v)+H(t)+\|u_t\|_2^2+\|v_t\|_2^2). \tag{4.51}$$

Combining (4.51) and (4.46), we deduce that

$$G'(t)\ge \Gamma G^{1/(1-\sigma)}(t),\ \text{for all } t>0,$$

where $\Gamma = \frac{\varepsilon h}{c_{20}}$. A simple integration over $(0,t)$ yields

$$G^{\sigma/(1-\sigma)}(t) \geq \frac{1}{G^{\frac{-\sigma}{1-\sigma}}(0) - \frac{\sigma\Gamma t}{1-\sigma}},$$

which implies that $G(t) \to +\infty$, as $t \to T^*$, where $T^* \leq \frac{1-\sigma}{\sigma\Gamma[G^{\frac{\sigma}{(1-\sigma)}}(0)]}$. Consequently, the solution of problem (1.1) blows up in finite time. □

5. Numerical tests

In this section, we show some numerical experiments to illustrate the theoretical results in Theorem 4.7. We solve the system (1.1) under specific initial data and Dirichlet boundary conditions. We use a numerical scheme based on the finite element method in space and the Newmark method in time [24, 23].

We consider problem (1.1) in two space-dimensions and take the functions m, r and p fulfilling the assumptions (1.2), (1.3) and (1.6). Precisely, we have

$$m(x,y) = 2 + \frac{1}{1+x^2}, \quad r(x,y) = 2 + \frac{1}{1+y^2}, \quad p(x,y) = 3 + \frac{2}{1+x^2+y^2},$$

and the source terms are given by (1.4) and (1.5) with $a = b = 1$. Whereas, the matrices A and B are given as follows

$$A = (1+e^{-t})\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = (1 + \frac{1}{1+t})\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}.$$

Test 1. We consider the circular domain $\Omega_1 = \{(x,y) : x^2 + y^2 < 1\}$ with a triangulation discretization (see the mesh-grid in Figure 1) which consists of 281 triangles and 162 degrees of freedoms [20] and use the initial conditions

$$u_0(x,y) = 2(1-x^2-y^2), \quad v_0(x,y) = 3(1-x^2-y^2), \quad u_1 = v_1 = 0.$$

We run our code with a time step $\Delta t = 10^{-3}$, which is small enough to catch the below-up behavior.

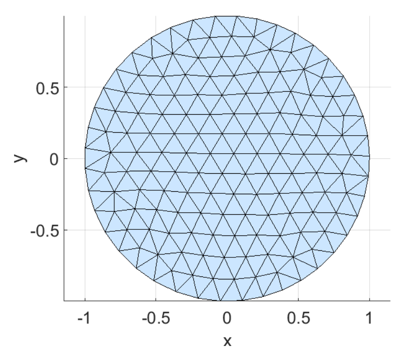
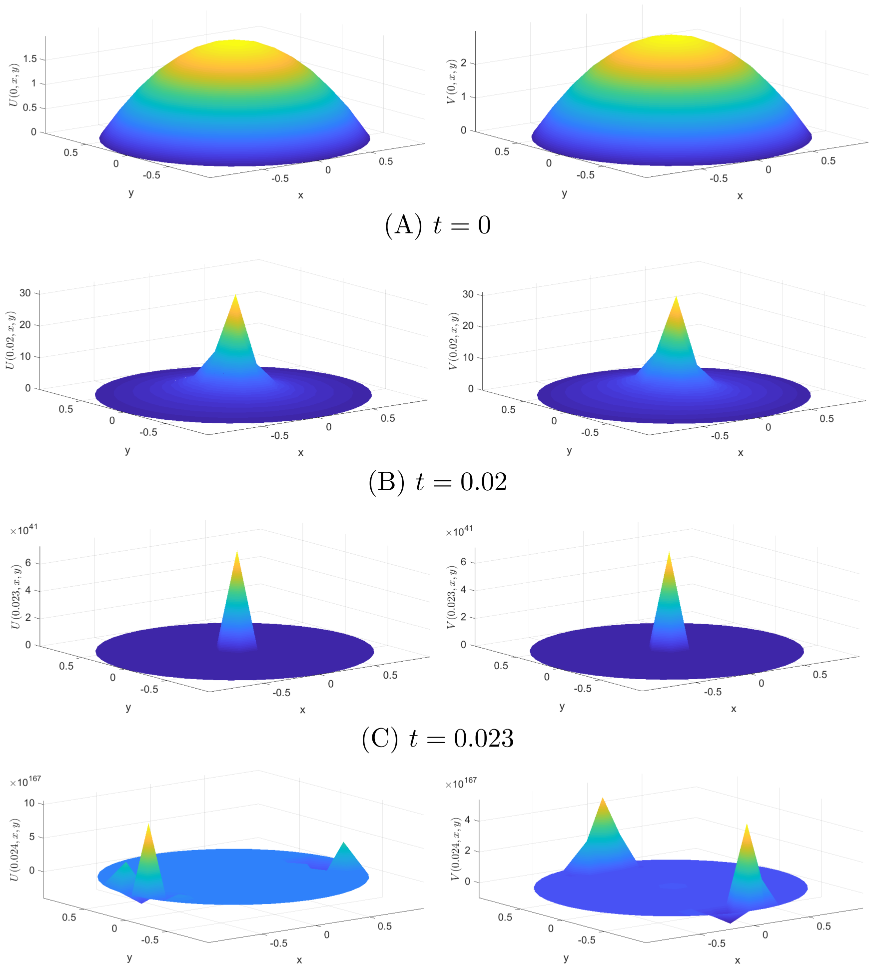


Figure 1. Uniform mesh grid of Ω_1.

Figure 2 shows the approximate numerical results of the solution (u,v) at different time iterations $t = 0$, $t = 0.02$, $t = 0.023$, and $t = 0.024$, where the left column shows the approximate values of u and the right column shows the approximate values of v. Note that the blow-up is occurring at instant $t = 0.024$.



(A) $t = 0$

(B) $t = 0.02$

(C) $t = 0.023$

(D) $t = 0.024$

FIGURE 2. Numerical results of Test 1 at different times.

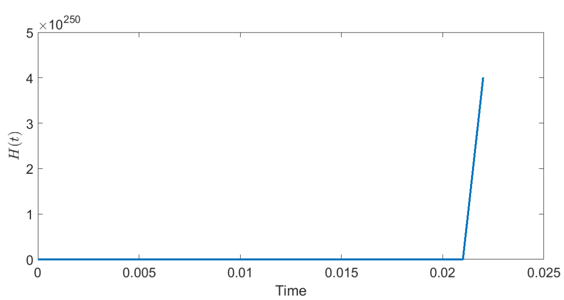


FIGURE 3. Test 1: Blow-up of H in finite time.

Figure 3 presents the numerical values of the functional $H(t)$ defined by (4.18) during the time iterations. It shows the blow-up of the energy of system (1.1).

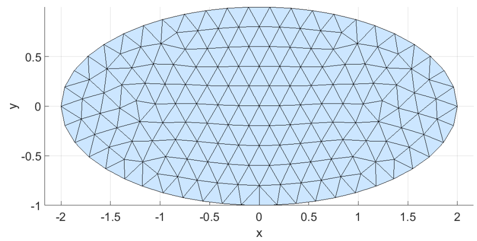
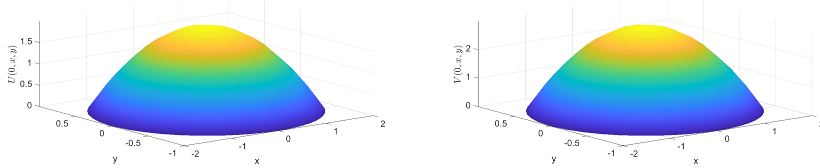
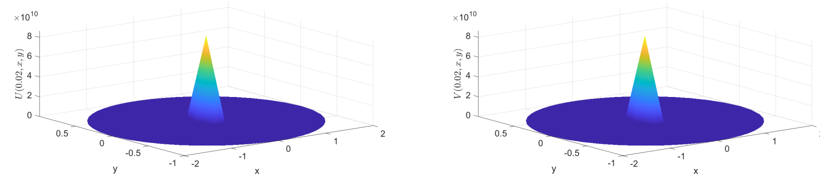


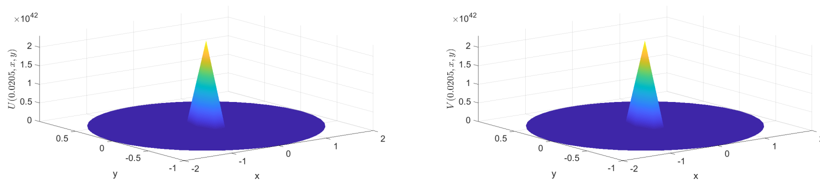
FIGURE 4. Uniform mesh grid of Ω_2.



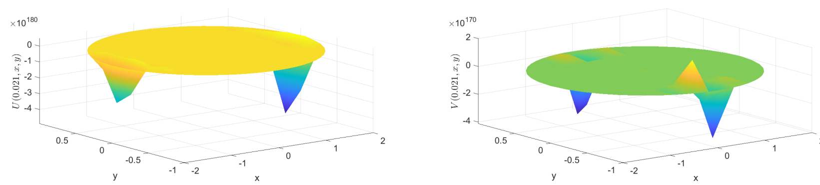
(A) $t = 0$



(B) $t = 0.02$



(C) $t = 0.0205$



(D) $t = 0.021$

FIGURE 5. Numerical results of Test 2 at different times.

Test 2. We consider the elliptical domain $\Omega_2 = \{(x, y) : \frac{x^2}{4} + y^2 < 1\}$ with a triangulation discretization (see the mesh-grid in Figure 4) which consists of 311 triangles and 180 degrees of freedom [20] and take the initial conditions

$$u_0(x, y) = 2(1 - \frac{x^2}{4} - y^2), \quad v_0(x, y) = 3(1 - \frac{x^2}{4} - y^2), \quad u_1 = v_1 = 0.$$

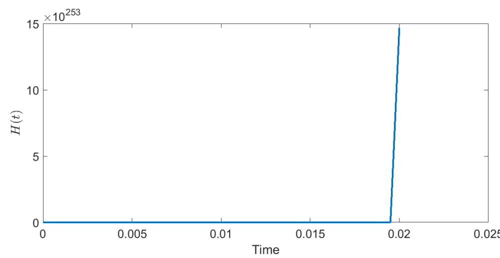


FIGURE 6. Test 2: Blow-up of H in finite time.

We run our code with a time step $\Delta t = 5 \cdot 10^{-4}$, which is small enough to catch the below-up behavior.

In Figure 5, we show the approximate numerical results of the solution (u, v) at different time iterations $t = 0$, $t = 0.02$, $t = 0.0205$ and $t = 0.021$, where the left column shows the approximate values of u and the right column shows the approximate values of v. Note that the blow-up takes place at instant $t = 0.021$.

For Test 2, the numerical values of the functional $H(t)$ are presented in Figure 6. Observe the blow-up of the energy from $t = 0.02$.

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