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POISSON MEASURES ON SEMI-DIRECT PRODUCTS OF INFINITE-DIMENSIONAL HILBERT SPACES

RICHARD C. PENNEY, ROMAN URBAN

Abstract. Let $G = X \rtimes A$ where X and A are Hilbert spaces considered as additive groups and the A-action on G is diagonal in some orthonormal basis. We consider a particular second order left-invariant differential operator $\mathcal{L}$ on G which is analogous to the Laplacian on $\mathbb{R}^n$. We prove the existence of "heat kernel" for $\mathcal{L}$ and give a probabilistic formula for it. We then prove that X is a "Poisson boundary" in a sense of Furstenberg for $\mathcal{L}$ with a (not necessarily) probabilistic measure ν on X called the "Poisson measure" for the operator $\mathcal{L}$.

1. Introduction

In recent years, there has been considerable interest in the study of second order differential equations on infinite dimensional Hilbert spaces and their generalizations. See for example Da Prato [3, 7] and the references contained therein. In this work we study a specific class of second order linear differential equations, the "Laplacians", on a very specific set of Hilbert spaces, the meta-abelian solvmanifolds. (See (1.1) below.)

The form of the differential operator that we are considering in this work, as well as the form of the algebraic structure of the space on which this operator acts has its origin in the analysis on Lie groups [18, 21]. In a sense, the context considered here can be treated as a generalisation of the (finite dimensional) setting considered in a series of papers on estimates of Poisson kernels for the second-order left-invariant differential operators on NA Lie groups, i.e. on the semi-direct products of nilpotent and Abelian Lie groups $A = \mathbb{R}^D$, $D \geq 1$ (see [4, 5, 6, 10, 11, 12, 13, 14, 15, 16]).

All these analytical problems have their source in probabilistic considerations (see e.g. [2, 8, 17]) of harmonic (with respect to a probability measure) functions on groups.

Our techniques are probabilistic and are generalizations of known results in the finite dimensional Lie group case. The fact that these techniques are useful in this context is in itself an interesting result. More specifically, for $p > 0$, and $d \in \mathbb{N} \cup \{\infty\}$, let

$$\ell_p^d = \begin{cases} \ell_p(\{1, \dots, d\}) & \text{if } d < \infty, \\ \ell_p(\mathbb{N}) & \text{if } d = \infty \end{cases}$$

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with respect to the counting measure. We identify ℓ_p^d with the subspace of ℓ_p^∞ of elements supported in $\{1,\dots,d\}$. We often omit d in our notation so that ℓ_p denotes ℓ_p^d for some fixed d. We let ℓ_p^+ be the set of elements of ℓ_p with positive entries and $\overline{\ell}_p^+$ the set of elements with non-negative entries.

For $u \in \mathbb{R}^d$, $d \in \mathbb{N} \cup \{\infty\}$, we let $[u] = \operatorname{diag} u$ be the corresponding $d \times d$ diagonal matrix.

Let $A = X = \ell_2$ considered as Abelian groups. The general element of A and X are denoted respectively by

$$a = (a_1, a_2, \dots),$$
$$x = (x_1, x_2, \dots).$$

We define an action of A on X by

$$x^a = \operatorname{Ad}_a x = e^{\operatorname{ad}(a)}(x) = (e^{\lambda_1(a)}x_1, \dots, e^{\lambda_d(a)}x_n, \dots),$$

where $\lambda_i : A \to \mathbb{R}$ is given by

$$\lambda_i(a) = a_i.$$

Recall that $\ell_2 \subset \ell_\infty$, so that for $a \in \ell_2$, both a and e^a are bounded. Then for a and x in ℓ_2, the x^a defined above is also in ℓ_2.

We consider the corresponding semi-direct product $G = X \rtimes A$ which is $X \times A$ with the product

$$(x,a)(y,b) = (x + y^a, a + b). \tag{1.1}$$

For $g \in G$ we let $x(g) = x \in X$ and $a(g) = a \in A$ denote the components of g in this product so that $g = (x,a)$.

Let $q, \beta \in \ell_2^+ \cap \ell_1$ and let $\alpha \in \overline{\ell}_2^+$. We consider the differential operators

$$\mathcal{L} = \Delta_\alpha + \mathcal{L}_\beta^a, \tag{1.2}$$

where

$$\mathcal{L}_\beta^a = \sum_j e^{2\lambda_j(a)} \beta_j \partial_{x_j}^2 \quad \text{and} \quad \Delta_\alpha = \Delta_\alpha^q = \sum_j q_j \partial_{a_j}^2 - \sum_j 2\alpha_j \partial_{a_j}.$$

We do not typically carry q in our notation as we consider it fixed. However, when necessary, we will indicate the dependence with a superscript q.

Following Da Prato and Zabczyk [7], we consider our operators as densely defined operators on the respective spaces $UC_b(G)$, $UC_b(A)$, and $UC_b(X)$ of uniformly continuous, bounded functions on the given space, depending on context. We also let $C_b(\cdot)$ denote the space of bounded continuous functions on the given space.

We require some technical assumptions on the growth rates of the coefficients. Explicitly, we assume that there is a constant $c > 0$ such that, for all $j \geq 1$,

$$\frac{2\alpha_j}{\beta_j} > c. \tag{1.3}$$

We assume additionally that there is a $\gamma \in \ell_2$ such that

$$[q]\gamma = \alpha. \tag{1.4}$$

Recall that $[q]$ is a matrix whose diagonal entries are equal to q.

For $d \in \mathbb{N}$ we let the superscript d denote the corresponding sequences obtained by considering only elements of $\ell_p^d \subset \ell_p$.

Our main results are in Theorem 5.1, which firstly proves the existence of a heat semigroup $P_t^{\mathcal{L}}$ for $\mathcal{L}$, and secondly provides a probabilistic formula for the semigroup. This formula is based on the existence of a diffusion kernel for $\mathcal{L}_\beta^{\sigma_t}$ (σ_t is a continuous trajectory of the process generated by Δ_α) which provides the unique solution to system (4.1) below, which is our second main result.

Finally, we prove that X is a "Poisson boundary" for $\mathcal{L}$ in the sense that there is a "Poisson measure" ν on X for $\mathcal{L}$. That is to say, there exists a probability measure ν on X such that

$$\mu_t^{\mathcal{L}} * \nu = \nu,$$

where $\mu_t^{\mathcal{L}}$ is a semi-group of probability measures generated by $\mathcal{L}$ and the convolution is defined by the action of A on X, i.e. if λ is a probability measure on G and ρ is a probability measure on X, then

$$\int_X f(x)(\lambda * \rho)(dx) = \int_X \int_G f(x + y^a)\lambda(dyda)\rho(dx).$$

To prove existence of the "Poisson measure" ν we give its construction. The outline of the paper is as follows. In Section 2 we recall the basic properties of Gaussian measure in the Hilbert space ℓ_2 which is the basic component in our results. In Section 3 we define a vertical component, i.e. a stochastic process on A generated by Δ_α. The vertical component is one of the two components of the skew-product formula (proved in Section 5) for the heat semigroup $P_t^{\mathcal{L}}$. The second component, called a horizontal component, is a diffusion on X generated by a time-dependent operator $\mathcal{L}_\beta^{\sigma_t}$. The horizontal component is considered in Section 4. Finally, in Section 6 we construct the Poisson measure ν on X for $\mathcal{L}$.

2. Gaussian measures on ℓ_2

The concept of Gaussian measures is fundamental to our results. A detailed and extensive discussion of Gaussian measures in the infinite-dimensional Hilbert and Banach spaces can be found e.g. in [3, 7, 1].

We restrict to the mean 0 case since this is what we require. For $\lambda \geq 0$, the corresponding Gaussian measure on $\mathbb{R}$ is then

$$N_\lambda(dx) = \begin{cases} (2\pi\lambda)^{-1/2} e^{-\frac{x^2}{2\lambda}} dx, & \lambda > 0, \\ \delta_0(dx), & \lambda = 0. \end{cases}$$

Let $\lambda \in \mathbb{R}^d$ with $\lambda_j \geq 0$, where $1 \leq d \leq \infty$. Let $[\lambda]$ denote the corresponding $d \times d$ diagonal matrix. The corresponding Gaussian measure on $\mathbb{R}^d$ is by definition

$$N_{[\lambda]}(dx) = \prod_{j=1}^d N_{\lambda_j}(dx_j). \tag{2.1}$$

The product measure exists as a measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ since each of the components is a probability measure [9, p. 157, Theorem B]. Let $|A|$ denote the cardinality of a set A, and let

$$\mathbb{R}^m = \prod_{\{j:\lambda_j \neq 0\}} \mathbb{R}, \quad \text{where } m = |\{j : \lambda_j \neq 0\}|,$$
$$\mathbb{R}^k = \prod_{\{j:\lambda_j = 0\}} \mathbb{R}, \quad \text{where } k = |\{j : \lambda_j = 0\}|.$$

Then we may write $\mathbb{R}^d = \mathbb{R}^m \oplus \mathbb{R}^k$. For $x \in \mathbb{R}^d$ let $x = x^m + x^k$ in this decomposition.

Lemma 2.1. *Let notation be as above. Then*

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)N_{[\lambda]}(dx) &= \int_{\mathbb{R}^m} f(x^m,0)N_{[\lambda^m]}(dx^m)\\ &= \int_{\mathbb{R}^m} f\big|_{\mathbb{R}^m}(x)N_{[\lambda^m]}(dx^m)\\ &= \int_{\mathbb{R}^d} f(x)\chi_{\mathbb{R}^m}(x)N_{[\lambda]}(dx),\end{aligned}$$

where $dx^m = \prod_{j=1}^m dx_j$.

Proof. By definition

$$N_{[\lambda]}(dx) = \prod_{\{j:\lambda_j\neq 0\}} N_{\lambda_j}(dx_j) \times \prod_{\{j:\lambda_j=0\}} N_{\lambda_j}(dx_j)$$

which is equivalent to the statement in the lemma. □

We say that two sequences λ^1 and λ^2 in $\mathbb{R}^d$ are disjoint if $\lambda_i^1\lambda_i^2 = 0$ for all i. Equivalently, λ^1 and λ^2 are disjoint if the sets $\{i : \lambda_i^1 \neq 0\}$ and $\{i : \lambda_i^2 \neq 0\}$ are disjoint. The following lemma is clear.

Lemma 2.2. *Assume that* $\lambda \in \mathbb{R}^d$ *where* $\lambda_j > 0$ *for all* j. *Suppose that* λ^1 *and* λ^2 *are disjoint elements of* $\mathbb{R}^d$ *and* $\lambda = \lambda^1 + \lambda^2$. *Then*

$$\begin{gathered}\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2},\\ N_{[\lambda]} = N_{[\lambda^1]} \otimes N_{[\lambda^2]},\end{gathered}$$

where $N_{[\lambda^i]}$ *are measures on* $\mathbb{R}^{d_i}$ *and* $d_i = |\{j : \lambda_j^i \neq 0\}|$.

The statement in [7, Theorem 1.2.1] implies that if $[\lambda]$ is trace class, i.e. $\lambda \in \ell_1$, then $N_{[\lambda]}(dx)$ restricts uniquely to a measure on ℓ_2. In our case, trace class will always be assumed, so we consider $N_{[\lambda]}(dx)$ as a measure on ℓ_2. The following lemma is simple and it is left to the reader to prove it.

Lemma 2.3. *Let* P *be a bounded, symmetric, positive operator on* ℓ_2 *which is diagonal in the standard basis* $\{e_j\}$ *of* ℓ_2 *with eigenvalues* p_j. *Then*

$$\int_{\ell_2} f(x)N_{P[\lambda]}(dx) = \int_{\ell_2} f(P^{1/2}x)N_{[\lambda]}dx.$$

3. Vertical component

By definition, the vertical component is the diffusion on A generated by Δ_α. This is the Markov process with transition kernel $P_{t,s}^\alpha(a, db)$ which gives the solution to the following initial problem on A:

$$\begin{gathered}\partial_t f(t,a) = \Delta_\alpha f(t,a), \quad t > s \geq 0,\\ f(s,a) = f_o(a).\end{gathered} \tag{3.1}$$

In the $\alpha = 0$ case, from [7, (3.1.9)], the solution for $f_o \in UC_b(A)$ is

$$\begin{gathered}f(t,x) = \int_A f_o(x+y)N_{(t-s)[q]}(dy) = \int_A f_o(y)R(-x)N_{(t-s)[q]}(dy),\\ P_{t,s}^0(x,dy) = R(-x)N_{(t-s)[q]}(dy).\end{gathered} \tag{3.2}$$

Furthermore as an operator on $UC_b(A)$, $t \mapsto P^0_{t,0}$ defines a continuous semi-group.

In the $\alpha \neq 0$ case, by hypothesis (1.4) there is a $\gamma \in \ell_2$ such that $[q]\gamma = \alpha$. Note also that from [7, Proposition 1.2.5], the measure in the second equation in (3.3) below is finite.

Corollary 3.1. *In the case $\alpha \in \ell_2$ (arbitrary case), the solution to* (3.1) *for $f_o \in UC_b(A)$ is*

$$\begin{aligned} f(x,t) &= e^{-t(\gamma,\alpha)} \int_{\ell_2} f_o(x+y) e^{-(\gamma,y)} N_{(t-s)[q]}(dy), \\ P^\alpha_{s,t}(x,dy) &= e^{-t(\gamma,\alpha)} R(-x)\big(e^{-(\gamma,y)} N_{(t-s)[q]}(dy)\big). \end{aligned} \tag{3.3}$$

Furthermore as an operator on $UC_b(A)$, $t \mapsto P^\alpha_{0,t}$ defines a continuous semi-group.

Proof. A simple computation shows that

$$\Delta^q_\alpha = e^{(\gamma,x)} \Delta^q_0 e^{-(\gamma,x)} + (\alpha,\gamma).$$

This suggests that as operators on $UC_b(A)$

$$P^\alpha_{s,t} = P^{\alpha,q}_{s,t} = e^{(t-s)(\alpha,\gamma)} e^{(\gamma,x)} P^{0,q}_{s,t} e^{-(\gamma,x)}$$

which is equivalent with the stated identity. A rigorous proof can be constructed either using finite rank approximations as in [7] or using the stated formula for Δ^q_α. We omit the details. □

4. Horizontal component

Let $\sigma \in C([0,\infty)), X)$ and $f_o \in UC_b(X)$. For $k \in \mathbb{N}$, let $UC^k_b(X)$ denote the set of elements of $UC_b(X)$, all of whose derivatives up to order k belong to $UC_b(X)$. Consider the initial value problem on $\mathbb{R}^+ \times X$,

$$\begin{gathered} \partial_t f(t,b) = \mathcal{L}^{\sigma_t}_\beta f(t,b), \quad t > s, \\ f(s,b) = f_o(b). \end{gathered} \tag{4.1}$$

For $s < t$, let

$$A^\sigma(s,t) = \int_s^t e^{2\sigma(u)} du := \Big(\int_s^t e^{2\sigma(u)_1} du, \dots, \int_s^t e^{2\sigma(u)_d} du\Big)^T \in \mathbb{R}^d,$$

that is we consider $A^\sigma(s,t)$ as a column vector, and

$$A^\sigma_j(s,t) = \int_s^t e^{2\sigma(u)_j} du.$$

Note that for $a \in A$,

$$(\mathrm{Ad}_a A^\sigma(s,t))_j = e^{a_j} A^\sigma_j(s,t) = \int_s^t e^{2\sigma(u)_j + a_j} du = A^{\sigma + a/2}(s,t)_j. \tag{4.2}$$

Theorem 4.1 below is one of our main results. We prove it in a series of propositions. Assume first that $d < \infty$. Under the Fourier transform in x, (4.1) implies

$$\begin{gathered} \partial_t \hat{f}(t,\xi) = \Big(-\sum_{j=1}^d e^{2\lambda_j(\sigma_t)} \beta_j \xi_j^2\Big) \hat{f}(\xi,t), \\ \hat{f}(0,\xi) = \hat{f}_o(\xi). \end{gathered}$$

This ODE is easily solved,

$$
\begin{aligned}
\hat{f}(t,\xi) &= \exp\Big(-\sum_{j=1}^{d} A_j^\sigma(s,t)\beta_j\xi_j^2\Big)\hat{f}_o(\xi)\\
&= e^{-\frac{1}{2}(2[A^\sigma(s,t)][\beta]\xi,\xi)}\hat{f}_o(\xi).
\end{aligned}
$$

If $d<\infty$, the next follows immediately from [7, Theorem 1.2.1]. The general case is Proposition 4.3 below.

Theorem 4.1. *Assume that $[\beta]$ is trace class. Let*

$$
P_{s,t}^{\sigma,\beta}(dy) = N_{2[\beta]A^\sigma(s,t)}(dy). \tag{4.3}
$$

Then for $f_o \in C_b(X)$ ($UC_b(X)$, resp.) the unique solution to (4.1) in $C_b(X)$ ($UC_b(X)$, resp.) is

$$
U^\sigma(s,t)f_o(x) = \int_X f_o(x+y)P_{s,t}^{\sigma,\beta}(dy) = \int_X f_o(x+[A^\sigma(s,t)]^{1/2}y)N_{2[\beta]}(dy). \tag{4.4}
$$

In the finite dimensional case, the following properties follow from Theorem 4.1 and are well known (see [10, 19, 20]). The process corresponding to the transition kernel (4.3) is called the *horizontal component*.

Corollary 4.2. *Assume $d<\infty$. Then*

(i) $U^\sigma(s,r)U^\sigma(r,t) = U^\sigma(s,t)$, $0\le s\le r\le t$,
(ii) $\partial_t U^\sigma(s,t)f = U^\sigma(t,s)\mathcal{L}^{\sigma(t)}f$,
(iii) $\partial_s U^\sigma(s,t)f = -\mathcal{L}^{\sigma(s)}U^\sigma(s,t)f$,
(iv) $U^\sigma(s,t): C_b^2(\mathbb{R}^d)\to C_b^2(\mathbb{R}^d)$, $s\le t$.

Assume now that $d=\infty$. For $m\in\mathbb{N}$ and β as in (1.2) let

$$
\beta^m = (\beta_1,\dots,\beta_m,0,0,\dots)\in\mathbb{R}^\infty, \quad P_{s,t}^{\sigma,m}(dy) = N_{2[\beta^m]A^{\sigma,m}(s,t)}(dy),
$$

where

$$
A^{\sigma,m}(s,t) = (A^{\sigma_1}(s,t),\dots,A^{\sigma_m}(s,t),0,0,\dots)^T.
$$

Then $m\mapsto P_{s,t}^{\sigma,m}(dy)$ is referred to as the *finite rank approximation* to $P_{s,t}^\sigma(dy)$.

We apply Lemma 2.2 with $\lambda=2\beta$, $\lambda^1=2\beta^m$, $\lambda^2=2\beta-2\beta^m$. Then in this lemma $d_1=m$ and $d_2=\infty$. Restricting to ℓ_2, this lemma implies that

$$
\begin{gathered}
\ell_2 = \mathbb{R}^m\times\ell_2,\\
N_{2[\beta]A^\sigma(s,t)} = N_{2[\beta^m]A^\sigma(s,t)}\otimes N_{[\lambda^2]A^\sigma(s,t)}.
\end{gathered}
$$

For $x\in\ell_2$, let x^m and x^∞ be the components of x with respect to the above decomposition. Let

$$
\begin{aligned}
u_m^\sigma(s,t) &= \int_{\ell_2} f_o(x+y)\chi_{\mathbb{R}^m}(y)N_{2[\beta]A^\sigma(s,t)}(dy)\\
&= \int_{\mathbb{R}^m\times\ell_2} f_o(x^m+y^m, x-x^m+y^\infty)\chi_{\mathbb{R}^m}(y^m,y^\infty)\\
&\quad\times N_{2[\beta^m]}A^\sigma(s,t)(dy_m)N_{[\lambda^2]A^\sigma(s,t)}(dy^\infty)\\
&= \int_{\mathbb{R}^m\times\ell_2} f_o(x^m+y^m, x-x^m+0)N_{2[\beta^m]A^\sigma(s,t)}(dy^m)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^m \times \ell_2} f_o(x+y^m) N_{[\beta^m] A^\sigma(s,t) 2}(dy^m) \\
&= \int_{\mathbb{R}^m} f_o(x+y^m) P^{\sigma,m}_{s,t}(dy^m).
\end{aligned}
$$

Recall that for a set B, χ_B is the indicator function of B, i.e. $\chi_B(x)=1$ if $x\in B$ and $\chi_B(x)=0$ if $x\notin B$. From the dominated convergence theorem it follows that

$$
\lim_{m\to\infty} P^{\sigma,m}_{s,t}(f_o) = P^{\sigma,\beta}_{s,t}(f_o).
$$

In particular from Corollary 4.2, for $0\le s\le r\le t$,

$$
P^{\sigma,\beta}_{s,r} P^{\sigma,\beta}_{r,t} = P^{\sigma,\beta}_{s,t}. \tag{4.5}
$$

Theorem 4.1 (for $d=\infty$) now follows from the next proposition.

Proposition 4.3. *For $f_o \in UC^2_b(X)$, $u(s,t)=U^\sigma(s,t)f_o$ is the unique solution to the initial value problem* (4.1).

Proof. We use the second formula in (4.4). Let

$$
B = 2[\beta], \quad C = [A^\sigma(s,t)]^{1/2}, \quad S = C^2(D^2 f_o)(x).
$$

From the mean value theorem for integrals applied in each variable

$$
\begin{aligned}
C^2 &= \Big[\Big(\int_s^t e^{2\sigma(u)_1} du, \int_s^t e^{2\sigma(u)_2} du, \dots\Big)^T\Big] \\
&= (t-s)[(e^{2\sigma(u_1)}, e^{2\sigma(u_2)}, \dots)^T] \\
&= (t-s) e^{2\sigma(u)}, u\in\mathbb{R}^d,
\end{aligned} \tag{4.6}
$$

where $u_i \in (s,t)$ for all i.

From Taylor's Theorem applied to the function $t\mapsto f_o(x+ty)$ at $t=1$,

$$
f_o(x+y) = f_o(x) + (Df_o(x), y) + \frac{1}{2}\left(D^2 f_o(x+\xi y)y, y\right)
$$

where differentiation is with respect to x and $\xi=\xi(x,y)\in[0,1]$. Then

$$
f_o(x+Cy) = f_o(x) + (Df_o(x), Cy) + \frac{1}{2}(C^2 D^2 f_o(x+\xi Cy)y, y). \tag{4.7}
$$

From [7, Proposition 1.2.4],

$$
\begin{gathered}
\int (f_o(x), Cy) N_B(dy) = 0, \\
\int S_{ij} y_i y_j N_B(dy) = 2 S_{ij}\beta_{ij},
\end{gathered}
$$

$$
\begin{aligned}
\int (Sy,y) N_B(dy) &= \sum_{ij} 2 S_{ij}\beta_{ij} \\
&= 2(t-s)\operatorname{Tr}([\beta](e^{2\sigma(u)})D^2)(f_o) \\
&= 2(t-s)\sum_i \beta_i e^{2\sigma(u_i)} \partial_i^2 f_o(x).
\end{aligned}
$$

We integrate (4.7) with $\xi=0$ against $N_{2[\beta]}(dy)$ using (4.6) and rearrange to find that

$$
U^\sigma(s,t)(x) - f_o(x) - (t-s)\sum_i \beta_i e^{2\sigma(u_i)} \partial_i^2 f_o(x)
$$

$$= \frac{t-s}{2} \int [e^{\sigma(u_i)}] \Big((D^2 f_o(x + (t-s)^{1/2} [e^{2\sigma(u_i)}] \xi y) - D^2 f_o(x)) y, y \Big) N_{2[\beta]}(dy).$$

Hence, as $t \to s^+$,

$$\frac{U^\sigma(s,t)(x) - f_o(x)}{t-s} - \sum_i \beta_i e^{2\sigma(u_i)} \partial_i^2 f_o(x) \to 0,$$

that is,

$$\partial_t U^\sigma(s,t)\big|_{t=s^+} = \mathcal{L}_\beta^{\sigma_s} f_o(x).$$

This proposition now follows as in the proof of [7, Theorem 3.2.3] using Corollary 4.2 (i). The uniqueness follows from the uniqueness in the finite dimensional case. $\square$

4.1. Ad**-invariance.** For f a C^∞ function on either G or X and $a \in A$ let

$$\mathrm{ad}_a f = f \circ \mathrm{ad}_a .$$

For $\mathcal{X}$ in the Lie algebra $\mathcal{G}$ of G we have

$$\mathrm{ad}_{-a} \circ \mathcal{X} \circ \mathrm{ad}_a = \mathrm{Ad}_a(\mathcal{X}).$$

Consequently, as an operator on X,

$$\begin{gathered} \mathcal{L}_\beta^{\sigma_t} = \sum_{j=1}^\infty e^{2\lambda_j(\sigma_t)} \beta_j \partial_{x_j}^2 = \mathrm{Ad}_{\sigma_t} \Big(\sum_{j=1}^\infty \beta_j \mathcal{X}_j^2 \Big), \\ \mathcal{L}_\beta^{\sigma_t + a} = \mathrm{Ad}_a(\mathcal{L}_\beta^{\sigma_t}) = \mathrm{Ad}_{-a} \circ (\mathcal{L}_\beta^{\sigma_t}) \circ \mathrm{Ad}_a . \end{gathered}$$

Corollary 4.4. *As operators on $C_b(X)$, for $a \in A$,*

$$U^{\sigma+a}(s,t) = \mathrm{Ad}_{-a} \circ U^\sigma(s,t) \circ \mathrm{Ad}_a .$$

Proof. From Corollary 4.2, integration against the quantity on the right solves the initial value problem (4.1) with σ replaced by $\sigma + a$. $\square$

Corollary 4.5. *For $f_o \in C_b(X)$,*

$$U^{\sigma+a}(s,t) f_o(x) = \int_X N_{2[\beta] A^{\sigma+a}(s,t)} f_o(x+y) dy.$$

Proof. From (4.4) and (2.3),

$$\begin{aligned} U^\sigma(s,t)(\mathrm{Ad}_a(f_o))(x) &= \int_X (f_o \circ \mathrm{Ad}_a)(y) P_{s,t}^\sigma(x,dy) \\ &= \int_X N_{2[\beta] A^\sigma(s,t)} f_o(\mathrm{Ad}_a(x+y))(dy) \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{Ad}_{-a} \circ U^\sigma(s,t) \circ \mathrm{Ad}_a(f_o)(x) &= \int_X N_{2[\beta] A^\sigma(s,t)} f_o(x + \mathrm{Ad}_a y) dy \\ &= \int_X N_{2[\beta] \mathrm{Ad}_a^2 A^\sigma(s,t)} f_o(x+y) dy \\ &= \int_X N_{2[\beta] A^{\sigma+a}(s,t)} f_o(x+y) dy. \end{aligned}$$

$\square$

5. A skew-product formula

Here we present our second main result. Let $P^{\sigma}_{s,t}$ be the measure defined in (4.3). We refer to the formula given by the next theorem as to a *skew-product formula.* It gives both the existence of and a disintegration of the heat semigroup $T_t^{\mathcal{L}}$ of $\mathcal{L}$ into a "horizontal" diffusion $\tau(t)$ defined on $X = \ell_2$ and a "vertical" one on $A = \ell_2$ generated by Δ_α. In finite dimensions, this is a special case of [6, Theorem 3.1], see also [12].

Theorem 5.1. *For $f \in UC_b^2(G)$ and $t \geq 0$ we have*

$$T_t^{\mathcal{L}} f(x,a) = \mathbf{E}_a \int f(x+y, \sigma_t) N_{2[\beta]A^\sigma(0,t)}(dy) \equiv v(t,x,a), \tag{5.1}$$

where the expectation is taken with respect to the distribution of the process σ in ℓ_2 generated by Δ_α, and starting from a, i.e., $\sigma_0 = a$.

Proof. We claim first that for $f \in UC_b^2(G)$, $v(t,x,a)$ defined in(5.1) is a solution of the integral equation

$$v(t,x,a) = \mathbf{E}_a f(x,\sigma_t) + \mathbf{E}_a \int_0^t \mathcal{L}_\beta^{\sigma_{t-s}}\big|_x v(s,x,\sigma_{t-s}) d\, s \tag{5.2}$$

We claim first that the only question is the second term. Let $f \in UC_b^2(G)$. Then from (4.1) for $0 < s < t$,

$$\begin{aligned}\int_0^t \mathbf{E}_a \mathcal{L}_\beta^{\sigma_{t-s}} v(s,x,\sigma_{t-s}) &= \mathbf{E}_a \int_0^t \mathcal{L}_\beta^{\sigma_{t-s}} U^\sigma(0,s) f(x,\sigma_{t-s}) \\ &= \int_0^t \mathbf{E}_a \int_{\ell_2} \sum_{j=1}^\infty e^{2\lambda_j(\sigma_{t-s})} \beta_j \partial^2_{x_j} v(x+y) N_{2[\beta]A^\sigma(s,0)}(dy)\, ds.\end{aligned}$$

Since

$$\begin{aligned}\big|\mathbf{E}_a \int_0^t e^{2\lambda_j(\sigma_{t-s})}\, ds\big| &= \big|\mathbf{E}_a \int_0^t e^{2\lambda_j(\sigma_u)}\, du\big| \\ &= \big|\mathbf{E}_a \int_0^t e^{2(\sigma_u)_j}\, du\big| \\ &= \big|\mathbf{E}_a \int_0^t e^{2(b_u)_j - \alpha_j t}\, du\big| \\ &\leq \Big|\mathbf{E}_a \int_0^t e^{2(b_u)_j}\, du\Big| = C_t\end{aligned}$$

we obtain (since $\beta \in \ell_1$) that

$$\begin{aligned}|\mathcal{L}^a_\beta v(s,x,a)| &\leq \Big|\|v\|^2_\infty \sum_{j=1}^\infty \beta_j \mathbf{E}_a \int_0^t e^{2\lambda_j(\sigma_{t-s})}\, ds\Big| \\ &\leq \|v\|^2_\infty C_t \big|\sum_{j=1}^\infty \beta_j\big| \\ &\leq C_t \|v\|_\infty \|\beta\|_{\ell_1}.\end{aligned}$$

Let $d\mathbf{W}_a(b)$ be a "Wiener measure", i.e. probability measure one $C([0,\infty), A)$ such that, for every $a \in A$, $\mathbf{W}_a(b : b(0) = a) = 1$.

We calculate

$$
\begin{aligned}
&\mathbf{E}_a \mathcal{L}_\beta^{b_{t-s}} v(s,x,b_{t-s}) \\
&= \int \mathcal{L}_\beta^{b_{t-s}} v(s,x,b_{t-s})\, d\mathbf{W}_a(b) \\
&= \int \mathcal{L}_\beta^{b_{t-s}} \int U^\sigma(0,s) f(x,\sigma_s) d\,\mathbf{W}_{b_{t-s}}(\sigma) d\,\mathbf{W}_a(b) \\
&= \int\int \mathcal{L}_\beta^{b_{t-s}} U^\sigma(0,s) f(x,\sigma_s) d\,\mathbf{W}_{b_{t-s}}(\sigma) d\,\mathbf{W}_a(b) \\
&= \int \mathcal{L}_\beta^{b_{t-s}} U^\sigma(t-s,t) f(x,b_t) d\,\mathbf{W}_a(b),
\end{aligned}
\tag{5.3}
$$

where in the last equality we have used the Markov property of the process.

We apply Fubini's theorem to obtain

$$
\int_0^t \mathbf{E}_a \mathcal{L}_\beta^{b_{t-s}} u(s,x,b_{t-s}) d\,s = \int\int_0^t \mathcal{L}_\beta^{b_{t-s}} U^b(t-s,t) f(x,b_t) d\,s d\mathbf{W}_a(b),
$$

but

$$
\int_0^t \mathcal{L}_\beta^{b_{t-s}} U^{b_{t-s}}(t-s,t) f(x,b_t) d\,s = U^b(0,t) f(x,b_t) - f(x,b_t).
$$

Indeed by property (iii) of U^b we obtain

$$
\begin{aligned}
\frac{d}{d\,s} U^b(t-s,t) f(x,b_t) &= -\frac{d}{d\,s} U^b(\cdot,t) f(x,b_t)\Big|_{t-s} \\
&= -\Big(-\mathcal{L}_\beta^{b_{t-s}} U^b(t-s,t) f(x,b_t)\Big) \\
&= \mathcal{L}_\beta^{b_{t-s}} U^b(t-s,t) f(x,b_t).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_0^t \mathbf{E}_a \mathcal{L}_\beta^{b_{t-s}} u(s,x,b_{t-s}) d\,s \\
&= \int U^b(0,t) f(x,b_t) d\,\mathbf{W}_a(b) - \int f(x,b_t) d\,\mathbf{W}_a(b) \\
&= u(t,x,a) - \mathbf{E}_a f(x,b_t).
\end{aligned}
$$

Thus

$$
T_t^{\mathcal{L}} f(x,a) = \mathbf{E}_a\Big(\int f(x+y,\sigma_t) N_{2[\beta]A^\sigma(0,t)}(dy)\Big) \tag{5.4}
$$

as claimed. □

6. Construction of the Poisson measure

Let operators be defined as in (1.2) and $T_t^{\mathcal{L}}$ be defined by (5.1). For $b \in A$ we let $T_t^{\mathcal{L},b}$ be defined by replacing σ_t with $\sigma_t + b$ in the initial value problem (4.1). For a measure μ on G we define

$$
\check{\mu}(\varphi) = \int_G \varphi(g^{-1}) \mu(dg).
$$

We fix $t > 0$ and consider a Markov chain R_n with the starting point $R_0 = e = (0,0)$ and the transition kernel $p(\cdot,\cdot) = \check{p}_t(\cdot,\cdot)$. We have

$$
R_n = (0,0)(X_1,A_1)\dots(X_n,A_n)
$$

$$= (X_1 + X_2^{A_1} + \cdots + X_n^{A_1+\cdots+A_{n-1}}, A_1 + \cdots + A_n).$$

The distribution of R_n is equal to $p^{(n)}(e, dg)$ - the n-folded composition of kernel p.

For $i = 1, 2$, let

$$\pi_i : X \rtimes A = \ell_2 \rtimes \ell_2 \to \ell_2$$

be the projection on the i-th variable, i.e.

$$\pi_1(x, a) = x, \quad \pi_2(x, a) = a.$$

Then

$$\pi_1(R_n) = X_1 + X_2^{A_1} + \cdots + X_n^{A_1+\cdots+A_{n-1}}.$$

For $i = 1, 2, \dots$, we denote

$$X_i = (X_{i,1}, X_{i,2}, \dots) \in \ell_2, \quad A_i = (A_{i,1}, A_{i,2}, \dots) \in \ell_2.$$

Theorem 6.1. *The limit*

$$\lim_{n\to\infty} \pi_1(R_n) = Z \tag{6.1}$$

exists a.e., and the distribution $\nu(0, dy)$ of the random variable Z is the Poisson kernel for $\mathcal{L}$.

Proof. The proof is based on ideas from [17, 8]. First we prove the existence of (6.1) and then we show the claim about the Poisson measure. To show the existence of (6.1) it is sufficient to show that

$$\limsup_{n\to\infty} \|X_n^{A_1+\cdots+A_{n-1}}\|_{\ell_2}^{1/n} < 1 \quad \text{a.e.}$$

We have,

$$\begin{aligned}
\|X_n^{A_1+\cdots+A_{n-1}}\|_{\ell_2} &= \|(e^{\lambda_1(A_1\cdots+A_{n-1})}X_{n,1}, e^{\lambda_2(A_1\cdots+A_{n-1})}X_{n,2}, \dots)\|_{\ell_2} \\
&= \|(e^{A_{1,1}\cdots+A_{n-1,1}}X_{n,1}, e^{A_{1,2}\cdots+A_{n-1,2}}X_{n,2}, \dots)\|_{\ell_2} \\
&= \|\operatorname{Ad}_{A_1+\cdots+A_{n-1}} X_n\|_{\ell_2} \\
&\le \|\operatorname{Ad}_{A_1+\cdots+A_{n-1}}\|_{\ell_2\to\ell_2} \|X_n\|_{\ell_2}.
\end{aligned}$$

Thus it suffices to show that

$$\lim_{n\to\infty} \|\operatorname{Ad}_{A_1+\cdots+A_n}\|_{\ell_2\to\ell_2}^{1/n} < 1 \quad \text{a.e.} \tag{6.2}$$

and

$$\limsup_{n\to\infty} \|X_{n+1}\|_{\ell_2}^{1/n} \le 1 \quad \text{a.e.} \tag{6.3}$$

First we prove (6.2). Clearly,

$$\|\operatorname{Ad}_a x\|_{\ell_2} \le \sup_{j\ge1} e^{\lambda_j(a)} \|x\|_{\ell_2}.$$

Therefore

$$\|\operatorname{Ad}_a\|_{\ell_2\to\ell_2} \le \sup_{j\ge1} e^{\lambda_j(a)} = \sup_{j\ge1} e^{a_j}$$

and consequently

$$\|\operatorname{Ad}_{A_1+\cdots+A_n}\|_{\ell_2\to\ell_2}^{1/n} \le \sup_{j\ge1} e^{\frac{1}{n}(A_{1,j}+\cdots+A_{n,j})}.$$

Thus to prove (6.2) we have to show that

$$0 < \sup_{j\ge1} \lim_{n\to\infty} e^{\frac{1}{n}\lambda_j(A_1+\cdots+A_n)} = \sup_{j\ge1} \lim_{n\to\infty} e^{\frac{1}{n}(A_{1,j}+\cdots+A_{n,j})} < 1, \quad \text{a.e.}$$

Equivalently we have to show that

$$-\infty < \sup_{j\geq 1} \lim_{n\to\infty} \frac{1}{n} (A_{1,j} + \dots + A_{n,j}) < 0.$$

Let

$$\pi_{2,j}(x,a) = a_j.$$

Then $\pi_{2,j}\mu_t$ is a gaussian semigroup of measures with the generator

$$q_j \partial^2_{a_j} - 2\alpha_j \partial_{a_j} = q_j \Big(\partial^2_{a_j} - \frac{2\alpha_j}{q_j} \partial_{a_j} \Big);$$

that is,

$$\pi_{2,j}\mu_t(da_j) = \frac{1}{\sqrt{4\pi q_j t}} \exp\Big(- \frac{(a_j - 2\alpha_j q_j t/q_j)^2}{4q_j t} \Big) da_j.$$

Thus by the assumption (1.3), there is a constant $c > 0$ such that

$$\int_{\ell_2} \int_{\ell_2} \pi_{2,j}(x,a) \mu(dx\, da) = \int_{\mathbb{R}} a_j \pi_{2,j}\mu_t(da) = -\frac{2\alpha_j}{q_j} < -c.$$

Notice that for every $j \geq 1$, $A_{1,j}, A_{2,j}, \dots$ is a sequence of i.i.d. random variables with values in $\mathbb{R}$ and with gaussian distribution $\pi_{2,j}\mu_t(da_j)$. By the strong law of large numbers

$$\lim_{n\to\infty} \frac{1}{n} (A_{1,j} + \dots + A_{n,j}) = \int_{\mathbb{R}} a_j \pi_{2,j}\mu_t(da_j) = -\frac{2\alpha_j}{q_j} < -c, \text{ a.e.}$$

Hence we obtain (6.2).

To prove (6.3) we proceed as follows. Let

$$f(x,a) = \log(1 + \|x\|).$$

We will prove that $T^{\mathcal{L}}_t f$ is finite. By the skew-product formula (5.1),

$$T^{\mathcal{L}}_t f(x,a) = \mathbf{E}_0 \Big(\int \log(1 + \|x+y\|) N_{A^{\sigma+a}(0,t)}(dy) \Big),$$

(σ is involved in the Gaussian measure $N_{A^{\sigma}(0,t)}(dy)$). Clearly there is $C > 0$ such that

$$\begin{aligned}
&\mathbf{E}_0 \int \log(1 + \|x+y\|) N_{A^{\sigma+a}(0,t)}(dy) \\
&\leq \mathbf{E}_0 \int \left(\log(1 + \|x\|) + \log(1 + \|y\|) \right) N_{A^{\sigma+a}(0,t)}(dy) \\
&\leq \mathbf{E}_0 \log(1 + \|x\|) + C \mathbf{E}_0 \int (1 + \|y\|^2) N_{A^{\sigma+a}(0,t)}(dy) < +\infty.
\end{aligned}$$

Hence from [7, Proposition 1.2.4],

$$\begin{aligned}
&\int_{\ell_2 \rtimes \ell_2} \log(1 + \|\pi_1(g)\|_{\ell_2}) p(e, dg) \\
&= \int_{\ell_2 \rtimes \ell_2} \log(1 + \|x\|_{\ell_2}) p(e, dg) < +\infty, \quad g = (x,a).
\end{aligned} \tag{6.4}$$

Notice that the real random variable Y is integrable if and only if

$$\sum_{n=1}^{\infty} \mathbf{P}(|Y| \geq n) < +\infty.$$

It follows from (6.4), that for every $r > 0$ the random variable

$$\frac{1}{r}\|\pi_1(g)\|_{\ell_2} = \frac{1}{r}\|X_n\|_{\ell_2}$$

is integrable. Therefore,

$$\begin{aligned}&\sum_{n=1}^{\infty} p\left(e, \{g \in G : \log(1 + \|X_n\|_{\ell_2}) \geq nr\}\right)\\ &= \sum_{n=1}^{\infty} p(e, \{g \in G : \log(1 + \|X_1\|_{\ell_2}) \geq nr\}) < +\infty \quad \forall r > 0.\end{aligned}$$

Let

$$A_n = \{\log(1 + \|X_n\|_{\ell_2}) \geq nr\}.$$

By Borel-Cantelli lemma

$$\limsup_{n\geq 1} A_n := \cap_{n=1}^{\infty}\Big(\cup_{m=n}^{\infty} A_m\Big) = \emptyset \quad \text{a.e.}$$

Thus,

$$\limsup_{n\geq 1}(1 + \|X_n\|_{\ell_2})^{1/n} \leq e^r \quad \text{a.e.}$$

and consequently

$$\limsup_{n\geq 1} \|X_n\|_{\ell_2}^{1/n} \leq e^r, \quad \text{a.e.}$$

Now taking $r \to 0$ inequality (6.3) is proved.

Now we prove that the object we have defined is the Poisson kernel. The argument is standard and is taken from [17, 4, 8]. The operator $\mathcal{L}$ generates semigroup of probability measures μ_t. Consider probability measure $\nu_n(dy) = \nu_n(0, dy)$. That is ν_n is the distribution of R_n with $R_0 = (0, 0)$. Recall that the transition kernel for R_n was defined as μ_t for some fixed t. Therefore, for the random walk R_n we have

$$\mu * \nu_n = \nu_{n+1}.$$

Obviously for every bounded continuous function f defined on X we have

$$\lim_{n\to\infty}(f, \nu_n) = (f, \nu).$$

Consequently $\mu * \nu = \nu$ and the proof is complete. □

Remark 6.2. Using methods of [4, 5] it is possible to prove that in fact $\nu_n(x, dy)$ and $\nu(x, dy)$ do not depend on x.

Now we recall a version of Doob's theorem [5, p. 17].

Theorem 6.3. *Let X_t be an almost surely continuous stochastic process. Suppose that for every sequence $0 < t_n \to \infty$ such that*

$$\lim_{n\to\infty}\frac{t_n}{n} = 1$$

there exists a limit

$$\lim_{t_n\to\infty} X_{t_n} = Z \quad a.e.$$

which does not depend on $\{t_n\}$. Then

$$\lim_{t\to\infty} X_t = Z \quad a.e.$$

By Doob's theorem we obtain the following corollary from Theorem 6.1.

Corollary 6.4. *Let* $\mu_t = P_t^{\mathcal{L}}(dy)$ *be the semigroup of probability measures on* $G = X \rtimes A = \ell_2 \rtimes \ell_2$ *generated by* $\mathcal{L}$*. Then for* $f \in C_b(X)$*,*

$$\lim_{t\to\infty}(\pi_1(\check{\mu}_t), f) = (\nu(\cdot), f),$$

where $(\check{\mu}, h) = (\mu, \check{h})$*,* $\check{h}(g) = h(g^{-1})$*.*

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Richard C. Penney
Department of Mathematics, Purdue University, 150 N. University St, West Lafayette, IN 47907, USA
Email address: rcp@math.purdue.edu

Roman Urban
Institute of Mathematics, Wroclaw University, Plac Grunwaldzki 2/4, 50-384 Wroclaw, Poland
Email address: roman.urban@math.uni.wroc.pl