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UNIQUENESS FOR OPTIMAL CONTROL PROBLEMS OF TWO-DIMENSIONAL SECOND GRADE FLUIDS

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Abstract. We study an optimal control problem with a quadratic cost functional for non-Newtonian fluids of differential type. More precisely, we consider the system governing the evolution of a second grade fluid filling a two-dimensional bounded domain, supplemented with a Navier slip boundary condition. Under certain assumptions on the size of the initial data and parameters of the model, we prove second-order sufficient optimality conditions. Furthermore, we establish a global uniqueness result for the solutions of the first-order optimality system.

1. Introduction

The optimization of evolutionary phenomena is crucial in several branches of the knowledge, for instance in finance, biology, rcology, aviation etc. [4, 5, 14, 17]. The optimal control of fluid flows is a major problem in mathematical physics, with relevant consequences in industrial applications. In the past decades, extensive research work has been carried out on the control of fluid flows described by the Navier-Stokes equations. However, many incompressible viscous fluids present in the nature and used in the industry do not satisfy the Newton's law of viscosity, and consequently cannot be described by the Navier-Stokes equations. Among these fluids, called non-Newtonian fluids, we can find colloidal suspensions and emulsions, some industrial oils, ink-jet prints, geological flows, biological fluids, body care fluids, some materials arising in polymer processing as well as in food processing, and many others.

In this article, we study second grade fluids, which belong to the class of non-Newtonian complex viscoelastic fluids of differential type. To understand the physical principles associated to the second grade fluid equations, as well as the physical properties of these fluids, we refer to [1, 20, 21, 22, 28].

From the mathematical point of view, the equations governing the evolution of second grade fluids are strongly nonlinear partial differential equations. The existence and uniqueness problems with a Dirichlet boundary condition were established in the pioneering works [6, 15, 16, 25] Despite the most usual boundary condition to be the non-slip Dirichlet boundary condition, practical studies show that some viscoelastic fluids slip against the boundary surface. Let us refer, for instance, [32] on capillary flow of highly entangled polyethylene (PE) melts, and [23] on microgel

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pastes and concentrated emulsions exhibiting a generic slip behavior at low stresses when sheared near smooth surfaces. Therefore, to accurately describe certain physical systems, a slip boundary condition should be considered [8, 9, 10, 11]. Article [7] established the well-posedness for the second grade fluid equations under a Navier slip boundary condition. Referring to the stochastic framework, the existence and uniqueness results have been investigated in [26, 27, 12, 29, 30] under non-slip and slip boundary conditions.

In the deterministic context, the optimal control problem for the second grade fluid equations was studied in the articles [2, 3], while the optimal control of the stochastic dynamic has been addressed in [13, 18]. The authors established the existence of an optimal solution for the control problem, and by analyzing the linearized state equation as well as the adjoint equation deduced the first-order optimality conditions.

In this article, we perform a second-order analysis by studying the second-order derivative of the objective functional, in addition, we obtain a global uniqueness result for the optimal solution. Essentially, if the fluid material is filling a bounded two-dimensional domain, the sufficient second-order optimality condition is achieved for sufficiently elastic and viscous fluid material with small size initial data. Alternatively, the same result can be obtained by taking an objective functional with strong intensity of the cost.

We should emphasize that the second-order sufficient optimality conditions are crucial to numerically solve the control problem, being necessary for the stability of the optimal solution, as well as to prove the convergence of the numerical approximations. As far as we know, the second-order analysis and the uniqueness problem for the optimization of second grade fluids is being addressed here for the first time.

In Section 2, we formulate the problem and recall some known results in the literature. We write key estimates for the solutions of the state equation, linearized equation and adjoint equation that will be necessary in the following sections. Section 3 establishes the second-order sufficient optimality conditions. Section 5 is devoted to the global uniqueness problem.

2. Formulation of the problem and preliminary results

We consider an optimal control problem associated with a non-stationary viscous, incompressible, second grade fluid. We assume that $\mathcal{O} \subset \mathbb{R}^2$ is a bounded, simply connected domain, having a sufficiently smooth boundary Γ. The fluid dynamic on a time interval $[0,T]$ is described by the following state equations

$$\begin{gathered} \frac{\partial}{\partial t}\upsilon(y) = \nu\Delta y - \operatorname{curl}\upsilon(y)\times y - \nabla\pi + u, \\ \operatorname{div} y = 0 \quad \text{in } Q = (0,T)\times\mathcal{O}, \\ y\cdot n = 0, \quad (n\cdot Dy)\cdot\tau = 0 \quad \text{on } \Sigma = (0,T)\times\Gamma, \\ y(0) = y_0 \quad \text{in } \mathcal{O}, \end{gathered} \tag{2.1}$$

where $y = (y_1, y_2)$ is the velocity field of the fluid,

$$Dy = \frac{\nabla y + (\nabla y)^T}{2}$$

corresponds to the symmetric part of the velocity gradient, and

$$\upsilon(y) = y - \alpha\Delta y,$$

where $\alpha > 0$ is a viscoelastic parameter. Moreover π denotes the hydrodynamic pressure $\nu > 0$ is the viscosity of the fluid, $n = (n_1, n_2)$ is the unit normal to the boundary Γ, $\tau = (-n_2, n_1)$ is the tangent vector to Γ and u represents an external mechanical force, which acts on the system as the control variable. Here $\upsilon = \upsilon(y)$ and y are two-dimensional vectors. To perform a three-dimensional calculus in the term $\operatorname{curl} \upsilon(y) \times y$ (or in a similar ones) we consider the usual identifications

$$\operatorname{curl} \upsilon = \frac{\partial \upsilon_2}{\partial x_1} - \frac{\partial \upsilon_1}{\partial x_2} \equiv (0, 0, \frac{\partial \upsilon_2}{\partial x_1} - \frac{\partial \upsilon_1}{\partial x_2}), \quad y = (y_1, y_2) \equiv (y_1, y_2, 0).$$

To formulate the problem and establish the results, we introduce the convenient functional spaces and some useful notation. We denote by $L^p = L^p(\mathcal{O})$, $1 \le p \le \infty$, the Lebesgue spaces, endowed with their natural norms $\|\cdot\|_p$ and consider the usual notations for the scalar products on the finite dimensional spaces $\mathbb{R}^2$ and $\mathbb{R}^{2\times 2}$,

$$u \cdot v = \sum_{i=1}^{2} u_i v_i, \quad u, v \in \mathbb{R}^2; \quad \eta : \zeta = \sum_{i,j=1}^{2} \eta_{ij} \zeta_{ij}, \quad \eta, \zeta \in \mathbb{R}^{2\times 2},$$

as well as for the scalar products in L^2

$$(u, v) = \int_{\mathcal{O}} u(x) \cdot v(x)\, dx, \quad (\eta, \zeta) = \int_{\mathcal{O}} \eta(x) : \zeta(x)\, dx.$$

We consider the standard Sobolev spaces $W^{k,p} = W^{k,p}(\mathcal{O})$, $1 \le p \le \infty$, endowed with their natural norms $\|\cdot\|_{W^{k,p}}$. In the particular case $p = 2$, we set $H^k = W^{k,2}$ and $\|\cdot\|_{H^k} = \|\cdot\|_{W^{k,2}}$.

Let us introduce the following divergence-free spaces:

$$\begin{gathered} H = \{v \in L^2 : \operatorname{div} v = 0 \text{ in } \mathcal{D}'(\mathcal{O}) \text{ and } v \cdot n = 0 \text{ on } H^{-1/2}(\Gamma)\}, \\ V = \{v \in H^1 : \operatorname{div} v = 0 \text{ in } \mathcal{O} \text{ and } v \cdot n = 0 \text{ on } \Gamma\}, \\ W = \{v \in V \cap H^2 : (n \cdot Dv) \cdot \tau = 0 \text{ on } \Gamma\}. \end{gathered}$$

On the space V, we consider the following scalar product and the corresponding norm

$$(u, v)_V = (u, v) + 2\alpha(Du, Dv), \quad \|u\|_V = (u, u)_V^{1/2}, \quad \forall u, v \in V.$$

Throughout the article, we denote by $C, C_1, \dots, C_4$ and K, $\widetilde{K}$, $\widehat{K}$, the constants which depend only on the domain $\mathcal{O}$.

Now, we recall the usual Korn inequality

$$\|y\|_{H^1}^2 \le K(\|y\|_2^2 + \|Dy\|_2^2), \quad \forall y \in H^1. \tag{2.2}$$

Let us define the operator

$$\mathbb{A}y = \mathbb{P}\triangle y \quad \text{for } y \in W,$$

where $\mathbb{P} : L^2 \to H$ is the Helmholtz projector in L^2. From [2, Lemma 2.3], we have the inequality

$$\|y\|_{H^2}^2 \le \widetilde{K}(\|y\|_2^2 + \|\mathbb{A}y\|_2^2), \quad \forall y \in W \cap H^3. \tag{2.3}$$

Considering the trilinear form

$$b(\phi, z, y) = (\phi \cdot \nabla z, y),$$

the nonlinear term of the equations can be written as

$$(\operatorname{curl} \upsilon(y) \times z, \phi) = b(\phi, z, \upsilon(y)) - b(z, \phi, \upsilon(y)), \quad \forall y, z \in W \cap H^3,\ \phi \in V. \tag{2.4}$$

Now, let us state a property of the nonlinear term, which will be useful in the following considerations.

Lemma 2.1. *For all z, $\phi \in H^2$, we have*

$$|(\operatorname{curl} \upsilon(z) \times z, \phi)| \le \widehat{K} \|\phi\|_{H^2} \|z\|_{H^2}^2. \tag{2.5}$$

Proof. We satisfy that

$$\begin{aligned} |(\operatorname{curl} \upsilon(z) \times z, \phi)| &\le |b(\phi, z, \upsilon(z))| + |b(z, \phi, \upsilon(z))| \\ &\le \|\phi\|_4 \|z\|_{W^{1,4}} \|z\|_{H^2} + \|z\|_4 \|\phi\|_{W^{1,4}} \|z\|_{H^2} \\ &\le \widehat{K} \|\phi\|_{H^2} \|z\|_{H^2}^2. \end{aligned}$$

□

In this article, we assume that the external mechanical force u and the initial data y_0 satisfy

$$u \in L^2(0, T; H^1) \quad \text{and} \quad y_0 \in W \cap H^3. \tag{2.6}$$

Lemma 2.2. *Under assumptions* (2.6) *there exists a unique solution*

$$y \in L^\infty(0, T; W \cap H^3) \text{ with } \frac{\partial y(t)}{\partial t} \in L^2(0, T; V)$$

of the problem (2.1)*, which is understood in the distributional sense*

$$\begin{aligned} &\big(\frac{\partial y(t)}{\partial t}, \phi\big) + 2\alpha\big(D\frac{\partial y(t)}{\partial t}, D\phi\big) + 2\nu(Dy(t), D\phi) \\ &+ (\operatorname{curl} \upsilon(y(t)) \times y(t), \phi) = (u(t), \phi), \quad \forall \phi \in V. \end{aligned} \tag{2.7}$$

Moreover, the solution satisfies

$$\|y\|_{L^\infty(0,T;H^3)}^2 \le \frac{C_1^2 \lambda_1^2}{\alpha^2} \tag{2.8}$$

with

$$\lambda_1^2 = (1 + 4K\widehat{\alpha})(\|y_0\|_{H^3}^2 + \|u\|_{L^1(0,T;H^1)}^2),$$

where the constant K is defined by (2.2)*, and $\widehat{\alpha} = \max((2\alpha)^{-1}, 2\alpha)$.*

Proof. The solvability of the problem (2.1) is shown in [7]. Let us write in a convenient form the estimates for the state variable y with respect to the initial data y_0 and the control variable u. The estimates (3.4) and (3.5) of Proposition 3.2 in [2] give

$$\|y\|_{L^\infty(0,T;H^3)}^2 \le \frac{C^2}{\alpha^2}\Big(\|y\|_{L^\infty(0,T;H^1)}^2 + \|\operatorname{curl} \sigma(y_0)\|_{L^2}^2 + \|\operatorname{curl} u\|_{L^1(0,T;L^2)}^2\Big). \tag{2.9}$$

In addition, relation (3.3) of Proposition 3.2 in [2] for y in H^1, and the Korn's inequality (2.2) yield

$$\|y\|_{L^\infty(0,T;H^1)}^2 \le 4K \frac{\max(1, 2\alpha)}{\min(1, 2\alpha)}\big(\|y_0\|_{H^1}^2 + \|u\|_{L^1(0,T;L^2)}^2\big). \tag{2.10}$$

Combining inequalities (2.9) and (2.10) we derive (2.8). □

Let us introduce the so-called *control-to-state* mapping $S : u \to y$, namely $y = S(u)$ is the solution of the equation (2.1) corresponding to the control u, and consider the reference pair $(u, y = S(u))$. In the next lemma, we recall a stability result for the solution of (2.1), which was proved in [2, Proposition 4.4].

Lemma 2.3. *Let us consider the initial data y_0 and two different control variables u_1, u_2 satisfying the assumptions* (2.6)*. Let $y_1 = S(u_1)$, $y_2 = S(u_2)$ be the corresponding solutions of* (2.1) *with the same initial data y_0. Then the difference $\overline{y} = y_2 - y_1$ satisfies the estimate*

$$\|\overline{y}\|^2_{L^\infty(0,T;H^2)} \le \lambda_2^2 \|\overline{u}\|^2_{L^2(Q)} \tag{2.11}$$

with $\overline{u} = u_2 - u_1$ and

$$\lambda_2^2 = \widetilde{K}\Big[\Big(1 + C_2T(1+\alpha^{-1})\frac{C_1\lambda_1}{\alpha}\Big)e^{C_2T(1+(1+\alpha^{-1})C_1\lambda_1)} + (\alpha\nu)^{-1}\Big]$$

for some positive constant C_1, C_2 depending only on $\mathcal{O}$.

Proof. By the first estimate of [2, Proposition 4.4] and (2.8), we deduce

$$\|\overline{y}\|^2_{L^\infty(0,T;L^2)} + 2\alpha\|D\overline{y}\|^2_{L^\infty(0,T;H^1)} \le e^{C_2T(1+(1+\alpha^{-1})C_1\lambda_1)}\|\overline{u}\|^2_{L^2(Q)}.$$

Using the second estimate of [2, Proposition 4.4] and (2.3), we obtain

$$\begin{aligned}
&\|\overline{y}\|^2_{L^\infty(0,T;H^2)} \\
&\le \widetilde{K}\Big[\|\overline{y}\|^2_{L^\infty(0,T;L^2)} + \frac{1}{\alpha}\Big(\frac{1}{\nu}\|\overline{u}\|^2_{L^2(Q)} + C_2T(1+\alpha^{-1})C_1\lambda_1\|\overline{y}\|^2_{L^\infty(0,T;L^2)}\Big)\Big] \\
&\le \widetilde{K}\Big[\Big(1 + C_2T(1+\alpha^{-1})\frac{C_1\lambda_1}{\alpha}\Big)\|\overline{y}\|^2_{L^\infty(0,T;L^2)} + (\alpha\nu)^{-1}\|\overline{u}\|^2_{L^2(Q)}\Big].
\end{aligned}$$

These two inequalities imply (2.11). □

According to [2, Proposition 4.5], the first-order Gâteaux derivative $z = S'(u)[w]$ of the mapping S, at the point u, in the direction w, is given by the solution of the linearized state equation at (u, y)

$$\begin{gathered}
\frac{\partial \upsilon(z)}{\partial t} - \nu\Delta z + \operatorname{curl}\upsilon(z)\times y + \operatorname{curl}\upsilon(y)\times z + \nabla\pi = w, \\
\nabla\cdot z = 0 \quad \text{in } Q, \\
z\cdot n = 0, \quad (n\cdot Dz)\cdot\tau = 0 \quad \text{on } \Sigma, \\
z(0) = 0 \quad \text{in } \mathcal{O},
\end{gathered} \tag{2.12}$$

which is well-posed in the Sobolev space H^2. The next lemma establishes a suitable estimate for the solution z.

Lemma 2.4. *Under assumptions* (2.6) *there exists a unique solution $z \in L^\infty(0,T;W)$ of* (2.12)*, such that*

$$\|z\|^2_{L^\infty(0,T;H^2)} \le \lambda_3^2\|w\|^2_{L^2(Q)}, \tag{2.13}$$

where

$$\begin{aligned}
\lambda_3^2 = \widetilde{K}\Big[&(\alpha\nu)^{-1}e^{C_3TC_1\lambda_1(\frac{1+\alpha}{\alpha^2})}\Big(1 + \frac{2}{\alpha}C_3TC_1\lambda_1(1+\alpha^{-1})e^{C_3TC_1\lambda_1(\frac{1+\alpha}{\alpha^2})}\Big) \\
&\times e^{C_3T(1+C_1\lambda_1(1+\alpha^{-1}))}\Big]
\end{aligned}$$

with λ_1 defined in (2.8)*.*

Proof. From the first estimate of [2, Proposition 4.3] and (2.8), we have

$$\|z\|^2_{L^\infty(0,T;L^2)} \le \|w\|^2_{L^2(Q)}e^{C_3T(1+C_1\lambda_1(1+\alpha^{-1}))}. \tag{2.14}$$

In addition, the second estimate of [2, Proposition 4.3] and (2.8) yield

$$\begin{aligned} &\|\mathbb{A}z\|^2_{L^\infty(0,T;L^2)} \\ &\leq \frac{1}{\alpha}\Big[\frac{1}{\nu}\|w\|^2_{L^2(Q)} + 2C_3TC_1\lambda_1(1+\alpha^{-1})\|z\|^2_{L^\infty(0,T;L^2)}\Big]e^{C_3TC_1\lambda_1(\frac{1+\alpha}{\alpha^2})} = R. \end{aligned} \tag{2.15}$$

Therefore, taking into account (2.3) and (2.15), we deduce that

$$\begin{aligned} \|z\|^2_{L^\infty(0,T;H^2)} &\leq \widetilde{K}[\|z\|^2_{L^\infty(0,T;L^2)} + R] \\ &= \widetilde{K}\Big[\Big(1 + \frac{1}{\alpha}2C_3TC_1\lambda_1(1+\alpha^{-1})e^{C_3TC_1\lambda_1(\frac{1+\alpha}{\alpha^2})}\Big) \\ &\quad \times \|z\|^2_{L^\infty(0,T;L^2)} + (\alpha\nu)^{-1}e^{C_3TC_1\lambda_1(\frac{1+\alpha}{\alpha^2})}\|w\|^2_{L^2(Q)}\Big]. \end{aligned}$$

Using (2.14), we obtain (2.13). □

Now, we formulate the control problem (see [2]). The space $\mathcal{U}_{ad}$ of the admissible control variables is a bounded, closed and convex subset of $L^2(0,T;H^1)$. Namely, there exists a constant $L > 0$ such that

$$\|u\|_{L^2(0,T;H^1)} \leq L, \quad \forall u \in \mathcal{U}_{ad}. \tag{2.16}$$

The control on the evolution of the physical system is imposed through a distributed mechanical force $u \in \mathcal{U}_{ad}$, aiming to match a desired target velocity profile

$$y_d \in L^2(Q).$$

The control u and the state $y = S(u)$ are constrained to satisfy the system (2.1), and the optimal control problem reads

$$\text{minimize}_u\{J(u,y) : u \in \mathcal{U}_{ad},\ y = S(u)\}, \tag{2.17}$$

where the objective functional is

$$J(u,y) = \frac{1}{2}\int_0^T\int_{\mathcal{O}} |y - y_d|^2\,dxdt + \frac{\lambda}{2}\int_0^T\int_{\mathcal{O}} |u|^2\,dx\,dt, \quad u \in \mathcal{U}^b_{ad}, \tag{2.18}$$

and $\lambda \geq 0$ is a fixed cost coefficient.

Let us recall that the first-order optimality conditions, at a locally optimal pair (u^*, y^*), can be formally deduced through the Lagrange's multipliers method. More precisely, considering the Lagrange function

$$L(u,y,p) = J(u,y) - \int_0^T \Big(p, -\frac{\partial}{\partial t}\upsilon(y) + \nu\Delta y - \operatorname{curl}\upsilon(y)\times y - \nabla\pi + u\Big)\,dt,$$

the optimal pair (u^*, y^*) should satisfy the relations

$$L'_y(u^*, y^*, p)[z] = 0, \tag{2.19}$$

$$L'_u(u^*, y^*, p)[w] \geq 0, \quad \forall w \in \mathcal{U}_{ad}. \tag{2.20}$$

Equation (2.19) yields the adjoint equation of the state equation linearized at the point $y = y^*$, and the system composed by equations (2.19), (2.20) and the state equations (2.1) for (u^*, y^*) constitute the so-called first-order optimality conditions.

The adjoint equation of the state equation linearized at the point $y = y^*$ arising from (2.19) is

$$\begin{gathered}\frac{\partial}{\partial t}\upsilon(p) = -\nu\Delta p - \operatorname{curl}\upsilon(y)\times p + \operatorname{curl}\upsilon(y\times p) + \nabla\pi - (y - y_d),\\ \operatorname{div} p = 0 \quad \text{in } Q,\\ p\cdot n = 0, \quad (n\cdot Dp)\cdot\tau = 0 \quad \text{on } \Sigma,\\ p(T) = 0 \quad \text{in } \mathcal{O}.\end{gathered} \tag{2.21}$$

Here, we recall the main result in [2], which shows the existence of an optimal solution and establishes the first-order optimality conditions.

Theorem 2.5 ([2]). *Under assumptions* (2.6) *and* (2.16), *problem* (2.17) *admits at least one solution*

$$(u, y) = (u, S(u)) \in \mathcal{U}_{ad} \times L^\infty(0, T; W \cap H^3).$$

Furthermore, there exists a unique solution

$$p = S^*(u) \in L^\infty(0, T; H^2)$$

of the adjoint system (2.21), *such that the following optimality condition holds*

$$\int_0^T (\lambda u + p, \Psi - u)\,dt \geq 0, \quad \forall\Psi \in \mathcal{U}_{ad}. \tag{2.22}$$

A triplet (u^*, y^*, p^*) obtained as a solution of the coupled system (constituted by the state equation (2.1), the adjoint equation (2.21) and the variational inequality (2.22)) is a candidate for an optimal solution, but not necessarily an optimal solution. The goal of this article is to analyze the conditions (on the initial data or on the parameters of the model) which guarantee that the solution of the coupled system is optimal and unique. This is a crucial step towards the implementation of the numerical methods to approximate the optimal control. To perform this task, we start by establishing convenient estimates for the adjoint state p.

Lemma 2.6. *Under the assumptions of Theorem 2.5, the adjoint state p satisfies*

$$\begin{aligned}&\|p\|^2_{L^\infty(0,T;H^2)}\\ &\leq \lambda_4^2\\ &= 2\widetilde{K}\Big[\Big(1 + \frac{1}{\alpha}C_4TC_1\lambda_1(1+\alpha^{-1})e^{C_4TC_1\lambda_1(\frac{1+\alpha}{\alpha^2})}\Big)\\ &\quad\times e^{C_4T(1+C_1\lambda_1(1+\alpha^{-1}))} + (\alpha\nu)^{-1}e^{C_4TC_1\lambda_1\frac{(1+\alpha)^2}{\alpha}}\Big]\Big(\frac{C_1^2\lambda_1^2}{\alpha^2} + \|y_d\|^2_{L^2(Q)}\Big),\end{aligned} \tag{2.23}$$

where the constants $\widetilde{K}$ and λ_1 are defined by (2.3) *and* (2.8), *respectively.*

Proof. Taking into account the first estimate of [2, Proposition 5.4], we have

$$\|p\|^2_{L^\infty(0,T;L^2)} \leq \|y - y_d\|^2_{L^2(Q)}e^{C_4T(1+C_1\lambda_1(1+\alpha^{-1}))}. \tag{2.24}$$

The second estimate of the same proposition gives

$$\begin{aligned}&\|\mathbb{A}p\|^2_{L^\infty(0,T;L^2)}\\ &\leq \frac{1}{\alpha}\big(\frac{1}{\nu}\|y - y_d\|^2_{L^2(Q)} + C_4TC_1\lambda_1(1+\alpha^{-1})\|p\|^2_{L^\infty(0,T;L^2)}\big)e^{C_4TC_1\lambda_1\frac{(1+\alpha)^2}{\alpha}}.\end{aligned} \tag{2.25}$$

Therefore, using the inequalities (2.24) and (2.25) and (2.3), we can apply the same reasoning as in the proof of Lemma 2.4 to obtain the claimed result. □

3. Uniqueness results for the control problem (2.17)

This section establishes the main results of the article. First, by analyzing the second-order derivative of the control-to-state mapping, as well as the second-order derivative of the cost functional, we deduce a sufficient second-order optimality condition, which guarantees that any triplet (u^*, y^*, w) obtained as a solution of the first-order coupled optimality system will produce a locally optimal pair (u^*, y^*).

Next, we will be able to prove that the solution of the first-order coupled optimality system is unique. This uniqueness result conjugated with the result of Theorem 2.5 yields the uniqueness of the solution for the optimal control problem (2.17). Our results will be achieved under some natural assumptions relying on the size of the initial data and the parameters of the model, or on the intensity of the cost. Essentially, if the fluid material is sufficiently viscous and elastic and the initial condition is small enough, or instead if the intensity of the cost is big enough, the solution of the first-order optimality system is unique, and corresponds to the unique global solution of the optimal control problem (2.17).

3.1. **Sufficient second-order optimality conditions.** In this section, we perform a second-order analysis. Let $y = S(u)$ be the solution of (2.1) for u (and y_0) satisfying (2.6). Let us consider some w_i, $i = 1, 2$, satisfying (2.6). Denoting

$$z_i = S'(u)[w_i], \quad i = 1, 2,$$

we can use standard arguments to show that the second-order Gâteaux derivative

$$\widetilde{z} = S''(u)[w_1, w_2] = \frac{d}{d\epsilon}\Big|_{\epsilon=0} S'(u + \epsilon w_2)[w_1]$$

solves the system

$$\begin{gathered}
\frac{\partial \upsilon(\widetilde{z})}{\partial t} - \nu \triangle \widetilde{z} + \operatorname{curl} \upsilon(\widetilde{z}) \times y + \operatorname{curl} \upsilon(y) \times \widetilde{z} + \nabla \widetilde{\pi} \\
= - \operatorname{curl} \upsilon(z_1) \times z_2 - \operatorname{curl} \upsilon(z_2) \times z_1, \\
\nabla \cdot \widetilde{z} = 0 \quad \text{in } Q, \\
\widetilde{z} \cdot n = 0, \quad (n \cdot D\widetilde{z}) \cdot \tau = 0 \quad \text{on } \Sigma, \\
\widetilde{z}(0) = 0 \quad \text{in } \mathcal{O}.
\end{gathered} \tag{3.1}$$

Definition 3.1. The control problem (2.17) is said to satisfies the second-order optimality condition at an optimal pair $(u, S(u))$, if there exists $\delta > 0$, such that the coercivity condition

$$J''(u)[w, w] > \delta \|w\|^2_{L^2(Q)}, \quad \forall w \in \mathcal{U}_{ad}, \tag{3.2}$$

holds.

Theorem 3.2. *Assume the hypothesis of Theorem 2.5 hold. Then the control problem* (2.17) *satisfies the second-order optimality condition at the optimal pair* $(u, S(u))$ *under the assumption that*

$$\lambda > 2\widehat{K}\lambda_3^2\lambda_4, \tag{3.3}$$

where $\widehat{K}, \lambda_3, \lambda_4$ *are defined in Lemmas 2.1, 2.4 and 2.6.*

Proof. Denoting $J(u) = J(u, S(u))$, we have

$$J'(u)[w_1] = \int_0^T ((z_1, y - y_d) + \lambda(u, w_1))dt$$

by [2, Proposition 4.5]. By calculations similar to the mentioned proposition, we can satisfy that

$$J''(u)[w_1,w_2] = \int_0^T ((z_1,z_2) + (\widetilde{z}, y - y_d) + \lambda(w_2,w_1))dt. \tag{3.4}$$

Then system (3.1), and the duality relation proved in [2, Proposition 5.5] yield

$$J''(u)[w_1,w_2] = \int_0^T ((z_1,z_2) - (p, \operatorname{curl}\upsilon(z_1)\times z_2 + \operatorname{curl}\upsilon(z_2)\times z_1) + \lambda(w_2,w_1))dt.$$

Hence taking $w_1 = w_2$, we have

$$J''(u)[w_1,w_1] = \int_0^T (\|z_1\|_2^2 + \lambda\|w_1\|_2^2 - 2(p, \operatorname{curl}\upsilon(z_1)\times z_1))dt.$$

By Lemmas 2.1, 2.4 and 2.6, we deduce the estimate

$$\begin{aligned}
\int_0^T |(p, \operatorname{curl}\upsilon(z_1)\times z_1)|dt &\le \widehat{K}\int_0^T \|z_1\|_{H^2}^2\|p\|_{H^2}dt \\
&\le \widehat{K}\lambda_4 \int_0^T \|z_1\|_{H^2}^2 dt \\
&\le \widehat{K}\lambda_3^2\lambda_4 \int_0^T \|w_1\|_2^2 dt.
\end{aligned}$$

Therefore,

$$J''(u)[w_1,w_1] > (\lambda - 2\widehat{K}\lambda_3^2\lambda_4)\int_0^T \|w_1\|_2^2 dt,$$

which implies the result of the theorem. □

3.2. **Uniqueness of the global optimal solution.** Now, we are able to show that the solution of the coupled system is unique, and provides the unique global optimal solution for the non-convex optimal control problem (2.17).

Theorem 3.3. *For any $\lambda > 2\widehat{K}\lambda_2^2\lambda_4$, the optimal control problem* (2.17) *has a unique global solution, where $\widehat{K}$, λ_3, λ_4 are defined in Lemmas 2.1, 2.3, and 2.6.*

Proof. We assume that u_1 and u_2 are two optimal control variables for problem (2.17). Let $y_i = S(u_i)$, $i = 1,2$, be the corresponding optimal states with the adjoint states $p_i = S^*(u_i)$, $i = 1,2$. Let us set $\overline{u} = u_2 - u_1$, $\overline{\pi} = \pi_2 - \pi_1$. The differences $\overline{y} = y_2 - y_1$ and $\overline{u} = u_2 - u_1$ solve the system

$$\begin{gathered}
\frac{\partial}{\partial t}\upsilon(\overline{y}) = \nu\Delta\overline{y} - \operatorname{curl}\upsilon(\overline{y})\times y_1 - \operatorname{curl}\upsilon(y_1)\times\overline{y} \\
- \operatorname{curl}\upsilon(\overline{y})\times\overline{y} - \nabla\overline{\pi} + \overline{u}, \\
\operatorname{div}\overline{y} = 0 \quad \text{in } Q, \\
\overline{y}\cdot n = 0, \quad (n\cdot D\overline{y})\cdot\tau = 0 \quad \text{on } \Sigma, \\
\overline{y}(0) = 0 \quad \text{in } \mathcal{O}.
\end{gathered} \tag{3.5}$$

The functions y_1, y_2 are the weak solutions of (2.1), which satisfies eq2.7). Therefore, considering the test function $\phi = p_1$ in eq2.7), we show that $\overline{y}$ verifies the variational equality

$$\big(\frac{\partial\overline{y}(t)}{\partial t}, p_1\big)_V = -2\nu(Dp_1(t), D\overline{y}) - b(p_1, y_1, \upsilon(\overline{y})) + b(y_1, p_1, \upsilon(\overline{y}))$$

$$- b(p_1, \overline{y}, \upsilon(y_1)) + b(\overline{y}, p_1, \upsilon(y_1)) - (\operatorname{curl} \upsilon(\overline{y}) \times \overline{y}, p_1) + (\overline{u}, p_1).$$

Writing the adjoint system (2.21) for p_1 in a respective variational form with the test function $\phi = \overline{y}$, we obtain the equality

$$\begin{aligned}\Big(\frac{\partial p_1}{\partial t}, \overline{y}(t)\Big)_V &= 2\nu(Dp_1(t), D\overline{y}) - b(\overline{y}, p_1, \upsilon(y_1)) + b(p_1, \overline{y}, \upsilon(y_1)) \\ &\quad + b(p_1, y_1, \upsilon(\overline{y})) - b(y_1, p_1, \upsilon(\overline{y})) - (y_1 - y_d, \overline{y}).\end{aligned}$$

Therefore summing the last two equalities, we obtain

$$\frac{\partial}{\partial t}(\overline{y}(t), p_1)_V = -(\operatorname{curl} \upsilon(\overline{y}) \times \overline{y}, p_1) - (y_1 - y_d, \overline{y}) + (\overline{u}, p_1).$$

The integration over $t \in (0, T)$ and the initial and final conditions for $\overline{y}$ and p_1 give

$$0 = \int_0^T -(\operatorname{curl} \upsilon(\overline{y}) \times \overline{y}, p_1) - (y_1 - y_d, \overline{y}) + (\overline{u}, p_1)\, dt. \tag{3.6}$$

By the symmetry, we can easily satisfy that the difference $y_1 - y_2 = -\overline{y}$ solves the system

$$\begin{gathered}-\frac{\partial}{\partial t}\upsilon(\overline{y}) = -\nu\Delta\overline{y} + \operatorname{curl}\upsilon(\overline{y}) \times y_2 + \operatorname{curl}\upsilon(y_2) \times \overline{y} - \operatorname{curl}\upsilon(\overline{y}) \times \overline{y} + \nabla\overline{\pi} - \overline{u}, \\ \operatorname{div}\overline{y} = 0 \quad \text{in } Q, \\ \overline{y} \cdot n = 0, \quad (n \cdot D\overline{y}) \cdot \tau = 0 \quad \text{on } \Sigma, \\ \overline{y}(0) = 0 \quad \text{in } \mathcal{O}.\end{gathered}$$

Using the same reasoning as above, applied for $\overline{y}$, we deduce that $y_1 - y_2$ satisfies the relation

$$0 = \int_0^T -(\operatorname{curl} \upsilon(\overline{y}) \times \overline{y}, p_2) + (y_2 - y_d, \overline{y}) - (\overline{u}, p_2)\, dt. \tag{3.7}$$

The sum of the equalities (3.6) and (3.7) yields

$$\int_0^T \|\overline{y}\|_2^2\, dt + \int_0^T (\overline{u}, p_1 - p_2) dt = \int_0^T (\operatorname{curl} \upsilon(\overline{y}) \times \overline{y}, p_1 + p_2)\, dt. \tag{3.8}$$

The optimality condition (2.22) for the optimal control u_1 with $\Psi = u_2$ reads

$$\int_0^T \{\lambda(u_1, \overline{u}) + (\overline{u}, p_1)\}\, dt \geq 0.$$

Analogously, the optimality condition (2.22) for the optimal control u_2 with $\Psi = u_1$ yields

$$-\int_0^T \{(\lambda u_2, \overline{u}) + (\overline{u}, p_2)\}\, dt \geq 0.$$

Adding the last two inequalities, we deduce that

$$\lambda \int_0^T \|\overline{u}\|_2^2\, dt \leq \int_0^T (\overline{u}, p_1 - p_2)\, dt. \tag{3.9}$$

Introducing this relation in (3.8), we obtain

$$\lambda \int_0^T \|\overline{u}\|_2^2\, dt + \int_0^T \|\overline{y}\|_2^2\, dt \leq \int_0^T (\operatorname{curl} \upsilon(\overline{y}) \times \overline{y}, p_1 + p_2)\, dt.$$

By Lemmas 2.1, 2.3, and 2.6 we have

$$\begin{aligned}\int_0^T |(\operatorname{curl} \upsilon(\overline{y}) \times \overline{y}, p_1 + p_2)| dt &\leq \int_0^T \|\overline{y}\|_{H^2}^2 (\|p_1\|_{H^2} + \|p_2\|_{H^2}) dt \\ &\leq 2\widehat{K}\lambda_4 \int_0^T \|\overline{y}\|_{H^2}^2 dt \\ &\leq 2\widehat{K}\lambda_2^2\lambda_4 \int_0^T \|\overline{u}\|_2^2 dt.\end{aligned}$$

Therefore,

$$(\lambda - 2\widehat{K}\lambda_2^2\lambda_4) \int_0^T \|\overline{u}\|_2^2 + \int_0^T \|\overline{y}\|_2^2 \, dt \leq 0,$$

which implies the result of the theorem. □

Let us remark that in Theorem 3.3, the constants λ_2, λ_4 just depend on the bounded set $\mathcal{U}_{ad}$ of the admissible control variables and the target and initial velocities y_d, y_0. Hence, from practical point of view, the motion of a second grade fluid can be optimally controlled by taking a sufficiently large intensity of the cost λ.

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