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MULTIPLE SOLUTIONS FOR $p(x)$-KIRCHHOFF TYPE PROBLEMS WITH ROBIN BOUNDARY CONDITIONS

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Abstract. This article considers $p(x)$-Kirchhoff type problems with Robin boundary conditions. Using the mountain pass theorem, the Ekeland's variational principle, and Krasnoselskii's genus theory, we prove that the problem has at least two nontrivial weak solutions or infinitely many nontrivial weak solutions under some suitable conditions on the nonlinearities. The main results improve and generalize the previous ones introduced in [2, 7].

1. Introduction

In this article, we study the existence of weak solutions for $p(x)$-Kirchhoff type problems with Robin boundary conditions

$$\begin{gathered}
-M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\Big)\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big)\\
= f(x,u)+\lambda g(x),\quad x\in\Omega, \\
|\nabla u|^{p(x)-2}\frac{\partial u}{\partial\nu}+\beta(x)|u|^{p(x)-2}u=0,\quad x\in\partial\Omega,
\end{gathered}\tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial\nu}$ is the outer normal derivative, $d\sigma_x$ is the measure on the boundary $\partial\Omega$, $\beta\in L^\infty(\partial\Omega)$, $\beta^-:=\inf_{x\in\partial\Omega}\beta(x)>0$, $p\in C_+(\overline{\Omega})$, $1<p^-:=\inf_{x\in\overline{\Omega}}p(x)\le p^+:=\max_{x\in\overline{\Omega}}p(x)<N$, λ is a nonnegative parameter, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and $M:\mathbb{R}^+:=[0,+\infty)\to\mathbb{R}^+$ are two continuous functions, $g:\Omega\to\mathbb{R}$ is a measurable function.

Problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\,dx\Big)\frac{\partial^2u}{\partial x^2}=0,\tag{1.2}$$

presented by Kirchhoff in 1883 as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings, see [22]. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. Problem (1.2) is often called a nonlocal problem because it contains an integral over Ω. This causes some mathematical difficulties which make the study of such a problem particularly interesting. The nonlocal problem

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models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [8].

Kirchhoff type problems have been studied in many papers in the previous decades. In [7, 13, 21, 24, 27, 28], using various methods the authors study the existence and multiplicity of solutions for Kirchhoff type problems involving the p-Laplacian operator $-\Delta_p(\cdot) = -\operatorname{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$. The $p(x)$-Laplacian operator where $p(\cdot)$ is a continuous function possesses more complicated properties than the p-Laplacian operator, mainly due to the fact that it is not homogeneous. The study of various mathematical problems with variable exponent are interesting in applications and raise many difficult mathematical problems, see [23, 25]. For this reason, ordinary differential and partial differential equations with nonstandard growth conditions have received specific attention in recent years, we refer to some results on $p(x)$-Kirchhoff type problems with Dirichlet or Neumann boundary conditions [5, 9, 10, 12, 14, 15, 20]. Relatively speaking, Kirchhoff type problems with Robin boundary conditions have rarely been considered. Robin boundary conditions are a weighted combination of Dirichlet and Neuman boundary conditions and it is also called impedance boundary conditions, from their application in electromagnetic problems or convective boundary conditions from their application in heat transfer problems. Moreover, Robin conditions are commonly used in solving Sturm-Liouville problems which appear in many contexts in sciences and engineering, see [16]. To the best of our knowledge, Allaoui [2] first introduced the $p(x)$-Kirchhoff type problems involving Robin boundary conditions and studied problem (1.1) in the case $\lambda = 0$ by using the mountain pass theorem, the fountain theorem and some properties of $(S)_+$ type operator. Regarding the $p(x)$-Laplacian problems with the Robin boundary conditions in the local case when $M(t) \equiv 1$, we refer to some papers [1, 3, 16, 19, 26], in which some existence and multiplicity results were obtained by using variational methods. Motivated by above mentioned papers and the results on the Kirchhoff type problem involving Laplace operator $-\Delta(\cdot)$ in [7], the purpose of this article is to consider Robin problem (1.1) with perturbation g and parameter λ. More precisely, under some suitable conditions on the nonlinear term f and the Kirchhoff function M, we prove that problem (1.1) has at least two weak solutions if $\lambda > 0$ small enough, see Theorem 2.2. In the case when $\lambda = 0$, we prove problem (1.1) with subcritical growth condition has infinitely many solutions, see Theorem 2.7. Our proofs are essentially based on the mountain pass theorem [4], the Ekeland variational principle [18] and Krasnoselskii's genus theory [11]. We emphasize that the results introduced here are new even in the case when $p(\cdot)$ is a constant and we do not need the non-degenerate condition on the Kirchhoff function M as in [2, 7], see assumption (A1).

Next we recall some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the books [17, 25] and the papers [1, 16, 19, 23]. Set

$$C_+(\overline{\Omega}) := \{h;\ h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\overline{\Omega}} h(x), \quad h^- = \inf_{x\in\overline{\Omega}} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \big\{u \text{ measurable real-valued functions such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\big\}.$$

We recall the following so-called *Luxemburg norm* on this space defined by the formula

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} := \inf\Big\{\lambda > 0 : \int_\Omega \big|\frac{u(x)}{\lambda}\big|^{p(x)}\,dx \le 1\Big\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \le p^+ < +\infty$ and continuous functions are dense if $p^+ < +\infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < +\infty$ and p_1, p_2 are variable exponents so that $p_1(x) \le p_2(x)$ a.e. $x \in \Omega$ then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequalities

$$\big|\int_\Omega uv\,dx\big| \le \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}$$

hold. An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < +\infty$ then the following relations hold

$$|u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}, \tag{1.3}$$

provided $|u|_{p(x)} > 1$ while

$$|u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}, \tag{1.4}$$

provided $|u|_{p(x)} < 1$ and

$$|u_n - u|_{p(x)} \to 0 \iff \rho_{p(x)}(u_n - u) \to 0. \tag{1.5}$$

If $p \in C_+(\overline{\Omega})$ the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$, consisting of functions $u \in L^{p(x)}(\Omega)$ whose distributional gradient ∇u exists almost everywhere and belongs to $[L^{p(x)}(\Omega)]^N$, endowed with the norm

$$\|u\| := \inf\Big\{\lambda > 0 : \int_\Omega \Big[\big|\frac{\nabla u(x)}{\lambda}\big|^{p(x)} + \big|\frac{u(x)}{\lambda}\big|^{p(x)}\Big]\,dx \le 1\Big\}$$

or

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)},$$

is a separable and reflexive Banach space. The space of smooth functions are in general not dense in $W^{1,p(x)}(\Omega)$, but if the exponent $p \in C_+(\overline{\Omega})$ is logarithmic Hölder continuous, that is,

$$|p(x) - p(y)| \le -\frac{M}{\log(|x-y|)}, \quad \forall x, y \in \Omega,\ |x-y| \le \frac{1}{2},$$

then the smooth functions are dense in $W^{1,p(x)}(\Omega)$. The space $\big(W_0^{1,p(x)}(\Omega), \|\cdot\|\big)$ is a separable and Banach space. We note that if $s \in C_+(\overline{\Omega})$ and $s(x) < p^*(x)$ for all $x \in \overline{\Omega}$ then the embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$$

is compact and continuous, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) > N$. If $s \in C_+(\partial\Omega)$ and $s(x) < p_*(x)$ for all $x \in \partial\Omega$ then the trace embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\partial\Omega)$$

is compact and continuous, where $p_*(x) = \frac{(N-1)p(x)}{N-p(x)}$ if $p(x) < N$ or $p_*(x) = +\infty$ if $p(x) > N$. Moreover, for any $u \in W^{1,p(x)}(\Omega)$, let us define

$$\|u\|_\partial := |\nabla u|_{L^{p(x)}(\Omega)} + |u|_{L^{p(x)}(\partial\Omega)},$$

then $\|u\|_\partial$ is a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to the norm $\|u\|$, see [16, Theorem 2.1].

Now, let us introduce a norm which will be used later. Let $\beta \in L^\infty(\partial\Omega)$ with $\beta^- = \inf_{x\in\partial\Omega} \beta(x) > 0$, and for any $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_{\beta(x)} := \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{\nabla u(x)}{\lambda}\Big|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)\Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\,d\sigma_x\,dx \le 1\Big\},$$

where $d\sigma_x$ is the measure on the boundary $\partial\Omega$. Then $\|u\|_{\beta(x)}$ is also a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to $\|\cdot\|$ and $\|\cdot\|_\partial$. Let

$$I_{\beta(x)}(u) = \int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)}\,d\sigma_x,$$

we have

$$\|u\|_{\beta(x)}^{p^-} \le I_{\beta(x)}(u) \le \|u\|_{\beta(x)}^{p^+} \tag{1.6}$$

provided $\|u\|_{\beta(x)} > 1$ while

$$\|u\|_{\beta(x)}^{p^+} \le I_{\beta(x)}(u) \le \|u\|_{\beta(x)}^{p^-} \tag{1.7}$$

provided $\|u\|_{\beta(x)} < 1$ and

$$\|u_n - u\|_{\beta(x)} \to 0 \iff I_{\beta(x)}(u_n - u) \to 0. \tag{1.8}$$

Proposition 1.1 (see [19]). *For $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega} \beta(x) > 0$, let us define the functional $L_{\beta(x)} : W^{1,p(x)}(\Omega) \to \mathbb{R}$ by*

$$L_{\beta(x)}(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x \tag{1.9}$$

for all $u \in W^{1,p(x)}(\Omega)$. Then $L_{\beta(x)} \in C^1(W^{1,p(x)}(\Omega), \mathbb{R})$ and its derivative is given by

$$L'_{\beta(x)}(u)(v) = \int_\Omega |\nabla u|^{p(x)-2}uv\,dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv\,d\sigma_x. \tag{1.10}$$

Moreover, we have the following assertions

(i) *$L'_{\beta(x)} : W^{1,p(x)}(\Omega) \to W^{-1,p(x)}(\Omega)$ is a continuous, bounded and strictly monotone operator;*

(ii) *$L'_{\beta(x)} : W^{1,p(x)}(\Omega) \to W^{-1,p(x)}(\Omega)$ is a mapping of type $(S)_+$, i.e. if $\{u_n\}$ converges weakly to u in $W^{1,p(x)}(\Omega)$ and $\limsup_{n\to\infty} L'_{\beta(x)}(u_n)(u_n-u) \le 0$, then $\{u_n\}$ converges strongly to u in $W^{1,p(x)}(\Omega)$.*

In the rest of this section, we introduce some notions and results on Krasnoselskii's genus theory, the readers can consult [6, 11]. Let Y be a real Banach space. Let us denote by $\mathcal{R}$ the class of all closed subsets $A \subset X\backslash\{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$, i.e.

$$\mathcal{R} = \{A \subset Y\backslash\{0\} : A \text{ is compact and } A = -A\}.$$

Definition 1.2. Let $A \in \mathcal{R}$ and $Y = \mathbb{R}^N$. The genus $\gamma(A)$ of A is defined by

$$\gamma(A) = \min\{k \geq 1 : \text{ there exists an odd continuous mapping } \phi : A \to \mathbb{R}^k\backslash\{0\}\}.$$

If such a mapping ϕ does not exist for any $k > 0$, we set $\gamma(A) = +\infty$.

Note that if A is a subset, which consists of finitely many pairs of points, then $\gamma(A) = 1$. Moreover, from the above definition, $\gamma(\emptyset) = 0$. A typical example of a set of genus k is a set, which is homeomorphic to a $(k-1)$ dimensional sphere via an odd map.

Proposition 1.3. *Let $Y = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then we have $\gamma(\partial\Omega) = N$.*

Let us denote by S the unit sphere in Y. It follows from Proposition 1.3 that $\gamma(S^{N-1}) = N$. If Y is of infinite dimension and separable then $\gamma(S) = +\infty$. We now recall an application of Palais-Smale "compactness" criterion, which was introduced by Clark [11].

Proposition 1.4. *Let $J \in C^1(Y, \mathbb{R})$ be a functional satisfying the Palais-Smale condition. Furthermore, let us suppose that*

(i) *J is bounded from below and even;*
(ii) *There is a compact set $K \in \mathcal{R}$ such that $\gamma(K) = k$ and $\sup_{x\in K} J(x) < J(0)$.*

Then J possesses at least k pairs of distinct critical points, and their corresponding critical values are less than $J(0)$.

2. Main results

2.1. **Existence of at least two solutions.** In this part, we consider problem (1.1) in the case when $\lambda > 0$. Under suitable conditions on the nonlinear term f and the Kirchhoff function M, we prove that (1.1) has at least two nontrivial weak solutions in the space X. Our idea is to apply the mountain pass theorem in [4] combined with Ekeland's variational principle in [18] to the energy functional J_λ associated to problem (1.1) when $\lambda > 0$ small enough. For this purpose, let us assume that $M : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and introduce the following conditions:

(A1) There exist constants $m_1, m_2 > 0$ and $1 < \alpha < q^-/p^+$ such that
$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1}$$
for all $t \in \mathbb{R}^+ := [0, +\infty)$, where $q^- = \inf_{x\in\overline{\Omega}} q(x)$, $q \in C_+(\overline{\Omega})$ is given by assumption (A2);

(A2) There exists a positive constant C such that
$$|f(x,t)| \leq C(1 + |t|^{q(x)-1}), \quad \forall (x,t) \in \Omega \times \mathbb{R},$$
where $q \in C_+(\overline{\Omega})$, $p(x) < q(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ for all $x \in \overline{\Omega}$;

(A3) $f(x,t) = o(|t|^{\alpha p^+ -1})$, $t \to 0$, uniformly a.e. $x \in \Omega$;

(A4) There exists a constant $\mu > \frac{m_2\alpha(p^+)^\alpha}{m_1(p^-)^{\alpha-1}}$ such that

$$\mu F(x,t) := \mu \int_0^t f(x,s)\,ds \le f(x,t)t, \quad \forall (x,t) \in \Omega \times \mathbb{R};$$

(A5) $\inf_{\{x\in\Omega;\ |t|=1\}} F(x,t) > 0$.

Definition 2.1. We say that $u \in W^{1,p(x)}(\Omega)$ is a weak solution of problem (1.1) if

$$\begin{aligned} &M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\Big) \\ &\times \Big(\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv\,d\sigma_x\Big) \\ &- \int_\Omega f(x,u)v\,dx - \lambda\int_\Omega g(x)v\,dx = 0 \end{aligned}$$

for all $v \in W^{1,p(x)}(\Omega)$.

The first result of this article is stated as follows.

Theorem 2.2. *Suppose that $g \in L^{\frac{\alpha p^+}{\alpha p^+-1}}(\Omega)$ and $g \not\equiv 0$. Let conditions* (A1)–(A5) *hold, then there exists $\lambda^* > 0$ such that* (1.1) *has at least two nontrivial weak solutions when $\lambda \in (0,\lambda^*)$.*

Let us denote by X the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ and consider the energy functional $J_\lambda : X \to \mathbb{R}$ given by

$$\begin{aligned} J_\lambda(u) = \widehat{M}\Big(&\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\Big) \\ &- \int_\Omega F(x,u)\,dx - \lambda\int_\Omega g(x)u\,dx. \end{aligned}$$

Then by (A2) and the continuous embeddings, we can show that that the functional J_λ is well-defined on X and $J_\lambda \in C^1(X,\mathbb{R})$ with the derivative given by

$$\begin{aligned} J'_\lambda(u)(v) = M\Big(&\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\Big) \\ &\times \Big(\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv\,d\sigma_x\Big) \\ &- \int_\Omega f(x,u)v\,dx - \lambda\int_\Omega g(x)v\,dx \end{aligned}$$

for all $u,v \in X$. Hence, we can find weak solutions of (1.1) as the critical points of the functional J_λ in the space X.

Lemma 2.3. *Assume that* (A2), (A4) *hold and that $g \in L^{\frac{\alpha p^+}{\alpha p^+-1}}(\Omega)$. Then there exist constants $\rho, r, \lambda^* > 0$ such that $J_\lambda(u) \ge r$ for all $u \in X$ with $\|u\|_{\beta(x)} = \rho$, when $\lambda \in (0,\lambda^*)$.*

Proof. Since $\alpha p^+ < q^- \le q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, the embeddings

$$X \hookrightarrow L^{\alpha p^+}(\Omega), \quad X \hookrightarrow L^{q(x)}(\Omega)$$

are continuous, and there exists two constants $C_1, C_2 > 0$ such that

$$|u|_{\alpha p^+} \le C_1\|u\|_{\beta(x)}, \quad |u|_{q(x)} \le C_2\|u\|_{\beta(x)}. \tag{2.1}$$

Let $0<\epsilon<\frac{m_1}{2\alpha(p^+)^\alpha C_1^{\alpha p^+}}$, where C_1 is given by (2.1). From the assumptions (A3), (A4), there exists a constant $C(\epsilon)$ depending on ϵ such that

$$F(x,t)\le \epsilon|t|^{\alpha p^+}+C(\epsilon)|t|^{q(x)},\quad \forall (x,t)\in\Omega\times\mathbb{R}, \tag{2.2}$$

Let $u\in X$ with $\|u\|_{\beta(x)}<1$ sufficiently small. From (1.6) and (2.1)-(2.2), applying the Hölder inequality we have

$$\begin{aligned}
&J_\lambda(u)\\
&=\widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\Big)-\int_\Omega F(x,u)\,dx-\lambda\int_\Omega g(x)u\,dx\\
&\ge \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{\beta(x)}^{\alpha p^+}-\epsilon\int_\Omega |u|^{\alpha p^+}\,dx-C(\epsilon)\int_\Omega |u|^{q(x)}\,dx-\lambda|g|_{\frac{\alpha p^+}{\alpha p^+-1}}|u|_{\alpha p^+}\\
&\ge \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{\beta(x)}^{\alpha p^+}-\epsilon C_1^{\alpha p^+}\|u\|_{\beta(x)}^{\alpha p^+}-C(\epsilon)C_2^{q^-}\|u\|_{\beta(x)}^{q^-}-\lambda C_1|g|_{\frac{\alpha p^+}{\alpha p^+-1}}\|u\|_{\beta(x)}\\
&\ge \Big(\frac{m_1}{2\alpha(p^+)^\alpha}\|u\|_{\beta(x)}^{\alpha p^+-1}-C(\epsilon)C_2^{q^-}\|u\|_{\beta(x)}^{q^--1}-\lambda C_1|g|_{\frac{\alpha p^+}{\alpha p^+-1}}\Big)\|u\|_{\beta(x)},
\end{aligned}$$

where $C_1,C_2>0$ are given by (2.1). Consider the functions $\gamma_1:[0,+\infty)\to\mathbb{R}$ is given by

$$\gamma_1(\tau)=\frac{m_1}{2\alpha(p^+)^\alpha}\tau^{\alpha p^+-1}-C(\epsilon)C_2^{q^-}\tau^{q^--1}.$$

Since $q^->\alpha p^+$, there exists a constant $\tau=\rho>0$ obeying the relationship $\gamma_1(\rho)=\max_{\tau\in[0,+\infty)}\gamma_1(\tau)>0$. Taking $\lambda^*=\frac{\gamma_1(\rho)}{2C_1|g|_{\frac{\alpha p^+}{\alpha p^+-1}}}>0$, it then follows that, if $\lambda\in(0,\lambda^*)$, we can choose r and $\rho>0$ such that $J_\lambda(u)\ge r>0$ for all $u\in X$ with $\|u\|_{\beta(x)}=\rho$. $\square$

Lemma 2.4. *Assume that* (A1), (A4), (A5) *hold. Then there exists a function $e\in X$ with $\|e\|_{\beta(x)}>\rho$ such that $J_\lambda(e)<0$, where ρ is given by Lemma 2.3.*

Proof. For each $x\in\Omega$ and $t\in\mathbb{R}$, let us define the function $\gamma_2(\tau)=\tau^{-\mu}F(x,\tau t)-F(x,t)$ for all $\tau\ge 1$. Then we deduce from (A4) that

$$\gamma_2'(\tau)=\tau^{-\mu-1}\left(f(x,\tau t)\tau t-\mu F(x,\tau t)\right)\ge 0,\quad \forall\tau\ge 1$$

of the function γ_2 is increasing on $[1,+\infty)$ and $\gamma_2(\tau)\ge\gamma_2(1)=0$ for all $\tau\in[1,+\infty)$. Hence,

$$F(x,\tau t)\ge \tau^\mu F(x,t),\quad \forall x\in\Omega,\quad t\in\mathbb{R},\ \tau\ge 1. \tag{2.3}$$

Let $\varphi\in C_0^\infty(\Omega)$ and $\varphi\not\equiv 0$ such that $\int_\Omega F(x,\varphi)\,dx>0$, by (A1) we have

$$\begin{aligned}
J_\lambda(\tau\varphi)&=\widehat{M}\Big(\int_\Omega\frac{1}{p(x)}|\nabla\tau\varphi|^{p(x)}\,dx+\int_{\partial\Omega}\frac{\beta(x)}{p(x)}|\tau\varphi|^{p(x)}\,d\sigma_x\Big)\\
&\quad-\int_\Omega F(x,\tau\varphi)\,dx-\lambda\int_\Omega g(x)\tau\varphi\,dx\\
&\le \frac{m_2\tau^{\alpha p^+}}{\alpha}\Big(\int_\Omega\frac{1}{p(x)}|\nabla\varphi|^{p(x)}\,dx+\int_{\partial\Omega}\frac{\beta(x)}{p(x)}|\varphi|^{p(x)}\,d\sigma_x\Big)^\alpha\\
&\quad-\tau^\mu\int_\Omega F(x,\varphi)\,dx-\lambda\tau\int_\Omega g(x)\varphi\,dx\to-\infty,
\end{aligned}$$

as $\tau\to+\infty$ since $\mu>\frac{m_2\alpha(p^+)^\alpha}{m_1(p^-)^{\alpha-1}}\ge\alpha p^+$. Therefore, there exists a constant $\tau_0>0$ such that $\|\tau_0\varphi\|_{\beta(x)}>\rho$ and $J_\lambda(\tau_0\varphi)<0$. Let $e=\tau_0\varphi$ the proof is complete. $\square$

Lemma 2.5. *Assume that* (A1)–(A4) *hold. Then the functional* J_λ *satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset X$ be such that

$$J_\lambda(u_n) \to c \in \mathbb{R}, \quad J'_\lambda(u_n) \to 0 \text{ in } X^*, \tag{2.4}$$

where X^* is the dual space of X.

We will prove that $\{u_n\}$ is bounded in X. Indeed, assume by contradiction that $\|u_n\|_{\beta(x)} \to +\infty$ as $n \to \infty$. By the conditions (A1), (A4) and (1.5), (2.4), applying the Hölder inequality we deduce for n large enough that $\|u_n\|_{\beta(x)} > 1$ and

$$\begin{aligned}
&c + 1 + \|u_n\|_{\beta(x)} \\
&\geq J_\lambda(u_n) - \frac{1}{\mu} J'_\lambda(u_n)(u_n) \\
&= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u_n|^{p(x)}\,d\sigma_x \Big) - \int_\Omega F(x,u_n)\,dx \\
&\quad - \lambda \int_\Omega g(x) u_n\,dx - M\Big(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u_n|^{p(x)}\,d\sigma_x \Big) \\
&\quad \times \Big(\int_\Omega |\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x) |u_n|^{p(x)}\,d\sigma_x \Big) + \frac{1}{\mu} \int_\Omega f(x,u_n) u_n\,dx \\
&\quad + \frac{\lambda}{\mu} \int_\Omega g(x) u_n\,dx \\
&\geq \Big(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{\mu(p^-)^{\alpha-1}} \Big) \Big(\int_\Omega |\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x) |u_n|^{p(x)}\,d\sigma_x \Big)^\alpha \\
&\quad + \int_\Omega \Big(\frac{1}{\mu} f(x,u_n) u_n - F(x,u_n) \Big)\,dx - \lambda \big(1 - \frac{1}{\mu}\big) \int_\Omega g(x) u_n\,dx \\
&\geq \Big(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{\mu(p^-)^{\alpha-1}} \Big) \|u_n\|_{\beta(x)}^{\alpha p^-} - \lambda C_1 \big(1 - \frac{1}{\mu}\big) |g|_{\frac{\alpha p^+}{\alpha p^+ - 1}} \|u\|_{\beta(x)},
\end{aligned}$$

where $\mu > m_2 \alpha (p^+)^\alpha / m_1 (p^-)^{\alpha-1}$. Dividing by $\|u\|_{\beta(x)}^{\alpha p^-}$ in the above inequality and passing to the limit as $n \to \infty$, we obtain a contradiction. This follows that the sequence $\{u_n\}$ is bounded in X.

Now, since the Banach space X is reflexive, there exists $u \in X$ such that passing to a subsequence, still denoted by $\{u_n\}$, it converges weakly to u in X and converges strongly to u in the spaces $L^{q(x)}(\Omega)$. Using the condition (A2) and Hölder inequality, we have

$$\begin{aligned}
\Big| \int_\Omega f(x,u_n)(u_n - u)\,dx \Big| &\leq \int_\Omega |f(x,u_n)| |u_n - u|\,dx \\
&\leq C \int_\Omega (1 + |u_n|^{q(x)-1}) |u_n - u|\,dx \\
&\leq 2C \Big(1 + ||u_n|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}} \Big) |u_n - u|_{q(x)} \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which yields

$$\lim_{m \to \infty} \int_\Omega f(x,u_n)(u_n - u)\,dx = 0. \tag{2.5}$$

Moreover, we have

$$\begin{aligned} \Big|\int_\Omega g(x)(u_n-u)\,dx\Big| &\le \int_\Omega |g(x)||u_n-u|\,dx \\ &\le 2|g|_{\frac{\alpha p^+}{\alpha p^+-1}}|u_n-u|_{\alpha p^+} \to 0 \quad \text{as } n\to\infty. \end{aligned} \tag{2.6}$$

Since $\{u_n\}$ converges weakly to u in X, by (2.4) we have $J'_\lambda(u_n)(u_n-u)\to 0$ as $n\to\infty$ or

$$\begin{aligned} &M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma_x\Big) \\ &\times \Big(\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx + \int_{\partial\Omega}\beta(x)|u_n|^{p(x)-2}u_n(u_n-u)\,d\sigma_x\Big) \\ &- \int_\Omega f(x,u_n)(u_n-u)\,dx - \lambda\int_\Omega g(x)(u_n-u)\,dx \to 0, \end{aligned}$$

which leads from (2.5)-(2.6) to

$$\begin{aligned} &M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma_x\Big) \\ &\times \Big(\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx + \int_{\partial\Omega}\beta(x)|u_n|^{p(x)-2}u_n(u_n-u)\,d\sigma_x\Big) \to 0. \end{aligned}$$

If $\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,\sigma_x \to 0$ then we have $\int_\Omega |\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega}\beta(x)|u_n|^{p(x)}\,d\sigma_x \to 0$ as $n\to\infty$ and thus $u_n\to 0$ strongly in X as $n\to\infty$. If $\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma_x \to t_0>0$ as $n\to\infty$ then it follows from the continuity of M that

$$M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma_x\Big) \to M(t_0)>0,$$

so that

$$M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma_x\Big) \ge \frac{1}{2}M(t_0)>0$$

for all n large enough. Hence,

$$\lim_{n\to\infty}\Big(\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx + \int_{\partial\Omega}\beta(x)|u_n|^{p(x)-2}u_n(u_n-u)\,d\sigma_x\Big) = 0$$

or

$$\lim_{n\to\infty} L'_{\beta(x)}(u_n)(u_n-u) = 0,$$

where $L_{\beta(x)}$ and $L'_{\beta(x)}$ are given by formulas (1.9) and (1.10).

From Proposition 1.1, the sequence $\{u_n\}$ converges strongly to u as $n\to\infty$. Thus, the functional J_λ satisfies the Palais-Smale condition. $\square$

Lemma 2.6. *Assume that $g\in L^{\frac{\alpha p^+}{\alpha p^+-1}}(\Omega)$ with $g\not\equiv 0$, and that* (A2)-(A5) *hold. Then there exists a function $\psi\in X$, $\psi\not\equiv 0$ such that $J_\lambda(\tau\psi)<0$ for all $\tau>0$ small enough.*

Proof. For $(x,t)\in\Omega\times\mathbb{R}$, set $\gamma_3(\tau)=F(x,\tau^{-1}t)\tau^\mu$, $\tau\ge 1$. By (A4), we have

$$\begin{aligned} \gamma_3'(\tau) &= f(x,\tau^{-1}t)\big(-\frac{t}{\tau^2}\big)\tau^\mu + F(x,\tau^{-1}t)\mu\tau^{\mu-1} \\ &= \tau^{\mu-1}[\mu F(x,\tau^{-1}t) - \tau^{-1}tf(x,\tau^{-1}t)] \le 0 \end{aligned}$$

so, $\gamma_3(t)$ is non-increasing. Thus, for any $|t| \geq 1$, we have $\gamma_3(1) \geq \gamma_3(|t|)$, that is

$$F(x,t) \geq F(x,|t|^{-1}t)|t|^{\mu} \geq C_3|t|^{\mu}, \tag{2.7}$$

where $C_3 = \inf_{x\in\Omega,|t|=1} F(x,t) > 0$ by (A5). From (A3), there exists a constant $\eta > 0$ such that

$$\Big|\frac{f(x,t)t}{|t|^{\alpha p^+}}\Big| = \Big|\frac{f(x,t)}{|t|^{\alpha p^+-1}}\Big| \leq 1, \tag{2.8}$$

for all $x \in \Omega$ and all $0 < |t| \leq \eta$. By (A2), for all $x \in \Omega$ and all $\eta \leq |t| \leq 1$, there exists $C_4 > 0$ such that

$$\Big|\frac{f(x,t)t}{|t|^{\alpha p^+}}\Big| \leq \frac{C(1+|t|^{q(x)-1})|t|}{|t|^{\alpha p^+}} \leq C_4. \tag{2.9}$$

From (2.8) and (2.9), we deduce that

$$f(x,t)t \geq -(C_4+1)|t|^{\alpha p^+},$$

for all $x \in \Omega$ and all $|t| \in [0,1]$. Using the equality $F(x,t) = \int_0^1 f(x,\tau t)t\,d\tau$, we obtain

$$F(x,t) \geq -\frac{1}{\alpha p^+}(C_4+1)|t|^{\alpha p^+}, \tag{2.10}$$

for all $x \in \Omega$ and all $|t| \in [0,1]$. Taking $C_5 = \frac{1}{\alpha p^+}(C_4+1) + C_3$, we then get from (2.7) and (2.10) that

$$F(x,t) \geq C_3|t|^{\mu} - C_5|t|^{\alpha p^+}, \tag{2.11}$$

for all $x \in \Omega$ and all $t \in \mathbb{R}$.

We now prove that there exists a function $\psi \in X$ such that $J_\lambda(\tau\psi) < 0$ for all $\tau > 0$ small enough. Since $g \in L^{\frac{\alpha p^+}{\alpha p^+-1}}(\Omega)$ and $g \not\equiv 0$, we can choose a function $\psi \in X$ be such that

$$\int_\Omega g(x)\psi(x)\,dx > 0,$$

then by (2.11) we have

$$\begin{aligned}
J_\lambda(\tau\psi) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla\tau\psi|^{p(x)}\,dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|\tau\psi|^{p(x)}\,d\sigma_x\Big)\\
&\quad - \int_\Omega F(x,\tau\psi)\,dx - \lambda\int_\Omega g(x)\tau\psi\,dx\\
&\leq \frac{m_2}{\alpha(p^-)^\alpha}\tau^{\alpha p^-}\Big(\int_\Omega|\nabla\psi|^{p(x)}\,dx + \int_{\partial\Omega}\beta(x)|\psi|^{p(x)}\,d\sigma_x\Big)^\alpha\\
&\quad - C_5\tau^{\mu}\int_\Omega|\psi|^{\mu}\,dx + C_3\tau^{\alpha p^+}\int_\Omega|\psi|^{\alpha p^+}\,dx - \lambda\tau\int_\Omega g(x)\psi\,dx < 0,
\end{aligned}$$

for all $\tau > 0$ small enough. $\square$

Proof of Theorem 2.2. By Lemmas 2.3–2.5, there exists $\lambda^* > 0$ such that for if $\lambda \in (0,\lambda^*)$, all assumptions of the mountain pass theorem by Ambrosetti-Rabinowitz [4] hold. Then, there exists a critical point $u_1 \in X$ of the functional J_λ, i.e. $J_\lambda'(u_1) = 0$ and thus, problem (1.1) has a nontrivial weak solution $u_1 \in X$ with positive energy

$$J_\lambda(u_1) = \overline{c} := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda(\gamma(t)) > 0,$$

where $\Gamma := \{\gamma \in C([0,1],X) :\ \gamma(0) = 0,\ \gamma(1) = e\}$ and the function e is given by Lemma 2.4.

We will show the existence of the second nontrivial weak solution $u_2 \in X$ and $u_2 \neq u_1$ by using the Ekeland variational principle. Indeed, by Lemma 2.3, it follows that on the boundary of the ball centered at the origin and of radius ρ in X, denoted by $B_\rho(0)$, we have

$$\inf_{u\in\partial B_\rho(0)} J_\lambda(u) > 0.$$

On the other hand, by Lemma 2.3 again, the functional J_λ is bounded from below on $B_\rho(0)$. Moreover, by Lemma 2.6, there exists $\varphi \in X$ such that $J_\lambda(\tau\varphi) < 0$ for all τ small enough. It follows that

$$-\infty < \underline{c} = \inf_{u\in\overline{B}_\rho(0)} J_\lambda(u) < 0.$$

Let us choose $\epsilon > 0$ such that

$$0 < \epsilon < \inf_{u\in\partial B_\rho(0)} J_\lambda(u) - \inf_{u\in\overline{B}_\rho(0)} J_\lambda(u).$$

Applying the Ekeland variational principle in [18] to the functional $J_\lambda : \overline{B}_\rho(0) \to \mathbb{R}$, it follows that there exists $u_\epsilon \in \overline{B}_\rho(0)$ such that

$$J_\lambda(u_\epsilon) < \inf_{u\in\overline{B}_\rho(0)} J_\lambda(u) + \epsilon,$$
$$J_\lambda(u_\epsilon) < J_\lambda(u) + \epsilon\|u - u_\epsilon\|_{\beta(x)}, \quad u \neq u_\epsilon,$$

then, we have $J_\lambda(u_\epsilon) < \inf_{u\in\partial B(0)} J_\lambda(u)$ and thus, $u_\epsilon \in B_\rho(0)$.

Now, we define the functional $I_\lambda : \overline{B}_\rho(0) \to \mathbb{R}$ by $I_\lambda(u) = J_\lambda(u) + \epsilon\|u - u_\epsilon\|_{\beta(x)}$. It is clear that u_ϵ is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + \tau v) - I_\lambda(u_\epsilon)}{t} \geq 0$$

for all $\tau > 0$ small enough and all $v \in B_\rho(0)$. The above information shows that

$$\frac{J_\lambda(u_\epsilon + \tau v) - J_\lambda(u_\epsilon)}{\tau} + \epsilon\|v\|_{\beta(x)} \geq 0.$$

Letting $\tau \to 0^+$, we deduce that

$$\langle J'_\lambda(u_\epsilon), v\rangle \geq -\epsilon\|v\|_{\beta(x)}.$$

It should be noticed that $-v$ also belongs to $B_\rho(0)$, so replacing v by $-v$, we obtain

$$\langle J'_\lambda(u_\epsilon), -v\rangle \geq -\epsilon\| - v\|_{\beta(x)}$$

or

$$\langle J'_\lambda(u_\epsilon), v\rangle \leq \epsilon\|v\|_{\beta(x)},$$

which helps us to deduce that $\|J'_\lambda(u_\epsilon)\|_{X^*} \leq \epsilon$. Therefore, there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$J_\lambda(u_n) \to \underline{c} = \inf_{u\in\overline{B}_\rho(0)} J_\lambda(u) < 0 \text{ and } J'_\lambda(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty. \tag{2.12}$$

Based on Lemma 2.6, the sequence $\{u_n\}$ converges strongly to some u_2 as $n \to \infty$. Moreover, since $J_\lambda \in C^1(X, \mathbb{R})$, by (2.12) it follows that $J'_\lambda(u_2) = 0$. Thus, u_2 is a non-trivial weak solution of problem (1.1) with negative energy $J_\lambda(u_2) = \underline{c} < 0$.

Finally, we point out the fact that $u_1 \neq u_2$ since $J_\lambda(u_1) = \overline{c} > 0 > \underline{c} = J_\lambda(u_2)$. The proof is complete. □

2.2. **Existence of infinitely many solutions.** The purpose of this part is to consider problem (1.1) in the case $\lambda = 0$. Under some suitable conditions on M and f, we prove the existence of infinitely many solutions for problem (1.1) by using the Krasnoselskii's genus theory [11], see Proposition 1.4. Let us introduce the following conditions:

- (A6) $f : \Omega \to \mathbb{R}$ is a continuous function such that
$$D_1 h(x)|t|^{r(x)-1} \le f(x,t) \le D_2 h(x)|t|^{r(x)-1}, \quad \forall (x,t) \in \Omega \times \mathbb{R}^+,$$
where $D_1, D_2 > 0$ are positive constants and $r \in C_+(\overline{\Omega})$ such that $1 < r(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ for all $x \in \overline{\Omega}$, the function $h \equiv 1$ if $p(x) \le r(x) < p^*(x)$ for all $x \in \overline{\Omega}$ while $h \in L_+^{r_0(x)}(\Omega)$ with $r_0(x) = \frac{p(x)}{p(x)-r(x)}$ if $1 < r(x) < p(x)$ for all $x \in \overline{\Omega}$;
- (A7) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$;

We have the following result.

Theorem 2.7. *Let* (A1), (A6), (A7) *hold. If* $p(x) \le r(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$ *with* $r^+ < \alpha p^-$ *or* $1 < r(x) < p(x)$ *for all* $x \in \overline{\Omega}$*, then* (1.1) *with* $\lambda = 0$ *has infinitely many weak solutions.*

With similar arguments as those used in the proof of Theorem 2.2, by assumption (A6), we can show that the functional $J_0 : X \to \mathbb{R}$ defined by
$$J_0(u) = \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\Big) - \int_\Omega F(x,u)\,dx$$
is of C^1 on X and its derivative is
$$\begin{aligned} J_0'(u)(v) = M\Big(&\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\Big) \\ &\times \Big(\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv\,d\sigma_x\Big) - \int_\Omega f(x,u)v\,dx, \end{aligned}$$
for all $u, v \in X$. Thus, weak solutions of problem (1.1) with $\lambda = 0$ are exactly the critical points of J_0.

Lemma 2.8. *Assume that* (A1), (A6) *hold. If* $p(x) \le r(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$ *with* $r^+ < \alpha p^-$ *or* $1 < r(x) < p(x)$ *for all* $x \in \overline{\Omega}$*, then the functional* J_0 *is bounded from below on* X *and satisfies the Palais-Smale condition.*

Proof. Since $1 < r(x) < p^*(x)$ for all $x \in \overline{\Omega}$, the embedding $X \hookrightarrow L^{r(x)}(\Omega)$ is continuous and compact, then there exists $C_6 > 0$ such that
$$|u|_{r(x)} \le C_6 \|u\|_{\beta(x)}, \quad \forall u \in X.$$
If $p(x) \le r(x) < p^*(x)$ for all $x \in \overline{\Omega}$ then $h \equiv 1$, by (A1) and (A6), it follows from the definition of the functional J_0, $(\lambda = 0)$ that
$$\begin{aligned} J_0(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\Big) - \int_\Omega F(x,u)\,dx \\ &\ge \frac{m_1}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\Big)^\alpha - \frac{D_2}{r^-}\int_\Omega |u|^{r(x)}\,dx \\ &\ge \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{\beta(x)}^{\alpha p^-} - \frac{D_2 C_6^{r^+}}{r^-}\|u\|_{\beta(x)}^{r^+}, \end{aligned}$$

for all $u \in X$ with $\|u\|_{\beta(x)} > 1$ large enough. Since we always have that $r^+ < \alpha p^-$, J_0 is coercive, i.e. $J_0(u) \to +\infty$ as $\|u\|_{\beta(x)} \to +\infty$ and bounded from below on X.

Similarly, if $1 < r(x) < p(x)$ for all $x \in \overline{\Omega}$ then $h \in L^{r_0(x)}(\Omega)$ with $r_0(x) = \frac{p(x)}{p(x)-r(x)}$. Applying the Hölder inequality and embedding theorem, we also have

$$\begin{aligned} J_0(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\Big) - \int_\Omega F(x,u)\,dx \\ &\geq \frac{m_1}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\Big)^\alpha - \frac{D_2}{r^-}\int_\Omega h(x)|u|^{r(x)}\,dx \\ &\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|_{\beta(x)}^{\alpha p^-} - \frac{2D_2C_6^{r^+}}{r^-}|h|_{r_0(x)}\|u\|_{\beta(x)}^{r^+}, \end{aligned}$$

for all $u \in X$ with $\|u\|_{\beta(x)} > 1$ large enough. Since $r^+ < p^- < \alpha p^-$, J_0 is coercive and bounded from below on X.

From these statements, if $\{u_n\}$ is a Palais-Smale sequence for the functional J_0, i.e. $J(u_n) \to \overline{c}$, $J'(u_n) \to 0$ in X^*, then $\{u_n\}$ is bounded in X. Since X is a reflexive Banach space, $\{u_n\}$ has a subsequence, still denoted by $\{u_n\}$, that converges weakly to some $u \in X$. Moreover, the embedding $X \hookrightarrow L^{r(x)}(\Omega)$ is continuous and compact, using (A6) and the Hölder inequality, we have

$$\begin{aligned} \Big|\int_\Omega f(x,u_n)(u_n-u)\,dx\Big| &\leq \int_\Omega |f(x,u_n)||u_n-u|\,dx \\ &\leq D_2\int_\Omega |u_n|^{r(x)-1}|u_n-u|\,dx \\ &\leq 2D_2\big||u_n|^{r(x)-1}\big|_{\frac{r(x)}{r(x)-1}}|u_n-u|_{r(x)} \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

if $p(x) \leq r(x) < p^*(x)$ for all $x \in \overline{\Omega}$, or

$$\begin{aligned} \Big|\int_\Omega f(x,u_n)(u_n-u)\,dx\Big| &\leq \int_\Omega |f(x,u_n)||u_n-u|\,dx \\ &\leq D_2\int_\Omega h(x)|u_n|^{r(x)-1}|u_n-u|\,dx \\ &\leq 3D_2|h|_{r_0(x)}\big||u_n|^{r(x)-1}\big|_{\frac{p(x)}{r(x)-1}}|u_n-u|_{p(x)} \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

if $1 < r(x) < p(x)$ for all $x \in \overline{\Omega}$, which yields

$$\lim_{m\to\infty}\int_\Omega f(x,u_n)(u_n-u)\,dx = 0. \tag{2.13}$$

From (2.13), with similar arguments as those presented in the proof of Lemma 2.6, we can show that $\{u_n\}$ converges strongly to $u \in X$ and thus, the functional J_0 satisfies the Palais-Smale condition. □

Proof of Theorem 2.7. We have known that for $p \in C_+(\overline{\Omega})$, $1 < p^- \leq p^+ < N$, $X = W^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space, then there exist $\{e_n\} \subset X$ and $\{e_n^*\} \subset X^*$ such that

$$\langle e_i^*, e_j\rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$X = \overline{\operatorname{span}\{e_n : n = 1, 2, \dots\}}, \quad X^* = \overline{\operatorname{span}\{e_n^* : n = 1, 2, \dots\}}.$$

For each $k \in \mathbb{N}$, consider $X_k = \operatorname{span}\{e_1, e_2, \dots, e_k\}$, the subspace if X spanned by the vectors $e_1, e_2, \dots, e_k$. Let $h(x) \equiv 1$ if $p(x) \le r(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $h \in L^{r_0(x)}(\Omega)$ if $1 < r(x) < p(x)$ for all $x \in \overline{\Omega}$, we define a norm $|\cdot|_{L^{r(x)}(\Omega,h(x))}$ on the space X_k sa follows

$$|u|_{L^{r(x)}(\Omega,h(x))} := \inf\Big\{\lambda > 0;\ \int_\Omega h(x)|\frac{u(x)}{\lambda}|^{r(x)}\,dx \le 1\Big\}. \tag{2.14}$$

Note that the embedding $X_k \hookrightarrow L^{s(x)}(\Omega)$, $1 < s(x) < p^*(x)$ is continuous. Since all norms on the finite dimensional space X_k are equivalent, so are the norms $\|\cdot\|_{\beta(x)}$ and $|\cdot|_{L^{r(x)}(\Omega,h(x))}$. Moreover, for any $u \in X_k$, it follows that

$$\begin{aligned}
J_0(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\Big) - \int_\Omega F(x,u)\,dx \\
&\le \frac{m_2}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma_x\Big)^\alpha - \frac{D_1}{r^+}\int_\Omega h(x)|u|^{r(x)}\,dx \\
&\le \frac{m_2}{\alpha(p^-)^\alpha}\|u\|_{\beta(x)}^{\alpha p^-} - \frac{D_1}{r^+}C(k)\|u\|_{\beta(x)}^{r^+} \\
&= \|u\|_{\beta(x)}^{r^+}\Big(\frac{m_2}{\alpha(p^-)^\alpha}\|u\|_{\beta(x)}^{\alpha p^- - r^+} - \frac{D_1}{r^+}C(k)\Big),
\end{aligned}$$

where $C(k)$ is a positive constant depending on k. For each $k \in \mathbb{N}$ as before, let us denote by R_k the positive constant such that

$$\frac{m_2}{\alpha(p^-)^\alpha} r_k^{\alpha p^- - r^+} < \frac{D_1}{r^+}C(k),$$

then, for all $0 < \rho_k < R_k$, and $u \in S_{\rho_k} := \{u \in X_k :\ \|u\|_{\beta(x)} = \rho_k\}$, S_{r_k} is a closed subset of $X\backslash\{0\}$ that is symmetric with respect to the origin, we obtain

$$\begin{aligned}
J_0(u) &\le \rho_k^{r^+}\Big(\frac{m_2}{\alpha(p^-)^\alpha}\rho_k^{\alpha p^- - r^+} - \frac{D_1}{r^+}C(k)\Big) \\
&\le R_k^{r^+}\Big(\frac{m_2}{\alpha(p^-)^\alpha}R_k^{\alpha p^- - r^+} - \frac{D_1}{r^+}C(k)\Big) < 0 = J_0(0),
\end{aligned}$$

which implies

$$\sup_{u \in S_{\rho_k}} J_0(u) < J_0(0).$$

Because X_k and $\mathbb{R}^k$ are isomorphic and S_{r_k} and S^{k-1} are homeomorphic, we conclude that $\gamma(S_{r_k}) = k$. Moreover, by assumption (A7), J_0 is even. By Proposition 1.4, the functional J_0 has at least k pair of different critical points. Since k is arbitrary, we obtain infinitely many critical points of J_0 and thus problem (1.1) with $\lambda = 0$ has infinitely many weak solutions. $\square$

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