BOUNDARY-DOMAIN INTEGRAL EQUATIONS FOR DIRICHLET DIFFUSION PROBLEMS WITH NON-SMOOTH COEFFICIENT

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ABSTRACT. We obtain a system of boundary-domain integral equations (BDIE) equivalent to the Dirichlet problem for the diffusion equation in non-homogeneous media. We use an extended version of the boundary integral method for PDEs with variable coefficients for which a parametrix is required. We generalize existing results for this family of parametrices considering a non-smooth variable coefficient in the PDE and source term in $H^{s-2}(\Omega)$, $1/2 < s < 3/2$ defined on a Lipschitz domain. The main results concern the equivalence between the original BVP and the corresponding BDIE system, as well as the well-posedness of the BDIE system.

1. INTRODUCTION

The popularity of the boundary integral equation method (BIE), also called the potential method, is owed to the reduction of dimension of a boundary value problem (BVP) with constant coefficients and homogeneous right hand side defined on a domain of $\mathbb{R}^n$. By applying the BIE method, one can reformulate the original BVP in terms of an equivalent integral equation defined exclusively on the boundary of the domain. This method has already been extensively studied for many boundary value problems, for instance: Laplace, Helmholtz, Stokes, Lamé in [13, 14, 24]. This approach requires an explicit formula for the fundamental solution of the PDE operator in the BVP which is not always available when the BVP has variable coefficients [6, 21].

The overcome this issue, one can construct a parametrix (Levi function) [6, 21] for the PDE operator and use it to derive an equivalent system of Boundary-Domain Integral Equations following a similar approach as for the BIE method. However, the reduction of dimension no longer applies as volume integrals will appear in the new formulation as a result of the remainder term. This is also the case for non-homogeneous problems with constant coefficients [13, Chapter 1 and 2].

To preserve the reduction of dimension, one can use the radial integration method (RIM) which allows to transform volume integrals into boundary only integrals [11]. This method has been successfully implemented to solve boundary-domain integral

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equations derived from BVPs with variable coefficients [1, 3] and has the ability to remove singularities appearing in the domain integrals.

Recent developments on numerical approximation of the solution of BDIEs show that there are effective and fast algorithms able to compute the solution. For example: the collocation method [22, 23] which, although leads to fully populated matrices, can be further enhanced by using hierarchical matrix compression and adaptive methods as shown in [12] to reduce the computational cost. Localised approaches to reduce the matrix dimension and storage have also been developed [15] which lead to sparse matrices.

Moreover, reformulating the original BVP in the Boundary Domain Integral Equation form can be beneficial, for instance, in inverse problems with variable coefficients [5].

In this article, we consider the diffusion equation in non homogeneous media, i.e.

$$\mathcal{A}u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\Big(a(x)\frac{\partial u(x)}{\partial x_i}\Big), \tag{1.1}$$

A possible parametrix is given by

$$P^y(x,y) = P(x,y;a(y)) = \frac{-1}{4\pi a(y)|x-y|},$$

where the superscript indicates $P^y(x,y)$ is a function of the variable coefficient depending on y. This parametrix has been extensively studied [6, 7, 18] and more references therein. However, a parametrix for a given partial differential operator is not necessarily unique. For instance, another possible parametrix for the same operator $\mathcal{A}$ is given by

$$P^x(x,y) = P(x,y;a(x)) = \frac{-1}{4\pi a(x)|x-y|}.$$

In this case, the parametrix depends on the variable coefficient $a(x)$. This parametrix was introduced in [19] for the mixed problem for the operator $\mathcal{A}$ in Lipschitz 3D domains with smooth coefficient in [19] and for the Dirichlet problem with smooth coefficient in Lipschitz domains and $f \in H^{-1}(\Omega)$ in [10]. In this article, we generalize these results for the Dirichlet problem with non-smooth coefficient.

The study of new families of parametrices is helpful at the time of constructing parametrices for systems of PDEs. For instance, for the Stokes system. In this case, the fundamental solution for the pressure does not present any relationship with the viscosity coefficient, whereas the parametrix for the pressure depends on two variable viscosity coefficients: one depending on y and another depending on x, see also [20].

However, most of the numerical methods to solve BDIEs aforementioned are tested for the Dirichlet problem [12, 22, 23, 1] with smooth coefficient. However, there are applications in Science and Engineering of elliptic PDEs of this type with non-smooth coefficient, see for example [2, 9]. In order to compare the performance of the parametrices $P^x(x,y)$ and $P^y(x,y)$, one needs first to prove the equivalence between the original Dirichlet BVP and the system of BDIEs as well as the uniqueness of the solution (well-posedness) of the system of BDIEs, which corresponds to another of the various purposes of this article.

2. Preliminaries

2.1. **Function spaces.** Let $\Omega = \Omega^+$ be a bounded simply connected open Lipschitz domain, $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}^+$ the complementary (unbounded) domain. We assume that their common boundary $\partial\Omega$ is a compact and simply connected surface.

In what follows $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ denotes the space of test functions with compact support, and $\mathcal{D}^*(\Omega)$ denotes its topological dual space, i.e. the space of Schwartz distributions or generalized functions. Furthermore, we use the Bessel potential spaces on the domain Ω and its boundary $\partial\Omega$ denoted respectively by $H^s(\Omega) = \{r_\Omega g : g \in H^s(\mathbb{R}^3)\}$ and $H^s(\partial\Omega)$, where $s \in \mathbb{R}$ (see e.g. [14, 24] for more details). We recall that the spaces H^s coincide with the Sobolev-Slobodetski spaces $W^{s,2}$ for any non-negative s [4].

By $\widetilde{H}^s(\Omega)$ we denote the closure of $\mathcal{D}(\Omega)$ in $H^s(\mathbb{R}^3)$. Furthermore, for Lipschitz domains, the following characterization applies

$$\widetilde{H}^s(\Omega) := \{g \in H^s(\mathbb{R}^3) : \operatorname{supp} g \subset \overline{\Omega}\}.$$

Sometimes, we will also require the space

$$\widetilde{H}^s_\bullet(\Omega) := r_\Omega \widetilde{H}^s(\Omega) := \{r_\Omega g : g \in \widetilde{H}^s(\Omega)\} \subset H^s(\Omega),$$

where r_Ω denotes the restriction operator on Ω.

Moreover, we shall introduce the fundamental concepts and notation of the Hölder spaces which will be useful for the treatment of non-smooth coefficients.

Let θ be a real number with $0 < \theta \leq 1$. Then a function $f : \Omega \to \mathbb{R}$ is said to be Hölder continuous with exponent θ if its Hölder coefficient $|f|_{0,\theta}$, defined as

$$|f|_{0,\theta} := \sup_{x \neq y;\ x,y \in \Omega} \frac{|f(x) - f(y)|}{\|x - y\|^\theta},$$

is finite. The set of all the Hölder continuous functions with exponent θ is denoted by $\mathcal{C}^{0,\theta}(\Omega)$. This space is a locally convex topological vector space endowed with the seminorm given by the Hölder coefficient

With this definition in mind, we can generalize the definition of the Hölder space for functions in $\mathcal{C}^{m,\theta}(\overline{\Omega})$ as the subset of $\mathcal{C}^m(\overline{\Omega})$ whose derivatives of order m are Hölder continuous with exponent θ on the closure of $\overline{\Omega}$. The space $\mathcal{C}^{m,\theta}(\overline{\Omega})$ is a Banach space endowed with the norm

$$\|f\|_{m,\theta} := \|f\|_{\mathcal{C}^m(\overline{\Omega})} + \max_{\alpha = m} |D^\alpha f|_{0,\theta}$$

where α ranges over multiindices.

Following the notation from [18, Definition 2.5], given μ a non negative real number, we define the space $\mathcal{C}^\mu_+(\overline{\Omega})$

$$\mathcal{C}^\mu_+(\overline{\Omega}) := \begin{cases} L^\infty(\overline{\Omega}) & \mu = 0 \\ \mathcal{C}^{\mu-1,1}(\overline{\Omega}) & \mu \in \mathbb{Z}^+ \\ \mathcal{C}^{m,\theta+\varepsilon}(\overline{\Omega}) & \text{where } \varepsilon > 0 \text{ when } \mu = m + \theta,\ m \in \mathbb{Z}^+, \theta \in (0,1). \end{cases}$$

2.2. **Partial differential equation.** Let us consider the scalar elliptic partial differential equation

$$\mathcal{A}u(x) := \sum_{i=1}^3 \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u(x)}{\partial x_i}\right) = f(x), \quad x \in \Omega, \tag{2.1}$$

where the coefficient $a(x) \in C_+^{|s-1|}(\overline{\Omega})$ with $s \in (1/2, 3/2)$, is bounded and satisfies the assumption

$$0 < a_0 \le a(x) \le a_1,$$

for some constants $a_0, a_1 \in \mathbb{R}^+$. Additionally, this equation can be written in weak form as

$$\langle \mathcal{A}u, v \rangle_\Omega := -\mathcal{E}_\Omega(u, v), \quad \forall v \in \mathcal{D}(\Omega), \tag{2.2}$$

where $u \in H^s(\Omega)$ and the bilinear functional $\mathcal{E}_\Omega$ is given by

$$\mathcal{E}_\Omega(u, v) := \langle a\nabla u, \nabla v \rangle_\Omega := \sum_{i=1}^{3} \langle a\partial_i u, \partial_i v \rangle_\Omega.$$

Indeed, because of the deseness of $\mathcal{D}(\Omega)$ in $\widetilde{H}^{2-s}(\Omega)$, the operator $\mathcal{A}$ from (2.2) defines a continuous mapping $\mathcal{A} : H^s(\Omega) \to H^{s-2}(\Omega) = [\widetilde{H}^{2-s}(\Omega)]^*$, (cf. [16, Section 3.1])

$$\langle \mathcal{A}u, v \rangle_\Omega := -\mathcal{E}_\Omega(u, v) \quad \forall u \in H^s(\Omega), v \in \widetilde{H}^{2-s}(\Omega).$$

Let us introduce the operator, $\check{\mathcal{A}} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega) = [H^{2-s}(\Omega)]^*$, defined by

$$\langle \check{\mathcal{A}}u, v \rangle_\Omega := -\check{\mathcal{E}}_\Omega(u, v) = \langle \nabla \cdot \mathring{E}_\Omega(a\nabla u), v \rangle_\Omega, \quad \forall u \in H^s(\Omega), \ v \in H^{2-s}(\Omega), \tag{2.3}$$

where $\mathring{E}_\Omega$ denotes the extension operator of functions defined in Ω, by zero outside Ω in $\mathbb{R}^3$. Alternatively, the operator $\check{\mathcal{A}}$ can also be defined as

$$\check{\mathcal{A}}u := \nabla \cdot \mathring{E}_\Omega r_\Omega(a\nabla u), \tag{2.4}$$

where r_Ω is the restriction operator to Ω. Let us note the operator $\check{\mathcal{A}}$ is bounded as well as a result of the boundedness of $\mathcal{A}$, $\mathring{E}_\Omega$ and r_Ω.

2.3. **Traces and conormal derivatives.** The Trace Theorem [8, Lemma 3.7] states that if a distribution u belongs to $H^s(\Omega)$, with $1/2 < s < 3/2$, then $\gamma^+ u \in H^{s-\frac{1}{2}}(\partial\Omega)$, where $\gamma^+ := \gamma_{\partial\Omega}^+$ is the trace operator on $\partial\Omega$ from Ω. Let $\gamma^{-1} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\mathbb{R}^3)$ denote a (non-unique) continuous right inverse to the trace operator γ^+, i.e., $\gamma_{\partial\Omega}^+ \gamma^{-1} w = w$ for any $w \in H^{s-\frac{1}{2}}(\partial\Omega)$.

For $u \in H^s(\Omega)$, $s > 3/2$, and $a \in C(\overline{\Omega})$, we can define on $\partial\Omega$ the co-normal derivative operator $T^\pm$, in the classical sense as

$$T_x^\pm u := \sum_{i=1}^{3} a(x)\gamma^\pm \left(\frac{\partial u}{\partial x_i}\right) n_i^\pm(x),$$

where $n^+(x)$ is the exterior unit normal vector directed *outwards* from the interior domain Ω at a point $x \in \partial\Omega$. Similarly, $n^-(x)$ is the unit normal vector directed *inwards* to the interior domain Ω at a point $x \in \partial\Omega$. Sometimes, we will also use the notation $T_x^\pm u$ or $T_y^\pm u$ to emphasise with respect to which variable we are differentiating. When the variable of differentiation is obvious or is a dummy variable, we will simply use the notation $T^\pm u$.

However, the classical co-normal derivative operator is generally not well defined if $u \in H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, see [18, Appendix A]. To overcome this issue, one can use the generalized co-normal derivative instead; see for instance [16, Definition 3.1].

Definition 2.1. Let $1/2 < s < 3/2$, $u \in H^s(\Omega)$, $a \in C_+^{|s-1|}(\overline{\Omega})$ and $r_\Omega \mathcal{A}u = r_\Omega \tilde{f}$ for some $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$. Furthermore, let $(\gamma^{-1})^* : H^{-s}(\mathbb{R}^n) \to H^{-s+\frac{1}{2}}(\partial\Omega)$ be the operator satisfying $\langle (\gamma^{-1})^*\varphi, w \rangle_{\partial\Omega} := \langle \varphi, \gamma^{-1}w \rangle_{\mathbb{R}^n}$ for any $w \in H^{s-\frac{1}{2}}(\partial\Omega)$ and

for any $\varphi \in H^{-s}(\mathbb{R}^n)$. Then, the *generalized co-normal derivative* $T^+(\tilde{f}, u) \in H^{s-\frac{3}{2}}(\partial\Omega)$ is defined by

$$\langle T^+(\tilde{f}, u), w \rangle_{\partial\Omega} := \langle \tilde{f}, \gamma^{-1}w \rangle_\Omega + \check{\mathcal{E}}_\Omega(u, \gamma^{-1}w) = \langle \tilde{f} - \check{\mathcal{A}}u, \gamma^{-1}w \rangle_\Omega, \qquad (2.5)$$

for all $w \in H^{\frac{3}{2}-s}(\partial\Omega)$, or alternatively by $T^+(\tilde{f}, u) := (\gamma^{-1})^*(\tilde{f} - \check{\mathcal{A}}u)$.

The key property of the generalized co-normal derivative $T^+(\tilde{f}; u)$ is its independence from the (non-unique) choice of the operator γ^{-1}, see [16, Theorem 5.3]. Moreover, the first Green identity holds in the following form for $u \in H^s(\Omega)$ with $r_\Omega \mathcal{A}u = r_\Omega \tilde{f}$ for some $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$,

$$\langle T^+(\tilde{f}, u), \gamma^+ v \rangle_{\partial\Omega} = \langle \tilde{f}, v \rangle_\Omega + \check{\mathcal{E}}_\Omega(u, v) = \langle \tilde{f} - \check{\mathcal{A}}u, v \rangle_\Omega, \quad \forall v \in H^{2-s}(\Omega). \quad (2.6)$$

Unfortunately, the generalized co-normal derivative operator is non-linear with respect to u for a given fixed $\tilde{f}$. Nevertheless, it is possible to gain linearity if the domain of the generalized co-normal derivative is constrained to an appropriate subspace of $H^s(\Omega)$, as the one given below.

Definition 2.2. Let $s \in \mathbb{R}$, and $\mathcal{A}_* : H^s(\Omega) \to \mathcal{D}^*(\Omega)$ be a linear operator. For $t \in \mathbb{R}$, we introduce the space

$$H^{s,t}(\Omega; \mathcal{A}_*) := \{g : g \in H^s(\Omega) : \mathcal{A}_* g \in \widetilde{H}^t_\bullet(\Omega)\},$$

endowed with the norm

$$\|g\|_{H^{s,t}(\Omega;\mathcal{A}_*)} := \left(\|g\|^2_{H^s(\Omega)} + \|\mathcal{A}_* g\|^2_{\widetilde{H}^t_\bullet(\Omega)} \right)^{1/2}.$$

The operator resulting from constraining the domain to a Sobolev space of the type defined above is the canonical co-normal derivative [16, Definition 6.5].

Definition 2.3. For $u \in H^{s,-1/2}(\Omega; \mathcal{A})$ and $a \in C_+^{|s-1|}(\overline{\Omega})$, $1/2 < s < 3/2$, we define the *canonical co-normal derivative* $T^+u \in H^{s-\frac{3}{2}}(\partial\Omega)$ as

$$\langle T^+u, w \rangle_{\partial\Omega} := \langle \tilde{\mathcal{A}}u, \gamma^{-1}w \rangle_\Omega + \check{\mathcal{E}}_\Omega(u, \gamma^{-1}w) = \langle \tilde{\mathcal{A}}u - \check{\mathcal{A}}u, \gamma^{-1}w \rangle_\Omega \qquad (2.7)$$

for all $w \in H^{\frac{3}{2}-s}(\partial\Omega)$; that is, $T^+u := (\gamma^{-1})^*(\tilde{\mathcal{A}}u - \check{\mathcal{A}}u)$.

By [16, Theorem 3.9] and [17, Theorem 6.6], the canonical co-normal derivative T^+u is independent of the (non-unique) choice of the operator γ^{-1}, the operator $T^+ : H^{s,-1/2}(\Omega; \mathcal{A}) \to H^{s-\frac{3}{2}}(\partial\Omega)$ is continuous, and the first Green identity holds in the form

$$\langle T^+u, \gamma^+ v \rangle_{\partial\Omega} := \langle \tilde{\mathcal{A}}u, v \rangle_\Omega + \check{\mathcal{E}}_\Omega(u, v) = \langle \tilde{\mathcal{A}}u - \check{\mathcal{A}}u, v \rangle_\Omega \quad \forall v \in H^{2-s}(\Omega). \quad (2.8)$$

The operator $T^+ : H^{s,t}(\Omega; \mathcal{A}) \to H^{s-\frac{3}{2}}(\partial\Omega)$ in Definition 2.5 is continuous for $t \geq -\frac{1}{2}$. The canonical co-normal derivative is defined by the function u and the operator $\mathcal{A}$ only, and thus does not depend separately on the right hand side $\tilde{f}$ (i.e. its behavior on the boundary), unlike the generalized co-normal derivative defined in (2.6). Additionally, the canonical co-normal derivative operator T^+ is linear in u.

For further reference on the generalized and canonical co-normal derivative, we refer the reader to [16, 17].

2.4. **Green identities.** Let $1/2 < s < 3/2$ and $a \in C_+^{|s-1|}(\overline{\Omega})$. If $u \in H^{s,-1/2}(\Omega; \mathcal{A})$, then Definitions (2.5) and (2.7) imply that the generalized co-normal derivative for arbitrary extension $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ of the distributions $r_\Omega \mathcal{A}u$ can be expressed as

$$\langle T^+(\tilde{f}, u), w \rangle_{\partial\Omega} := \langle T^+u, w \rangle_{\partial\Omega} + \langle \tilde{f} - \tilde{\mathcal{A}}u, \gamma^{-1}w \rangle_\Omega \quad \forall w \in H^{\frac{3}{2}-s}(\Omega).$$

If $u \in H^s(\Omega)$ and $v \in H^{2-s,-\frac{1}{2}}(\Omega; \mathcal{A})$, then swapping over the roles of u and v in (2.8), we obtain the first Green identity for v,

$$\check{\mathcal{E}}_\Omega(u, v) + \langle u, \tilde{\mathcal{A}}v \rangle_\Omega = \langle T^+v, \gamma^+u \rangle_{\partial\Omega}. \tag{2.9}$$

Furthermore, $r_\Omega \mathcal{A}u = r_\Omega \tilde{f}$ with $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$, then subtracting (2.9) from (2.6), the following second Green identity is obtained,

$$\langle \tilde{f}, v \rangle_\Omega - \langle u, \tilde{\mathcal{A}}v \rangle_\Omega = \langle T^+(\tilde{f}, u), \gamma^+v \rangle_{\partial\Omega} - \langle T^+v, \gamma^+u \rangle_{\partial\Omega}. \tag{2.10}$$

When $u \in H^{s,-1/2}(\Omega; \mathcal{A})$ and $v \in H^{2-s,-\frac{1}{2}}(\Omega; \mathcal{A})$, we arrive at the familiar form of the second Green identity,

$$\langle v, \tilde{\mathcal{A}}u \rangle_\Omega - \langle u, \tilde{\mathcal{A}}v \rangle_\Omega = \langle T^+u, \gamma^+v \rangle_{\partial\Omega} - \langle T^+v, \gamma^+u \rangle_{\partial\Omega}.$$

2.5. **Boundary value problem.** For $1/2 < s < 3/2$, we aim to derive boundary-domain integral equation systems for the following Dirichlet boundary value problem. Find a function $u \in H^s(\Omega)$ satisfying

$$\mathcal{A}u = f \quad \text{in } \Omega, \tag{2.11a}$$

$$\gamma^+u = \varphi_0 \quad \text{on } \partial\Omega, \tag{2.11b}$$

where $\varphi_0 \in H^{s-\frac{1}{2}}(\partial\Omega)$ and $f \in H^{s-2}(\Omega)$.

The following assertion is well known for $s = 1$ and follows from the first Green identity and the Lax Milgram Theorem, see [18, Theorem 5.1].

Theorem 2.4. *Let $1/2 < s < 3/2$ and $a \in C_+^{|s-1|}(\overline{\Omega})$. The Dirichlet problem* (2.11) *has at most one solution in $H^s(\Omega)$.*

3. Parametrices and remainders

One of the purposes of this article is to obtain a system of integral equations that is equivalent to (2.11). To obtain such a system, one needs first to obtain an integral representation of the solution in terms of surface and volume potentials. In the case of constant coefficients for the Laplace equation, one substitutes the fundamental solution in the second Green identity, see for example [13, 24]. However, when the PDE has variable coefficients, although the fundamental solution might exist, it may not always be available explicitly as required for numerical approximation of the solution. One possible way around this is to introduce a parametrix.

Definition 3.1. A distribution P is said to be a parametrix for a given differential operator $\mathcal{B}$ with remainder R if

$$\mathcal{B}P = \delta + R.$$

where $\delta(\cdot)$ is the Dirac distribution.

For a given operator $\mathcal{B}$, the parametrix might not be unique. For example, the parametrix

$$P^y(x, y) = \frac{1}{a(y)}P_\Delta(x - y), \quad x, y \in \mathbb{R}^3,$$

was employed in [15, 6] for the operator $\mathcal{A}$ defined in (2.1), where

$$P_\Delta(x - y) = \frac{-1}{4\pi|x - y|}$$

is the fundamental solution of the Laplace operator. The remainder corresponding to the parametrix P^y is

$$R^y(x, y) = \sum_{i=1}^{3} \frac{1}{a(y)} \frac{\partial a(x)}{\partial x_i} \frac{\partial}{\partial x_i} P_\Delta(x - y), \quad x, y \in \mathbb{R}^3.$$

In this article, for the same operator $\mathcal{A}$ defined in (2.1), we use another parametrix,

$$P(x, y) := P^x(x, y) = \frac{1}{a(x)} P_\Delta(x - y), \quad x, y \in \mathbb{R}^3, \tag{3.1}$$

which leads to the corresponding remainder

$$
\begin{aligned}
R(x, y) = R^x(x, y) &= -\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \Big(\frac{1}{a(x)} \frac{\partial a(x)}{\partial x_i} P_\Delta(x, y) \Big) \\
&= -\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \Big(\frac{\partial \ln a(x)}{\partial x_i} P_\Delta(x, y) \Big), \quad x, y \in \mathbb{R}^3.
\end{aligned}
\tag{3.2}
$$

Note that both the remainders R_x and R_y are weakly singular, i.e.,

$$R^x(x, y), \, R^y(x, y) \in \mathcal{O}(|x - y|^{-2}).$$

4. Domain and boundary integral operators

After replacing the parametrix in the Green identities, we obtain an integral representation formula of the solution in terms of potential-type operators whose kernel is somehow related to the parametrix. Therefore, these operators receive the name of surface or volume parametrix based potential operators depending on whether the integral operator is defined exclusively on the boundary or within the domain.

In this section, we introduce the parametrix-based volume and surface potentials and analyze their mapping properties.

4.1. **Volume parametrix-based potentials.** Let $\rho \in \mathcal{D}(\overline{\Omega})$, the volume potential and the remainder potential operator, corresponding to parametrix (3.1) and remainder (3.2) are defined as

$$\mathbf{P}\rho(y) := \int_{\mathbb{R}^3} P(x, y)\rho(x) \, dx, \quad y \in \mathbb{R}^3,$$

$$\mathcal{P}\rho(y) := \int_\Omega P(x, y)\rho(x) \, dx, \quad y \in \Omega,$$

$$\mathcal{R}\rho(y) = \int_\Omega R(x, y)\rho(x) \, dx, \quad y \in \Omega.$$

Since the parametrix and the remainder are related to the fundamental solution of the Laplace equation, as shown in (3.1)-(3.2), then the volume parametrix-based potentials also preserve a similar relation with their analogous volume potentials of the constant coefficient case, i.e. based on the fundamental solution, see [20].

$$\mathbf{P}\rho = \mathbf{P}_\triangle \Big(\frac{\rho}{a} \Big), \tag{4.1}$$

$$\mathcal{P}\rho = \mathcal{P}_\Delta\left(\frac{\rho}{a}\right), \tag{4.2}$$

$$\mathcal{R}\rho = \nabla \cdot [\mathcal{P}_\Delta(\rho \nabla \ln a)] - \mathcal{P}_\Delta(\rho \Delta \ln a). \tag{4.3}$$

Taking into account relations (4.1)-(4.3) and that the variable coefficient is continuous and bounded, one can prove the following mapping properties which result from applying the analogous mapping properties of the volume newtonian potential for the constant coefficient case given in [18, Lemma 3.1] using a similar argument as in [18, Theorem 3.2].

Theorem 4.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, the following operators are continuous:*

$$\mu\mathbf{P} : H^s(\mathbb{R}^3) \to H^{s+2}(\mathbb{R}^3), \quad s \in \mathbb{R},\ a \in C_+^{|s+2|}(\mathbb{R}^3),\ \forall \mu \in \mathcal{D}(\mathbb{R}^3);$$

$$\mathcal{P} : \widetilde{H}^s(\Omega) \to H^{s+2}(\Omega), \quad s \in \mathbb{R},\ a \in C_+^{|s+2|}(\overline{\Omega});$$

$$: H^s(\Omega) \to H^{s+2}(\Omega), \quad -\frac{1}{2} < s < \frac{1}{2},\ a \in C_+^{s+2}(\overline{\Omega});$$

$$\mathcal{R} : \widetilde{H}^s(\Omega) \to H^{s+1}(\Omega), \quad s \in \mathbb{R},\ a \in C_+^{|s+1|}(\overline{\Omega});$$

$$: H^s(\Omega) \to H^{s+1}(\Omega), \quad -\frac{1}{2} < s < \frac{1}{2},\ a \in C_+^{s+1}(\overline{\Omega});$$

$$: H^s(\Omega) \to H^{s,-1/2}(\Omega;\mathcal{A}), \quad \frac{1}{2} < s < \frac{3}{2},\ a \in C_+^{3/2}(\overline{\Omega}).$$

$$\mathcal{R} : H^s(\Omega) \to H^s(\Omega),\ \frac{1}{2} < s < \frac{3}{2},\ a \in C_+^s(\overline{\Omega});$$

$$\gamma^+\mathcal{R} : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega),\ \frac{1}{2} < s < \frac{3}{2},\ a \in C_+^s(\overline{\Omega}).$$

The following corollary follows directly from applying the Rellich Theorem on compact embeddings for Sobolev spaces [14, Theorem 3.27] and the Trace Theorem to the mapping properties given in the previous theorem for the $\mathcal{R}$ operator.

Corollary 4.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Then, the following operators are compact.*

$$\mathcal{R} : H^s(\Omega) \to H^s(\Omega), \quad \frac{1}{2} < s < \frac{3}{2},\ a \in C_+^s(\overline{\Omega});$$

$$\gamma^+\mathcal{R} : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad \frac{1}{2} < s < \frac{3}{2},\ a \in C_+^s(\overline{\Omega}).$$

4.2. **Surface parametrix-based potentials.** The parametrix-based single layer and double layer surface potentials are defined for $y \in \mathbb{R}^3 : y \notin \partial\Omega$, as

$$V\rho(y) := -\int_{\partial\Omega} P(x,y)\rho(x)\,dS(x), \quad W\rho(y) := -\int_{\partial\Omega} T_x^+ P(x,y)\rho(x)\,dS(x).$$

By relation (3.1), the operators V and W can also be written in terms of the surface potentials associated with the Laplace operator,

$$V\rho = V_\Delta\left(\frac{\rho}{a}\right), \tag{4.4}$$

$$W\rho = W_\Delta\rho - V_\Delta\left(\rho\frac{\partial \ln a}{\partial n}\right). \tag{4.5}$$

Similar to the previous section, one can derive the mapping properties of the single layer and double layer parametrix-based surface potential operators taking into

account relations (4.4)-(4.5) along with the analogous mapping properties of their counterpart operators of the constant coefficient case given in [18, Theorem 3.3] using a similar argument as in [18, Theorem 3.5].

Theorem 4.3. *Let Ω be a bounded Lipschitz domain.*

(i) *The following operators are continuous if $1/2 \le s \le 3/2$:*

$$\mu V : H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\mathbb{R}^3), \quad a \in C_+^s(\mathbb{R}^3), \ \forall \mu \in \mathcal{D}(\mathbb{R}^3);$$

$$\mu W : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega), \quad a \in C_+^s(\overline{\Omega});$$

$$\mu r_{\Omega_-} W : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega_-), \quad a \in C_+^s(\overline{\Omega}_-), \ \forall \mu \in \mathcal{D}(\mathbb{R}^3).$$

(ii) *The following operators are continuous if $1/2 < s \le 3/2$ and $a \in C_+^{3/2}(\overline{\Omega})$:*

$$r_\Omega V : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s,-1/2}(\Omega; \mathcal{A});$$

$$\mu r_{\Omega_-} V : H^{s-\frac{3}{2}}(\partial\Omega) \to H_{\mathrm{loc}}^{s,-1/2}(\Omega_-; \mathcal{A}), \quad \forall \mu \in \mathcal{D}(\mathbb{R}^3);$$

$$r_\Omega W : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s,-1/2}(\Omega; \mathcal{A});$$

$$\mu r_{\Omega_-} W : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s,-1/2}(\Omega_-; \mathcal{A}), \quad \forall \mu \in \mathcal{D}(\mathbb{R}^3);$$

(iii) *The following operators are continuous if $1/2 < s < 3/2$,*

$$\gamma^\pm V : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad a \in C_+^s(\overline{\Omega}_\pm);$$

$$\gamma^\pm W : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad a \in C_+^s(\overline{\Omega}_\pm).$$

The next theorem follows from the well-known jump properties of the boundary integral operators for the Laplace operator and relations (4.4)-(4.5), see also [14, Theorem 6.11] or [18, Theorem 3.3 (iii)].

Theorem 4.4. *Let $\partial\Omega$ be a compact Lipschitz boundary, $1/2 < s < 3/2$, $\varphi \in H^{s-\frac{1}{2}}(\partial\Omega)$ and $\psi \in H^{s-\frac{3}{2}}(\partial\Omega)$. Then*

$$\gamma^+ V\psi - \gamma^- V\psi = 0, \quad \gamma^+ W\varphi - \gamma^- W\varphi = -\varphi, \quad \text{if } a \in C_+^s(\mathbb{R}^3). \qquad (4.6)$$

The previous jump properties lead to the definition of two new boundary integral operators related with the traces of the single layer and double layer operators, also called *direct values of the single and double layer potentials.*

$$\mathcal{V} := \gamma^+ V, \quad \mathcal{W} := \frac{1}{2}(\gamma^+ W + \gamma^- W).$$

The mapping properties of $\mathcal{V}$ and $\mathcal{W}$ are given by the following result which is a direct consequence of applying the Trace Theorem to the mapping properties of the operator V and W given in Theorem 4.3.

Corollary 4.5. *Let $\partial\Omega$ be a compact Lipschitz boundary, $1/2 < s < 3/2$. The following operators are continuous:*

$$\mathcal{V} : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad a \in C_+^s(\overline{\Omega}_\pm); \qquad (4.7)$$

$$\mathcal{W} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad a \in C_+^s(\overline{\Omega}_\pm). \qquad (4.8)$$

Employing definitions (4.7) and (4.8), the jump properties (4.6) can be re-written as

$$\gamma^\pm V\psi = \mathcal{V}\psi, \quad \gamma^\pm W\varphi = \mp\frac{1}{2}\varphi + \mathcal{W}\varphi, \quad \text{if } a \in C_+^s(\mathbb{R}^3). \qquad (4.9)$$

By Corollary 4.5 and relations (4.4) and (4.5), the operators $\mathcal{V}$ and $\mathcal{W}$ can be expressed in terms of the volume and surface potentials and operators associated with the Laplace operator, as follows [20],

$$\mathcal{V}\rho = \mathcal{V}_\Delta \left(\frac{\rho}{a}\right),$$

$$\mathcal{W}\rho = \mathcal{W}_\Delta \rho - \mathcal{V}_\Delta \left(\rho \frac{\partial \ln a}{\partial n}\right).$$

A key result for proving uniqueness of the solution of the integral equation systems that we will derive in the upcoming sections is related to the invertibility of the single layer potential. This result is well known (see [14, Theorem 8.12 and Corollary 8.13] or [18, Lemma 4.8 (i)].) for the constant coefficient case, and can easily be extrapolated to the variable coefficient case thanks to the relation (4.4).

Lemma 4.6. *Let Ω be a bounded simply-connected Lipschitz domain, let $a \in \mathcal{C}_+^s(\overline{\Omega})$ and let $1/2 < s < 3/2$.*

(1) *The operator $\mathcal{V} : H^s(\partial\Omega) \to H^{s+1}(\partial\Omega)$ is an isomorphism.*
(2) *If $\Psi^* \in H^{s-\frac{3}{2}}(\partial\Omega)$ and*

$$r_\Omega V \Psi^*(y) = 0, \quad y \in \Omega$$

then $\Psi^*(y) = 0$.

4.3. **Integral representation of the solution of the BVP.** In this section, we apply the usual procedures of the boundary-domain integral equation method [6, 10] for obtaining an integral equation system derived from the integral representation formula which results from substituting the parametrix (3.1) in the first or second Green identities.

Let $u \in H^s(\Omega)$, $1/2 < s < 3/2$, $a \in C_+^s(\overline{\Omega})$, and let us substitute $v(x) = P(x,y)$, in the first Green identity (2.10). Then we obtain the following integral representation formula, also called *generalized third Green identity*, for the solution of (2.11),

$$u + \mathcal{R}u + W\gamma^+ u = \mathcal{P}\breve{A}u \quad \text{in } \Omega, \tag{4.10}$$

where the term in the right hand side, considering the definitions of the operator $\breve{A}$ given in (2.3) and (2.4), is given by

$$\mathcal{P}\breve{A}u(y) := \langle \breve{A}u, P(.,y)\rangle_\Omega = -\breve{\mathcal{E}}(u, P(.,y)) = -\langle \widetilde{E}_\Omega^{s-1}(a\nabla u), \nabla P(.,y)\rangle_\Omega, \quad y \in \Omega.$$

When $\mathcal{A}u = r_\Omega \tilde{f}$ in Ω, where $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$, the integral representation formula (4.10) can be reformulated as

$$u + \mathcal{R}u - VT^+(\tilde{f}, u) + W\gamma^+ u = \mathcal{P}\tilde{f} \quad \text{in} \quad \Omega. \tag{4.11}$$

Since we aim to obtain a system of integral equations equivalent to the Dirichlet problem (2.11), it will be necessary to obtain an integral representation formula for the trace of the solution. Then, by applying the trace operator to both sides of the identity (4.11) and taking into account the jump properties given in (4.9), we obtain

$$\frac{1}{2}\gamma^+ u + \gamma^+ \mathcal{R}u - VT^+(\tilde{f}, u) + \mathcal{W}\gamma^+ u = \gamma^+ \mathcal{P}\tilde{f}, \quad \text{on } \partial\Omega. \tag{4.12}$$

5. Boundary-domain integral equation system

We aim to obtain a system of integral equations equivalent to Dirichlet BVP (2.11). The integral equations will be written in terms of the surface and volume parametrix-based potentials which are defined on the boundary and in the domain respectively. Hence, this system of integral equations receives the name of Boundary-Domain Integral Equation System (BDIES). The BDIES is obtained following the boundary-domain integral equations method which extends the Boundary Integral Equation method for the variable coefficient case, see e.g., [19, 18, 6] for more details.

Let $a \in C^s_+(\overline{\Omega})$ with $s \in (\frac{1}{2}, \frac{3}{2})$. Furthermore, let $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ be an extension of $f \in H^{s-2}(\Omega)$ (i.e., $f = r_\Omega \tilde{f}$), which always exists [16, Lemma 2.15 and Theorem 2.16]. Then, to obtain the boundary domain integral equations of the system from the integral representation formulas (4.12)-(6.1), we introduce the unknown co-normal derivative $\psi := T^+(\tilde{f}, u) \in H^{s-\frac{3}{2}}(\partial\Omega)$ which shall be regarded as formally segregated from u. On the other hand, let us recall that the trace datum for the Dirichlet problem is known. Hence, we replace in (4.12)-(6.1) $\gamma^+ u = \varphi_0$ and we move all these terms to the right hand side. We obtain the following system consisting of two equations with two unknown functions $(u, \psi) \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$,

$$u + \mathcal{R}u - V\psi = F_0 \quad \text{in } \Omega, \tag{5.1a}$$

$$\gamma^+ \mathcal{R}u - \mathcal{V}\psi = \gamma^+ F_0 - \varphi_0 \quad \text{on } \partial\Omega, \tag{5.1b}$$

where

$$F_0 = \mathcal{P}\tilde{f} - W\varphi_0. \tag{5.2}$$

Note that for $\varphi_0 \in H^{s-\frac{1}{2}}(\partial\Omega)$, we have the inclusion $F_0 \in H^s(\Omega)$ if $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ due to the mapping properties of the surface and volume potentials.

The system (5.1), given by (5.1a)-(5.1b) can be written in matrix notation as

$$\mathcal{A}^1 \mathcal{U} = \mathcal{F}^1,$$

where $\mathcal{U}$ represents the vector containing the unknowns of the system,

$$\mathcal{U} = (u, \psi)^\top \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega),$$

the right-hand side vector is

$$\mathcal{F}^1 := [F_0, \gamma^+ F_0 - \varphi_0]^\top \in H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega),$$

and the matrix operator $\mathcal{A}^1$ is defined by

$$\mathcal{A}^1 = \begin{bmatrix} I + \mathcal{R} & -V \\ \gamma^+ \mathcal{R} & -\mathcal{V} \end{bmatrix}.$$

6. Main results

Let us prove that the Dirichlet boundary value problem (2.11) in Ω is equivalent to the system of the Boundary Domain Integral Equations (5.1a), (5.1b). Before proving the main equivalence theorem, let us prove an instrumental lemma which will simplify the proof of the main result.

The following statement is similar to [18, Lemma 4.2] in which a different parametrix was used. It extends [10, Lemma 4.1], where the corresponding assertion was proven for $\tilde{f} \in L_2(\Omega)$, $s = 1$, $a \in C^\infty(\Omega)$.

Lemma 6.1. *Let* $1/2 < s < 3/2$ *and* $a \in C_+^s(\overline{\Omega})$. *Let* $u \in H^s(\Omega)$, $\Psi \in H^{s-\frac{3}{2}}(\partial\Omega)$, $\Phi \in H^{s-\frac{1}{2}}(\partial\Omega)$, *and* $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ *satisfy*

$$u + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}\tilde{f} \quad in \ \Omega. \tag{6.1}$$

Then

$$\mathcal{A}u = r_\Omega \tilde{f} \quad in \ \Omega,$$

$$V(\Psi - T^+(\tilde{f}, u)) - W(\Phi - \gamma^+ u) = 0 \quad in \ \Omega.$$

Proof. We subtract (6.1) from the first Green identity (4.10) to obtain

$$V(\Psi) - W(\Phi - \gamma^+ u) = \mathcal{P}(\check{\mathcal{A}}u - \tilde{f}). \tag{6.2}$$

Then (6.2) can be rewritten in terms of the constant coefficient volume and surface potentials by applying the relations (4.2), (4.4), and (4.5), as

$$V_\Delta\Big(\frac{\Psi}{a}\Big) - W_\Delta(\Phi - \gamma^+ u) + V_\Delta\Big(\frac{\partial \ln a}{\partial n}\,(\Phi - \gamma^+ u)\Big) = \mathcal{P}_\Delta\Big(\frac{\check{\mathcal{A}}u - \tilde{f}}{a}\Big).$$

Then, we apply the Laplace operator to both sides of the previous equation taking into account that the terms with V_Δ and W_Δ are harmonic functions and hence vanish. Furthermore, the Laplace operator applied to the newtonian potential gives

$$\Delta \mathcal{P}_\Delta\Big(\frac{\check{\mathcal{A}}u - \tilde{f}}{a}\Big) = \frac{\check{\mathcal{A}}u - \tilde{f}}{a}.$$

Since the coefficient $a(x)$ is strictly positive,

$$\check{\mathcal{A}}u - \tilde{f} = 0 \quad in \ \Omega. \tag{6.3}$$

This implies that $r_\Omega \check{\mathcal{A}}u = \mathcal{A}u = r_\Omega \tilde{f}$, which completes the proof of part (i).

On the other hand, part (i) implies that $\tilde{f}$ is an extension of the distribution $\mathcal{A}u \in H^{s-2}(\Omega)$ to $\widetilde{H}^{s-2}(\Omega)$. Therefore, by applying the first Green identity (2.6) with $v = P$, we obtain (see [18, Lemma 4.2])

$$\mathcal{P}(\check{\mathcal{A}}_\Omega u - \tilde{f}) = VT^+(\tilde{f}, u) \quad in \ \Omega. \tag{6.4}$$

Finally, substituting (6.3) and (6.4) into (6.2), we arrive at the relation in part (ii). $\square$

The above lemma will simplify considerably the proof of the Equivalence Theorem, one of the main results of this paper. In particular, the lemma will help us showing that a solution of the BDIES (5.1) is a solution of the BVP.

Theorem 6.2. *Let* $1/2 < s < 3/2$ *and* $a \in C_+^s(\overline{\Omega})$. *Let* $\varphi_0 \in H^{s-\frac{1}{2}}(\partial\Omega)$, $f \in H^{s-2}(\Omega)$ *and* $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ *is such that* $r_\Omega \tilde{f} = f$.

   (i) *If a function* $u \in H^s(\Omega)$ *solves the Dirichlet BVP* (2.11), *then the pair* $(u, \psi)^\top \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$ *where*

$$\psi = T^+(\tilde{f}, u), \qquad on \quad \partial\Omega, \tag{6.5}$$

    solves the BDIE system (5.1).

   (ii) *If a pair* $(u, \psi)^\top \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$ *solves the BDIE system* (5.1) *then* u *solves the BVP and the functions* ψ *satisfy* (6.5).

Proof. (i) Let $u \in H^s(\Omega)$ be a solution of the boundary value problem (2.11) and let ψ be defined by (6.5), evidently implying $\psi \in H^{s-\frac{3}{2}}(\partial\Omega)$. Then, it immediately follows from Theorem 6.1 and relations (6.1) and (4.12) that the pair (u, ψ) solves BDIE system (5.1) with the right hand side $\mathcal{F}^1$ which completes the proof of item (i).

(ii). Let now the pair $(u, \psi)^\top \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$ solve the BDIE system. Taking the trace of the equation (5.1a) and subtract it from the equation (5.1b), we obtain

$$\gamma^+ u = \varphi_0, \quad \text{on } \partial\Omega. \tag{6.6}$$

i.e. u satisfies the Dirichlet condition (2.11b).

Equation (5.1b) and Lemma 6.1 with $\Psi = \psi, \Phi = \varphi_0$ imply that u is a solution of PDE (2.11a) and

$$V\Psi^* - W\Phi^* = 0 \quad \text{in } \Omega.$$

where $\Psi^* = \psi - T^+(\tilde{f}, u)$ and $\Phi^* = \phi - \gamma^+ u$. From (6.6), Lemma 4.6 implies $\Psi^* = 0$ which completes the proof of condition (6.6). Thus u obtained from solution of the BDIE system (5.1) solves the Dirichlet problem. $\square$

Theorem 6.3. *Let Ω be a bounded simply-connected Lipschitz domain, $1/2 < s < 3/2$, and $\sigma = \max\{1, s\}$. Then the operator*

$$\mathcal{A}^1 : H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega), \quad \text{if } a \in C_+^\sigma(\overline{\Omega}) \tag{6.7}$$

satisfies:

(i) *the operator $\mathcal{A}^1$ is bounded and Fredholm with zero index;*
(ii) *the operator $\mathcal{A}^1$ has a bounded inverse.*

Proof. (i) The operator is continuous because the mapping properties of the operators involved in the matrix $\mathcal{A}^1$, see Theorem 4.1 and Theorem 4.3. To prove the Fredholm property of operator (6.8), let us consider the operator

$$\mathcal{A}_0^1 : H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega), \tag{6.8}$$

given by

$$\mathcal{A}_0^1 := \begin{bmatrix} I & -V \\ 0 & -\mathcal{V} \end{bmatrix} \implies \mathcal{A}^1 - \mathcal{A}_0^1 = \begin{bmatrix} \mathcal{R} & 0 \\ \gamma^+\mathcal{R} & 0 \end{bmatrix}.$$

As a result of compactness properties of the operators $\mathcal{R}$ and $\gamma^+\mathcal{R}$ (cf. Corollary 4.2), the operator $\mathcal{A}_0^1$ is a compact perturbation of operator (6.8), i.e. the operator $\mathcal{A}^1 - \mathcal{A}_0^1$ is compact. The operator $\mathcal{A}_0^1$ is an upper triangular matrix operator with scalar diagonal invertible operators $I : H^s(\Omega) \to H^s(\Omega)$ and $\mathcal{V} : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$ where the invertibility of the operator $\mathcal{V}$ follows from Theorem 4.6. This implies the invertibility of the operator $\mathcal{A}_0^1$. As a result, the operator (6.8) is Fredholm with index zero.

To prove part (ii) which we consider two cases: (a) $1 \leq s < 3/2$ and (b) $1/2 < s \leq 1$.

(a) Since $1 \leq s < 3/2$, we have $\sigma = s$. From the Equivalence Theorem (6.2) (ii) and the uniqueness of the BVP provided by Theorem (2.4), we can deduce the injectivity of the operator (6.7). Precisely, considering the homogeneous system $\mathcal{A}\mathcal{U} = 0$, the zero right hand side $\mathcal{F}^1 = 0$ can be represented as in (5.2) in terms of $\tilde{f} = 0$ and $\varphi_0 = 0$. By applying the Equivalence Theorem (6.2) (ii), then the solution of this homogeneous system can be represented as $\mathcal{U} = (u, T^+(0; u))^T$, where u is a solution of the Dirichlet problem (2.11) with right hand sides $f = 0$

and $\varphi_0 = 0$. However, the Dirichlet BVP has only the trivial solution $u = 0$, because of Theorem (2.4). Hence, the only solution of the homogeneous BDIES (5.1) is the trivial solution. Therefore, the Fredholm index of the operator (6.7), provided by part (i), implies its invertibility for $1 \le s < 3/2$.

(b) Since $s \in (1/2, 1]$, we have $\sigma = 1$, and hence $a \in C_+^1(\overline{\Omega})$. By Theorem (i), the operator (6.7) is also Fredholm with zero index. Since the invertibility has already been proved for the case $s = 1$, the result [18, Lemma 7.5] implies that the kernels (null-spaces) of the operator (6.8) with $s = 1$ and the operator (6.8) for $s \in (\frac{1}{2}, 1)$ consist of only the zero element for any $s \in (1/2, 1]$, which implies that the operator is invertible for all $s \in (1/2, 1]$. □

Remark 6.4. For a given function $f \in H^{s-2}(\Omega)$, its extension $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ is not unique. Nevertheless, since the solution of a Dirichlet BVP (2.11) does not depend on this extension, equivalence Theorem (6.2) (ii) implies that u in the solution of BDIE systems $\mathcal{A}^1$ does not depend on the particular choice of extension $\tilde{f}$, however, ψ obviously does, see (6.5).

7. Conclusions

A new parametrix for the diffusion equation in non homogeneous media (with variable coefficient) with Lipschitz boundary has been analyzed in this paper. Mapping properties of the corresponding parametrix-based surface and volume potentials have been shown in corresponding Sobolev spaces.

A BDIE system, based on a new parametrix for the originalBVP has been obtained. Equivalence of the BDIE system to the original BVP was proved in the case when the PDE right-hand side function from $H^{s-2}(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$ and the Dirichlet data is from the space $H^{s-\frac{3}{2}}(\partial\Omega)$. The invertibility of the matrix operator defining the BDIE was proved in the corresponding Sobolev space.

Now, we have obtained an analogous system to the BDIE (5.1) of [6, 19] with a new family of parametrices which is uniquely solvable. Hence, further investigation about the numerical advantages of using one family of parametrices over another will follow.

Analyzing BDIEs for different parametrices, i.e. depending on the variable coefficient $a(x)$ or $a(y)$, is crucial in order to understand the analysis of BDIEs derived with parametrices that depend on the variable coefficient $a(x)$ and $a(y)$ at the same, as is the case for the Stokes system, see [19]. Therefore, applying a similar approach, one could extend the results of this paper to elliptic and strongly elliptic PDE systems.

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