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SECOND ORDER SOBOLEV REGULARITY FOR p-HARMONIC FUNCTIONS IN SU(3)

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ABSTRACT. Let u be a weak solution to the degenerate subelliptic p-Laplacian equation

$$\Delta_{\mathcal{H},p}u(x) = \sum_{i=1}^{6} X_i(|\nabla_{\mathcal{H}}u|^{p-2}X_iu) = 0,$$

where $\mathcal{H}$ is the orthogonal complement of a Cartan subalgebra in SU(3) and its orthonormal basis is composed of the vector fields $X_1, \dots, X_6$. We prove that when $1 < p < 7/2$, the solution u has the second order horizontal Sobolev $W^{2,2}_{\mathcal{H},\mathrm{loc}}$-regularity.

1. Introduction

We consider the group SU(3), that is, the special unitary group of 3×3 complex matrices endowed with a horizontal vector field $\nabla_{\mathcal{H}} = \{X_1, X_2, \dots, X_6\}$. Let Ω be a domain in SU(3) and $1 < p < \infty$. We call a function u as a p-harmonic function in Ω if $u \in W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$ is a weak solution to the degenerate subelliptic p-Laplacian equation

$$\Delta_{\mathcal{H},p}u(x) = \sum_{i=1}^{6} X_i(|\nabla_{\mathcal{H}}u|^{p-2}X_iu) = 0 \quad \text{in } \Omega, \tag{1.1}$$

that is,

$$\int_\Omega \sum_{i=1}^{6} |\nabla_{\mathcal{H}}u|^{p-2}X_iuX_i\phi dx = 0, \quad \phi \in C_0^\infty(\Omega),$$

where $\nabla_{\mathcal{H}}u = (X_1u, X_2u, \dots, X_6u)$ is the horizontal gradient of a function $u \in C^1(\Omega)$, $W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega;\mathbb{R})$ is the first order p-th integrable horizontal local Sobolev space, that is, all functions $u \in L^p_{\mathrm{loc}}(\Omega)$ with its distributional horizontal gradient $\nabla_{\mathcal{H}}u \in L^p_{\mathrm{loc}}(\Omega)$, see Section 2 for more details.

When $p = 2$, the p-harmonic functions in SU(3) are usually called as harmonic functions, and are always smooth as proved by Hörmander [8]. When $p \neq 2$, for p-harmonic functions u in SU(3) satisfying

$$0 < M^{-1} \le |\nabla_{\mathcal{H}}u|(x) \le M \quad \text{a.e. in } \Omega, \tag{1.2}$$

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Domokos-Manfredi [4] also proved that $u \in C^\infty$. However without assumption (1.2), one can not expect that $u \in C^\infty$. Recently, for general p-harmonic function in SU(3), Domokos-Manfredi [3] built the $C^{0,1}$-regularity and, when $2 \le p < \infty$, the $C^{1,\alpha}$-regularity.

This article aims to establish the following second order Sobolev regularity for p-harmonic functions u in SU(3) as below, that is, $u \in W^{2,2}_{\mathcal{H},\mathrm{loc}}(\Omega)$. Here for any function v we say $v \in W^{2,2}_{\mathcal{H},\mathrm{loc}}(\Omega)$ if $v \in W^{1,2}_{\mathcal{H},\mathrm{loc}}(\Omega)$ and its second order distributional horizontal derivative $\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}v = (X_iX_jv)_{1\le i,j\le 6} \in L^2_{\mathrm{loc}}(\Omega)$. For convenience, for $\phi \in C_0^\infty(\Omega)$ we write

$$K_\phi = 1 + \|\nabla_{\mathcal{H}}\phi\|^2_{L^\infty(\Omega)} + \|\phi\nabla_{\mathcal{T}}\phi\|_{L^\infty(\Omega)}. \tag{1.3}$$

Theorem 1.1. *Let $1 < p < 7/2$. If u is a p-harmonic function in a domain $\Omega \subset \mathrm{SU}(3)$, then $u \in W^{2,2}_{\mathcal{H},\mathrm{loc}}(\Omega)$. Moreover, when $1 < p \le 2$, for any $\phi \in C_0^\infty(\Omega)$ with $0 \le \phi \le 1$, we have*

$$\int_\Omega \phi^2|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u|^2dx \le c\int_{spt(\phi)}|\nabla_{\mathcal{H}}u|^{2-p}dx + cK_\phi^2\int_{spt(\phi)}|\nabla_{\mathcal{H}}u|^{p+2}dx; \tag{1.4}$$

when $2 < p < 7/2$, for any $\phi \in C_0^\infty(\Omega)$ with $0 \le \phi \le 1$, we have

$$\begin{aligned}\int_\Omega \phi^6|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u|^2dx &\le cK_\phi^3\int_{spt(\phi)}|\nabla_{\mathcal{H}}u|^{p+2}dx + cK_\phi\int_\Omega\phi^4|\nabla_{\mathcal{H}}u|^{p-2}dx\\ &\quad + c\int_\Omega\phi^6|\nabla_{\mathcal{H}}u|^{4-p}dx,\end{aligned} \tag{1.5}$$

where $c = c(p)$ is a positive constant.

Recall that, for p-harmonic functions in Euclidean spaces, their $C^{1,\alpha}$-regularity has been established by [18, 17, 7, 9, 16]. Their Sobolev $W^{2,2}$-regularity with $1 < p < 3+\frac{2}{n-2}$ was proved in [12] (see also [6]). In particular, for p-harmonic functions in $\mathbb{R}^6$, the range of p to get their Sobolev $W^{2,2}_{\mathrm{loc}}$-regularity is also $1 < p < 7/2$, but when $\frac{7}{2} \le p < \infty$, it remains open to get their $W^{2,2}_{\mathrm{loc}}$-regularity; see [6] for more details. Moreover, for p-harmonic functions in Heisenberg group $\mathbb{H}^n$, their $C^{0,1}$ and $C^{1,\alpha}$-regularity has been established in [2, 5, 11, 13, 15, 19, 14]. If $1 < p \le 4$ when $n = 1$ and $1 < p < 3 + \frac{1}{n-1}$ when $n \ge 2$, their horizontal Sobolev $HW^{2,2}_{\mathrm{loc}}$-regularity was established in [5, 10].

To prove Theorem 1.1, it is standard to consider the regularized equation of subelliptic p-Laplacian equation as did in [3]. To be precise, let u be a p-harmonic function in Ω. Given any smooth domain $U \Subset \Omega$ and $\delta \in (0,1]$, denote by $u^\delta \in W^{1,p}_{\mathcal{H}}(U)$ the weak solution to the regularized equation

$$\sum_{i=1}^6 X_i[(\delta + |\nabla_{\mathcal{H}}v|^2)^{\frac{p-2}{2}}X_iv] = 0 \quad \text{in } U,\ v - u \in W^{1,p}_{\mathcal{H},0}(U). \tag{1.6}$$

As for the existence, uniqueness and C^∞-regularity of u^δ, we refer the reader to [4, 3] and references therein. It was proved by Domokos-Manfredi [3] (see Theorem 2.3 below) that $\nabla_{\mathcal{H}}u^\delta \in L^\infty_{\mathrm{loc}}(U)$ uniformly in $\delta \in (0,1]$ and also that $u^\delta \to u$ in $C^0(U)$ as $\delta \to 0$.

To show Theorem 1.1, it suffices to prove that $\{u^\delta\}_{\delta\in(0,1]}$ have the following $W^{2,2}_{\mathcal{H},\mathrm{loc}}(\Omega)$-regularity uniformly in $\delta \in (0,1]$. Indeed, sending $\delta \to 0$, from which one can conclude Theorem 1.1 in a standard way.

Theorem 1.2. *Let $1 < p < 7/2$. If $u^\delta \in W^{1,p}_{\mathcal{H},\rm loc}(U)$ is the weak solution to* (1.6)*, then $u^\delta \in W^{2,2}_{\mathcal{H},\rm loc}(U)$ uniformly in $\delta \in (0,1]$. Moreover, when $1 < p \le 2$, for any $\phi \in C_0^\infty(U)$ with $0 \le \phi \le 1$, we have*

$$\begin{aligned}\int_U \phi^2 |\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\delta|^2 dx \le c \int_{spt(\phi)} (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{2-p}{2}} dx \\ + cK_\phi^2 \int_{spt(\phi)} (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p+2}{2}} dx;\end{aligned} \tag{1.7}$$

when $2 < p < 7/2$, for any $\phi \in C_0^\infty(U)$ with $0 \le \phi \le 1$, we have

$$\begin{aligned}&\int_U \phi^6 |\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\delta|^2 dx \\ &\le cK_\phi^3 \int_{spt(\phi)} (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p+2}{2}} dx + cK_\phi \int_U \phi^4 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} dx \\ &\quad + c \int_U \phi^6 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{4-p}{2}} dx,\end{aligned} \tag{1.8}$$

where K_ϕ is as in (1.3) *and the constant $c = c(p) > 0$.*

Below, we outline the idea for proving Theorem 1.2. Our proof is based on several a priori estimates for u^δ established in [3]; see Lemmas 2.1 and 2.2. We consider two cases: $1 < p \le 2$ and $2 < p < 7/2$.

When $1 < p \le 2$, we conclude (1.7) from Lemmas 2.1 and 2.2 in a direct way.

In the case $2 < p \le 7/2$, to obtain (1.8) we use some ideas from [6, 10] to decompose the horizontal Hessian matrix and then combine a priori estimates in [3]. We proceed as below. For simplicity we write the subelliptic 2-Laplacian as $\Delta_0 v = \Delta_{0,2} v$, and write the symmetrization of horizontal hessian $\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} v = (X_i X_j v)_{1\le i,j\le 6}$ as

$$D_0^2 v := \Big(\frac{X_iX_jv + X_jX_iv}{2}\Big)_{1\le i,j\le 6}.$$

First, the following lemma gives a pointwise estimate of $|D_0^2 u^\delta|^2$, which is inferred from a fundamental inequality in [6, Lemma 2.1]. See Section 4 for details.

Lemma 1.3. *Let $1 < p < 7/2$. If $u^\delta \in W^{1,p}_{\mathcal{H},\rm loc}(U)$ is the weak solution to* (1.6)*. Then*

$$|D_0^2 u^\delta|^2 \le c[|D_0^2 u^\delta|^2 - (\Delta_0 u^\delta)^2] \quad \text{in } U, \tag{1.9}$$

where the constant $c = c(p) > 0$.

Next, we bound the integral of the right-hand side of (1.9); see Section 3 for details. We denote by $\nabla_{\mathcal{T}} v := (X_7 v, X_8 v)$ the vertical derivative of v.

Lemma 1.4. *For any $v \in C^\infty(U)$ and any $\phi \in C_0^\infty(U)$, we have*

$$\begin{aligned}&\Big|\int_U [|D_0^2 v|^2 - (\Delta_0 v)^2]\phi^6 dx\Big| \\ &\le c\int_U |\nabla_{\mathcal{H}} v|^2 \phi^6 dx + c\int_U |\nabla_{\mathcal{H}} v||\nabla_{\mathcal{H}}\nabla_{\mathcal{T}} v|\phi^6 dx \\ &\quad + c\int_U |\nabla_{\mathcal{H}} v||\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} v||\phi|^5[|\nabla_{\mathcal{H}}\phi| + |\phi|]dx,\end{aligned} \tag{1.10}$$

where c is a positive constant.

In regards to the term

$$\int_U \phi^6 |\nabla_{\mathcal{H}} u^\delta||\nabla_{\mathcal{H}}\nabla_{\mathcal{T}} u^\delta| dx$$

appearing in the right hand side of (1.10), applying some Caccoippoli type inequalities established in [3] (see Lemmas 2.1 and 2.2), we have the following upper bound.

Lemma 1.5. *Let $2 < p \le 4$. If $u^\delta \in W^{1,p}_{\mathcal{H},\mathrm{loc}}(U)$ is the weak solution to (1.6), then for any $\phi \in C_0^\infty(U)$ with $0 \le \phi \le 1$, we have*

$$\begin{aligned} &\int_U \phi^6 |\nabla_{\mathcal{H}} u^\delta||\nabla_{\mathcal{H}}\nabla_{\mathcal{T}} u^\delta| dx \\ &\le cK_\phi^3 \int_{spt(\phi)} (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p+2}{2}} dx + cK_\phi \int_U \phi^4 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} dx \\ &\quad + \int_U \phi^6 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{4-p}{2}} dx, \end{aligned} \tag{1.11}$$

where K_ϕ is as in (1.3) and the constant $c = c(p) > 0$.

On the other hand, we are going to bound $|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} v|^2$ via $|D_0^2 v|^2$ from above. Denote by Mv the difference between $\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} v$ and $D_0^2 v$, that is

$$Mv := \nabla_{\mathcal{H}}\nabla_{\mathcal{H}} v - D_0^2 v = \Big(\frac{X_iX_j v - X_jX_i v}{2}\Big)_{1\le i,j\le 6} = \Big(\frac{[X_i, X_j]v}{2}\Big)_{1\le i,j\le 6}.$$

Since M is an anti-symmetric matrix ($m_{i,j} = -m_{j,i}$), we obtain

$$|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} v|^2 = |D_0^2 v|^2 + |Mv|^2.$$

We bound the integration of $|Mv|^2$ as follows.

Lemma 1.6. *For any $v \in C^\infty(U)$ and any $\phi \in C_0^\infty(U)$, we have*

$$\begin{aligned} \int_U |Mv|^2 \phi^6 dx &\le 6 \int_U |\nabla_{\mathcal{H}} v||\nabla_{\mathcal{H}}\nabla_{\mathcal{T}} v|\phi^6 dx + 36 \int_U |\nabla_{\mathcal{H}} v||\nabla_{\mathcal{T}} v||\phi^5 \nabla_{\mathcal{H}}\phi| dx \\ &\quad + \int_U |\nabla_{\mathcal{H}} v|^2 \phi^6 dx. \end{aligned} \tag{1.12}$$

Finally, combining Lemmas 1.3, 1.4, 1.5 and 1.6, we conclude (1.8) for $2 < p < 7/2$.

2. Preliminaries

We recall the special unitary group of 3×3 complex matrices

$$\{g \in \mathrm{GL}(3, \mathbb{C}) : g \cdot g^* = I, \det g = 1\}$$

as the group SU(3) and define its Lie algebra by

$$\mathrm{su}(3) := \{X \in \mathrm{gl}(3, \mathcal{C}) : X + X^* = 0, \mathrm{tr}\, X = 0\}.$$

From this, we give the inner product on SU(3) by

$$\langle X, Y\rangle := -\frac{1}{2}\,\mathrm{tr}(XY).$$

On the other hand, we note that the two-dimensional maximal torus on SU(3) is given by the set

$$\mathbb{T} := \Big\{ \begin{pmatrix} e^{ia_1} & 0 & 0 \\ 0 & e^{ia_2} & 0 \\ 0 & 0 & e^{ia_3} \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R}, a_1 + a_2 + a_3 = 0 \Big\}.$$

Then we choose its Lie algebra as the Cartan subalgebra, that is,

$$\mathcal{T} := \Big\{ \begin{pmatrix} ia_1 & 0 & 0 \\ 0 & ia_2 & 0 \\ 0 & 0 & ia_3 \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R}, a_1 + a_2 + a_3 = 0 \Big\}.$$

According to the definition of SU(3), we can obtain its orthonormal basis composed of the following Gell Mann matrices $\mathcal{G}$:

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$T_1 = \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -i/\sqrt{3} & 0 & 0 \\ 0 & -i/\sqrt{3} & 0 \\ 0 & 0 & 2i/\sqrt{3} \end{pmatrix}.$$

Note that T_1 and T_2 can be generated by the following two vector fields:

$$X_7 = -[X_1, X_2] = \begin{pmatrix} -2i & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_8 = -[X_3, X_4] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix},$$

which form an orthonormal basis of the Cartan subalgebra $\nabla_{\mathcal{T}} = \{X_7, X_8\}$. Table 1 provides all the commutators of the vector fields $X_1, X_2, \dots, X_8$.

TABLE 1. Commutators in SU(3)

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	$-X_7$	X_5	$-X_6$	$-X_3$	X_4	$4X_2$	$2X_2$
X_2	X_7	0	X_6	X_5	$-X_4$	$-X_3$	$-4X_1$	$-2X_1$
X_3	$-X_5$	$-X_6$	0	$-X_8$	X_1	X_2	$2X_4$	$4X_4$
X_4	X_6	$-X_5$	X_8	0	X_2	$-X_1$	$-2X_3$	$-4X_3$
X_5	X_3	X_4	$-X_1$	$-X_2$	0	$X_8 - X_7$	$2X_6$	$-2X_6$
X_6	$-X_4$	X_3	$-X_2$	X_1	$X_7 - X_8$	0	$-2X_5$	$2X_5$
X_7	$-4X_2$	$4X_1$	$-2X_4$	$2X_3$	$-2X_6$	$2X_5$	0	0
X_8	$-2X_2$	$2X_1$	$-4X_4$	$4X_3$	$2X_6$	$-2X_5$	0	0

Consider the orthonormal basis of the horizontal subspace $\mathcal{H}$ in SU(3); that is,

$$\nabla_{\mathcal{H}} = \{X_1, X_2, \dots, X_6\}.$$

Note that the matrices $\mathcal{G}$ are left-invariant vector fields. According to Table 1, the basis $\nabla_{\mathcal{H}}$ satisfies the Hörmander condition at every point of SU(3) and produces the horizontal distribution of a sub-Riemannian manifold.

We say that the curve $\gamma : [0,T] \to \mathrm{SU}(3)$ is subunitary associated to $\nabla_{\mathcal{H}}$ if the following two conditions are met: the curve γ is an absolutely continuous function; there are measurable functions $\{\alpha_i \in L^\infty[0,T]\}_{1\le i\le 6}$ such that

$$\gamma'(t) = \sum_{i=1}^{6} \alpha_i(t) X_i(\gamma(t)) \quad \text{and} \quad \sum_{i=1}^{6} \alpha_i^2(t) \le 1 \quad \text{for a.e. } t \in [0,T].$$

Since at every point of SU(3) the basis $\nabla_{\mathcal{H}}$ satisfies the Hörmander condition, by [1], for any two given points $x, y \in \mathrm{SU}(3)$ there exist subunitary curves γ connecting them. As a result, we define the Carnot-Carathéodory distance in regard to $\nabla_{\mathcal{H}}$ by

$$\begin{aligned} d(x,y) = \inf \big\{ & T \ge 0 : \text{there exists a subunitary curve } \gamma : [0,T] \to \mathrm{SU}(3) \\ & \text{connecting } x \text{ and } y \big\}. \end{aligned}$$

With respect to this distance d, we define the Carnot-Carathéodory balls centered at $x \in \mathrm{SU}(3)$ with radius $r > 0$ by

$$B_r(x) = \{ y \in \mathrm{SU}(3) : d(x,y) < r \}.$$

We denote by dx the bi-invariant Harr-measure, by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathrm{SU}(3)$ and by

$$⨍_E f\,dx = \frac{1}{|E|} \int_E f\,dx$$

the average of an integrable function f over set E.

In the rest of this section, we recall several a priori uniform estimates for regularized equation by Domokos-Manfredi [3]; see [3, Corollary 4.1]. Let u be a p-harmonic function in a domain $\Omega \subset \mathrm{SU}(3)$, where $1 < p < \infty$. Given any smooth domain $U \Subset \Omega$ and $\delta \in (0,1]$, denote by $u^\delta \in W^{1,p}_{\mathcal{H}}(U)$ the weak solution to the regularized equation (1.6). We have the following result.

Lemma 2.1. *For any $\phi \in C_0^\infty(U)$ with $0 \le \phi \le 1$, the followings hold:*

(i) *If $\beta \ge 0$, then*

$$\begin{aligned} &\int_U \phi^2 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u^\delta|^{2\beta} |\nabla_{\mathcal{H}} \nabla_{\mathcal{T}} u^\delta|^2 dx \\ &\le c \int_U |\nabla_{\mathcal{H}} \phi|^2 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u^\delta|^{2\beta+2} dx \\ &\quad + c(\beta+1)^2 \int_U \phi^2 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p}{2}} |\nabla_{\mathcal{T}} u^\delta|^{2\beta} dx. \end{aligned} \tag{2.1}$$

(ii) *If $\beta \ge 0$, then*

$$\begin{aligned} &\int_U \phi^2 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}+\beta} |\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u^\delta|^2 dx \\ &\le c(\beta+1)^4 \int_U \phi^2 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}+\beta} |\nabla_{\mathcal{T}} u^\delta|^2 dx \\ &\quad + c(\beta+1)^2 K_\phi \int_{spt(\phi)} (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p}{2}+\beta} dx, \end{aligned} \tag{2.2}$$

(iii) *If $\beta \geq 1$, then*

$$\begin{aligned}&\int_U \phi^{2\beta+2}(\delta+|\nabla_{\mathcal{H}}u^\delta|^2)^{\frac{p-2}{2}}|\nabla_{\mathcal{T}}u^\delta|^{2\beta}|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u^\delta|^2dx\\ &\leq c^\beta(\beta+1)^{4\beta}\|\nabla_{\mathcal{H}}\phi\|^{2\beta}_{L^\infty(U)}\int_U \phi^2(\delta+|\nabla_{\mathcal{H}}u^\delta|^2)^{\frac{p-2}{2}+\beta}|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u^\delta|^2dx.\end{aligned}\tag{2.3}$$

(iv) *If $\beta \geq 1$, then*

$$\begin{aligned}&\int_U \phi^2(\delta+|\nabla_{\mathcal{H}}u^\delta|^2)^{\frac{p-2}{2}+\beta}|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u^\delta|^2dx\\ &\leq c(\beta+1)^{12}K_\phi\int_{spt(\phi)}(\delta+|\nabla_{\mathcal{H}}u^\delta|^2)^{\frac{p}{2}+\beta}dx.\end{aligned}\tag{2.4}$$

Above K_ϕ is as in (1.3) and constants $c = c(p) > 0$.

Combining (2.3) and (2.4), we obtain the following result.

Lemma 2.2. *For any $\beta \geq 1$ and any $\phi \in C_0^\infty(U)$ with $0 \leq \phi \leq 1$, we have*

$$\begin{aligned}&\int_U \phi^{2\beta+2}(\delta+|\nabla_{\mathcal{H}}u^\delta|^2)^{\frac{p-2}{2}}|\nabla_{\mathcal{T}}u^\delta|^{2\beta}|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u^\delta|^2dx\\ &\leq c^\beta(\beta+1)^{12+4\beta}K_\phi^{\beta+1}\int_{spt(\phi)}(\delta+|\nabla_{\mathcal{H}}u^\delta|^2)^{\frac{p}{2}+\beta}dx,\end{aligned}\tag{2.5}$$

where K_ϕ is as in (1.3) and the constant $c = c(p) > 0$.

Moreover, Domokos-Manfredi [3] further established the following uniform gradient estimate and also convergence. We also write $u^0 = u$.

Theorem 2.3. *We have $\nabla_{\mathcal{H}}u^\delta \in L^\infty_{\rm loc}(U;\mathbb{R}^6)$ uniformly in $\delta \in [0,1)$ and, for any ball $B_{2r} \subset U$,*

$$\|\nabla_{\mathcal{H}}u^\delta\|_{L^\infty(B_r)} \leq c(p)\Big(\fint_{B_{2r}}(\delta+|\nabla_{\mathcal{H}}u^\delta|^2)^{\frac{p}{2}}\Big)^{1/p}.\tag{2.6}$$

Moreover, $u^\delta \to u$ in $C^0(\bar{U})$.

3. Proofs of lemmas

In this section, we prove Lemmas 1.3, 1.4, 1.5, and 1.6. To prove Lemma 1.3 we need the following pointwise inequality from [6, Lemma 2.1]. To simplify the following proofs, we write the subelliptic ∞-Laplacian $\Delta_{0,\infty}v$ of $v \in C^\infty$ as

$$\Delta_{0,\infty}v = \sum_{i,j=1}^6 X_ivX_iX_jvX_jv = (\nabla_{\mathcal{H}}v)^T\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}v\nabla_{\mathcal{H}}v = (\nabla_{\mathcal{H}}v)^TD_0^2v\nabla_{\mathcal{H}}v.$$

Lemma 3.1. *For any $v \in C^\infty(U)$, we have*

$$\begin{aligned}&\Big||D_0^2v\nabla_{\mathcal{H}}v|^2-\Delta_0v\Delta_{0,\infty}v-\frac{1}{2}[|D_0^2v|^2-(\Delta_0v)^2]|\nabla_{\mathcal{H}}v|^2\Big|\\ &\leq 2[|D_0^2v|^2|\nabla_{\mathcal{H}}v|^2-|D_0^2v\nabla_{\mathcal{H}}v|^2]\quad \text{in } U.\end{aligned}\tag{3.1}$$

Proof. For each point $\bar{x} \in U$, we assume that $\nabla_{\mathcal{H}}v(\bar{x}) \neq 0$ below, otherwise (3.1) obviously holds. In the case $\nabla_{\mathcal{H}}v(\bar{x}) \neq 0$, we may also assume that $|\nabla_{\mathcal{H}}v(\bar{x})| = 1$ below, otherwise we divide both sides by $|\nabla_{\mathcal{H}}v(\bar{x})|^2$.

At $\bar{x}$, since $D_0^2 v(\bar{x})$ is a symmetric matrix, by the linear algebra theory, we obtain a set of eigenvalues based on the matrix $D_0^2 v(\bar{x})$, that is, $\{\lambda_i\}_{i=1}^6 \subset \mathbb{R}$. Then according to the linear algebra theory again, there is an orthogonal matrix $O \in \mathbf{O}(6)$ such that

$$O^T D_0^2 v O = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_6\}.$$

Noting that $O^{-1} = O^T$, we have

$$|D_0^2 v|^2 = |O^T D_0^2 v O|^2 = \sum_{i=1}^6 (\lambda_i)^2, \quad \Delta_0 v = \sum_{i=1}^6 \lambda_i.$$

For simplicity, we write $O^T \nabla_{\mathcal{H}} v = \sum_{i=1}^6 a_i \mathbf{e}_i =: \vec{a}$. Thus

$$\Delta_{0,\infty} v = (\nabla_{\mathcal{H}} v)^T D_0^2 v \nabla_{\mathcal{H}} v = (O^T \nabla_{\mathcal{H}} v)^T (O^T D_0^2 v O)(O^T \nabla_{\mathcal{H}} v) = \sum_{i=1}^6 \lambda_i (a_i)^2,$$

$$|D_0^2 v \nabla_{\mathcal{H}} v|^2 = |(O^T D_0^2 v O)(O^T \nabla_{\mathcal{H}} v)|^2 = \sum_{i=1}^6 (\lambda_i)^2 (a_i)^2.$$

By [6, Lemma 2.2] with $\vec{\lambda} := (\lambda_1, \lambda_2, \dots, \lambda_6)$ and $\vec{a} := O^T \nabla_{\mathcal{H}} v$, we have

$$\begin{aligned}
&\Big| |D_0^2 v \nabla_{\mathcal{H}} v|^2 - \Delta_0 v \Delta_{0,\infty} v - \frac{1}{2}[|D_0^2 v|^2 - (\Delta_0 v)^2] |\nabla_{\mathcal{H}} v|^2 \Big| \\
&= \Big| \sum_{i=1}^6 (\lambda_i)^2 (a_i)^2 - \Big(\sum_{i=1}^6 \lambda_i\Big) \Big[\sum_{j=1}^6 \lambda_j (a_j)^2\Big] - \frac{1}{2}\Big[\sum_{i=1}^6 (\lambda_i)^2 - (\sum_{i=1}^6 \lambda_i)^2\Big] \Big| \\
&\le 2\Big[\sum_{i=1}^6 (\lambda_i)^2 - \sum_{i=1}^6 (\lambda_i)^2 (a_i)^2\Big] \\
&= 2[|D_0^2 v|^2 |\nabla_{\mathcal{H}} v|^2 - |D_0^2 v \nabla_{\mathcal{H}} v|^2],
\end{aligned}$$

which implies (3.1). □

Now we apply Lemma 3.1 to prove Lemma 1.3.

Proof of Lemma 1.3. Noting that $u^\delta \in C^\infty(U)$, dividing both sides of (1.6) by $(\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-4}{2}}$, we have

$$(p-2)\Delta_{0,\infty} u^\delta + (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)\Delta_0 u^\delta = 0 \quad \text{in } U. \tag{3.2}$$

For any point $\bar{x} \in U$, we consider two cases: $\nabla_{\mathcal{H}} u^\delta(\bar{x}) = 0$ and $\nabla_{\mathcal{H}} u^\delta(\bar{x}) \neq 0$. In the case $\nabla_{\mathcal{H}} u^\delta(\bar{x}) = 0$, since

$$\Delta_{0,\infty} u^\delta(\bar{x}) = (\nabla_{\mathcal{H}} u^\delta(\bar{x}))^T \nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u^\delta(\bar{x}) \nabla_{\mathcal{H}} u^\delta(\bar{x}) = 0,$$

Equality (3.2) implies that $\Delta_0 u^\delta(\bar{x}) = 0$. Thus (1.9) holds.

Now we prove (1.9) in the case $\nabla_{\mathcal{H}} u^\delta(\bar{x}) \neq 0$. Applying Lemma 3.1 with $v = u^\delta$ and multiplying both sides by $(p-2)^2$, at $\bar{x}$ we have

$$\begin{aligned}
&(p-2)^2 |D_0^2 u^\delta \nabla_{\mathcal{H}} u^\delta|^2 - (p-2)^2 \Delta_0 u^\delta \Delta_{0,\infty} u^\delta \\
&- \frac{(p-2)^2}{2}[|D_0^2 u^\delta|^2 - (\Delta_0 u^\delta)^2] |\nabla_{\mathcal{H}} u^\delta|^2 \\
&\le 2(p-2)^2 [|D_0^2 u^\delta|^2 |\nabla_{\mathcal{H}} u^\delta|^2 - |D_0^2 v \nabla_{\mathcal{H}} u^\delta|^2].
\end{aligned} \tag{3.3}$$

Combining (3.3) and (3.2), at $\bar{x}$ we have

$$\begin{aligned}&(p-2)^2|D_0^2u^\delta\nabla_{\mathcal{H}}u^\delta|^2+(p-2)(\Delta_0u^\delta)^2[|\nabla_{\mathcal{H}}u^\delta|^2+\delta]\\&-\frac{(p-2)^2}{2}[|D_0^2u^\delta|^2-(\Delta_0u^\delta)^2]|\nabla_{\mathcal{H}}u^\delta|^2\\&\le 2(p-2)^2[|D_0^2u^\delta|^2|\nabla_{\mathcal{H}}u^\delta|^2-|D_0^2u^\delta\nabla_{\mathcal{H}}u^\delta|^2].\end{aligned}$$

By dividing both sides by $|\nabla_{\mathcal{H}}u^\delta(\bar{x})|^2$, at $\bar{x}$ we have

$$\begin{aligned}&3(p-2)^2\frac{|D_0^2u^\delta\nabla_{\mathcal{H}}u^\delta|^2}{|\nabla_{\mathcal{H}}u^\delta|^2}+(p-2)\frac{(\Delta_0u^\delta)^2}{|\nabla_{\mathcal{H}}u^\delta|^2}[|\nabla_{\mathcal{H}}u^\delta|^2+\delta]\\&\le\frac{(p-2)^2}{2}[|D_0^2u^\delta|^2-(\Delta_0u^\delta)^2]+2(p-2)^2|D_0^2u^\delta|^2.\end{aligned}\tag{3.4}$$

Recalling that

$$\Delta_{0,\infty}u^\delta=(\nabla_{\mathcal{H}}u^\delta)^TD_0^2u^\delta\nabla_{\mathcal{H}}u^\delta,$$

by Hölder's inequality and (3.2), at $\bar{x}$ we have

$$(p-2)^2\frac{|D_0^2u^\delta\nabla_{\mathcal{H}}u^\delta|^2}{|\nabla_{\mathcal{H}}u^\delta|^2}\ge(p-2)^2\frac{|\Delta_{0,\infty}u^\delta|^2}{|\nabla_{\mathcal{H}}u^\delta|^4}\ge\frac{(\Delta_0u^\delta)^2}{|\nabla_{\mathcal{H}}u^\delta|^2}[|\nabla_{\mathcal{H}}u^\delta|^2+\delta].\tag{3.5}$$

Here we apply Hölder's inequality to estimate the first inequality in (3.5), and apply (3.2) to estimate the second inequality.

Combining (3.4) and (3.5), we have

$$(p+1)\Big(\frac{\Delta_0u^\delta}{|\nabla_{\mathcal{H}}u^\delta|}\Big)^2[|\nabla_{\mathcal{H}}u^\delta|^2+\delta]\le\frac{(p-2)^2}{2}[|D_0^2u^\delta|^2-(\Delta_0u^\delta)^2]+2(p-2)^2|D_0^2u^\delta|^2.$$

Thus

$$(p+1)(\Delta_0u^\delta)^2\le\frac{(p-2)^2}{2}[|D_0^2u^\delta|^2-(\Delta_0u^\delta)^2]+2(p-2)^2|D_0^2u^\delta|^2.$$

From this, subtracting $[(p+1)(\Delta_0u^\delta)^2-(p+1-2(p-2)^2)|D_0^2u^\delta|^2]$ from both sides, we have

$$[p+1-2(p-2)^2]|D_0^2u^\delta|^2\le\big[p+1+\frac{(p-2)^2}{2}\big][|D_0^2u^\delta|^2-(\Delta_0u^\delta)^2].$$

Noting that $1<p<7/2$ implies

$$p+1-2(p-2)^2=(p-1)(7-2p)>0,$$

we conclude (1.9). □

Proof of Lemma 1.4. For simplicity we write the right-hand side of (1.10) as

$$\begin{aligned}R:=&c\int_U|\nabla_{\mathcal{H}}v|^2\phi^6dx+c\int_U|\nabla_{\mathcal{H}}v||\nabla_{\mathcal{H}}\nabla_{\mathcal{T}}v|\phi^6dx\\&+c\int_U|\nabla_{\mathcal{H}}v||\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}v||\phi|^5[|\nabla_{\mathcal{H}}\phi|+|\phi|]dx.\end{aligned}\tag{3.6}$$

Recall that

$$D_0^2v=\Big(\frac{X_iX_jv+X_jX_iv}{2}\Big)_{1\le i,j\le 6},\quad \Delta_0v=\sum_{i=1}^6X_iX_iv.$$

Then

$$\begin{aligned}
&[|D_0^2v|^2-(\Delta_0 v)^2]\\
&=\sum_{i,j=1}^{6}\Big(\frac{X_iX_jv+X_jX_iv}{2}\Big)^2-\Big(\sum_{i=1}^{6}X_iX_iv\Big)^2\\
&=\sum_{i,j=1}^{6}\Big[\frac{1}{4}[(X_iX_jv)^2+(X_jX_iv)^2+2X_iX_jvX_jX_iv]-X_iX_ivX_jX_jv\Big]\\
&=\frac{1}{4}\sum_{i,j=1}^{6}[(X_iX_jv)^2-X_iX_ivX_jX_jv]+\frac{1}{4}\sum_{i,j=1}^{6}[(X_jX_iv)^2-X_iX_ivX_jX_jv]\\
&\quad+\frac{1}{2}\sum_{i,j=1}^{6}[X_iX_jvX_jX_iv-X_iX_ivX_jX_jv]\\
&=\frac{1}{2}\sum_{i,j=1}^{6}[(X_iX_jv)^2-X_iX_ivX_jX_jv]\\
&\quad+\frac{1}{2}\sum_{i,j=1}^{6}[X_iX_jvX_jX_iv-X_iX_ivX_jX_jv].
\end{aligned}\tag{3.7}$$

By this, to prove (1.10), we only need to prove that, for $1\le i,j\le 6$,

$$\int_U[(X_iX_jv)^2-X_iX_ivX_jX_jv]\phi^6dx\le R, \tag{3.8}$$

$$\int_U[X_iX_jvX_jX_iv-X_iX_ivX_jX_jv]\phi^6dx\le R, \tag{3.9}$$

where R is as in (3.6).

First, we prove (3.8). Integrating by parts, we have

$$\int_U(X_iX_jv)^2\phi^6dx=-\int_U X_jvX_iX_iX_jv\phi^6dx-6\int_U X_jvX_iX_jv\phi^5X_i\phi dx.$$

Since $X_iX_j=X_jX_i+[X_i,X_j]$, we have

$$\int_U X_jvX_iX_iX_jv\phi^6dx=\int_U X_jvX_iX_jX_iv\phi^6dx+\int_U X_jvX_i[X_i,X_j]v\phi^6dx.$$

Combining the above two equalities, since $X_iX_j=X_jX_i+[X_i,X_j]$ again, we have

$$\begin{aligned}
\int_U(X_iX_jv)^2\phi^6dx=&-\int_U X_jvX_jX_iX_iv\phi^6dx-6\int_U X_jvX_iX_jv\phi^5X_i\phi dx\\
&-\int_U X_jv[X_i,X_j]X_iv\phi^6dx-\int_U X_jvX_i[X_i,X_j]v\phi^6dx.
\end{aligned}$$

Integrating by parts again, we have

$$\begin{aligned}
\int_U(X_iX_jv)^2\phi^6dx=&\int_U X_jX_jvX_iX_iv\phi^6dx+6\int_U X_jvX_iX_iv\phi^5X_j\phi dx\\
&-6\int_U X_jvX_iX_jv\phi^5X_i\phi dx-\int_U X_jv[X_i,X_j]X_iv\phi^6dx\\
&-\int_U X_jvX_i[X_i,X_j]v\phi^6dx.
\end{aligned}\tag{3.10}$$

Table 1 shows that

$$[X_i, X_j] = \sum_{k=1}^{8} c_{i,j}^k X_k \text{ for any } i, j \in \{1, 2, \dots, 8\} \tag{3.11}$$

and that

$$\begin{aligned}
[X_i, X_j]X_i &= \sum_{k=1}^{8} c_{i,j}^k X_k X_i \\
&= \sum_{k=1}^{8} c_{i,j}^k \left(X_i X_k + [X_k, X_i]\right) \\
&= \sum_{k=1}^{8} c_{i,j}^k \Big(X_i X_k + \sum_{m=1}^{8} c_{k,i}^m X_m\Big) \quad \text{for } i, j \in \{1, 2, \dots, 8\},
\end{aligned} \tag{3.12}$$

where $c_{i,j}^k$ and $c_{k,i}^m$ are constants and are completely determined by Table 1. Combining (3.10), (3.11) and (3.12), then subtracting $\int_U X_i X_i v X_j X_j v \phi^6 dx$ from both sides, by the fact

$$|\nabla_{\mathcal{T}} v|^2 \le 2|\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} v|^2,$$

we obtain (3.8).

Finally, we prove (3.9) in a similar way. Integrating by parts, we have

$$\int_U X_i X_j v X_j X_i v \phi^6 dx = -\int_U X_j v X_i X_j X_i v \phi^6 dx - 6\int_U X_j v X_j X_i v \phi^5 X_i \phi dx.$$

Since $X_i X_j = X_j X_i + [X_i, X_j]$, we have

$$\int_U X_j v X_i X_j X_i v \phi^6 dx = \int_U X_j v X_j X_i X_i v \phi^6 dx + \int_U X_j v [X_i, X_j] X_i v \phi^6 dx.$$

Combining the above two equalities, by integration by parts again, we have

$$\begin{aligned}
&\int_U X_i X_j v X_j X_i v \phi^6 dx \\
&= \int_U X_j X_j v X_i X_i v \phi^6 dx - \int_U X_j v [X_i, X_j] X_i v \phi^6 dx \\
&\quad + 6\int_U X_j v X_i X_i v \phi^5 X_j \phi dx - 6\int_U X_j v X_j X_i v \phi^5 X_i \phi dx.
\end{aligned} \tag{3.13}$$

We combine (3.12) and (3.13). Then subtracting $\int_U X_i X_i v X_j X_j v \phi^6 dx$ from both sides, by the fact that

$$|\nabla_{\mathcal{T}} v|^2 \le 2|\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} v|^2,$$

we obtain (3.9). ☐

Proof of Lemma 1.5. Since $2 < p \le 4$, by Young's inequality, we have

$$\begin{aligned}
&\int_U \phi^6 |\nabla_{\mathcal{H}} u^\delta| |\nabla_{\mathcal{H}} \nabla_{\mathcal{T}} u^\delta| dx \\
&\le \int_U \phi^6 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} \nabla_{\mathcal{T}} u^\delta|^2 dx + \int_U \phi^6 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{4-p}{2}} dx.
\end{aligned} \tag{3.14}$$

By (2.1) in Lemma 2.1 with $\beta = 0$ and $\phi \to \phi^3$ therein, we have

$$\begin{aligned} &\int_U \phi^6 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} \nabla_{\mathcal{T}} u^\delta|^2 dx \\ &\le c \|\nabla_{\mathcal{H}} \phi\|^2_{L^\infty(U)} \int_U \phi^4 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u^\delta|^2 dx \\ &\quad + c \int_U \phi^6 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p}{2}} dx. \end{aligned} \tag{3.15}$$

By Young's inequality again, that $|\nabla_{\mathcal{T}} u^\delta|^2 \le 2|\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u^\delta|^2$ and Lemma 2.2 with $\beta = 1$ therein, we have

$$\begin{aligned} &\int_U \phi^4 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u^\delta|^2 dx \\ &\le \int_U \phi^4 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u^\delta|^4 dx + \int_U \phi^4 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} dx \\ &\le c K_\phi^2 \int_{spt(\phi)} (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p+2}{2}} dx + \int_U \phi^4 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} dx. \end{aligned} \tag{3.16}$$

Here we apply Young's inequality to estimate the first inequality in (3.16), and apply the fact $|\nabla_{\mathcal{T}} u^\delta|^2 \le 2|\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u^\delta|^2$ and Lemma 2.2 to estimate the second inequality.

We combine (3.15) and (3.16). Then by Young's inequality, we have

$$\begin{aligned} &\int_U \phi^6 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} \nabla_{\mathcal{T}} u^\delta|^2 dx \\ &\le c K_\phi^3 \int_{spt(\phi)} (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p+2}{2}} dx + c K_\phi \int_U \phi^4 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} dx. \end{aligned} \tag{3.17}$$

Combining (3.17) and (3.14), we conclude (1.11). $\square$

Proof of Lemma 1.6. Recall that

$$Mv = \Big(\frac{[X_i, X_j] v}{2} \Big)_{1 \le i,j \le 6}.$$

According to Table 1, we have

$$\begin{aligned} |Mv|^2 &= \frac{1}{2} [(X_7 v)^2 + (X_8 v)^2 + (X_8 v - X_7 v)^2] + |\nabla_{\mathcal{H}} v|^2 \\ &= (X_7 v)^2 + (X_8 v)^2 - X_7 v X_8 v + |\nabla_{\mathcal{H}} v|^2. \end{aligned}$$

Since

$$2|X_7 v X_8 v| \le (X_7 v)^2 + (X_8 v)^2,$$

it remains to bound the integration of $(X_7 v)^2$ and the integration of $(X_8 v)^2$.

First, we bound the integration of $(X_7 v)^2$. Since $X_7 = -[X_1, X_2]$, integration by parts yields

$$\begin{aligned} \int_U (X_7 v)^2 \phi^6 dx &= \int_U (X_2 X_1 v - X_1 X_2 v) X_7 v \phi^6 dx \\ &= \int_U X_2 v X_1 X_7 v \phi^6 dx - \int_U X_1 v X_2 X_7 v \phi^6 dx \\ &\quad + 6 \int_U X_2 v X_7 v \phi^5 X_1 \phi dx - 6 \int_U X_1 v X_7 v \phi^5 X_2 \phi dx. \end{aligned}$$

Thus

$$\int_U (X_7 v)^2 \phi^6 dx \le 2\int_U |\nabla_{\mathcal{H}} v||\nabla_{\mathcal{H}}\nabla_{\mathcal{T}} v|\phi^6 dx + 12\int_U |\nabla_{\mathcal{H}} v||\nabla_{\mathcal{T}} v||\phi^5 \nabla_{\mathcal{H}}\phi| dx.$$

Finally, we bound the integration of $(X_8 v)^2$ in the same way. Combining these together, we conclude (1.12). $\square$

4. Proofs of main results

Proof of Theorem 1.2. We consider two cases: $1 < p \le 2$ and $2 < p < \infty$. When $1 < p \le 2$, applying (2.2) in Lemma 2.1 with $\beta = (2-p)/2 \ge 0$, we have

$$\int_U \phi^2 |\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\delta|^2 dx \le cK_\phi \int_{spt(\phi)} (\delta + |\nabla_{\mathcal{H}} u^\delta|^2) dx + c\int_U \phi^2 |\nabla_{\mathcal{T}} u^\delta|^2 dx. \tag{4.1}$$

By Young's inequality, the fact $|\nabla_{\mathcal{T}} u^\delta| \le 2|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\delta|$ and Lemma 2.2 with $\beta = 1$, we have

$$\begin{aligned}
&\int_U \phi^2 |\nabla_{\mathcal{T}} u^\delta|^2 dx \\
&= \int_U \phi^2 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{2-p}{4}} (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{4}} |\nabla_{\mathcal{T}} u^\delta|^2 dx \\
&\le \int_{spt(\phi)} (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{2-p}{2}} dx + \int_U \phi^4 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{T}} u^\delta|^4 dx \\
&\le \int_{spt(\phi)} (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{2-p}{2}} dx + cK_\phi^2 \int_{spt(\phi)} (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p}{2}+1} dx.
\end{aligned} \tag{4.2}$$

Here we apply Young's inequality to estimate the first inequality in (4.2), and apply Lemma 2.2 to estimate the second inequality. Combining (4.1) and (4.2), by Young's inequality therein, we obtain (1.7).

Now, we consider the case $2 \le p < 7/2$. Recalling that

$$|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\delta|^2 = |D_0^2 u^\delta|^2 + |Mu^\delta|^2,$$

by Lemmas 1.3, 1.4, and 1.6, we have

$$\begin{aligned}
\int_U |\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\delta|^2 \phi^6 dx &\le c\int_U |\nabla_{\mathcal{H}} u^\delta|^2 \phi^6 dx + c\int_U |\nabla_{\mathcal{H}} u^\delta||\nabla_{\mathcal{H}}\nabla_{\mathcal{T}} u^\delta|\phi^6 dx \\
&\quad + c\int_U |\nabla_{\mathcal{H}} u^\delta||\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\delta||\phi|^5 [|\nabla_{\mathcal{H}}\phi| + |\phi|] dx.
\end{aligned} \tag{4.3}$$

To obtain (1.8), it remains to estimate the second term in the right-hand of (4.3). By Lemma 1.5, (4.3) becomes

$$\begin{aligned}
&\int_U |\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\delta|^2 \phi^6 dx \\
&\le c\int_U |\nabla_{\mathcal{H}} u^\delta|^2 \phi^6 dx + c\int_U |\nabla_{\mathcal{H}} u^\delta||\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\delta||\phi|^5 [|\nabla_{\mathcal{H}}\phi| + |\phi|] dx \\
&\quad + cK_\phi^3 \int_{spt(\phi)} (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p+2}{2}} dx + cK_\phi \int_U \phi^4 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{p-2}{2}} dx \\
&\quad + c\int_U \phi^6 (\delta + |\nabla_{\mathcal{H}} u^\delta|^2)^{\frac{4-p}{2}} dx.
\end{aligned}$$

By Young's inequality, we obtain (1.8). $\square$

Proof of Theorem 1.1. Let Ω be a domain in SU(3). Consider any p-harmonic function $u \in W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$. Given any smooth domain $U \Subset \Omega$, for $p \in (1,\infty)$ and $\delta \in (0,1]$, we let $u^\delta \in W^{1,p}_{\mathcal{H}}(U)$ be a weak solution to (1.6). By Theorem 1.2, we have that

$$\nabla_{\mathcal{H}} u^\delta \in W^{1,2}_{\mathcal{H},\mathrm{loc}}(U) \quad \text{uniformly in } \delta \in (0,1]. \tag{4.4}$$

Theorem 2.3 shows that

$$u^\delta \to u \quad \text{in } C^0(U) \text{ as } \delta \to 0, \tag{4.5}$$

$$\nabla_{\mathcal{H}} u^\delta \in L^\infty(U) \quad \text{uniformly in } \delta \in (0,1]. \tag{4.6}$$

Combining (4.4) and (4.5), we have

$$\nabla_{\mathcal{H}} u^\delta \to \nabla_{\mathcal{H}} u \quad \text{weakly in } W^{1,2}_{\mathcal{H},\mathrm{loc}}(U) \text{ and in } L^2_{\mathrm{loc}}(U) \text{ as } \delta \to 0. \tag{4.7}$$

By (4.6) and Hölder's inequality, (4.7) implies that

$$\nabla_{\mathcal{H}} u^\delta \to \nabla_{\mathcal{H}} u \quad \text{in } L^q_{\mathrm{loc}}(U) \text{ for } 0 < q < \infty \text{ as } \delta \to 0.$$

By letting $\delta \to 0$ in (1.7) and (1.8), we can obtain (1.4) and (1.5). $\square$

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