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INITIAL LAYER ASSOCIATED WITH BOUSSINESQ SYSTEMS FOR THERMOSOLUTAL CONVECTION

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Abstract. This article concerns the behavior of the initial layer appearing at large Prandtl number in Boussinesq equations with the ill initial data. By using the asymptotic expansion methods of singular perturbation theory, we establish an approximate solution and the rate of convergence as the Prandtl number tends to infinity. Our results improve the existing ones concerning thermosolutal convection.

1. Introduction

Double diffusive convection has been considered a fundamental fluid dynamical phenomenon that can occur when a layer of fluid with a dissolved solute, is heated from below. This phenomena was first studied in the ocean [14], where heat and salt are relevant properties; so the process is called "thermohaline" or "thermosolutal" convection. Many investigations have been studied in the stratification of magma chambers [2] and convection in the sun, in meteorology, geophysics, and astrophysics.

In this article, we consider the thermosolutal convection setting of a horizontal layer of fluid confined by two parallel planes a distance h apart. The warm salty fluid tends to rise and salty fluid being heavier than fresh fluid tends to fall. The Boussinesq system with rotation for thermosolutal convection is stated as follows:

$$\begin{gathered}
\partial_t u + (u\cdot\nabla)u = \frac{1}{\rho_0}(\nabla p + \rho g k) + \sqrt{Ta}k\times u + \nu\Delta u,\\
\nabla\cdot u = 0,\\
\partial_t T + u\cdot\nabla T - \kappa_T\Delta T = 0,\\
\partial_t S + u\cdot\nabla S - \kappa_S\Delta S = 0,\\
u|_{z=0,h} = 0,\\
T|_{z=0} = T_0, \quad T|_{z=h} = T_1,\\
S|_{z=0} = S_0, \quad S|_{z=h} = S_1,
\end{gathered}$$

where $T_0 > T_1$ and $S_0 > S_1$. The unknown functions $u = (u_1, u_2, u_3)^\top$, p, T and S represent the velocity field, the scalar pressure, the scalar temperature field and the solute concentration of the fluid, respectively. ν, κ_T and κ_S are the kinematic

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viscosity, the thermal and solute diffusivity, respectively. ρ is the density, $\rho_0 > 0$ is the fluid density at the lower surface $z = 0$ and g is the gravity acceleration constant, $k := (0,0,1)^\top$ is the unit upward vector. $Ta = \frac{4\Omega^2 h^2}{\nu^2}$ is the Taylor number, Ω is the rotation rate and h is the distance between two plates.

The lower (upper) boundary is maintained at a constant temperature T_0 (T_1) and a constant solute concentration S_0 (S_1). For simplicity, we consider the periodic boundary conditions in the horizontal directions:

$$(u,T,S)(x,t) = (u,T,S)(x+k_1L_1, y+k_2L_2, z, t), \quad \text{for all } k_1,k_2 \in \mathbb{Z}.$$

By non-dimensionalization and the Oberbeck-Boussinesq approximation, Park [25] obtained the dimensionless form

$$\lambda[\partial_t u^\lambda + (u^\lambda\cdot\nabla)u^\lambda] + \nabla p^\lambda = \Delta u^\lambda + \sqrt{Ta}k\times u^\lambda + (R_TT^\lambda - R_SS^\lambda)k, \tag{1.1}$$

$$\nabla\cdot u^\lambda = 0, \tag{1.2}$$

$$\partial_t T^\lambda + u^\lambda\cdot\nabla T^\lambda = \Delta T^\lambda, \tag{1.3}$$

$$\partial_t S^\lambda + u^\lambda\cdot\nabla S^\lambda = \tau\Delta S^\lambda, \tag{1.4}$$

in the dimensionless domain $D := (0,L_1)\times(0,L_2)\times[0,1]$ ($L_1, L_2 > 0$), $t\in(0,T_*)$, $T_* > 0$. $\lambda = \frac{1}{Pr}$, $Pr = \frac{\nu}{\kappa_T}$ is the Prandtl number, $R_T = \frac{\alpha g(T_0-T_1)h^3}{\kappa_T\nu}$ is the thermal Rayleigh number, $R_S = \frac{\gamma g(S_0-S_1)h^3}{\kappa_T\nu}$ is the salinity Rayleigh number, $\tau = \frac{\kappa_S}{\kappa_T}$ is the Lewis number. α and γ denote the coefficients of thermal expansion and its analogous compositional counterpart, respectively. The Lewis number τ for magmatic substances is less than 10^{-1}, here we consider $0 \ll \tau < 1$ for simplicity.

We complement the above system with the boundary conditions:

$$u^\lambda|_{z=0,1} = 0, \quad (x,y,t)\in X\times(0,T_*), \tag{1.5}$$

$$T^\lambda, S^\lambda|_{z=0} = 1, \quad T^\lambda, S^\lambda|_{z=1} = 0, \quad (x,y,t)\in X\times(0,T_*), \tag{1.6}$$

for $X := (0,L_1)\times(0,L_2)$, and provide an initial condition

$$(u^\lambda, T^\lambda, S^\lambda)(t=0) = (u_0^\lambda, T_0^\lambda, S_0^\lambda)(x,y,z). \tag{1.7}$$

The mathematical analysis of the thermosolutal convection has attracted much attention (see [1, 4, 8, 11, 13, 16, 18, 23, 24, 26, 29, 31]). Hallstrom [10] discussed the global existence and uniqueness of smooth solutions for the infinite Prandtl number limit of the Boussinesq equations. Park [24] considered the thermosolutal convection with or without rotation in the infinite Prandtl number, and investigated that there are bifurcating solutions for 2-D and 3-D Boussinesq equations. Park [25] studied the infinite Prandtl number limit and derived the convergence of Boussinesq system for the thermosolutal convection to the infinite Prandtl number system.

Our purpose is to study the asymptotic behavior of the thermosolutal convection in the infinite limit of Prandtl number. For one fluid model as a vanishing limit case, there have been large investigations, see for instance [3, 20, 21, 30]. Motivated by [28] and the related studies, see [6, 7, 9, 15, 17], we view the Boussinesq system as a small perturbation of the infinite Prandtl number model. Firstly, we derive the appearance of initial layer in detail. Secondly, by using the multi-scale approach [12, 19] and the matched asymptotic expansion methods [5], we construct an approximation solution to (1.1)–(1.7) as the combination of inner and initial expansions. Finally, we study the convergence of (1.1)–(1.7) to the infinite Prandtl number system as the Prandtl number tends to ∞. Our results improve the existing results in [25].

This article is organized as follows. Section 2 is dedicated to the derivation of initial layer. In Section 3, we construct an approximate solution. The main result and its proof are presented in Section 4.

Let C be a positive generic constant, independent on λ, while C may depend on S for fixed $T_* > 0$.

2. Derivation of initial layer

Let $\left(u^\lambda, p^\lambda, T^\lambda, S^\lambda\right)$ be the global weak solution of the system (1.1)–(1.6) in the Leray's sense. The initial data has

$$u^\lambda(t=0) = \left(u_0^0 + \lambda u_0^1 + u_{0E}^\lambda\right)(x,y,z), \tag{2.1}$$

$$T^\lambda(t=0) = \left(T_0^0 + \lambda T_0^1 + T_{0E}^\lambda\right)(x,y,z), \tag{2.2}$$

$$S^\lambda(t=0) = \left(S_0^0 + \lambda S_0^1 + S_{0E}^\lambda\right)(x,y,z), \tag{2.3}$$

where u_0^0, u_0^1, T_0^0, T_0^1, S_0^0 and S_0^1 are $C^\infty(D)$ functions, u_{0E}^λ, T_{0E}^λ, S_{0E}^λ denote the remainders for the initial data, u_{0E}^λ, T_{0E}^λ, $S_{0E}^\lambda \in C^\infty(D)$ satisfies

$$\|(u_{0E}^\lambda, T_{0E}^\lambda, S_{0E}^\lambda)(x,y,z)\|_{L^2(D)} \le C\lambda^2, \tag{2.4}$$

and u_0^0, u_0^1, T_0^0, T_0^1, S_0^0, S_0^1 satisfy the following compatibility conditions:

$$\nabla \cdot u_0^0 = \nabla \cdot u_0^1 = 0, \tag{2.5}$$

$$(u_0^0, u_0^1)\mid_{z=0,1} = 0, \tag{2.6}$$

$$(T_0^0, S_0^0)|_{z=0} = (1,1), \tag{2.7}$$

$$(T_0^0, S_0^0)|_{z=1} = (0,0), \tag{2.8}$$

$$(T_0^1, S_0^1)\big|_{z=0,1} = (0,0). \tag{2.9}$$

In this section, by employing the Stokes operator [19] and singular perturbation theory [22, 27], we consider the initial layer behavior of the solution when the Prandtl number tends to infinity, i.e., λ tends to 0.

Letting $\lambda = 0$ in the system (1.1)–(1.6), we obtain that

$$\Delta u^0 - \nabla p^0 + \sqrt{Ta}k \times u^0 + (R_T T^0 - R_S S^0)k = 0, \tag{2.10}$$

$$\nabla \cdot u^0 = 0, \tag{2.11}$$

$$\partial_t T^0 + (u^0 \cdot \nabla) T^0 = \Delta T^0, \tag{2.12}$$

$$\partial_t S^0 + (u^0 \cdot \nabla) S^0 = \tau \Delta S^0, \tag{2.13}$$

$$u^0|_{z=0,1} = 0, \tag{2.14}$$

$$T^0, S^0|_{z=0} = 1, \quad T^0, S^0|_{z=1} = 0, \tag{2.15}$$

for $(x,y,z,t) \in D \times (0,T_*)$, $T_* > 0$.

The initial condition of T^0, S^0 is

$$(T^0, S^0)(t=0) = (T_0^0, S_0^0)(x,y,z), (x,y,z) \in D, \tag{2.16}$$

where $(T_0^0, S_0^0)(x,y,z)$ stands for the limit of $(T_0^\lambda, S_0^\lambda)(x,y,z)$ as $\lambda \to 0$.

Because of the singularity of perturbation, the limit of $u_0^\lambda(x,y,z)$ as $\lambda \to 0$ can not be satisfied by the velocity $u^0(t=0)$ in the limit system (2.10)–(2.16).

Restricting (2.10), (2.11) and (2.14) to $t=0$ gives

$$\begin{gathered}
\Delta u^0(t=0)-\nabla p^0(t=0)+\sqrt{Ta}k\times u^0(t=0)\\
+(R_T T^0(t=0)-R_S S^0(t=0))k=0,\\
\nabla\cdot u^0(t=0)=0,\\
u^0|_{z=0,1}(t=0)=0.
\end{gathered}$$

By solving the above system, we know that the value of $u^0(t=0)$ is determined by initial value of T^0 and S^0, while $\lim_{\lambda\to 0}u_0^\lambda$ is given arbitrarily and independently of $T^0(t=0)$ and $S^0(t=0)$, so $\lim_{\lambda\to 0}u_0^\lambda\neq u^0(t=0)$. Comparing (1.7) with (2.16) leads to the appearance of an initial layer.

3. Approximate solution

In this section, we give a rigorous proof of infinite lint of the Prandtl for system (1.1)–(1.6). Firstly, we establish the approximate solution. We seek an expansion of the form

$$\begin{gathered}
\left(u^\lambda,p^\lambda,T^\lambda,S^\lambda\right)(x,y,z,t)\sim\sum_{i=0}^{\infty}(\lambda)^i(u^{In,i}(x,y,z,t)+u^{I,i}(x,y,z,\eta),\\
p^{In,i}(x,y,z,t)+p^{I,i}(x,y,z,\eta),\\
T^{In,i}(x,y,z,t)+T^{I,i}(x,y,z,\eta),S^{In,i}(x,y,z,t)+S^{I,i}(x,y,z,\eta),
\end{gathered}\tag{3.1}$$

where λ stands for the length of the initial layer and $\eta=t/\lambda$ is the fast time variable. $\left(u^{In,i},p^{In,i},T^{In,i},S^{In,i}\right)(x,y,z,t)$ denotes the inner functions for the velocity field, the scalar pressure, the temperature field, and the solute concentration, respectively, independent of λ. $(u^{I,i},p^{I,i},T^{I,i},S^{I,i})(x,y,z,\eta)$ stands for the initial layer functions and the initial layer functions satisfying

$$(u^{I,i},p^{I,i},T^{I,i},S^{I,i})(\eta\to+\infty)=0.\tag{3.2}$$

We assume that the asymptotic expansion of the system (1.1)–(1.7) including the initial corrections is of the form

$$\begin{gathered}
\left(u_a^\lambda,p_a^\lambda,T_a^\lambda,S_a^\lambda\right)(x,y,z,t)=\sum_{i=0}^{1}\lambda^i(u^{In,i}(x,y,z,t)+u^{I,i}(x,y,z,\eta),\\
p^{In,i}(x,y,z,t)+p^{I,i}(x,y,z,\eta),\\
T^{In,i}(x,y,z,t)+T^{I,i}(x,y,z,\eta),S^{In,i}(x,y,z,t)+S^{I,i}(x,y,z,\eta)).
\end{gathered}\tag{3.3}$$

Moreover, to match the boundary and initial conditions (1.5)–(1.7), we shall impose the following restrictions

$$\begin{gathered}
u_a^\lambda|_{z=0,1}=(u_{In}^\lambda+u_I^\lambda)|_{z=0,1}=u_I^\lambda|_{z=0,1}=\sum_{i=0}^{1}\lambda^i u^{I,i}|_{z=0,1},\\
T_a^\lambda|_{z=0}=(T_{In}^\lambda+T_I^\lambda)|_{z=0}=1+\sum_{i=0}^{1}\lambda^i T^{I,i}|_{z=0},\\
T_a^\lambda|_{z=1}=(T_{In}^\lambda+T_I^\lambda)|_{z=1}=\sum_{i=0}^{1}\lambda^i T^{I,i}|_{z=1},
\end{gathered}$$

$$
\begin{aligned}
S_a^\lambda|_{z=0} &= (S_{In}^\lambda + S_I^\lambda)|_{z=0} = 1 + \sum_{i=0}^{1} \lambda^i S^{I,i}|_{z=0}, \\
S_a^\lambda|_{z=1} &= (S_{In}^\lambda + S_I^\lambda)|_{z=1} = \sum_{i=0}^{1} \lambda^i S^{I,i}|_{z=1}, \\
u_a^\lambda(t=0) &= \sum_{i=0}^{1} \lambda^i (u^{In,i}(t=0) + u^{I,i}(\eta=0)), \\
T_a^\lambda(t=0) &= \sum_{i=0}^{1} \lambda^i (T^{In,i}(t=0) + T^{I,i}(\eta=0)), \\
S_a^\lambda(t=0) &= \sum_{i=0}^{1} \lambda^i (S^{In,i}(t=0) + S^{I,i}(\eta=0)).
\end{aligned}
$$

i.e.,

$$(u^{In,0} + u^{I,0})|_{z=0,1} = 0, \tag{3.4}$$

$$(u^{In,1} + u^{I,1})|_{z=0,1} = 0, \tag{3.5}$$

$$(T^{In,0} + T^{I,0})|_{z=0} = 1, \quad (T^{In,1} + T^{I,1})|_{z=0} = 0, \tag{3.6}$$

$$(T^{In,0} + T^{I,0})|_{z=1} = 0, \quad (T^{In,1} + T^{I,1})|_{z=1} = 0, \tag{3.7}$$

$$(S^{In,0} + S^{I,0})|_{z=0} = 1, \quad (S^{In,1} + S^{I,1})|_{z=0} = 0, \tag{3.8}$$

$$(S^{In,0} + S^{I,0})|_{z=1} = 0, \quad (S^{In,1} + S^{I,1})|_{z=1} = 0, \tag{3.9}$$

$$u^{In,0}(t=0) + u^{I,0}(\eta=0) = u_0^0, \quad u^{In,1}(t=0) + u^{I,1}(\eta=0) = u_0^1, \tag{3.10}$$

$$T^{In,0}(t=0) + T^{I,0}(\eta=0) = T_0^0, \quad T^{In,1}(t=0) + T^{I,1}(\eta=0) = T_0^1, \tag{3.11}$$

$$S^{In,0}(t=0) + S^{I,0}(\eta=0) = S_0^0, \quad S^{In,1}(t=0) + S^{I,1}(\eta=0) = S_0^1. \tag{3.12}$$

We discuss the construction of the inner functions and initial layer functions

$$
\begin{aligned}
\left(u_a^\lambda, p_a^\lambda, T_a^\lambda, S_a^\lambda\right) := &\left(u_{In}^\lambda, p_{In}^\lambda, T_{In}^\lambda, S_{In}^\lambda\right)(x,y,z,t) \\
&+ \left(u_I^\lambda, p_I^\lambda, T_I^\lambda, S_I^\lambda\right)(x,y,z,\eta), \quad \eta = \frac{t}{\lambda},
\end{aligned} \tag{3.13}
$$

where

$$\left(u_{In}^\lambda, p_{In}^\lambda, T_{In}^\lambda, S_{In}^\lambda\right) = \sum_{i=0}^{1} \lambda^i \left(u^{In,i}, p^{In,i}, T^{In,i}, S^{In,i}\right), \tag{3.14}$$

$$\left(u_I^\lambda, p_I^\lambda, T_I^\lambda, S_I^\lambda\right) = \sum_{i=0}^{1} \lambda^i \left(u^{I,i}, p^{I,i}, T^{I,i}, S^{I,i}\right). \tag{3.15}$$

First, we study inner expansion away from $t = 0$ in Section 3.1. Then, we study the initial layer expansion near $t = 0$ in Section 3.2. Finally, we consider the approximate solution in Section 3.3.

3.1. **Inner expansion.** Away from $t = 0$ in (3.1), the solution to system (1.1)–(1.6) has the expansion

$$\left(u^\lambda, p^\lambda, T^\lambda, S^\lambda\right)(x,y,z,t) \sim \sum_{i=0}^{\infty} \lambda^i (u^{In,i}, p^{In,i}, T^{In,i}, S^{In,i})(x,y,z,t).$$

First, inserting the above expansion into (1.1)–(1.6) and using direct calculations, we obtain that

$$
\begin{gathered}
\sum_{i=0}^{\infty}\lambda^i(\lambda[\partial_t u^{In,i}+\sum_{j=0}^{i}u^{In,j}\cdot\nabla u^{In,i-j}]+\nabla p^{In,i}-\Delta u^{In,i}-\sqrt{Ta}k\times u^{In,i} \\
-(R_T T^{In,i}-R_S S^{In,i})k)=0, \\
\sum_{i=0}^{\infty}\lambda^i\nabla\cdot u^{In,i}=0, \\
\sum_{i=0}^{\infty}\lambda^i\Big(\partial_t T^{In,i}+\sum_{j=0}^{i}u^{In,j}\cdot\nabla T^{In,i-j}-\Delta T^{In,i}\Big)=0, \\
\sum_{i=0}^{\infty}\lambda^i\Big(\partial_t T^{In,i}+\sum_{j=0}^{i}u^{In,j}\cdot\nabla S^{In,i-j}-\tau\Delta S^{In,i}\Big)=0, \\
\sum_{i=0}^{\infty}\lambda^i u^{In,i}|_{z=0,1}=0, \\
\sum_{i=0}^{\infty}\lambda^i(T^{In,i},S^{In,i})|_{z=0}=(1,1), \\
\sum_{i=0}^{\infty}\lambda^i(T^{In,i},S^{In,i})|_{z=1}=(0,0).
\end{gathered}
$$

Then $\big(u_{In}^{\lambda},p_{In}^{\lambda},T_{In}^{\lambda},S_{In}^{\lambda}\big)$ satisfies

$$
\begin{aligned}
&\lambda[\partial_t u_{In}^{\lambda}+(u_{In}^{\lambda}\cdot\nabla)u_{In}^{\lambda}]+\nabla p_{In}^{\lambda} \\
&=\Delta u_{In}^{\lambda}+\sqrt{Ta}k\times u_{In}^{\lambda}+(R_T T_{In}^{\lambda}-R_S S_{In}^{\lambda})k+R_{In,u}^{\lambda},
\end{aligned}\tag{3.16}
$$

$$
\nabla\cdot u_{In}^{\lambda}=0,\tag{3.17}
$$

$$
\partial_t T_{In}^{\lambda}+(u_{In}^{\lambda}\cdot\nabla)T_{In}^{\lambda}=\Delta T_{In}^{\lambda}+R_{In,T}^{\lambda},\tag{3.18}
$$

$$
\partial_t S_{In}^{\lambda}+(u_{In}^{\lambda}\cdot\nabla)S_{In}^{\lambda}=\tau\Delta S_{In}^{\lambda}+R_{In,S}^{\lambda},\tag{3.19}
$$

$$
u_{In}^{\lambda}|_{z=0,1}=0,\tag{3.20}
$$

$$
(T_{In}^{\lambda},S_{In}^{\lambda})|_{z=0}=(1,1),\quad (T_{In}^{\lambda},S_{In}^{\lambda})|_{z=1}=(0,0),\tag{3.21}
$$

where the remainders $R_{In,u}^{\lambda}$, $R_{In,T}^{\lambda}$ and $R_{In,S}^{\lambda}$ are

$$
\begin{aligned}
R_{In,u}^{\lambda}=&-\sum_{i=2}^{\infty}\lambda^i(\lambda[\partial_t u^{In,i}+\sum_{j=0}^{i}u^{In,j}\cdot\nabla u^{In,i-j}]+\nabla p^{In,i} \\
&-\Delta u^{In,i}-\sqrt{Ta}k\times u^{In,i}-(R_T T^{In,i}-R_S S^{In,i})k)+\lambda^3 u^{In,1}\cdot\nabla u^{In,1}, \\
R_{In,T}^{\lambda}=&-\sum_{i=2}^{\infty}\lambda^i\Big(\partial_t T^{In,i}+\sum_{j=0}^{i}u^{In,j}\cdot\nabla T^{In,i-j}-\Delta T^{In,i}\Big)+\lambda^2 u^{In,1}\cdot\nabla T^{In,1}, \\
R_{In,S}^{\lambda}=&-\sum_{i=2}^{\infty}\lambda^i\Big(\partial_t S^{In,i}+\sum_{j=0}^{i}u^{In,j}\cdot\nabla S^{In,i-j}-\tau\Delta S^{In,i}\Big)+\lambda^2 u^{In,1}\cdot\nabla S^{In,1}.
\end{aligned}
$$

Thus, $R^\lambda_{In,u}$, $R^\lambda_{In,T}$ and $R^\lambda_{In,S}$ satisfy

$$\|(R^\lambda_{In,u}, R^\lambda_{In,T}, R^\lambda_{In,S})\|_{L^\infty(0,T_*;H^s(D))} \le C\lambda^2, \tag{3.22}$$

for $T_* > 0$ and $s \ge 1$.

Now, we set the coefficient of $O(\lambda^0)$ in system (3.16)-(3.19) as zero and use (3.20)-(3.21), (2.16). Then for $\big(u^{In,0}, p^{In,0}, T^{In,0}, S^{In,0}\big)$, we have

$$\Delta u^{In,0} + \sqrt{Ta}k \times u^{In,0} + (R_T T^{In,0} - R_S S^{In,0})k - \nabla p^{In,0} = 0, \tag{3.23}$$

$$\nabla \cdot u^{In,0} = 0, \tag{3.24}$$

$$\partial_t T^{In,0} + (u^{In,0} \cdot \nabla) T^{In,0} = \Delta T^{In,0}, \tag{3.25}$$

$$\partial_t S^{In,0} + (u^{In,0} \cdot \nabla) S^{In,0} = \tau \Delta S^{In,0}, \tag{3.26}$$

$$u^{In,0}|_{z=0,1} = 0, \tag{3.27}$$

$$(T^{In,0}, S^{In,0})|_{z=0} = (1,1),\ (T^{In,0}, S^{In,0})|_{z=1} = (0,0), \tag{3.28}$$

$$(T^{In,0}, S^{In,0})(t=0) = (T^0_0, S^0_0). \tag{3.29}$$

Similarly, we consider the $O(\lambda^1)$ terms. We set the coefficient of $O(\lambda^1)$ in the system (3.16)-(3.19) as zero and use (3.20)-(3.21), the initial conditions (3.11)-(3.12).

At first, $(u^{In,1}, p^{In,1}, T^{In,1}, S^{In,1})$ satisfy

$$\begin{aligned} &\Delta u^{In,1} + \sqrt{Ta}k \times u^{In,1} + (R_T T^{In,1} - R_S S^{In,1})k \\ &- \nabla p^{In,1} - \partial_t u^{In,0} - (u^{In,0} \cdot \nabla) u^{In,0} = 0, \end{aligned} \tag{3.30}$$

$$\nabla \cdot u^{In,1} = 0, \tag{3.31}$$

$$\partial_t T^{In,1} + u^{In,0} \cdot \nabla T^{In,1} + u^{In,1} \cdot \nabla T^{In,0} = \Delta T^{In,1}, \tag{3.32}$$

$$\partial_t S^{In,1} + u^{In,0} \cdot \nabla S^{In,1} + u^{In,1} \cdot \nabla S^{In,0} = \tau \Delta S^{In,1}, \tag{3.33}$$

$$u^{In,1}|_{z=0,1} = 0, \tag{3.34}$$

$$(T^{In,1}, S^{In,1})|_{z=0,1} = (0,0), \tag{3.35}$$

$$T^{In,1}(t=0) = T^1_0 - T^{I,1}(\eta=0), \quad S^{In,1}(t=0) = S^1_0 - S^{I,1}(\eta=0). \tag{3.36}$$

We have the compatibility condition

$$(T^1_0 - T^{I,1}(\eta=0), S^1_0 - S^{I,1}(\eta=0))\big|_{z=0,1} = (0,0). \tag{3.37}$$

The rotating system (3.23)-(3.29) has stationary Stokes equations by a buoyancy force proportional to the temperature and the solute concentration coupled with heat advection-diffusion equations of the temperature and the solute concentration. The system (3.30)-(3.36) can be considered as one linear system of Stokes equations coupled with a linearized heat advection-diffusion equations. So, the existence of the smooth solutions is the same to those of the incompressible Stokes equations. Since the proof is basic, we omit the details.

Proposition 3.1. *Assume that* T^0_0, T^1_0, S^0_0, S^1_0, $T^{In,1}(t=0)$, $S^{In,1}(t=0) \in C^\infty(D)$ *satisfy the suitable compatibility conditions like* (2.7)–(2.9), (3.37) *etc. There is a unique and global* $C^\infty(D \times [0,+\infty))$ *smooth solution of system* (3.23)–(3.29) *and* (3.30)–(3.36), *respectively.*

3.2. **Initial layer expansion.** We now derive the systems satisfying the initial layer functions.

Near $t=0$, inserting (3.13) into system (1.1)–(1.4), one obtains

$$
\begin{aligned}
&\lambda[\partial_t u_a^\lambda + (u_a^\lambda\cdot\nabla)u_a^\lambda] + \nabla p_a^\lambda - \Delta u_a^\lambda - \sqrt{Ta}k\times u_a^\lambda - (R_T T_a^\lambda - R_S S_a^\lambda)k\\
&= \lambda[\partial_t(u_{In}^\lambda + u_I^\lambda) + ((u_{In}^\lambda + u_I^\lambda)\cdot\nabla)(u_{In}^\lambda + u_I^\lambda)] + \nabla(p_{In}^\lambda + P_I^\lambda)\\
&\quad - \Delta(u_{In}^\lambda + u_I^\lambda) - \sqrt{Ta}k\times(u_{In}^\lambda + u_I^\lambda) - (R_T(T_{In}^\lambda + T_I^\lambda) - R_S(S_{In}^\lambda + S_I^\lambda))k\\
&= R_{In,u}^\lambda + \lambda[\partial_t u_I^\lambda + (u_{In}^\lambda\cdot\nabla)u_I^\lambda + u_I^\lambda\cdot\nabla(u_{In}^\lambda + u_I^\lambda)] + \nabla p_I^\lambda - \Delta u_I^\lambda\\
&\quad - \sqrt{Ta}k\times u_I^\lambda - (R_T T_I^\lambda - R_S S_I^\lambda)k,
\end{aligned}
\tag{3.38}
$$

$$
\nabla\cdot u_a^\lambda = \nabla\cdot(u_{In}^\lambda + u_I^\lambda) = \nabla\cdot u_I^\lambda = \sum_{i=0}^{1}\lambda^i\nabla\cdot u^{I,i}, \tag{3.39}
$$

$$
\begin{aligned}
&\partial_t T_a^\lambda + (u_a^\lambda\cdot\nabla)T_a^\lambda - \Delta T_a^\lambda\\
&= \partial_t(T_{In}^\lambda + T_I^\lambda) + ((u_{In}^\lambda + u_I^\lambda)\cdot\nabla)(T_{In}^\lambda + T_I^\lambda) - \Delta(T_{In}^\lambda + T_I^\lambda)\\
&= R_{In,T}^\lambda + \partial_t T_I^\lambda + u_{In}^\lambda\cdot\nabla T_I^\lambda + (u_I^\lambda\cdot\nabla)(T_{In}^\lambda + T_I^\lambda) - \Delta T_I^\lambda,
\end{aligned}
\tag{3.40}
$$

$$
\begin{aligned}
&\partial_t S_a^\lambda + (u_a^\lambda\cdot\nabla)S_a^\lambda - \tau\Delta S_a^\lambda\\
&= \partial_t(S_{In}^\lambda + S_I^\lambda) + ((u_{In}^\lambda + u_I^\lambda)\cdot\nabla)(S_{In}^\lambda + S_I^\lambda) - \tau\Delta(S_{In}^\lambda + S_I^\lambda)\\
&= R_{In,S}^\lambda + \partial_t S_I^\lambda + u_{In}^\lambda\cdot\nabla S_I^\lambda + (u_I^\lambda\cdot\nabla)(S_{In}^\lambda + S_I^\lambda) - \tau\Delta S_I^\lambda.
\end{aligned}
\tag{3.41}
$$

We consider

$$
\begin{aligned}
u^{In,i}(x,y,z,t) &= u^{In,i}(x,y,z,\lambda\eta) = u^{In,i}(x,y,z,0) + \lambda\partial_t u^{In,i}(t=0)\eta + \cdots,\\
T^{In,i}(x,y,z,t) &= T^{In,i}(x,y,z,\lambda\eta) = T^{In,i}(x,y,z,0) + \lambda\partial_t T^{In,i}(t=0)\eta + \cdots,\\
S^{In,i}(x,y,z,t) &= S^{In,i}(x,y,z,\lambda\eta) = S^{In,i}(x,y,z,0) + \lambda\partial_t S^{In,i}(t=0)\eta + \cdots.
\end{aligned}
$$

Next we compare the coefficients of $O(\lambda^i)$, $i\geq 0$ in the resulting system and derive the systems satisfying the initial layer functions.

Taking the coefficient of $O(\lambda^{-1})$ in (3.40)–(3.41) as zero, it follows that $\partial_\eta T^{I,0}=0$, $\partial_\eta S^{I,0}=0$, By this and (3.2), we obtain

$$
T^{I,0}(x,y,z,\tau) = 0, \tag{3.42}
$$

$$
S^{I,0}(x,y,z,\tau) = 0, \tag{3.43}
$$

which indicates that the temperature and the solute concentration have no zero order initial layer.

Setting the coefficients of $O(\lambda^0)$ in (3.38)−(3.41) as zero and using (3.42)-(3.43), we have

$$
\partial_\eta u^{I,0} + \nabla p^{I,0} - \Delta u^{I,0} - \sqrt{Ta}k\times u^{I,0} = 0, \tag{3.44}
$$

$$
\nabla\cdot u^{I,0} = 0, \tag{3.45}
$$

$$
\partial_\eta T^{I,1} + u^{I,0}\cdot\nabla(T^{In,0}(t=0)) = 0, \tag{3.46}
$$

$$
\partial_\eta S^{I,1} + u^{I,0}\cdot\nabla(S^{In,0}(t=0)) = 0. \tag{3.47}
$$

and using the boundary conditions (3.4), (3.27), and the initial condition (3.10), one obtains

$$u^{I,0}|_{z=0,1} = 0, \tag{3.48}$$

$$u^{I,0}(\eta = 0) = u_0^0 - u^{In,0}(t = 0). \tag{3.49}$$

Then $(u^{I,0}, p^{I,0}, T^{I,0}, S^{I,0})$ satisfy the system (3.42)-(3.45), (3.48), and (3.49) as

$$\begin{gathered}
T^{I,0}(x, y, z, \tau) = 0,\\
S^{I,0}(x, y, z, \tau) = 0,\\
\partial_\eta u^{I,0} + \nabla p^{I,0} - \Delta u^{I,0} - \sqrt{Ta}k \times u^{I,0} = 0,\\
\nabla \cdot u^{I,0} = 0,\\
u^{I,0}|_{z=0,1} = 0,\\
u^{I,0}(\eta = 0) = u_0^0 - u^{In,0}(t = 0).
\end{gathered}$$

Now, we investigate the initial and boundary conditions of $T^{I,1}$, $S^{I,1}$. Using (3.46)-(3.47) and the decay condition (3.2), we obtain

$$T^{I,1} = -\int_\eta^\infty [u^{I,0} \cdot \nabla(T^{In,0}(t = 0))](s) ds, \tag{3.50}$$

$$T^{S,1} = -\int_\eta^\infty [u^{I,0} \cdot \nabla(S^{In,0}(t = 0))](s) ds. \tag{3.51}$$

Then, we restrict (3.50)-(3.51) to $\eta = 0$, and replace the right term of result by $\overline{T}^{I,1}$, $\overline{S}^{I,1}$. Namely,

$$T^{I,1}(\eta = 0) = \overline{T}^{I,1}, \tag{3.52}$$

$$S^{I,1}(\eta = 0) = \overline{S}^{I,1}. \tag{3.53}$$

By restricting (3.50)-(3.51) to $z = 0, 1$ and using (3.48), one obtains

$$T^{I,1}|_{z=0,1} = 0, \tag{3.54}$$

$$S^{I,1}|_{z=0,1} = 0. \tag{3.55}$$

Moreover, from (3.11)-(3.12) it follows that

$$T^{I,1}(\tau = 0) = T_0^1 - T^{In,1}(t = 0), \tag{3.56}$$

$$T^{S,1}(\tau = 0) = S_0^1 - S^{In,1}(t = 0). \tag{3.57}$$

Setting the coefficients of $O(\lambda^1)$ in (3.38), (3.39) as zero, we have

$$\begin{aligned}
&\partial_\eta u^{I,1} + \nabla p^{I,1} - \Delta u^{I,1} - \sqrt{Ta}k \times u^{I,1} - (R_T T^{I,1} - R_S S^{I,1})k\\
&= -(u^{I,0} \cdot \nabla) u^{In,0}(t = 0) - (u^{In,0}(t = 0) \cdot \nabla) u^{I,0} - (u^{I,0} \cdot \nabla) u^{I,0},
\end{aligned} \tag{3.58}$$

$$\nabla \cdot u^{I,1} = 0. \tag{3.59}$$

From the boundary conditions and the initial condition from (3.5), (3.34), and (3.10), we deduce that

$$u^{I,1}\big|_{z=0,1} = 0, \tag{3.60}$$

$$u^{I,1}(\eta = 0) = u_0^1 - u^{In,1}(t = 0). \tag{3.61}$$

Hence, we obtain the system

$$
\begin{gathered}
\partial_\eta T^{I,1} + u^{I,0}\cdot\nabla(T^{In,0}(t=0)) = 0,\\
\partial_\eta S^{I,1} + u^{I,0}\cdot\nabla(S^{In,0}(t=0)) = 0,\\
\partial_\eta u^{I,1} + \nabla p^{I,1} - \Delta u^{I,1} - \sqrt{Ta}k\times u^{I,1} - (R_T T^{I,1} - R_S S^{I,1})k\\
= -(u^{I,0}\cdot\nabla)u^{In,0}(t=0) - (u^{In,0}(t=0)\cdot\nabla)u^{I,0} - (u^{I,0}\cdot\nabla)u^{I,0},\\
\nabla\cdot u^{I,1} = 0,\\
u^{I,1}\big|_{z=0,1} = 0,\\
u^{I,1}(\eta=0) = u_0^1 - u^{In,1}(t=0).
\end{gathered}
$$

In view of [28], we obtain the following result, whose proof is easy.

Proposition 3.2. *Assume that (2.1)-(2.3) hold, u_0^0, u_0^1, T_0^0, T_0^1, S_0^0, $S_0^1\in C^\infty(D)$ satisfy the compatibility conditions (2.5)-(2.9). There exist a unique and smooth solution $(u^{I,0}, p^{I,0})$ to the system (3.42)-(3.45), (3.48) and (3.49) and a unique and smooth solution $(u^{I,1}, p^{I,1}, T^{I,1}, S^{I,1})$ to the system (3.46)-(3.47) and (3.58)-(3.61) satisfying the exponential decay to zero as $\eta\to\infty$, in the sense that*

$$
\|(u^{I,0}, u^{I,1}, T^{I,1}, S^{I,1})(\cdot,\eta)\|_{H^s(D)} \le Ce^{-\beta\eta},
$$

for some constant $\beta>0$ and any $s\ge 1$.

3.3. **Approximate solution.** Simple computations yield

$$
\begin{aligned}
&\lambda[\partial_t u_a^\lambda + (u_a^\lambda\cdot\nabla)u_a^\lambda] + \nabla p_a^\lambda - \Delta u_a^\lambda - \sqrt{Ta}k\times u_a^\lambda - (R_T T_a^\lambda - R_S S_a^\lambda)k\\
&= R_{In,u}^\lambda + (\partial_\eta u^{I,0} + \nabla p^{I,0} - \Delta u^{I,0} - \sqrt{Ta}k\times u^{I,0})\\
&\quad + \lambda[\partial_\eta u^{I,1} + \nabla p^{I,1} - \Delta u^{I,1} - \sqrt{Ta}k\times u^{I,1} - (R_T T^{I,1} - R_S S^{I,1})k\\
&\quad + (u^{I,0}\cdot\nabla)u^{In,0}(t=0) + (u^{In,0}(t=0)\cdot\nabla)u^{I,0} + (u^{I,0}\cdot\nabla)u^{I,0}]\\
&\quad + R_{I,u}^\lambda,
\end{aligned}\tag{3.62}
$$

$$
\begin{aligned}
&\partial_t T_a^\lambda + (u_a^\lambda\cdot\nabla)T_a^\lambda - \Delta T_a^\lambda\\
&= R_{In,T}^\lambda + \lambda^{-1}\partial_\eta T^{I,0} + (\partial_\eta T^{I,1} + u^{I,0}\cdot\nabla(T^{In,0}(t=0))\\
&\quad + (u^{In,0} + u^{I,0})\cdot\nabla T^{I,0} + R_{I,T}^\lambda,
\end{aligned}\tag{3.63}
$$

$$
\begin{aligned}
&\partial_t S_a^\lambda + (u_a^\lambda\cdot\nabla)S_a^\lambda - \tau\Delta S_a^\lambda\\
&= R_{In,S}^\lambda + \lambda^{-1}\partial_\eta S^{I,0} + (\partial_\eta S^{I,1} + u^{I,0}\cdot\nabla(S^{In,0}(t=0))\\
&\quad + (u^{In,0} + u^{I,0})\cdot\nabla S^{I,0} + R_{I,S}^\lambda,
\end{aligned}\tag{3.64}
$$

where the remainders $R_{I,u}^\lambda$, $R_{I,T}^\lambda$ and $R_{I,S}^\lambda$, caused by the initial layer, are shown by

$$
\begin{aligned}
&R_{I,u}^\lambda\\
&= \lambda^2[\eta(u^{I,0}\cdot\nabla)\partial_t u^{In,0}(\theta_1 t) + \eta\partial_t u^{In,0}(\theta_2 t)\cdot\nabla u^{I,0} + (u^{In,0}(t=0)\cdot\nabla)u^{I,1}\\
&\quad + (u^{In,1}\cdot\nabla)u^{I,0} + u^{I,0}\cdot\nabla(u^{In,1} + u^{I,1}) + u^{I,1}\cdot\nabla(u^{In,0}(t=0) + u^{I,0})]\\
&\quad + \lambda^3[\eta\partial_t u^{In,0}(\theta_3 t)\cdot\nabla u^{I,1} + \eta(u^{I,1}\cdot\nabla)\partial_t u^{In,0}(\theta_4 t)\\
&\quad + (u^{In,1}\cdot\nabla)u^{I,1} + u^{I,1}\cdot\nabla(u^{In,1} + u^{I,1})], 0<\theta_i<1,\quad i=1,2,3,4,
\end{aligned}\tag{3.65}
$$

$$\begin{aligned}
R_{I,T}^{\lambda} &= \lambda[\eta(u^{I,0}\cdot\nabla)\partial_t T^{In,0}(t=0) + u^{In,0}(t=0)\cdot\nabla T^{I,1} \\
&\quad + u^{I,0}\cdot\nabla(T^{In,1}(t=0)+T^{I,1}) + u^{I,1}\cdot\nabla T^{In,0}(t=0) \\
&\quad + (u^{In,1}+u^{I,1})\cdot\nabla T^{I,0} - \Delta T^{I,1}] \\
&\quad + \lambda^2[\frac{1}{2}\eta^2(u^{I,0}\cdot\nabla)\partial_{tt}T^{In,0}(\theta_5 t) + \eta(u^{I,1}\cdot\nabla)\partial_t T^{In,0}(\theta_6 t) \\
&\quad + \eta(u^{I,0}\cdot\nabla)\partial_t T^{In,1}(\theta_7 t) \\
&\quad + \eta\partial_t u^{In,0}(\theta_8 t)\cdot\nabla T^{I,1} + (u^{In,1}\cdot\nabla)T^{I,1} + u^{I,1}\cdot\nabla(T^{In,1}+T^{I,1})], \\
&\qquad 0<\theta_i<1, \quad i=5,6,7,8,
\end{aligned} \tag{3.66}$$

$$\begin{aligned}
R_{I,S}^{\lambda} &= \lambda[\eta(u^{I,0}\cdot\nabla)\partial_t S^{In,0}(t=0) + u^{In,0}(t=0)\cdot\nabla S^{I,1} \\
&\quad + u^{I,0}\cdot\nabla(S^{In,1}(t=0)+S^{I,1}) + u^{I,1}\cdot\nabla S^{In,0}(t=0) \\
&\quad + (u^{In,1}+u^{I,1})\cdot\nabla S^{I,0} - \tau\Delta S^{I,1}] \\
&\quad + \lambda^2[\frac{1}{2}\eta^2(u^{I,0}\cdot\nabla)\partial_{tt}S^{In,0}(\theta_9 t) + \eta(u^{I,1}\cdot\nabla)\partial_t S^{In,0}(\theta_{10} t) \\
&\quad + \eta(u^{I,0}\cdot\nabla)\partial_t S^{In,1}(\theta_{11} t) + \eta\partial_t u^{In,0}(\theta_{12} t)\cdot\nabla S^{I,1} \\
&\quad + (u^{In,1}\cdot\nabla)S^{I,1} + u^{I,1}\cdot\nabla(S^{In,1}+S^{I,1})], \\
&\qquad 0<\theta_i<1, \quad i=9,10,11,12.
\end{aligned} \tag{3.67}$$

Hence, the previous computations show that $(u_a^\lambda, p_a^\lambda, T_a^\lambda, S_a^\lambda)$ solves the initial-boundary problem

$$\begin{aligned}
&\lambda[\partial_t u_a^\lambda + (u_a^\lambda\cdot\nabla)u_a^\lambda] + \nabla p_a^\lambda \\
&= \Delta u_a^\lambda + \sqrt{Ta}k\times u_a^\lambda + (R_T T_a^\lambda - R_S S_a^\lambda)k + R_{In,u}^\lambda + R_{I,u}^\lambda,
\end{aligned} \tag{3.68}$$

$$\nabla\cdot u_a^\lambda = 0, \tag{3.69}$$

$$\partial_t T_a^\lambda + (u_a^\lambda\cdot\nabla)T_a^\lambda = \Delta T_a^\lambda + R_{In,T}^\lambda + R_{I,T}^\lambda, \tag{3.70}$$

$$\partial_t S_a^\lambda + (u_a^\lambda\cdot\nabla)S_a^\lambda = \tau\Delta S_a^\lambda + R_{In,S}^\lambda + R_{I,S}^\lambda, \tag{3.71}$$

$$u_a^\lambda|_{z=0,1} = 0, \tag{3.72}$$

$$(T_a^\lambda, S_a^\lambda)|_{z=0} = (1,1), \tag{3.73}$$

$$(T_a^\lambda, S_a^\lambda)|_{z=1} = (0,0), \tag{3.74}$$

$$(u_a^\lambda, T_a^\lambda, S_a^\lambda)(t=0) = (u_0^0+\lambda u_0^1, T_0^0+\lambda T_0^1, S_0^0+\lambda S_0^1), \tag{3.75}$$

where the remainders $R_{In,u}^\lambda$, $R_{In,T}^\lambda$, $R_{In,S}^\lambda$ satisfy (3.22) and $R_{I,u}^\lambda$, $R_{I,T}^\lambda$, $R_{I,S}^\lambda$ defined by (3.65)-(3.67) respectively satisfy

$$\begin{aligned}
&\|R_{I,u}^\lambda(.,\eta)\|_{L^\infty(D)} \le C\lambda^2(\eta+1)e^{-\beta\eta}, \\
&\|(R_{I,T}^\lambda, R_{I,S}^\lambda)(\cdot,\eta)\|_{L^\infty(D)} \le C\lambda(\eta^2+\eta+1)e^{-\beta\eta},
\end{aligned} \tag{3.76}$$

for some constant $\beta>0$ and for any $t\in[0,S]$ and any fixed $S>0$. The estimate (3.76) can be easily derived by the definitions of $R_{I,u}^\lambda$, $R_{I,T}^\lambda$, $R_{I,S}^\lambda$ and Proposition 3.2.

4. Main results

Theorem 4.1. *Assume that* (2.1)-(2.3) *hold. And assume that* u_0^0, u_0^1, T_0^0, T_0^1, S_0^0, $S_0^1 \in C^\infty(D)$ *satisfy the compatibility conditions* (2.5)-(2.9)*. Then, as* $\lambda\to 0$*, for* *any $0 < T_* < \infty$, we have*

$$\|(u^\lambda - u_a^\lambda, T^\lambda - T_a^\lambda, S^\lambda - S_a^\lambda)\|_{L^\infty(0,T_*;L^2(D))} \le C\lambda^{3/2}, \tag{4.1}$$

and we obtain

$$\|(u^\lambda - u_a^\lambda, T^\lambda - T_a^\lambda, S^\lambda - S_a^\lambda)\|_{L^2(0,T_*;H^1(D))} \le C\lambda^{3/2}, \tag{4.2}$$

where $H^1(D)$=$W^{1,2}(D)$, for some positive constants C independent of λ.

Proof. We use the classical L^2-energy method, and separate the proof into five steps.

Step 1. We define the error functions as

$$u_e^\lambda = u^\lambda - u_a^\lambda, \quad p_e^\lambda = p^\lambda - p_a^\lambda, \quad T_e^\lambda = T^\lambda - T_a^\lambda, \quad S_e^\lambda = S^\lambda - S_a^\lambda,$$

which satisfies

$$\begin{aligned} &\lambda[\partial_t u_e^\lambda + (u_a^\lambda \cdot \nabla) u_e^\lambda + (u_e^\lambda \cdot \nabla)(u_a^\lambda + u_e^\lambda)] + \nabla p_e^\lambda \\ &= \Delta u_e^\lambda + \sqrt{Ta} k \times u_e^\lambda + (R_T T_e^\lambda - R_S S_e^\lambda) k - R_{In,u}^\lambda - R_{I,u}^\lambda, \end{aligned} \tag{4.3}$$

$$\nabla \cdot u_e^\lambda = 0, \tag{4.4}$$

$$\partial_t T_e^\lambda + (u_a^\lambda \cdot \nabla) T_e^\lambda + (u_e^\lambda \cdot \nabla)(T_a^\lambda + T_e^\lambda) = \Delta T_e^\lambda - R_{In,T}^\lambda - R_{I,T}^\lambda, \tag{4.5}$$

$$\partial_t S_e^\lambda + (u_a^\lambda \cdot \nabla) S_e^\lambda + (u_e^\lambda \cdot \nabla)(S_a^\lambda + S_e^\lambda) = \tau \Delta S_e^\lambda - R_{In,S}^\lambda - R_{I,S}^\lambda, \tag{4.6}$$

$$(u_e^\lambda, T_e^\lambda, S_e^\lambda)|_{z=0,1} = (0,0,0) \tag{4.7}$$

$$(u_e^\lambda, T_e^\lambda, S_e^\lambda)(t=0) = (u_{0E}^\lambda, T_{0E}^\lambda, S_{0E}^\lambda)(x,y,z). \tag{4.8}$$

where $R_{In,u}^\lambda$, $R_{In,T}^\lambda$, $R_{In,S}^\lambda$, $R_{I,u}^\lambda$, $R_{I,T}^\lambda$ and $R_{I,S}^\lambda$ are the remainders, u_{0E}^λ, T_{0E}^λ and S_{0E}^λ are defined in Section 2.

Step 2. Taking the L^2-inner product of (4.5) with T_e^λ and integrating over D, one obtains

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|T_e^\lambda\|_{L^2(D)}^2 \\ &= \int_D \Delta T_e^\lambda T_e^\lambda \,dx\,dy\,dz - \int_D (R_{In,T}^\lambda + R_{I,T}^\lambda) T_e^\lambda \,dx\,dy\,dz \\ &\quad - \int_D \left(u_a^\lambda \cdot \nabla\right) T_e^\lambda T_e^\lambda \,dx\,dy\,dz - \int_D \left(u_e^\lambda \cdot \nabla\right)(T_a^\lambda + T_e^\lambda) T_e^\lambda \,dx\,dy\,dz \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.9}$$

Applying Green's first formula to I_1 and taking into account (4.7) yield

$$I_1 = \oint\oint_\Gamma T_e^\lambda \frac{\partial T_e^\lambda}{\partial n} dS - \int_D |\nabla T_e^\lambda|^2 \,dx\,dy\,dz = -\int_D |\nabla T_e^\lambda|^2 \,dx\,dy\,dz, \tag{4.10}$$

where Γ represents the boundary surface.

Next, we estimate I_2 by the estimates (3.22), (3.76). We obtain

$$\begin{aligned} |I_2| &\le \xi_1 \|T_e^\lambda\|_{L^2(D)}^2 + C(\xi_1) \|R_{In,T}^\lambda + R_{I,T}^\lambda\|_{L^2(D)}^2 \\ &\le \xi_1 \|T_e^\lambda\|_{L^2(D)}^2 + C(\xi_1)(C\lambda^4 + C\lambda^2(\eta^2 + \eta + 1)^2 e^{-2\beta\eta}), \end{aligned} \tag{4.11}$$

where the constant $C(\xi_1) > 0$ is independent of λ, and ξ_1 is a small constant.

According to (3.69), (3.72), and (4.7),

$$
\begin{aligned}
I_3 &= -\int_D u_a^\lambda \cdot \nabla \left(\frac{(T_e^\lambda)^2}{2} \right) dx\,dy\,dz \\
&= -\int_D \nabla \cdot \left(u_a^\lambda \frac{(T_e^\lambda)^2}{2} \right) dx\,dy\,dz + \int_D \nabla \cdot u_a^\lambda \frac{(T_e^\lambda)^2}{2}\, dx\,dy\,dz = 0.
\end{aligned} \tag{4.12}
$$

The last integral term I_4 can be established similarly as in estimating I_3. We obtain

$$
\begin{aligned}
I_4 &= -\int_D \left(u_e^\lambda \cdot \nabla \right) T_a^\lambda T_e^\lambda \, dx\,dy\,dz - \int_D (u_e^\lambda \cdot \nabla) T_e^\lambda T_e^\lambda \, dx\,dy\,dz \\
&= -\int_D (u_e^\lambda \cdot \nabla) T_a^\lambda T_e^\lambda \, dx\,dy\,dz \\
&\le | -\int_D (u_e^\lambda \cdot \nabla) T_a^\lambda T_e^\lambda \, dx\,dy\,dz | \\
&\le C(\xi_2) \|\nabla T_a^\lambda\|^2_{L^\infty(D)} \|u_e^\lambda\|^2_{L^2(D)} + \xi_2 \|T_e^\lambda\|^2_{L^2(D)},
\end{aligned} \tag{4.13}
$$

where the constant $C(\xi_2) > 0$ is independent of λ, and ξ_2 is a small constant.

Putting estimates (4.10)-(4.13) into (4.9) yields

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|T_e^\lambda\|^2_{L^2(D)} + \|\nabla T_e^\lambda\|^2_{L^2(D)} \\
&\le C(\xi_1)(C\lambda^4 + C\lambda^2(\eta^2 + \eta + 1)^2 e^{-2\beta\eta}) + (\xi_1 + \xi_2) \|T_e^\lambda\|^2_{L^2(D)} \\
&\quad + C(\xi_2) \|\nabla T_a^\lambda\|^2_{L^\infty(D)} \|u_e^\lambda\|^2_{L^2(D)}.
\end{aligned}
$$

Assuming that ξ_1 is small enough but independent of λ, it follows that

$$
\begin{aligned}
&\frac{d}{dt} \|T_e^\lambda\|^2_{L^2(D)} + \|\nabla T_e^\lambda\|^2_{L^2(D)} \\
&\le 2 \|\nabla T_a^\lambda\|^2_{L^\infty(D)} C(\xi_2) \|u_e^\lambda\|^2_{L^2(D)} \\
&\quad + 2C(\xi_1)(C\lambda^4 + C\lambda^2(\eta^2 + \eta + 1)^2 e^{-2\beta\eta}).
\end{aligned} \tag{4.14}
$$

Integrating (4.14) over $[0, t]$ for $t \in [0, T_*]$ and any fixed $T_* > 0$, we have

$$
\begin{aligned}
&\|T_e^\lambda(t)\|^2_{L^2(D)} + \int_0^t \|\nabla T_e^\lambda\|^2_{L^2(D)} dt \\
&\le \|T_e^\lambda(t = 0)\|^2_{L^2(D)} + C \int_0^t \|u_e^\lambda\|^2_{L^2(D)} dt + C\lambda^3.
\end{aligned} \tag{4.15}
$$

Step 3. Similarly, taking the L^2-inner product of (4.6) with S_e^λ and integrating over D yield

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|S_e^\lambda\|^2_{L^2(D)} &= \int_D \tau \Delta S_e^\lambda S_e^\lambda \, dx\,dy\,dz - \int_D (R^\lambda_{In,S} + R^\lambda_{I,S}) S_e^\lambda \, dx\,dy\,dz \\
&\quad - \int_D \left(u_a^\lambda \cdot \nabla \right) S_e^\lambda S_e^\lambda \, dx\,dy\,dz - \int_D \left(u_e^\lambda \cdot \nabla \right) (S_a^\lambda + S_e^\lambda) S_e^\lambda \, dx\,dy\,dz.
\end{aligned}
$$

Similarly, we derive

$$
\begin{aligned}
&\frac{d}{dt} \|S_e^\lambda\|^2_{L^2(D)} + 2\tau \|\nabla S_e^\lambda\|^2_{L^2(D)} \\
&\le 2 \|\nabla S_a^\lambda\|^2_{L^\infty(D)} C(\xi_4) \|u_e^\lambda\|^2_{L^2(D)} \\
&\quad + 2C(\xi_3)(C\lambda^4 + C\lambda^2(\eta^2 + \eta + 1)^2 e^{-2\beta\eta}).
\end{aligned} \tag{4.16}
$$

Integrating (4.16) over $[0,t]$ for $t\in[0,T_*]$ and any fixed $T_*>0$, we obtain

$$\begin{aligned} &\|S_e^\lambda(t)\|_{L^2(D)}^2+2\tau\int_0^t\|\nabla S_e^\lambda\|_{L^2(D)}^2dt\\ &\le\|S_e^\lambda(t=0)\|_{L^2(D)}^2+C\int_0^t\|u_e^\lambda\|_{L^2(D)}^2dt+C\lambda^3. \end{aligned}\tag{4.17}$$

Step 4. Similarly, testing (4.3) by u_e^λ and integrating over D, it follows that

$$\begin{aligned} &\int_D(\lambda[\partial_t u_e^\lambda+(u_a^\lambda\cdot\nabla)u_e^\lambda+(u_e^\lambda\cdot\nabla)(u_a^\lambda+u_e^\lambda)]+\nabla p_e^\lambda)u_e^\lambda\,dx\,dy\,dz\\ &=\int_D(\Delta u_e^\lambda+\sqrt{Ta}k\times u_e^\lambda\\ &\quad+(R_TT_e^\lambda-R_SS_e^\lambda)k-R_{In,u}^\lambda-R_{I,u}^\lambda)u_e^\lambda\,dx\,dy\,dz. \end{aligned}\tag{4.18}$$

First, taking into account divergence formula, Proposition 3.2, the approximate solution's property (3.69), (4.4) and the boundary condition (3.72), the left-hand side terms of (4.18) can be expressed as

$$\int_D\lambda\partial_t u_e^\lambda u_e^\lambda\,dx\,dy\,dz=\frac{\lambda}{2}\frac{d}{dt}\|u_e^\lambda\|_{L^2(D)}^2,$$

$$\begin{aligned} &\int_D\lambda(u_a^\lambda\cdot\nabla)u_e^\lambda u_e^\lambda\,dx\,dy\,dz\\ &=\int_D\lambda\nabla\cdot\Big(u_a^\lambda\frac{(u_e^\lambda)^2}{2}\Big)\,dx\,dy\,dz-\int_D\lambda\nabla\cdot u_a^\lambda\frac{\left(u_e^\lambda\right)^2}{2}\,dx\,dy\,dz=0, \end{aligned}$$

$$\begin{aligned} &\int_D\lambda(u_e^\lambda\cdot\nabla)(u_a^\lambda+u_e^\lambda)u_e^\lambda\,dx\,dy\,dz\\ &=\int_D\lambda(u_e^\lambda\cdot\nabla)u_a^\lambda u_e^\lambda\,dx\,dy\,dz+\int_D\lambda\nabla\cdot\Big(u_e^\lambda\frac{(u_e^\lambda)^2}{2}\Big)\,dx\,dy\,dz\\ &\quad-\int_D\lambda\nabla\cdot u_e^\lambda\frac{(u_e^\lambda)^2}{2}\,dx\,dy\,dz\\ &=\int_D\lambda\left(u_e^\lambda\cdot\nabla\right)u_a^\lambda u_e^\lambda\,dx\,dy\,dz\\ &\le\Big|\int_D\lambda\left(u_e^\lambda\cdot\nabla\right)u_a^\lambda u_e^\lambda\,dx\,dy\,dz\Big|\\ &\le\lambda\|\nabla u_a^\lambda\|_{L^\infty(D)}\|u_e^\lambda\|_{L^2(D)}^2, \end{aligned}$$

$$\begin{aligned} &\int_D\nabla p_e^\lambda u_e^\lambda\,dx\,dy\,dz\\ &=\int_D\nabla\cdot(p_e^\lambda u_e^\lambda)\,dx\,dy\,dz-\int_D\nabla\cdot u_e^\lambda p_e^\lambda\,dx\,dy\,dz=0. \end{aligned}$$

For the right-hand side terms of (4.18), we have

$$\begin{aligned} &\int_D\Delta u_e^\lambda u_e^\lambda\,dx\,dy\,dz\\ &=\int_D\sum_{i=1}^3(\partial_{xx}+\partial_{yy}+\partial_{zz})u_{ie}^\lambda u_{ie}^\lambda\,dx\,dy\,dz \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \int_0^{L_2} \sum_{i=1}^3 \Big(\partial_x u_{ie}^\lambda u_{ie}^\lambda \big|_{x=0}^{x=L_1} - \int_0^{L_1} (\partial_x u_{ie}^\lambda)^2 dx \Big)\, dy\, dz \\
&\quad + \int_0^1 \int_0^{L_1} \sum_{i=1}^3 \Big(\partial_y u_{ie}^\lambda u_{ie}^\lambda \big|_{y=0}^{y=L_2} - \int_0^{L_2} (\partial_y u_{ie}^\lambda)^2 dy \Big)\, dx\, dz \\
&\quad + \int_0^{L_2} \int_0^{L_1} \sum_{i=1}^3 \Big(\partial_z u_{ie}^\lambda u_{ie}^\lambda \big|_{z=0}^{z=1} - \int_0^{1} (\partial_z u_{ie}^\lambda)^2 dz \Big)\, dx\, dy \\
&= - \int_D (\nabla u_e^\lambda)^2 \, dx\, dy\, dz,
\end{aligned}
$$

where we use $u_e^\lambda = (u_{1e}^\lambda, u_{2e}^\lambda, u_{3e}^\lambda)$.

$$
\begin{aligned}
&\int_D \sqrt{Ta} k \times u_e^\lambda u_e^\lambda \, dx\, dy\, dz \\
&= \int_D \sqrt{Ta} (-u_{2e}^\lambda, u_{1e}^\lambda, 0)(u_{1e}^\lambda, u_{2e}^\lambda, u_{3e}^\lambda)^\top \, dx\, dy\, dz = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\int_D (R_T T_e^\lambda - R_S S_e^\lambda) k u_e^\lambda \, dx\, dy\, dz \\
&\le \Big| \int_D (R_T T_e^\lambda - R_S S_e^\lambda) k u_e^\lambda \, dx\, dy\, dz \Big| \\
&\le \xi_5 \|u_e^\lambda\|_{L^2(D)}^2 + C(\xi_5)(R_T^2 \|T_e^\lambda\|_{L^2(D)}^2 + R_S^2 \|S_e^\lambda\|_{L^2(D)}^2),
\end{aligned}
$$

$$
\begin{aligned}
&- \int_D (R_{In,u}^\lambda + R_{I,u}^\lambda) u_e^\lambda \, dx\, dy\, dz \\
&\le \Big| \int_D (R_{In,u}^\lambda + R_{I,u}^\lambda) u_e^\lambda \, dx\, dy\, dz \Big| \\
&\le \xi_6 \|u_e^\lambda\|_{L^2(D)}^2 + C(\xi_6) \|R_{In,u}^\lambda + R_{I,u}^\lambda\|_{L^2(D)}^2 \\
&\le \xi_6 \|u_e^\lambda\|_{L^2(D)}^2 + C(\xi_6)(C\lambda^4 + C\lambda^4(\eta+1)^2 e^{-2\beta\eta}),
\end{aligned}
$$

where the constants $C(\xi_i) > 0$, $(i = 5, 6)$ are independent on λ, and $\xi_i > 0$, $(i = 5, 6)$ are small constants.

Substituting the above derivation equations into (4.18) yields

$$
\begin{aligned}
&\frac{\lambda}{2} \frac{d}{dt} \|u_e^\lambda\|_{L^2(D)}^2 + \|\nabla u_e^\lambda\|_{L^2(D)}^2 \\
&\le \lambda \|\nabla u_a^\lambda\|_{L^\infty(D)} \|u_e^\lambda\|_{L^2(D)}^2 + (\xi_5 + \xi_6) \|u_e^\lambda\|_{L^2(D)}^2 \\
&\quad + C(\xi_5)(R_T^2 \|T_e^\lambda\|_{L^2(D)}^2 + R_S^2 \|S_e^\lambda\|_{L^2(D)}^2) + C(\xi_6)(C\lambda^4 + C\lambda^4(\eta+1)^2 e^{-2\beta\eta}).
\end{aligned}
$$

By restricting λ to be small enough satisfying $\lambda \|\nabla u_a^\lambda\|_{L^\infty(D)} \le C\lambda \le 1/4$ and taking ξ_5, ξ_6 to be small enough $(\xi_5 + \xi_6 = 1/4)$ but independent of λ, one obtains

$$
\begin{aligned}
\lambda \frac{d}{dt} \|u_e^\lambda\|_{L^2(D)}^2 + \|\nabla u_e^\lambda\|_{L^2(D)}^2 &\le 2C(\xi_5)(R_T^2 \|T_e^\lambda\|_{L^2(D)}^2 + R_S^2 \|S_e^\lambda\|_{L^2(D)}^2) \\
&\quad + 2C(\xi_6)(C\lambda^4 + C\lambda^4(\eta+1)^2 e^{-2\beta\eta}),
\end{aligned} \tag{4.19}
$$

that is,

$$
\begin{aligned}
\lambda \frac{d}{dt} \|u_e^\lambda\|_{L^2(D)}^2 + \|u_e^\lambda\|_{L^2(D)}^2 &\le 2C(\xi_5)(R_T^2 \|T_e^\lambda\|_{L^2(D)}^2 + R_S^2 \|S_e^\lambda\|_{L^2(D)}^2) \\
&\quad + 2C(\xi_6)(C\lambda^4 + C\lambda^4(\eta+1)^2 e^{-2\beta\eta}),
\end{aligned}
$$

i.e.,

$$\begin{aligned}\frac{d}{dt}(e^{\frac{t}{\lambda}}\|u_e^\lambda\|^2_{L^2(D)}) &\le [2C(\xi_5)(R_T^2\|T_e^\lambda\|^2_{L^2(D)}+R_S^2\|S_e^\lambda\|^2_{L^2(D)})\\ &\quad+2C(\xi_6)(C\lambda^4+C\lambda^4(\eta+1)^2e^{-2\beta\eta})]\lambda^{-1}e^{\frac{t}{\lambda}}.\end{aligned}\tag{4.20}$$

Integrating (4.20) over $[0,t]$ for $t\in[0,T_*]$ and any fixed $T_*>0$, we have

$$\begin{aligned}\|u_e^\lambda(t)\|^2_{L^2(D)} &\le \|u_e^\lambda(t=0)\|^2_{L^2(D)}+2C(\xi_5)(R_T^2\|T_e^\lambda(t)\|^2_{L^\infty(0,t;L^2(D))}\\ &\quad+R_S^2\|S_e^\lambda(t)\|^2_{L^\infty(0,t;L^2(D))})+2C(\xi_6)C\lambda^4.\end{aligned}\tag{4.21}$$

Step 5. Combining (4.15), (4.17) and (4.21), and restricting ξ_2 to be small enough independent of λ yields

$$\|u_e^\lambda(t)\|^2_{L^2(D)}\le C\|(u_e^\lambda,T_e^\lambda,S_e^\lambda)(t=0)\|^2_{L^2(D)}+C\|u_e^\lambda(t)\|^2_{L^2(0,t;L^2(D))}+C\lambda^3.$$

Using Gronwall's lemma and (2.4) yields

$$\|u_e^\lambda(t)\|^2_{L^2(0,t;L^2(D))}\le C\lambda^3,\tag{4.22}$$

$$\|u_e^\lambda(t)\|^2_{L^\infty(0,T_*;L^2(D))}\le C\lambda^3.\tag{4.23}$$

Inserting (4.22) into (4.15) and (4.17) yields

$$\|T_e^\lambda(t)\|^2_{L^2(D)}+\int_0^t\|\nabla T_e^\lambda\|^2_{L^2(D)}dt\le C\lambda^3,\tag{4.24}$$

$$\|S_e^\lambda(t)\|^2_{L^2(D)}+2\tau\int_0^t\|\nabla S_e^\lambda\|^2_{L^2(D)}dt\le C\lambda^3.\tag{4.25}$$

From the two inequalities above, we have

$$\|T_e^\lambda(t)\|^2_{L^\infty(0,T_*;L^2(D))}\le C\lambda^3,\tag{4.26}$$

$$\|S_e^\lambda(t)\|^2_{L^\infty(0,T_*;L^2(D))}\le C\lambda^3,\tag{4.27}$$

$$\|T_e^\lambda(t)\|^2_{L^2(0,T_*;H^1(D))}\le C\lambda^3,\tag{4.28}$$

$$\|S_e^\lambda(t)\|^2_{L^2(0,T_*;H^1(D))}\le C\lambda^3.\tag{4.29}$$

Simple computations yield

$$\begin{aligned}&\lambda\|u_e^\lambda(t)\|^2_{L^2(D)}+\int_0^t\|\nabla u_e^\lambda\|^2_{L^2(D)}dt\\ &\le \lambda\|u_e^\lambda(t=0)\|^2_{L^2(D)}+2C(\xi_5)(R_T^2\|T_e^\lambda(t)\|^2_{L^2(0,t;L^2(D))}\\ &\quad+R_S^2\|S_e^\lambda(t)\|^2_{L^2(0,t;L^2(D))})+C\lambda^4.\end{aligned}\tag{4.30}$$

Inserting (4.26)-(4.29) into (4.30) yields

$$\lambda\|u_e^\lambda(t)\|^2_{L^2(D)}+\int_0^t\|\nabla u_e^\lambda\|^2_{L^2(D)}dt\le C\lambda^3.$$

Hence,

$$\|u_e^\lambda(t)\|^2_{L^2(0,T_*;H^1(D))}\le C\lambda^3.\tag{4.31}$$

Estimates (4.23), (4.26)-(4.29) and (4.31) lead to (4.1)-(4.2) in Theorem 4.1. □

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