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WEAK SOLUTION BY THE SUB-SUPERSOLUTION METHOD FOR A NONLOCAL SYSTEM INVOLVING LEBESGUE GENERALIZED SPACES

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Abstract. We consider a system of nonlocal elliptic equations

$$\begin{aligned}
&-\mathcal{A}(x,|v|_{L^{r_1(x)}})\operatorname{div}(a_1(|\nabla u|^{p_1(x)})|\nabla u|^{p_1(x)-2}\nabla u)\\
&= f_1(x,u,v)|\nabla v|^{\alpha_1(x)}_{L^{q_1(x)}} + g_1(x,u,v)|\nabla v|^{\gamma_1(x)}_{L^{s_1(x)}},\\
&-\mathcal{A}(x,|u|_{L^{r_2(x)}})\operatorname{div}(a_2(|\nabla v|^{p_2(x)})|\nabla u|^{p_2(x)-2}\nabla u)\\
&= f_2(x,u,v)|\nabla u|^{\alpha_2(x)}_{L^{q_2(x)}} + g_2(x,u,v)|\nabla u|^{\gamma_2(x)}_{L^{s_2(x)}},
\end{aligned}$$

with Dirichlet boundary condition, where Ω is a bounded domain in $\mathbb{R}^N$ ($N > 1$) with C^2 boundary. Using sub-supersolution method, we prove the existence of at least one positive weak solution. Also, we study a generalized logistic equation and a sublinear system.

1. Introduction

Partial differential equations involving the $p(x)$-Laplacian arise in several areas of science and technology such as nonlinear elasticity, fluid mechanics, non-Newtonian fluids and image processing (see [8, 31, 37, 38]). In the previous decades there have been several works related to the p and $p(x)$-Laplacian operators; see [1, 2, 6, 16, 17, 19, 20, 21, 22, 23, 29, 30, 34, 35, 40] and the references therein.

Nonlocal problems including Laplace operator have been intensively studied since their first appearance in the work of Kirchhoff [27] who studied a wave equation which is a generalization of the D'Alembert equation. On this subject the reader may also consult Carrier [5] and Lions [28].

However, non-local problems are not restricted to mechanical motivations as in the aforementioned works. They also appear in a wide variety of applications as population dynamics [9, 10, 12], Ohmic heating [26], the formation of shear bands in materials [32], heat transfer in thermistors [25], combustion theory [33], microwave heating of ceramic materials [3].

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Here, we consider the nonlocal system

$$
\begin{aligned}
&-\mathcal{A}(x,|v|_{L^{r_1(x)}})\operatorname{div}\Big(a(|\nabla u|^{p_1(x)})|\nabla u|^{p_1(x)-2}\nabla u\Big)\\
&= f_1(x,u,v)|v|^{\alpha_1(x)}_{L^{q_1(x)}} + g_1(x,u,v)|v|^{\gamma_1(x)}_{L^{s_1(x)}} \quad \text{in } \Omega,\\
&-\mathcal{A}(x,|u|_{L^{r_2(x)}})\operatorname{div}\Big(a(|\nabla v|^{p_2(x)})|\nabla v|^{p_2(x)-2}\nabla v\Big)\\
&= f_2(x,u,v)|u|^{\alpha_2(x)}_{L^{q_2(x)}} + g_2(x,u,v)|u|^{\gamma_2(x)}_{L^{s_2(x)}} \quad \text{in } \Omega,\\
&\qquad u=v=0 \quad \text{on } \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N>1)$ with C^2 boundary, $|\cdot|_{L^{m(x)}}$ is the norm of the space $L^{m(x)}(\Omega)$, $-\Delta_{p(x)}u := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, $r_i, p_i, q_i, s_i, \alpha_i, \gamma_i : \Omega \to [0,\infty), i=1,2$ are measurable functions and $\mathcal{A}, f_1, f_2, g_1, g_2 : \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying certain conditions. To be more specific about the structure of the operator in (1.1), we consider functions $a:\mathbb{R}^+\to\mathbb{R}^+$ of class C^1 satisfying the following conditions:

(A1) There exist constants $k_1,k_2,k_3,k_4\geq 0$, $2<p_i\leq o_i<N$ such that

$$
k_1t^{p_i}+k_2t^{o_i}\leq a(t^{p_i})t^{p_i}\leq k_3t^{p_i}+k_4t^{o_i}, \quad \text{for all } t\geq 0.
$$

(A2) The function $t\mapsto A_i(t^{p_i})$ is strictly convex, where $A_i(t)=\int_0^t a_i(s)ds$.

(A3) The function $t\mapsto a_i(t^{p_i})t^{p_i-2}$ is increasing.

Various operators occurring in applications are included in models for the boundary value problem (1.1) as one can see from next examples. The following operators satisfy (A1)–(A3):

(i) If $a_i(t)=1$ for $i=1,2$, we obtain the p-Laplacian and problem (1.1) becomes

$$
\begin{gathered}
-\mathcal{A}(x,|v|_{L^{r_1(x)}})\Delta_{p_1}u = f_1(x,u,v)|\nabla v|^{\alpha_1(x)}_{L^{q_1(x)}} + g_1(x,u,v)|\nabla v_1|^{\gamma_1(x)}_{L^{s_1(x)}} \quad \text{in } \Omega,\\
-\mathcal{A}(x,|u|_{L^{r_2(x)}})\Delta_{p_2}v = f_2(x,u,v)|\nabla u|^{\alpha_2(x)}_{L^{q_2(x)}} + g_2(x,u,v)|\nabla u|^{\gamma_2(x)}_{L^{s_2(x)}} \quad \text{in } \Omega,\\
u=v=0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

with $q_i=p_i$, $k_1+k_2=1$ and $k_3+k_4=1$.

(ii) If $a_i(t)=1+t^{\frac{o_i-p_i}{p_i}}$ for $i=1,2$, we obtain the (p,o)-Laplacian or $p\&o$-Laplacian and problem (1.1) becomes

$$
\begin{gathered}
-\mathcal{A}(x,|v|_{L^{r_1(x)}})\left(\Delta_{p_1}u+\Delta_{o_1}u\right) = f_1(x,u,v)|\nabla v|^{\alpha_1(x)}_{L^{q_1(x)}} + g_1(x,u,v)|\nabla v_1|^{\gamma_1(x)}_{L^{s_1(x)}} \quad \text{in } \Omega,\\
-\mathcal{A}(x,|u|_{L^{r_2(x)}})\left(\Delta_{p_2}v+\Delta_{o_2}v\right) = f_2(x,u,v)|\nabla u|^{\alpha_2(x)}_{L^{q_2(x)}} + g_2(x,u,v)|\nabla u|^{\gamma_2(x)}_{L^{s_2(x)}} \quad \text{in } \Omega,\\
u=v=0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

with $k_1=k_2=k_3=k_4=1$.

(iii) If $a_i(t)=1+\frac{1}{(1+t)^{\frac{p_i-2}{p_i}}}$ for $i=1,2$, we obtain

$$
\begin{aligned}
&-\mathcal{A}(x,|v|_{L^{r_1(x)}})\operatorname{div}\Big|\nabla u|^{p_1-2}\nabla u + \frac{|\nabla u|^{p_1-2}\nabla u}{(1+|\nabla u|^{p_1})^{\frac{p_1-2}{p_1}}}\Big)\\
&= f_1(x,u,v)|\nabla v|^{\alpha_1(x)}_{L^{q_1(x)}} + g_1(x,u,v)|\nabla v|^{\gamma_1(x)}_{L^{s_1(x)}} \quad \text{in } \Omega,\\
&-\mathcal{A}(x,|u|_{L^{r_2(x)}})\operatorname{div}\Big(|\nabla v|^{p_2-2}\nabla v + \frac{|\nabla v|^{p_2-2}\nabla v}{(1+|\nabla v|^{p_2})^{\frac{p_2-2}{p_2}}}\Big)\\
&= f_2(x,u,v)|\nabla u|^{\alpha_2(x)}_{L^{q_2(x)}} + g_2(x,u,v)|\nabla u|^{\gamma_2(x)}_{L^{s_2(x)}} \quad \text{in } \Omega,
\end{aligned}
$$

$$u = v = 0 \quad \text{on } \partial\Omega,$$

with $q_i = p_i$, $k_1 + k_2 = 1$, and $k_3 + k_4 = 2$.

(iv) If $a_i(t) = 1 + t^{\frac{o_i - p_i}{p_i}} + \frac{1}{(1+t)^{\frac{p_i-2}{p_i}}}$ for $i = 1, 2$, we obtain

$$\begin{gathered}
\begin{aligned}
&- \mathcal{A}(x, |v|_{L^{r_1(x)}})\Big(\Delta_{p_1} u + \Delta_{o_1} u + \operatorname{div}\Big(\frac{|\nabla u|^{p_1-2}\nabla u}{(1+|\nabla u|^{p_1})^{\frac{p_1-2}{p_1}}}\Big)\Big) \\
&= f_1(x,u,v)|\nabla v|^{\alpha_1(x)}_{L^{q_1(x)}} + g_1(x,u,v)|\nabla v|^{\gamma_1(x)}_{L^{s_1(x)}} \quad \text{in } \Omega, \\
&- \mathcal{A}(x, |u|_{L^{r_2(x)}})\Big(\Delta_{p_2} v + \Delta_{o_2} v + \operatorname{div}\Big(\frac{|\nabla v|^{p_2-2}\nabla v}{(1+|\nabla v|^{p_2})^{\frac{p_2-2}{p_2}}}\Big)\Big) \\
&= f_2(x,u,v)|\nabla u|^{\alpha_2(x)}_{L^{q_2(x)}} + g_2(x,u,v)|\nabla u|^{\gamma_2(x)}_{L^{s_2(x)}} \quad \text{in } \Omega,
\end{aligned} \\
u = v = 0 \quad \text{on } \partial\Omega,
\end{gathered}$$

with $k_1 = k_2 = k_4 = 1$ and $k_3 = 2$.

Several works related to (1.1) in the p-Laplacian case, that is, with $p(x) = p$ (a constant) can be found in [4, 7, 14, 15, 18, 24, 39] and their references. Chen et al. [7] proved the existence of positive solutions for a class of nonvariational elliptic system with nonlocal source

$$\begin{gathered}
-\Delta u^m = f_1(x,u)|v|^{\alpha}_{L^p} \quad \text{in } \Omega, \\
-\Delta v^n = f_2(x,v)|u|^{\beta}_{L^q} \quad \text{in } \Omega, \\
u > 0, v > 0 \quad \text{in } \Omega, \\
u = v = 0 \quad \text{on } \partial\Omega.
\end{gathered}$$

using the Galerkin method, a fixed point theorem in finite dimensions, and sub-supersolution technique. Corrêa et al [14] studied the existence of positive solutions for the nonlocal problem

$$\begin{gathered}
-\Delta_{p_1} u = |v|^{\alpha_1}_{L^{q_1}} \quad \text{in } \Omega, \\
-\Delta_{p_2} v = |u|^{\alpha_2}_{L^{q_2}} \quad \text{in } \Omega, u = v = 0 \quad \text{on } \partial\Omega,
\end{gathered}$$

by using Rabinowitz's theorem [36]. Santos et al [39] studiedthe system

$$\begin{gathered}
-\mathcal{A}(x, |v|_{L^{r_1(x)}})\Delta u = f_1(x,u,v)|v|^{\alpha_1(x)}_{L^{q_1(x)}} + g_1(x,u,v)|v|^{\gamma_1(x)}_{L^{s_1(x)}} \quad \text{in } \Omega, \\
-\mathcal{A}(x, |u|_{L^{r_2(x)}})\Delta u = f_2(x,u,v)|u|^{\alpha_2(x)}_{L^{q_2(x)}} + g_2(x,u,v)|u|^{\gamma_2(x)}_{L^{s_2(x)}} \quad \text{in } \Omega, \\
u = v = 0 \quad \text{on } \partial\Omega,
\end{gathered}$$

where $\mathcal{A} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a function satisfying certain conditions. They use an abstract result involving sub-supersolution, whose proof is based on the Schaefer's fixed point theorem. Specifically, it was considered a sublinear system, a concave-convex problem and a system of logistic equations.

The scalar version of (1.1),

$$\begin{gathered}
-\mathcal{A}(x, |u|_{L^{r(x)}})\Delta_{p(x)} u = f(x,u)|u|^{\alpha(x)}_{L^{q(x)}} + g(x,u)|u|^{\gamma(x)}_{L^{s(x)}} \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.2}$$

was considered in [40]. The authors obtained an abstract result involving sub and super solutions for (1.1) that generalizes [39, Theorem 1]. As an application of such result the authors presented three applications of [39, Theorem 1] for the $p(x)$-Laplacian operator.

The main result of this paper proves the existence of at least one weak positive solution for (1.1) via sub-supersolution method. This is an extension of [39, Theorem 2] and [41, Theorem 1.1] for the $p(x)$-Laplacian operator.

2. Function spaces

Here, we introduce a suitable function space, where the solution of problem (1.1) make sense. Next we recall some facts about the known spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ (see [21] and the references therein for more details).

Let $\Omega \subset \mathbb{R}^N$, $(N \geq 1)$ be a bounded domain. Given $p \in L_+^\infty(\Omega)$, the generalized Lebesgue space is

$$L^{p(x)}(\Omega) := \big\{u \in \mathcal{S}(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < \infty\big\},$$

where $\mathcal{S}(\Omega) := \big\{u : \Omega \to \mathbb{R} : u \text{ is measurable}\big\}$. The $L^{p(x)}(\Omega)$ is a Banach space with the norm

$$|u|_{p(x)} := \inf \big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\big\}.$$

Given $m \in L^\infty(\Omega)$, we define

$$m^+ := \operatorname{ess\,sup}_\Omega m(x), \quad m^- := \operatorname{ess\,inf}_\Omega m(x).$$

Proposition 2.1. *Let $\rho(u) := \int_\Omega |u|^{p(x)} dx$. Then for $u, u_n \in L^{p(x)}(\Omega)$, and $n \in \mathbb{N}$, the following assertions hold*

(i) *Let $u \neq 0$ in $L^{p(x)}(\Omega)$, then $|u|_{L^{p(x)}} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$.*
(ii) *If $|u|_{L^{p(x)}} < 1$ $(= 1, > 1)$, then $\rho(u) < 1$ $(= 1, > 1)$.*
(iii) *If $|u|_{L^{p(x)}} > 1$, then $|u|^{p^-}_{L^{p(x)}} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}}$.*
(iv) *If $|u|_{L^{p(x)}} < 1$, then $|u|^{p^+}_{L^{p(x)}} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}}$.*
(v) *$|u_n|_{L^{p(x)}} \to 0 \Leftrightarrow \rho(u_n) \to 0$, and $|u_n|_{L^{p(x)}} \to \infty \Leftrightarrow \rho(u_n) \to \infty$.*

Theorem 2.2. *Assume $p, q \in L_+^\infty(\Omega)$. The following statements hold*

(i) *If $p^- > 1$ and $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ a.e. in Ω, then*
$$\Big|\int_\Omega uv dx\Big| \leq \big(\frac{1}{p^-} + \frac{1}{q^-}\big)|u|_{L^{p(x)}}|v|_{L^{q(x)}}.$$
(ii) *If $q(x) \leq p(x)$ a.e. in Ω and $|\Omega| < \infty$, then $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

One can define the generalized Sobolev space

$$W^{1,p(x)}(\Omega) := \big\{u \in L^{p(x)}(\Omega) : \frac{\partial u}{\partial x_j} \in L^{p(x)}(\Omega), j = 1, \dots, N\big\}$$

with the norm

$$\|u\|_* = |u|_{L^{p(x)}} + \sum_{j=1}^N \Big|\frac{\partial u}{\partial x_j}\Big|_{L^{p(x)}}.$$

The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_*$.

Theorem 2.3. *If $p^- > 1$, then $W^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space.*

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $p, q \in C(\overline{\Omega})$. Define the function $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = \infty$ if $N \geq p(x)$. The following statements are hold.*

(i) *(Poincaré inequality) If $p^- > 1$, then there is a constant $C > 0$ such that $|u|_{L^{p(x)}} \leq C|\nabla u|_{L^{p(x)}}$ for all $u \in W_0^{1,p(x)}(\Omega)$.*

(ii) *If $p^-, q^- > 1$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.*

By part (i) of Proposition 2.4, $\|u\| := |\nabla u|_{L^{p(x)}}$ defines a norm in $W_0^{1,p(x)}(\Omega)$ which is equivalent to the norm $\|\cdot\|_*$.

Definition 2.5. For $u, v \in W^{1,p(x)}(\Omega)$, we say that $-\Delta_{p(x)} u \leq -\Delta_{p(x)} v$, if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \leq \int_\Omega |\nabla v|^{p(x)-2} \nabla v \nabla \varphi,$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$ with $\varphi \geq 0$.

The following result appears in [23, Lemma 2.2] and [20, Proposition 2.3].

Proposition 2.6. *Let $u, v \in W^{1,p(x)}(\Omega)$. If $-\Delta_{p(x)} u \leq -\Delta_{p(x)} v$ and $u \leq v$ on $\partial\Omega$, (i.e., $(u-v)^+ \in W_0^{1,p(x)}(\Omega)$) then $u \leq v$ in Ω. If $u, v \in C(\overline{\Omega})$ and $S = \big\{x \in \Omega : u(x) = v(x)\big\}$ is a compact set of Ω, then $S = \emptyset$.*

Next we recall [20, Lemma 2.1].

Lemma 2.7. *Let $\lambda > 0$ be the unique solution of the problem*

$$\begin{gathered} -\Delta_{p(x)} z_\lambda = \lambda \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

Define $\rho_0 = \frac{p^-}{2|\Omega|^{\frac{1}{N}} C_0}$. If $\lambda \geq \rho_0$ then $|z_\lambda|_{L^\infty} \leq C^* \lambda^{\frac{1}{p^- - 1}}$, and $|z_\lambda|_{L^\infty} \leq C_* \lambda^{\frac{1}{p^+ - 1}}$ if $\lambda < \rho_0$. Here C^* and C_* are positive constants depending only on $p^+, p^-, N, |\Omega|$ and C_0, where C_0 is the best constant of the embedding $W_0^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$.

Regarding the function z_λ of the previous result, it follows from [19, Theorem 1.2] and [23, Theorem 1] that $z_\lambda \in C^1(\overline{\Omega})$ with $z_\lambda > 0$ in Ω. The proof of Theorem 3.5 is mainly based on the following result by Rabinowitz [36].

Theorem 2.8. *Let E be a Banach space and $\Phi : \mathbb{R}^+ \times E \to E$ a compact map such that $\Phi(0, u) = 0$ for all $u \in E$. Then the equation*

$$u = \Phi(\lambda, u)$$

possesses an unbounded continuum $\mathcal{C} \subset \mathbb{R}^+ \times E$ of solutions with $(0, 0) \in \mathcal{C}$.

We point out that a mapping $\Phi : E \to E$ is compact if it is continuous and for each bounded subset $U \subset E$, the set $\overline{\Phi(U)}$ is compact.

3. Existence of solutions

In this section, we prove Theorem 3.5 which shows the existence of at least one weak solution for system (1.1), via new sub-supersolution method. For this, we recall preliminaries.

Definition 3.1. The pair (u_1, u_2) is called a weak solution of (1.1), if $u_i \in W_0^{1,p_i(x)}(\Omega) \cap L^\infty(\Omega)$ and

$$\begin{aligned}
&\int_\Omega a_1(|\nabla u_1|^{p_1(x)})|\nabla u_1|^{p_1(x)-2}\nabla u_1 \nabla\varphi dx \\
&= \int_\Omega \Big(\frac{f_1(x,u_1,u_2)|u_2|_{L^{q_1(x)}}^{\alpha_1(x)}}{\mathcal{A}(x,|u_2|_{L^{r_1(x)}})} + \frac{g_1(x,u_1,u_2)|u_2|_{L^{s_1(x)}}^{\gamma_1(x)}}{\mathcal{A}(x,|u_2|_{L^{r_1(x)}})}\Big)\varphi dx, \\
&\int_\Omega a_2(|\nabla u_2|^{p_2(x)})|\nabla u_2|^{p_2(x)-2}\nabla u_2 \nabla\varphi dx \\
&= \int_\Omega \Big(\frac{f_2(x,u_1,u_2)|u_1|_{L^{q_2(x)}}^{\alpha_2(x)}}{\mathcal{A}(x,|u_1|_{L^{r_2(x)}})} + \frac{g_2(x,u_1,u_2)|u_1|_{L^{s_2(x)}}^{\gamma_2(x)}}{\mathcal{A}(x,|u_1|_{L^{r_2(x)}})}\Big)\varphi dx,
\end{aligned}$$

for all $\varphi \in W_0^{1,p_i(x)}(\Omega)$ and $i \neq j$ with $i, j = 1, 2$.

Given $u, v \in \mathcal{S}(\Omega)$ we write $u \leq v$ if $u(x) \leq v(x)$ a.e. in Ω. If $u \leq v$ we define

$$[u, v] := \big\{w \in \mathcal{S}(\Omega) : u(x) \leq w(x) \leq v(x) \text{ a.e. in } \Omega\big\}.$$

To simplify notation in the next definition we denote

$$\begin{aligned}
\widetilde{f}_1(x,t,s) = f_1(x,t,s), \quad \widetilde{g}_1(x,t,s) = g_1(x,t,s), \\
\widetilde{f}_2(x,t,s) = f_2(x,s,t), \quad \widetilde{g}_2(x,t,s) = g_2(x,s,t).
\end{aligned}$$

Definition 3.2. The pairs $(\underline{u}_i, \overline{u}_i)$, $i = 1, 2$ are called sub-supersolutions for (1.1) if $\underline{u}_i \in W_0^{1,p_i(x)}(\Omega) \cap L^\infty(\Omega)$, $\overline{u}_i \in W^{1,p_i(x)}(\Omega) \cap L^\infty(\Omega)$ with $\underline{u}_i \leq \overline{u}_i$, $\underline{u}_i = 0 \leq \overline{u}_i$ on $\partial\Omega$ and for all $\varphi \in W_0^{1,p_i(x)}(\Omega)$ with $\varphi \geq 0$ the following inequalities hold

$$\begin{aligned}
&\int_\Omega a_i(|\nabla \underline{u}_i|^{p_i(x)})|\nabla \underline{u}_i|^{p_i(x)-2}\nabla \underline{u}_i \nabla\varphi dx \\
&\leq \int_\Omega \Big(\frac{\widetilde{f}_i(x,\underline{u}_i,w)|\underline{u}_j|_{L^{q_i(x)}}^{\alpha_i(x)}}{\mathcal{A}(x,|w|_{L^{r_i(x)}})} + \frac{\widetilde{g}_i(x,\underline{u}_i,w)|\underline{u}_j|_{L^{s_i(x)}}^{\gamma_i(x)}}{\mathcal{A}(x,|w|_{L^{r_i(x)}})}\Big)\varphi dx, \\
&\int_\Omega |a_i(|\nabla \overline{u}_i|^{p_i(x)})|\nabla \overline{u}_i|^{p_i(x)-2}\nabla \overline{u}_i \nabla\varphi dx \\
&\geq \int_\Omega \Big(\frac{\widetilde{f}_i(x,\overline{u}_i,w)|\overline{u}_j|_{L^{q_i(x)}}^{\alpha_i(x)}}{\mathcal{A}(x,|w|_{L^{r_i(x)}})} + \frac{\widetilde{g}_i(x,\overline{u}_i,w)|\overline{u}_j|_{L^{s_i(x)}}^{\gamma_i(x)}}{\mathcal{A}(x,|w|_{L^{r_i(x)}})}\Big)\varphi dx,
\end{aligned} \tag{3.1}$$

for all $w \in [\underline{u}_j \overline{u}_j]$ where $i, j = 1, 2$ with $i \neq j$.

Remark 3.3. The space $L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$ with the norm

$$|(u,v)|_{1,2} = |u|_{L^{p_1(x)}} + |v|_{L^{p_2(x)}}.$$

is a Banach space.

In what follows, we study the existence and uniqueness of solution for

$$\begin{gathered}
-\operatorname{div}(a_1(|\nabla u|^{p_1(x)})|\nabla u|^{p_1(x)-2}\nabla u) = G_1(z_1, z_2) \quad \text{in } \Omega, \\
-\operatorname{div}(a_2(|\nabla v|^{p_2(x)})|\nabla v|^{p_2(x)-2}\nabla v) = G_2(z_1, z_2) \quad \text{in } \Omega, \\
u = v = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.2}$$

where $(z_1, z_2) \in L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth bounded domain and $a_i : \mathbb{R}^+ \to \mathbb{R}^+$ be a C^1 function satisfying (A1), (A2) and (a_3). Assume $G_i : L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega) \to L^{p_i'(x)}(\Omega)$, where $p_i'(x) = \frac{p_i(x)}{(p_i(x)-1)}$, $G_i(z_1, z_2)$ are continuous and $|G_i(z_1, z_2)| \leq K_i$ for all $(z_1, z_2) \in L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$. Then problem* (3.2) *has a unique solution $(u, v) \in W_0^{1,l_1(x)}(\Omega) \times W_0^{1,l_2(x)}(\Omega)$.*

Proof. Consider the functional $\mathfrak{I} : W_0^{1,l_1(x)}(\Omega) \times W_0^{1,l_2(x)}(\Omega) \to \mathbb{R}$ defined as

$$\begin{aligned} \mathfrak{I}(u,v) = \frac{1}{p_1} \int_\Omega A(|\nabla u|^{p_1(x)})dx + \frac{1}{p_2} \int_\Omega A(|\nabla v|^{p_2(x)})dx \\ - \int_\Omega G_1(z,z)udx - \int_\Omega G_2(z,z)vdx. \end{aligned} \tag{3.3}$$

From (A1) the functional (3.3) is well defined and so $\mathfrak{I} \in C^1(W_0^{1,q_1}(\Omega) \times W_0^{1,q_2}(\Omega), \mathbb{R})$. Notice that, (A2) implies that $\mathfrak{I}$ is strictly convex and weakly lower semicontinuous. Also, (A1), $|G_i(z_1, z_2)| \leq K_i$ and Hölder's inequality imply

$$\begin{aligned} \mathfrak{I}(u,v) \geq \frac{k_1}{p_1^-} \|u\|^{p_1(x)}_{W_0^{1,p_1(x)}(\Omega)} + \frac{k_1}{p_2^-} \|v\|^{p_2(x)}_{W_0^{1,p_2(x)}(\Omega)} + \frac{k_2}{l_1^-} \|u\|^{l_1(x)}_{W_0^{1,l_1(x)}(\Omega)} \\ + \frac{k_2}{l_2^-} \|v\|^{l_2(x)}_{W_0^{1,l_2(x)}(\Omega)} - K_0 C\big(\|u\|_{W_0^{1,l_1(x)}(\Omega)} + \|v\|_{W_0^{1,l_2(x)}(\Omega)}\big) \end{aligned}$$

for $C > 0$ and all $(u, v) \in W_0^{1,l_1(x)}(\Omega) \times W_0^{1,l_2(x)}(\Omega)$ with $\rho(|\nabla u|), \rho(|\nabla v|) \geq 1$, what shows that $\mathfrak{I}$ is coercive. Hence $\mathfrak{I}$ has a unique critical point (a global minimizer), which is the unique solution to (3.2). $\square$

To state the main result of this article we need the following assumptions on $r_i, p_i, q_i, s_i, \alpha_i, \gamma_i$ in (1.1):

(A4) $p_i \in C^1(\overline{\Omega}), r_i, q_i, s_i \in L^\infty_+(\Omega)$, where
$$L^\infty_+(\Omega) = \big\{m \in L^\infty(\Omega) \text{ with} \operatorname{ess\,inf} m(x) \geq 1\big\}$$
and for $i = 1, 2$, we have $\alpha_i, \gamma_i \in L^\infty(\Omega)$ and
$$1 < p_i^- := \inf_\Omega p_i(x) \leq p_i^+ := \sup_\Omega p_i(x) < N, \quad \alpha_i(x), \gamma_i(x) \geq 0 \quad \text{a.e. in } \Omega\,.$$

We set

$$\begin{gathered} \underline{\sigma} := \min\big\{|\underline{w}|_{L^{r_i(x)}}, \text{ for} i = 1, 2\big\}, \quad \overline{\sigma} := \max\big\{|\overline{w}|_{L^{r_i(x)}}, \text{ for } i = 1, 2\big\}, \\ \underline{w} := \min\{\underline{u}_i, \text{ for } i = 1, 2\}, \quad \overline{w} := \max\{\overline{u}_i, \text{ for } i = 1, 2\}. \end{gathered} \tag{3.4}$$

Theorem 3.5. *Assume*

- $r_i, p_i, q_i, s_i, \alpha_i$ *and* γ_i *satisfy* (A4),
- $(\underline{u}_i, \overline{u}_i)$ *is a sub-supersolution for* (1.1) *with* $\underline{u}_i > 0$ *a.e. in* Ω,
- $f_i(x,t,s), g_i(x,t,s) \geq 0$ *in* $\overline{\Omega} \times [0, |\overline{u}_1|_{L^\infty}] \times [0, |\overline{u}_2|_{L^\infty}]$,
- $\mathcal{A} : \overline{\Omega} \times (0, \infty) \to \mathbb{R}$ *is a continuous function with* $\mathcal{A}(x,t) > 0$ *in* $\overline{\Omega} \times \big[\underline{\sigma}, \overline{\sigma}\big]$,
- $a_i : \mathbb{R}^+ \to \mathbb{R}^+$ *is a* C^1 *function satisfying (A1), (A2) and* (a_3).

Then (1.1) *has at least one weak positive solution* (u_1, u_2) *with* $u_i \in [\underline{u}_i, \overline{u}_i]$, $i = 1, 2$.

Proof. For $i = 1, 2$ consider the operators $T_i : L^{p_i(x)}(\Omega) \to L^\infty(\Omega)$ defined by

$$T_i z(x) = \begin{cases} \underline{u}_i(x), & \text{if } z(x) \leq \underline{u}_i(x), \\ z(x), & \text{if } \underline{u}_i(x) \leq z(x) \leq \overline{u}_i(x), \\ \overline{u}_i(x), & \text{if } z(x) \geq \overline{u}_i(x). \end{cases}$$

Since $T_iz \in [\underline{u}_i, \overline{u}_i]$ and $\underline{u}_i, \overline{u}_i \in L^\infty(\Omega)$ it follows that the operators T_i are well-defined.

We define $p_i'(x) = p_i(x)/\big(p_i(x)-1\big)$ and consider the operators $H_i : [\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2] \to L^{p_i'(x)}(\Omega)$ given by

$$H_i(u_1,u_2)(x) = \frac{f_i(x,u_1(x),u_2(x))|u_j|^{\alpha_i(x)}_{L^{q_i(x)}}}{\mathcal{A}(x,|u_j|_{L^{r_i(x)}})} + \frac{g_i(x,u_1(x),u_2(x))|u_j|^{\gamma_i(x)}_{L^{s_i(x)}}}{\mathcal{A}(x,|u_j|_{L^{r_i(x)}})}$$

where $i \neq j$ with $i,j = 1,2$, and $|\cdot|_{L^{m(x)}}$ denotes the norm of the space $L^{m(x)}(\Omega)$.

Since $f_i, g_i, \mathcal{A}$ are continuous functions, $\mathcal{A}(x,t) > 0$ in the compact set $\overline{\Omega} \times \big[\underline{\sigma}, \overline{\sigma}\big]$, $T_iz_i \in [\underline{u}_i, \overline{u}_i]$ for all $z_i \in L^{p_i(x)}(\Omega)$, $\underline{u}_i, \overline{u}_i \in L^\infty(\Omega)$, and $|w|^{\theta(x)}_{L^{m(x)}} \leq |w|^{\theta^-}_{L^{m(x)}} + |w|^{\theta^+}_{L^{m(x)}}$ for all $w \in L^{m(x)}(\Omega)$ with $\theta \in L^\infty(\Omega)$, it follows that there are constants $K_i > 0$ such that

$$|H_i(T_1z_1, T_2z_2)| \leq K_i \tag{3.5}$$

for all $(z_1, z_2) \in L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$.

By the Lebesgue Dominated Convergence Theorem, the mappings $(z_1, z_2) \mapsto H_i(T_1z_1, T_2z_2)$ from $L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$ to $L^{p_i'(x)}(\Omega)$, $i = 1,2$, are continuous.

The operator $\Phi : \mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega) \to L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$ given by

$$\Phi(\lambda, z_1, z_2) = (u_1, u_2),$$

is well-defined, by [21, Theorem 4.1], where $(u_1, u_2) \in W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$ is the unique solution of

$$\begin{gathered}
-\operatorname{div}(a_1(|\nabla u_1|^{p_1(x)})|\nabla u_1|^{p_1(x)-2}\nabla u_1) = \lambda H_1(T_1z_1, T_2z_2) \quad \text{in } \Omega, \\
-\operatorname{div}(a_2(|\nabla u_2|^{p_2(x)})|\nabla u_2|^{p_2(x)-2}\nabla u_2) = \lambda H_2(T_1z_1, T_2z_2) \quad \text{in } \Omega, \\
u_1 = u_2 = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.6}$$

by Lemma 3.4, where $(z_1, z_2) \in L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$.

Claim 1: Φ is compact. Let $(\lambda_n, z_n^1, z_n^2) \subset \mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$ be a bounded sequence and consider $(u_n^1, u_n^2) = \Phi(\lambda_n, z_n^1, z_n^2)$. The definition of Φ implies that

$$\int_\Omega a_i(|\nabla u_n^i|^{p_i(x)-2})|\nabla u_n^i|^{p_i(x)-2}\nabla u_n^i \nabla\varphi = \lambda_n \int_\Omega H_i(T_1z_n^1, T_2z_n^2)\varphi,$$

for all $\varphi \in W_0^{1,p_i(x)}(\Omega)$, where $i,j = 1,2$ blue with $i \neq j$.

Considering the test function $\varphi = u_n^i$, the boundness of (λ_n) and inequality (3.5), we obtain

$$\int_\Omega |\nabla u_n^i|^{p_i(x)} \leq \overline{\lambda} K_i \int_\Omega |u_n^i|$$

for all $n \in \mathbb{N}$. Here $\overline{\lambda}$ is a constant that does not depend on $n \in \mathbb{N}$.

Since $p_i^- > 1$, the embedding $L^{p_i(x)}(\Omega) \hookrightarrow L^1(\Omega)$ is hold. Combining such embedding with the Poincaré inequality we obtain

$$\int_\Omega |\nabla u_n^i|^{p_i(x)} \leq CK_i\|u_n^i\|,$$

for all $n \in \mathbb{N}$. Suppose that $|\nabla u_n^i|_{L^{p_i(x)}} > 1$. Thus by Proposition 2.1 we have $\|u_n^i\|^{p^- -1} \leq CK_i$ for all $n \in \mathbb{N}$ where C is a constant that does not depend on n. Then (u_n^i) is bounded in $W_0^{1,p_i(x)}(\Omega)$. The reflexivity of $W_0^{1,p_i(x)}(\Omega)$ and the compact embedding $W_0^{1,p_i(x)}(\Omega) \hookrightarrow L^{p_i(x)}(\Omega)$ provides the result.

Claim 2: Φ is continuous. Consider a sequence $(\lambda_n, z_n^1, z_n^2)$ in $\mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$ converging to (λ, z^1, z^2) in $\mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$. Define $(u_n^1, u_n^2) = \Phi(\lambda_n, z_n^1, z_n^2)$ and $(u^1, u^2) = \Phi(\lambda, z^1, z^2)$. Using the definition of Φ we obtain

$$\int_\Omega a_i(|\nabla u_n^i|^{p_i(x)-2})|\nabla u_n^i|^{p_i(x)-2}\nabla u_n^i \nabla\varphi = \lambda_n \int_\Omega H_i(T_1 z_n^1, T_2 z_n^2)\varphi, \tag{3.7}$$

$$\int_\Omega a_i(|\nabla u^i|^{p_i(x)-2})|\nabla u^i|^{p_i(x)-2}\nabla u^i \nabla\varphi = \lambda \int_\Omega H_i(T_1 z^1, T_2 z^2)\varphi \tag{3.8}$$

for all $\varphi \in W_0^{1,p_i(x)}(\Omega)$ where $i, j = 1, 2$, and $i \neq j$.

Considering $\varphi = (u_n^i - u^i)$ in (3.7) and (3.8) and subtracting (3.8) from (3.7) we obtain

$$\begin{aligned}
&\int_\Omega \big\langle a(|\nabla u_n^i|^{p_i(x)-2})|\nabla u_n^i|^{p_i(x)-2}\nabla u_n^i \\
&- a(|\nabla u^i|^{p_i(x)-2})|\nabla u^i|^{p_i(x)-2}\nabla u^i, \nabla(u_n^i - u^i)\big\rangle \\
&= \int_\Omega \lambda_n H(T_1 z_n^1, T_2 z_n^2)(u_n^i - u^i) - \int_\Omega \lambda H(T_1 z^1, T_2 z^2)\big](u_n^i - u^i).
\end{aligned}$$

Using Hölder's inequality we have

$$\begin{aligned}
&\big| \int_\Omega \big\langle a(|\nabla u_n^i|^{p_i(x)-2})|\nabla u_n^i|^{p_i(x)-2}\nabla u_n^i \\
&- a(|\nabla u^i|^{p_i(x)-2})|\nabla u|^{p_i(x)-2}\nabla u^i, \nabla(u_n^i - u^i)\big\rangle\big| \\
&\leq |u_n^i - u^i|_{p_i(x)} |\lambda_n H_i(T_1 z_n^1, T_2 z_n^2) - \lambda H_i(T_1 z^1, T_2 z^2)|_{p_i'(x)}
\end{aligned}$$

The arguments above ensures that (u_n^i) is bounded in $W_0^{1,p_i(x)}(\Omega)$. Since $\lambda_n \to \lambda$ and $H_i(T_1 z_n^1, T_2 z_n^2) \to H_i(T_1 z^1, T_2 z^2)$ in $L^{p_i'(x)}(\Omega)$ for $i = 1, 2$ we have

$$\begin{aligned}
&\big| \int_\Omega \big\langle a(|\nabla u_n^i|^{p_i(x)-2})|\nabla u_n^i|^{p_i(x)-2}\nabla u_n^i \\
&- a(|\nabla u^i|^{p_i(x)-2})|\nabla u^i|^{p_i(x)-2}\nabla u^i, \nabla(u_n^i - u^i)\big\rangle\big| \to 0.
\end{aligned}$$

Therefore $u_n^i \to u^i$ in $L^{p_i(x)}(\Omega)$ for $i = 1, 2$ which proves the continuity of Φ.

Combining the fact that $\Phi(0, z_1, z_2) = (0, 0, 0)$ for all $(z_1, z_2) \in L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$ with the previous claims we have by Theorem 2.8 that the equation $\Phi(\lambda, u, v) = (u, v)$ possesses an unbounded continuum $\mathcal{C} \subset \mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$ of solutions with $(0, 0, 0) \in \mathcal{C}$.

Claim 3: $\mathcal{C}$ is bounded with respect to the parameter λ. Suppose that there exists $\lambda^* > 0$ such that $\lambda \leq \lambda^*$ for all $(\lambda, u^1, u^2) \in \mathcal{C}$. For $(\lambda, u^1, u^2) \in \mathcal{C}$ the definition of Φ imply that

$$\begin{gathered}
-\operatorname{div}(a(|\nabla u_1|^{p_1(x)-2})|\nabla u_1|^{p_1(x)-2}\nabla u_1) = \lambda H_1(T_1 u_1, T_2 u_2) \quad \text{in } \Omega, \\
-\operatorname{div}(a(|\nabla u_2|^{p_2(x)-2})|\nabla u_2|^{p_2(x)-2}\nabla u_2) = \lambda H_2(T_1 u_1, T_2 u_2) \quad \text{in } \Omega, \\
u_1 = u_2 = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.9}$$

Using the test function u_i in (3.9) and considering (3.5), we obtain

$$\int_\Omega |\nabla u_i|^{p_i(x)} \leq \lambda^* C |u_i|_{L^{p(x)}}\,.$$

Suppose that $|\nabla u_i|_{L^{p(x)}} > 1$. Then using Proposition 2.1 and the Poincaré inequality we obtain that

$$|u_i|^{p_i-1}_{L^{p_i(x)}} \leq \lambda^* C.$$

Thus $\mathcal{C}$ is bounded in $\mathbb{R}^+ \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$, which is a contradiction.

Considering $\lambda = 1$, by (3.9) we have

$$\begin{aligned} &\int_\Omega a(|\nabla u_i|^{p_i(x)-2})|\nabla u_i|^{p_i(x)-2}\nabla u_i \nabla\varphi \\ &= \int_\Omega \Big(\frac{f_i(x, T_1u_1, T_2u_2)|T_ju_j|^{\alpha_i(x)}_{L^{q_i(x)}}}{\mathcal{A}(x, |T_ju_j|_{L^{r_i(x)}})}\Big)\varphi \\ &\quad + \int_\Omega \Big(\frac{g_i(x, T_1u_1, T_2u_2)|T_ju_j|^{\gamma_i(x)}_{L^{s_i(x)}}}{\mathcal{A}(x, |T_ju_j|_{L^{r_i(x)}})}\Big)\varphi, \end{aligned} \tag{3.10}$$

for all $\varphi \in W_0^{1,p_i(x)}(\Omega)$ where $i, j = 1, 2$ with $i \neq j$.

Now we claim that $u_i \in [\underline{u}_i, \overline{u}_i]$ for $i = 1, 2$. To prove this claim we define

$$\begin{aligned} L_1(\underline{u}_1 - u_1)_+ := \int_{\{\underline{u}_1 \geq u_1\}} &\big\langle a(|\nabla \underline{u}_1|^{p_1(x)-2})|\nabla \underline{u}_1|^{p_1(x)-2}\nabla \underline{u}_1 \\ &- a(|\nabla u_1|^{p_1(x)-2})|\nabla u_1|^{p_1(x)-2}\nabla u_1, \nabla(\underline{u}_1 - u_1)\big\rangle dx. \end{aligned}$$

Using the facts that $T_2u_2 \in [\underline{u}_2, \overline{u}_2]$, $\underline{u}_i(x) > 0$ a.e. in Ω, $i = 1, j = 2$, considering $w = T_2u_2$ and $\varphi = (\underline{u}_1 - u_1)_+$ in the first inequality of (3.1) and combining with equation (3.10) we obtain

$$\begin{aligned} L_1(\underline{u}_1 - u_1)_+ \leq &\int_{\{\underline{u}_1 \geq u_1\}} \frac{f_1(x, \underline{u}_1, T_2u_2)(|\underline{u}_2|^{\alpha_1(x)}_{L^{q_1(x)}} - |T_2u_2|^{\alpha_1(x)}_{L^{q_1(x)}})}{\mathcal{A}(x, |T_2u_2|_{L^{r_1(x)}})}(\underline{u}_1 - u_1) \\ &+ \int_{\{\underline{u}_1 \geq u_1\}} \frac{g_1(x, \underline{u}_1, T_2u_2)(|\underline{u}_2|^{\gamma_1(x)}_{L^{s_1(x)}} - |T_2u_2|^{\gamma_1(x)}_{L^{s_1(x)}})}{\mathcal{A}(x, |T_2u_2|_{L^{r_1(x)}})}(\underline{u}_1 - u_1), \end{aligned}$$

which implies that

$$\begin{aligned} &\int_{\{\underline{u}_1 \geq u_1\}} \big\langle a(|\nabla \underline{u}_1|^{p_1(x)-2})|\nabla \underline{u}_1|^{p_1(x)-2}\nabla \underline{u}_1 \\ &- a(|\nabla u_1|^{p_1(x)-2}\nabla u_1)|\nabla u_1|^{p_1(x)-2}\nabla u_1, \nabla(\underline{u}_1 - u_1)\big\rangle \leq 0. \end{aligned}$$

Therefore $\underline{u}_1 \leq u_1$. The same reasoning imply the other inequalities. Since $u_i \in [\underline{u}_i, \overline{u}_i]$, we have $T_iu_i = u_i$. Therefore the pair (u_1, u_2) is a weak positive solution of (S). $\square$

4. Applications

The main goal of this section is to apply Theorem 3.5 to some nonlocal problems.

4.1. **A generalized logistic equation.** Here we present a generalization of the classic logistic equation studied in [11, 13, 39] and [39, Theorem 8]. We consider

$$\begin{gathered} -\mathcal{A}(x, |v|_{L^{r_1(x)}}) \operatorname{div}(a_1(|\nabla u|^{p_1(x)})|\nabla u|^{p_1(x)-2}\nabla u) = \lambda f_1(u)|v|^{\alpha_1(x)}_{L^{q_1(x)}} \quad \text{in } \Omega, \\ -\mathcal{A}(x, |u|_{L^{r_2(x)}}) \operatorname{div}(a_1(|\nabla v|^{p_2(x)})|\nabla v|^{p_2(x)-2}\nabla v) = \lambda f_2(v)|u|^{\alpha_2(x)}_{L^{q_2(x)}} \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.1}$$

where the function $\mathcal{A}(x,t)$ satisfies

$$\mathcal{A}(x,0) \geq 0, \quad \lim_{t\to 0^+} \mathcal{A}(x,t) = \infty, \quad \lim_{t\to +\infty} \mathcal{A}(x,t) = \pm\infty.$$

Assume that there are numbers $\theta_i > 0$, for $i = 1,2$ such that the functions $f_i : [0,\infty) \to \mathbb{R}$ satisfy the conditions:

(A5) $f_i \in C^0([0,\theta_i], \mathbb{R})$, for $i = 1,2$,
(A6) $f_i(0) = f_i(\theta_i) = 0$, $f_i(t) > 0$ in $(0,\theta_i)$ for $i = 1,2$.

Remark 4.1. Notice that $W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$ is a Banach space endowed with the norm

$$|(u,v)| := \max\left\{|\nabla u|_{p_1(x)}, |\nabla v|_{p_2(x)}\right\}.$$

Theorem 4.2. *Suppose that r_i, p_i, q_i, α_i satisfy* (A4). *Assume f_i satisfies* (A5), (A6) *and $a_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function satisfying (A1)–(A3) for $i = 1,2$. If $\mathcal{A}(x,t) > 0$ in $\overline{\Omega} \times \left(0, \max\{|\theta_1|_{L^{r_2(x)}}, |\theta_2|_{L^{r_1(x)}}\}\right]$, then there exists $\lambda_0 > 0$ such that* (4.1) *has a positive weak solution for $\lambda \geq \lambda_0$.*

Proof. Consider the functions $\widetilde{f}_i(t) = f_i(t)$ for $t \in [0,\theta_i]$, and $\widetilde{f}_i(t) = 0$ for $t \in \mathbb{R} \setminus [0,\theta_i]$, $i = 1,2$. The functional

$$\begin{aligned} J_\lambda(u,v) &= \int_\Omega \frac{1}{p_1(x)} A(|\nabla u|^{p_1(x)})dx + \int_\Omega \frac{1}{p_2(x)} A(|\nabla v|^{p_2(x)})dx \\ &\quad - \lambda \int_\Omega \widetilde{F}_1(u)dx - \lambda \int_\Omega \widetilde{F}_2(v)dx \\ &:= J_{1,\lambda}(u) + J_{2,\lambda}(v), \end{aligned}$$

where $\widetilde{F}_i(t) = \int_0^t \widetilde{f}_i(s)ds$ is of class $C^1(W_0^{1,p_1(x)} \times W_0^{1,p_2(x)}(\Omega), \mathbb{R})$.

Since $|\widetilde{f}_i(t)| \leq C$, $t \in \mathbb{R}$ for some constant which does not depends on $i = 1,2$ we have that J is coercive. Thus J has a minimum $(z_\lambda, w_\lambda) \in W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$ with

$$\begin{gathered} -\operatorname{div}(a_1(|\nabla z_\lambda|^{p_1(x)})|\nabla z_\lambda|^{p_1(x)-2}\nabla z_\lambda) = \lambda \widetilde{f}_1(z_\lambda) \quad \text{in } \Omega, \\ z_\lambda = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.2}$$

and

$$\begin{gathered} -\operatorname{div}(a_2(|\nabla w_\lambda|^{p_2(x)})|\nabla w_\lambda|^{p_2(x)-2}\nabla w_\lambda) = \lambda \widetilde{f}_2(w_\lambda) \quad \text{in } \Omega, \\ w_\lambda = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.3}$$

Note that the unique solutions of (4.2) and (4.3) are given by the minimum of functionals $J_{1,\lambda}$ and $J_{2,\lambda}$ respectively.

Consider a function $\varphi_0 \in W_0^{1,p_i(x)}(\Omega)$ for for $i = 1,2$, with $\tilde{F}_i(\varphi_0) > 0$, for $i = 1,2$. Define $(z_0, w_0) := (z_{\tilde{\lambda}_0}, w_{\tilde{\lambda}_0})$, where $\tilde{\lambda}_0$ satisfies

$$\int_\Omega \frac{1}{p_i(x)} A(|\nabla \varphi_0|^{p_i(x)})dx < \widetilde{\lambda}_0 \int_\Omega \widetilde{F}_i(\varphi_0)dx,$$

for $i = 1,2$. We have $J_{1,\tilde{\lambda}_0}(z_0) \leq J_{1,\tilde{\lambda}_0}(\varphi_0) < 0$ and also that $J_{2,\tilde{\lambda}_0}(z_0) < 0$. Therefore $z_0 \neq 0$ and $w_0 \neq 0$. Since $-\operatorname{div}(a_1(|\nabla z_0|^{p_1(x)})|\nabla z_0|^{p_1(x)-2}\nabla z_0)$ and $-\operatorname{div}(a_2(|\nabla w_0|^{p_2(x)})|\nabla w_0|^{p_2(x)-2}\nabla w_0)$ are nonnegative, we have $z_0, w_0 > 0$ in Ω. Note that by [22, Theorem 4.1] and [19, Theorem 1.2], we obtain that $z_0, w_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1]$.

Using the test function $\varphi = (z_0-\theta_1)^+ \in W_0^{1,p_1(x)}(\Omega)$ in (4.2) we obtain

$$\begin{aligned}
&\int_\Omega a_1(|\nabla z_0|^{p_1(x)})|\nabla z_0|^{p_1(x)-2}\nabla z_0 \nabla (z_0-\theta_1)^+ dx\\
&= \widetilde{\lambda}_0 \int_{\{z_0>\theta\}} \widetilde{f}_1(z_0)(z_0-\theta_1)dx = 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{\{z_0>\theta\}} \big\langle a_1(|\nabla z_0|^{p_1(x)})|\nabla z_0|^{p_1(x)-2}\nabla z_0\\
&- a_1(|\nabla \theta_1|^{p_1(x)})|\nabla \theta_1|^{p_1(x)-2}\nabla \theta_1, \nabla(z_0-\theta_1)\big\rangle dx = 0,
\end{aligned}$$

which imply $(z_0-\theta_1)_+ = 0$ in Ω. Thus $0<z_0\le\theta_1$. A similar reasoning provides $0<w_0\le\theta_2$.

Note that there is a constant $C>0$ such that $|z_0|_{L^{q_1(x)}}^{\alpha_1(x)}, |w_0|_{L^{q_2(x)}}^{\alpha_2(x)} \ge C$. Define

$$\begin{aligned}
\mathcal{A}_0 := \max\big\{&\mathcal{A}(x,t) : (x,t)\in\overline{\Omega}\times[\min\{|z_0|_{L^{r_2(x)}}, |w_0|_{L^{r_1(x)}}\},\\
&\max\{|\theta_1|_{L^{r_2(x)}}, |\theta_2|_{L^{r_1(x)}}\}]\big\}
\end{aligned}$$

and $\mu_0 = \mathcal{A}_0/C$. Then

$$\begin{aligned}
-\operatorname{div}(a_1(|\nabla z_\lambda|^{p_1(x)})|\nabla z_\lambda|^{p_1(x)-2}\nabla z_\lambda) &= \widetilde{\lambda}_0 f_1(z_0)\\
&= \frac{1}{\mathcal{A}_0}\widetilde{\lambda}_0\mu_0 f_1(z_0)|w_0|_{L^{q_1(x)}}^{\alpha_1(x)} \frac{\mathcal{A}_0}{\mu_0|z_0|_{L^{q_1(x)}}^{\alpha_1(x)}}\\
&\le \frac{1}{\mathcal{A}_0}\widetilde{\lambda}_0\mu_0 f_1(z_0)|w_0|_{L^{q_1(x)}}^{\alpha_1(x)}.
\end{aligned}$$

Thus for each $\lambda\ge\lambda_0 := \widetilde{\lambda}_0\mu_0$ and $w\in[w_0,\theta_2]$, we obtain

$$-\operatorname{div}(a_1(|\nabla z_\lambda|^{p_1(x)})|\nabla z_\lambda|^{p_1(x)-2}\nabla z_\lambda) \le \frac{1}{\mathcal{A}(x,|w|_{L^{r_1(x)}})}\lambda f_1(z_0)|w_0|_{L^{q_1(x)}}^{\alpha_1(x)}.$$

If necessary, we can consider a larger $\lambda_0>0$ such that

$$-\operatorname{div}(a_2(|\nabla w_0|^{p_2(x)})|\nabla w_0|^{p_2(x)-2}\nabla w_0) \le \frac{1}{\mathcal{A}(x,|w|_{L^{r_2(x)}})}\lambda f_2(w_0)|z_0|_{L^{q_2(x)}}^{\alpha_2(x)},$$

for all $\lambda\ge\lambda_0$ and $w\in[z_0,\theta_1]$. Since $f_i(\theta_i)=0$ for $i=1,2$, we have that (z_0,θ_1) and (w_0,θ_2) are sub-supersolutions pairs for (4.1). $\square$

4.2. **Sublinear problem.** Here, we study a nonlocal problem to generalize [39, Theorem 6]. We prove the following theorem.

Theorem 4.3. *Assume that*

- p_i, q_i, r_i, s_i *for* $i=1,2$ *satisfy* (A4)*;*
- $\alpha_i, \beta_i \in L^\infty(\Omega)$*, for* $i=1,2$*;*
- *for* $i=1,2$*, we have*

$$\begin{gathered}
0<\alpha_1^+ + \gamma_1^+ < p_i^- - 1, \quad 0 < \frac{\alpha_1^+}{p_2^- - 1} + \frac{\beta_1^+}{p_1^- - 1} < 1,\\
0<\alpha_2^+ + \gamma_2^+ < p_i^- - 1, \quad 0 < \frac{\alpha_2^+}{p_1^- - 1} + \frac{\beta_2^+}{p_2^- - 1} < 1;
\end{gathered}$$

- $a_0>0$ *is a positive constant;*
- *One of the following two conditions holds*

(A7) $\mathcal{A}(x,t) \geq a_0$ *on* $\overline{\Omega} \times [0,\infty)$,
(A8) $0 < \mathcal{A}(x,t) \leq a_0$ *on* $\overline{\Omega} \times (0,\infty)$, *and* $\lim_{t\to+\infty} \mathcal{A}(x,t) = a_\infty > 0$ *uniformly on* Ω.

Then the problem

$$
\begin{gathered}
-\mathcal{A}(x,|v|_{L^{r_1(x)}})\big(\Delta_{p_1(x)}u - \Delta u\big) = (u^{\beta_1(x)} + v^{\gamma_1(x)})|v|^{\alpha_1(x)}_{L^{q_1(x)}} \quad \text{in } \Omega,\\
-\mathcal{A}(x,|u|_{L^{r_2(x)}})\big(\Delta_{p_2(x)}v - \Delta v\big) = (u^{\beta_2(x)} + v^{\gamma_2(x)})|u|^{\alpha_2(x)}_{L^{q_2(x)}} \quad \text{in } \Omega,\\
u = v = 0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{4.4}
$$

has a positive solution.

Proof. Suppose that (A7) is hold, that is, $\mathcal{A}(x,t) \geq a_0$ in $\overline{\Omega} \times [0,+\infty)$. We start by constructing $(\overline{u},\overline{v})$. Let $\lambda > 0$ be a positive number, which will be chosen later and denote by $z_\lambda \in W_0^{1,p_1(x)}(\Omega) \cap L^\infty(\Omega)$ and $y_\lambda \in W_0^{1,p_2(x)}(\Omega) \cap L^\infty(\Omega)$ the unique solutions of (2.1) respectively.

For $\lambda > 0$ sufficiently large it follows from Lemma 2.7 that there is a constant $K > 1$ that does not depend on λ such that

$$
0 < z_\lambda(x) \leq K\lambda^{\frac{1}{p_1^- - 1}} \quad \text{in } \Omega, \tag{4.5}
$$

$$
0 < y_\lambda(x) \leq K\lambda^{\frac{1}{p_2^- - 1}} \quad \text{in } \Omega. \tag{4.6}
$$

Since $\alpha_1^+ + \gamma_1^+ < p_2^- - 1$ and $\frac{\alpha_1^+}{p_2^- - 1} + \frac{\beta_1^+}{p_1^- - 1} < 1$, it is possible to choose $\lambda > 1$ such that (4.5), (4.6) and

$$
\frac{1}{a_0}\big(K^{\beta_1^+}\lambda^{\frac{\beta_1^+}{p_1^- - 1} + \frac{\alpha_1^+}{p_2^- - 1}} + K^{\gamma_1^+}\lambda^{\frac{\alpha_1^+ + \gamma_1^+}{p_2^- - 1}}\big)\max\{|K|^{\alpha_1^-}_{L^{q_1(x)}}, |K|^{\alpha_1^+}_{L^{q_1(x)}}\} \leq \lambda \tag{4.7}
$$

occur. By (4.5), (4.6) and (4.7), we obtain

$$
\frac{1}{a_0}(z_\lambda^{\beta_1(x)} + w^{\gamma_1(x)})|y_\lambda|^{\alpha_1(x)}_{L^{q_1(x)}} \leq \lambda, w \in [0, y_\lambda].
$$

Thus for $w \in [0, y_\lambda]$ we obtain

$$
\begin{gathered}
-\Delta_{p_1(x)}z_\lambda - \Delta z_\lambda \geq \frac{1}{\mathcal{A}(x,|w|_{L^{r_1(x)}})}(z_\lambda^{\beta_1(x)} + w^{\gamma_1(x)})|y_\lambda|^{\alpha_1(x)}_{L^{q_1(x)}} \quad \text{in } \Omega,\\
z_\lambda = 0 \quad \text{on } \partial\Omega.
\end{gathered}
$$

Considering, if necessary, a larger $\lambda > 0$ the previous reasoning implies that

$$
\begin{gathered}
-\Delta_{p_2(x)}y_\lambda - \Delta y_\lambda \geq \frac{1}{\mathcal{A}(x,|w|_{L^{r_2(x)}})}(w^{\beta_2(x)} + {y_\lambda}^{\gamma_2(x)})|z_\lambda|^{\alpha_2(x)}_{L^{q_2(x)}} \quad \text{in } \Omega,\\
y_\lambda = 0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

for all $w \in [0, z_\lambda]$.

Now, we construct $(\underline{u}_i, \underline{v}_i), i = 1,2$. Since $\partial\Omega$ is C^2, there is a constant $\delta > 0$ such that $d \in C^2(\overline{\Omega_{3\delta}})$ and $|\nabla d(x)| \equiv 1$, where $d(x) := \operatorname{dist}(x,\partial\Omega)$ and $\overline{\Omega_{3\delta}} := \{x \in \overline{\Omega}; d(x) \leq 3\delta\}$. From [29, Page 12], for $\sigma \in (0,\delta)$ sufficiently small, the function $\phi_i = \phi_i(k,\sigma),\ i = 1,2$ defined by

$$
\phi_i(x) = \begin{cases}
e^{kd(x)} - 1 & \text{if } d(x) < \sigma,\\
e^{k\sigma} - 1 + \int_\sigma^{d(x)} ke^{k\sigma}\big(\frac{2\delta - t}{2\delta - \sigma}\big)^{\frac{2}{p_i^- - 1}}dt & \text{if } \sigma \leq d(x) < 2\delta,\\
e^{k\sigma} - 1 + \int_\sigma^{2\delta} ke^{k\sigma}\big(\frac{2\delta - t}{2\delta - \sigma}\big)^{\frac{2}{p_i^- - 1}}dt & \text{if } 2\delta \leq d(x),
\end{cases}
$$

belongs to $C_0^1(\overline{\Omega})$, where $k>0$ is an arbitrary number and that

$$-\Delta_{p_i(x)}(\mu\phi_i)=\begin{cases}-k(k\mu e^{kd(x)})^{p_i(x)-1}\Big[(p_i(x)-1)\\ +(d(x)+\frac{\ln k\mu}{k})\nabla p_i(x)\nabla d(x)+\frac{\Delta d(x)}{k}\Big] & \text{if } d(x)<\sigma,\\ \Big\{\frac{1}{2\delta-\sigma}\frac{2(p_i(x)-1)}{p_i^- -1}-\big(\frac{2\delta-d(x)}{2\delta-\sigma}\big)\times\Big[\ln k\mu e^{k\sigma}\\ \big(\frac{2\delta-d(x)}{2\delta-\sigma}\big)^{\frac{2}{p_i^- -1}}\nabla p_i(x)\nabla d(x)+\Delta d(x)\Big]\Big\}\\ \times(k\mu e^{k\sigma})^{p_i(x)-1}\big(\frac{2\delta-d(x)}{2\delta-\sigma}\big)^{\frac{2(p_i(x)-1)}{p_i^- -1}-1} & \text{if } \sigma<d(x)<2\delta,\\ 0 & \text{if } 2\delta<d(x),\end{cases}$$

and

$$-\Delta(\mu\phi_i)=\begin{cases}-k(k\mu e^{kd(x)})\big[1+\frac{\Delta d(x)}{k}\big] & \text{if } d(x)<\sigma,\\ \Big\{\frac{2}{2\delta-\sigma}-\big(\frac{2\delta-d(x)}{2\delta-\sigma}\big)\Delta d(x)\Big\}(k\mu e^{k\sigma})\big(\frac{2\delta-d(x)}{2\delta-\sigma}\big) & \text{if } \sigma<d(x)<2\delta,\\ 0 & \text{if } 2\delta<d(x),\end{cases}$$

for all $\mu>0$ and $i=1,2$.

Define $\mathcal{A}_\lambda:=\max\big\{\mathcal{A}(x,t):(x,t)\in\overline{\Omega}\times\big[0,\max\{|y_\lambda|_{L^{r_1(x)}}|z_\lambda|_{L^{r_2(x)}}\}\big]\big\}$. We have

$$a_0\le\mathcal{A}(x,|w|_{L^{r_1(x)}})\le\mathcal{A}_\lambda\quad\text{in }\Omega$$

for all $w\in[0,y_\lambda]$.

Let $\sigma=\frac{1}{k}\ln 2$ and $\mu=e^{-ak}$ where

$$a=\frac{\min\{p_1^- -1,p_2^- -1\}}{\max\{\max_{\overline{\Omega}}|\nabla p_1|+1,\max_{\overline{\Omega}}|\nabla p_2|+1\}}.$$

Then $e^{k\sigma}=2$ and $k\mu\le 1$ if $k>0$ is sufficiently large.

Let $x\in\Omega$ with $d(x)<\sigma$. If $k>0$ is large enough we have $|\nabla d(x)|=1$ and then we have

$$\begin{aligned}\Big|d(x)+\frac{\ln(k\mu)}{k}\Big||\nabla p_1(x)||\nabla d(x)|&\le\Big(|d(x)|+\frac{|\ln(k\mu)|}{k}\Big)|\nabla p_1(x)|\\ &\le\Big(\sigma-\frac{\ln(k\mu)}{k}\Big)|\nabla p_1(x)|\\ &=\Big(\frac{\ln 2}{k}-\frac{\ln k}{k}\Big)|\nabla p_1(x)|+a|\nabla p_1(x)|\\ &<p_1^- -1.\end{aligned}\tag{4.8}$$

Note that there exists a constant $A>0$, that does not depend on k, such that $|\Delta d(x)|<A$ for all $x\in\overline{\partial\Omega_{3\delta}}$. Using the above inequality and the expression of $-\Delta_{p_1(x)}(\mu\phi)$ and $-\Delta(\mu\phi)$, we obtain $-\Delta_{p_1(x)}(\mu\phi_1)-\Delta(\mu\phi_1)\le 0$ for $x\in\Omega$ with $d(x)<\sigma$ or $d(x)>2\delta$ for $k>0$ large enough. Therefore

$$\begin{aligned}-\Delta_{p_1(x)}(\mu\phi_1)-\Delta(\mu\phi_1)\le 0&\le\frac{1}{\mathcal{A}_\lambda}(\mu\phi_1)^{\beta_1(x)}|\mu\phi_2|_{L^{q_1(x)}}^{\alpha_1(x)}\\ &\le\frac{1}{\mathcal{A}_\lambda}((\mu\phi_1)^{\beta_1(x)}+w^{\gamma_1(x)})|\mu\phi_2|_{L^{q_1(x)}}^{\alpha_1(x)}\end{aligned}$$

for all $w \in L^\infty(\Omega)$ with $w \geq \mu\phi_2$ and $d(x) < \sigma$ or $2\delta < d(x)$. Using an idea in [29, estimate (3.10)], if $\sigma < d(x) < 2\delta$, then

$$\begin{aligned} -\Delta_{p_1(x)}(\mu\phi_1) - \Delta(\mu\phi_1) &\leq \tilde{C}(k\mu)^{p_1^- -1}|\ln k\mu| + \tilde{C}(k\mu)|\ln k\mu| \\ &= \tilde{C}((k\mu)^{p_1^- -1} + k\mu)\big|\ln \frac{k}{e^{ak}}\big| . \end{aligned} \tag{4.9}$$

[40, Theorem 2] and $\alpha_1^+ + \gamma_1^+ < p_1^- - 1$ imply

$$\lim_{k\to+\infty} \frac{\tilde{C}k^{p_1^- -1} + \tilde{C}k}{e^{ak(p_1^- -1-(\alpha_1^+ + \gamma_1^+))}}\big|\ln \frac{k}{e^{ak}}\big| = 0. \tag{4.10}$$

Note that $\phi_1(x) \geq 1$ if $\sigma \leq d(x) < 2\delta$ because $\phi_1(x) \geq e^{k\sigma} - 1$ and $e^{k\sigma} = 2$ for all $k > 0$. Thus, there is a constant $C_0 > 0$ that does not depend on k such that $|\phi_2|_{L^{q_1(x)}(\Omega)}^{\alpha_1(x)} \geq C_0$ if $\sigma < d(x) < 2\delta$. By (4.10), we can choose $k > 0$ large enough such that

$$\frac{\tilde{C}k^{p_1^- -1}\tilde{C}k}{e^{ak[(p_1^- -1)-(\alpha_1^+ + \beta_1^+)]}}\big|\ln \frac{k}{e^{ak}}\big| \leq \frac{C_0}{\mathcal{A}_\lambda}. \tag{4.11}$$

Therefore from (4.9) and (4.11), we have

$$-\Delta_{p_1(x)}(\mu\phi_1) - \Delta(\mu\phi_1) \leq \frac{1}{\mathcal{A}_\lambda}((\mu\phi_1)^{\beta_1(x)} + w^{\gamma_1(x)})|\mu\phi_2|_{L^{q_1(x)}}^{\alpha_1(x)},$$

for all $w \in L^\infty(\Omega)$ with $w \geq \mu\phi_2$ and $\sigma < d(x) < 2\delta$ for $k > 0$ large enough. Thus it is possible to conclude that

$$-\Delta_{p_1(x)}(\mu\phi_1) - \Delta(\mu\phi_1) \leq \frac{1}{\mathcal{A}_\lambda}((\mu\phi_1)^{\beta_1(x)} + w^{\gamma_1(x)})|\mu\phi_2|_{L^{q_1(x)}}^{\alpha_1(x)} \quad \text{in } \Omega.$$

Fix $k > 0$ satisfying the above property and the inequality $-\Delta_{p_1(x)}(\mu\phi_1) - \Delta(\mu\phi_1) \leq 1$. For $\lambda > 1$ we have $-\Delta_{p_1(x)}(\mu\phi_1) - \Delta(\mu\phi_1) \leq -\Delta_{p_1(x)}z_\lambda - \Delta z_\lambda$. Therefore $\mu\phi_1 \leq z_\lambda$.

Since $\alpha_2^+ + \gamma_2^+ < p_2^- - 1$, a similar reasoning imply that there is $\mu > 0$ small enough such that

$$-\Delta_{p_2(x)}(\mu\phi_2) - \Delta(\mu\phi_2) \leq \frac{1}{\mathcal{A}(x, |w|_{L^{r_2(x)}})}(w^{\beta_2} + (\mu\phi_2)^{\gamma_2})|\mu\phi_1|_{L^{q_2(x)}(\Omega)}^{\alpha_2(x)}$$

in Ω for all $w \in L^\infty(\Omega)$ with $w \geq \mu\phi_1$ and that $\mu_2\phi \leq y_\lambda$. The first part of the result is proved.

Now suppose that $0 < \mathcal{A}(x,t) \leq a_0$ in $\overline{\Omega} \times (0,\infty)$. Let $\delta, \sigma, \mu, a, \lambda, z_\lambda, y_\lambda$ and $\phi_i, i = 1,2$ as before. From the previous arguments there exist $k > 0$ large enough and $\mu > 0$ small such that

$$\begin{gathered} -\Delta_{p_1(x)}(\mu\phi_1) - \Delta(\mu\phi_1) \leq 1 \quad \text{in } \Omega, \\ -\Delta_{p_1(x)}(\mu\phi) - \Delta(\mu\phi) \leq \frac{1}{a_0}((\mu\phi_1)^{\beta_1(x)} + w^{\gamma_1(x)})|\mu\phi_2|_{L^{q_1(x)}}^{\alpha_1(x)} \quad \text{in } \Omega \end{gathered} \tag{4.12}$$

for all $w \in [\mu\phi_2, y_\lambda]$, also that

$$\begin{gathered} -\Delta_{p_2(x)}(\mu\phi_2) - \Delta(\mu\phi_2) \leq 1 \quad \text{in } \Omega \\ -\Delta_{p_2(x)}(\mu\phi_2) - \Delta(\mu\phi_2) \leq \frac{1}{a_0}(w^{\beta_2(x)} + (\mu\phi_2)^{\gamma_2(x)})|\mu\phi_1|_{L^{q_2(x)}}^{\alpha_2(x)} \quad \text{in } \Omega \end{gathered} \tag{4.13}$$

for all $w \in [\mu\phi_1, z_\lambda]$. Since $\lim_{t\to\infty} \mathcal{A}(x,t) = a_\infty > 0$ uniformly in Ω there is a large constant $a_1 > 0$ such that $\mathcal{A}(x,t) \geq \frac{a_\infty}{2}$ in $\overline{\Omega} \times (a_1, \infty)$. Let

$$m_k := \min\big\{\mathcal{A}(x,t) : (x,t) \in \overline{\Omega} \times [\min\{|\mu\phi_1|_{L^{r_1(x)}}, |\mu\phi_2|_{L^{r_2(x)}}\}, a_1]\big\} > 0\,,$$
$$\mathcal{A}_k := \min\big\{m_k, \frac{a_\infty}{2}\big\}.$$

We have $\mathcal{A}(x,t) \geq \mathcal{A}_k$ in $\overline{\Omega} \times [\min\{|\mu\phi_1|_{L^{r_1(x)}}, |\mu\phi_2|_{L^{r_2(x)}}\}, \infty)$.

Fix $k > 0$ satisfying (4.12) and (4.13). Consider $\lambda > 1$ such that (4.5), (4.6), and

$$\frac{1}{\mathcal{A}_k}\Big(K^{\beta_1^+} \lambda^{\frac{\beta_1^+}{p_1^- -1} + \frac{\alpha_1^+}{p_2^- -1}} + K^{\gamma_1^+} \lambda^{\frac{\alpha_1^+ + \gamma_1^+}{p_2^- -1}}\Big) \max\{|K|^{\alpha_1^-}_{L^{q_1(x)}}, |K|^{\alpha_1^+}_{L^{q_1(x)}}\} \leq \lambda,$$
$$\frac{1}{\mathcal{A}_k}\Big(K^{\beta_2^+} \lambda^{\frac{\beta_2^+ + \alpha_2^+}{p_1^- -1}} + K^{\gamma_2^+} \lambda^{\frac{\gamma_2^+}{p_2^- -1} + \frac{\alpha_2^+}{p_1^- -1}}\Big) \max\{|K|^{\alpha_2^+}_{L^{q_2(x)}}, |K|^{\alpha_2^-}_{L^{q_2(x)}}\} \leq \lambda$$

where $K > 1$ is a constant that does not depend on k and λ (see Lemma 2.7). Therefore,

$$-\Delta_{p_1(x)} z_\lambda - \Delta z_\lambda \leq \frac{1}{\mathcal{A}(x, |w|_{L^{r_1(x)}})} (z_\lambda^{\beta_1(x)} + w^{\gamma_1(x)}) |y_\lambda|^{\alpha_1(x)}_{L^{q_1(x)}}$$

in $\Omega, w \in [\mu\phi_2, y_\lambda]$. Arguing as before and considering a suitable choice for λ and k, we obtain

$$-\Delta_{p_2(x)} y_\lambda - \Delta y_\lambda \leq \frac{1}{\mathcal{A}(x, |w|_{L^{r_2(x)}})} (w^{\beta_2(x)} + y_\lambda^{\beta_2(x)}) |z_\lambda|^{\alpha_2(x)}_{L^{q_2(x)}}$$

in $\Omega, w \in [\mu\phi_1, z_\lambda]$. The comparison principle imply that $\mu\phi_1 \leq z_\lambda$ and $\mu\phi_2 \leq y_\lambda$ if μ is small. $\square$

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