DYNAMICAL BEHAVIOR IN A REACTION-DIFFUSION SYSTEM WITH PREY-TAXIS

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ABSTRACT. In this article, we study a diffusive predator-prey system with prey-taxis under homogeneous Neumann boundary conditions. We establish the existence and boundedness of nonnegative global solutions. Through comparison with the system without prey-taxis, we find that the positive constant equilibrium remains stable for positive prey-taxis, while negative prey-taxis makes it unstable.

1. INTRODUCTION

The interaction of predator and prey is one of the most fundamental relationships in complex biological systems. For some predator-prey systems, under certain circumstances, predators have to look for food, share food or compete for food among other predators. In some predator-prey models, the carrying capacity of predator is proportional to the densities of prey [2, 6, 7, 8, 14, 15, 18, 19, 21, 23, 24, 25].

The predator's movement to find prey is decided by the pheromone released by the prey in certain extent. Prey-taxis is the spatiotemporal variations of predators in response to prey gradient, and predator-prey systems with prey-taxis have captured considerable attention in various forms [3, 4, 9, 11, 12, 16, 17, 26].

In this article, we consider a spatial model with prey-taxis the form

$$\frac{\partial u}{\partial t} = d_1 \Delta u + ug(u) - p(u)v, \quad x \in \Omega, \ t > 0,$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v - \chi \nabla \cdot (\alpha(v)v\nabla u) + \sigma v(1 - \frac{v}{u}), \quad x \in \Omega, \ t > 0,$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega, \ t > 0,$$

$$u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) \geq 0, \not\equiv 0, \quad x \in \Omega, \tag{1.1}$$

where the habitat of both species Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$ with the smooth boundary $\partial\Omega$, ν is the outer normal vector and homogeneous Neumann boundary conditions (no flux boundary condition) is imposed on both u and v, so the system is closed. Here $u(x,t)$ and $v(x,t)$ denote the densities of prey and predator at place x and time t respectively, d_1 and d_2 are the dispersal rates of prey and predator, σ stands for the intrinsic growth rate of the predator, and the carrying capacity of the predator is proportional to the densities of prey [14], the

function $ug(u)$ refers to the net growth rate of the prey without predators, and $g(u)$ is per capita growth rate satisfying the following condition:

(A1) $g \in C^1([0, +\infty))$, there exists $k > 0$, such that $g(u)$ is positive for $0 < u < k$, and $g(u)$ is negative for $u > k$ and $g(k) = 0$.

As pointed in [14], the following four classical types of the functional response $p(u)$ are typical and useful:

$$p(x) = x \quad \text{(Leslie-Gower type)},$$

$$p(x) = \frac{x}{x + a} \quad \text{(Holling-Tanner type)},$$

$$p(x) = \frac{x^2}{(x + a)(x + b)} \quad \text{(Sigmodial type)},$$

$$p(x) = 1 - e^{-ax} \quad \text{(Ivlev functional response)}.$$

Note that the above four types of functional response $p(u)$ satisfy the following hypotheses:

(A2) $p \in C^1([0, +\infty))$, $p(0) = 0$, $p(u) > 0$ for $u > 0$ and $p'(u) > 0$ for $u \geq 0$. Moreover, there exists a positive constant $P > 0$ such that $p'(u) \leq P$ for all $u > 0$.

There are quite a few qualitative analyzes on predator-prey system (1.1), e.g., [8] for Leslie-Gower functional response and [6] for Holling-Tanner functional response.

The term $\chi \nabla \cdot (\alpha(v) v \nabla u)$ is the sensitivity of predator to prey, which quantifies the tendency of predator to move toward the direction of the increasing prey gradient. χ is the prey tactic coefficient, $\chi \geq 0$ measures the intensity of the directed motion of predator. As pointed in [27], $\alpha(v)$ can be taken as

$$\alpha(v) = \begin{cases} 1 - \frac{v}{N}, & 0 \leq v \leq N, \\ 0, & v > N, \end{cases} \tag{1.2}$$

where N represents the maximum carrying capacity of predators in a unit volume. If the number of predator exceeds the volume N, the trend of direct motion of predator will approach 0.

When the response function in the predator equation is typical Holling Type I and Holling Type II, Lee [16] considered the traveling wave solutions, and investigated the pattern formation under homogeneous Neumann boundary conditions in a bounded interval [17]. Chakraborty et al [3] showed that the mode of functional response function plays an important role in resolving spatial patterns through numerical simulation. The global existence of nonnegative solutions and the stability of steady-state solutions were presented in [9, 26]. For the constant prey tactic, the global existence of the nonnegative solution was discussed in [11], where two dimensions for any $\chi > 0$ were considered. The diffusive predator-prey system (1.1) without prey-taxis (i.e., $\chi=0$) has been extensively studied in [5, 8, 14]. In this paper, we focus on the dynamical behavior change under homogeneous Neumann boundary conditions from $\chi = 0$ to $\chi > 0$.

The rest of the paper is structured as follows. In Section 2, we show the global existence of non-negative solution in system (1.1). In Section 3, the stability/instability of positive constant steady state is investigated for different χ.

2. Existence of global solutions

We start with the existence of classical solutions of system (1.1) when $\chi = 0$, see e.g. [14].

Lemma 2.1. *Suppose that p, g satisfy* (A1) *and* (A2)*, $\sigma > 0$, $d_1, d_2 > 0$ and $\chi = 0$ in* (1.1)*. Let $u_0(x) > 0$, $v_0(x) \geq 0$, then*

$$0 < u(x,t) \leq m_1, \quad 0 < v(x,t) \leq m_2, \tag{2.1}$$

where

$$m_1 = \max\{\|u_0\|_\infty, k\}, \quad m_2 = \max\{\|v_0\|_\infty, m_1\}. \tag{2.2}$$

In the following part, we shall prove that for $\chi > 0$, the system (1.1) with prey-taxis still permit a global classical solution. It is known that k is the carrying capacity of prey in (1.1), and N indicates the maximal number of predators that can fill a unit volume. Based on the meaning of m_2 and N, the condition $N > m_2$ proposed in [20] always holds in our paper.

Note that $v = N$ is not a differentiable point of $\alpha(v)$. To obtain classical solutions, by the example in [28], we extend $\alpha(v)$ as follows:

$$\overline{\alpha}(v) = \begin{cases} > 1, & v < 0, \\ 1 - \frac{v}{N}, & 0 \leq v \leq N, \\ < 0, & v > N. \end{cases} \tag{2.3}$$

Then $\overline{\alpha}(v)$ is a smooth extension of $\alpha(v)$.

Now we narrow our attention to the existence of classical global solutions of the system

$$\frac{\partial u}{\partial t} = d_1 \Delta u + u g(u) - p(u)v, \quad x \in \Omega,\ t > 0,$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v - \chi \nabla \cdot (\overline{\alpha}(v)v\nabla u) + \sigma v(1 - \frac{v}{u}), \quad x \in \Omega,\ t > 0, \tag{2.4}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0,$$

$$u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) \geq 0, \not\equiv 0 \quad x \in \Omega.$$

It turns out that the existence of classical global solutions to system (2.4) implies the existence of classical global solutions to system (1.1). This is because $\alpha(v) = \overline{\alpha}(v)$ when $0 \leq v \leq N$ and we will show that $v \in [0, N]$ later, while $\alpha(v) = 0$ when $v > N$ and Lemma 2.1 gives the desirable results.

We define

$$X = \left\{\omega \in W^{1,p}(\Omega) : \frac{\partial \omega}{\partial \nu} = 0, x \in \partial\Omega\right\}.$$

Lemma 2.2. (1) *Assume $u_0, v_0 \in W^{1,p}(\Omega)$, where $p > n$, and $\chi > 0$. Suppose* (A1) *and* (A2) *hold. Then there exists a maximal existence time $T_{\max}$, such that system* (2.4) *has a unique nonnegative solution $u, v \in C([0, T_{\max}); W^{1,p}(\Omega)) \cap C^1((0, T_{\max}), C^1(\Omega))$, where $T_{\max}$ depends on initial data $(u_0, v_0) \in X^2$, and u,v satisfy*

$$u(x,t) > 0, \quad v(x,t) \geq 0, \quad x \in \overline{\Omega},\ 0 \leq t < T_{\max}. \tag{2.5}$$

(2) *If for every $T > 0$ there exists a constant containing $M_0(T)$ such that*

$$\|(u(t), v(t))\|_\infty \leq M_0(T), \quad 0 < t < \min\{T, T_{\max}\}, \tag{2.6}$$

where $M_0(T)$ is a constant depending on T and $\|(u(t), v(t))\|_{1,p}$, then $T_{\max} = +\infty$.

Proof. Let $\zeta = (u, v)$. We can rewrite equalities in (2.4) in the form

$$
\begin{aligned}
\zeta_t &= \nabla \cdot (a(\zeta)\nabla\zeta) + \Phi(\zeta), \quad x \in \Omega,\ t > 0, \\
\frac{\partial\zeta}{\partial\nu} &= 0, \quad x \in \partial\Omega,\ t > 0, \\
\zeta(\cdot, 0) &= (u_0, v_0), \quad x \in \Omega,
\end{aligned}
\tag{2.7}
$$

where

$$
a(\zeta) = \begin{pmatrix} d_1 & 0 \\ -\chi\overline{\alpha}(v)v & d_2 \end{pmatrix}, \quad \Phi(\zeta) = \begin{pmatrix} ug(u) - p(u)v \\ \sigma v(1 - \frac{v}{u}) \end{pmatrix}.
$$

By [1, Theorem 14.4 and 14.6], the local existence of solutions in (2.4) is obtained. Moreover, the diffusion matrix $a(\zeta)$ in (2.7) is lower-triangular, the result in (2) follows from [1], so we have $T_{\max} = \infty$. □

Theorem 2.3. *Assume that $\chi > 0$ and (A1) and (A2) hold. Then the solution $u(x, t)$ of system (1.1) satisfies*

$$
0 < u(x, t) \le \max\{\|u_0\|_\infty, k\} = m_1, \quad \limsup_{t\to\infty} u(x, t) \le k. \tag{2.8}
$$

Proof. From the first equation in (1.1), it holds

$$
\begin{aligned}
\frac{\partial u}{\partial t} - d_1\Delta u &= ug(u) - p(u)v \le ug(u), \quad x \in \Omega,\ t > 0, \\
\frac{\partial u}{\partial\nu} &= 0, \quad x \in \partial\Omega,\ t > 0. \\
u(x, 0) &= u_0(x), \quad x \in \Omega.
\end{aligned}
\tag{2.9}
$$

Let $u^*(t)$ be the solution of the ODE problem

$$
\begin{aligned}
\frac{du^*(t)}{dt} &= u^*(t)g(u^*(t)), \quad t > 0, \\
u^*(0) &= \|u_0\|_\infty.
\end{aligned}
\tag{2.10}
$$

Then hypothesis (A1) gives $u^*(t) \le m_1 = \max\{\|u_0\|_\infty, k\}$. Furthermore $u^*(t)$ is a super-solution of the PDE problem

$$
\begin{aligned}
\frac{\partial U}{\partial t} - d_1\Delta U &= Ug(U), \quad x \in \Omega,\ t > 0, \\
\frac{\partial U}{\partial\nu} &= 0, \quad x \in \partial\Omega,\ t > 0. \\
U(x, 0) &= u_0(x), \quad x \in \Omega.
\end{aligned}
\tag{2.11}
$$

Therefore,

$$
0 < U(x, t) \le u^*(t), \quad \text{for all } (x, t) \in \overline{\Omega} \times (0, \infty). \tag{2.12}
$$

By the comparison principle, we have

$$
0 < u(x, t) < U(x, t) \le u^*(t) \le m_1, \quad (x, t) \in \overline{\Omega} \times (0, \infty). \tag{2.13}
$$

Since $g(u) < 0$ for all $u > k$ by hypothesis (A1), we have that

$$
\limsup_{t\to+\infty} u^*(x, t) \le \lim_{t\to+\infty} u^*(t) = k,
$$

which along with (2.13) gives (2.8). □

Theorem 2.4. *Assume that $0 \leq v_0(x) \leq N$, then the solution $(u(x,t), v(x,t))$ of (2.4) for all $(x,t) \in \overline{\Omega} \times (0,T)$ satisfies $0 \leq v(x,t) \leq N$.*

Proof. We define

$$Lv = v_t - d_2 \Delta v + \chi \nabla \cdot (\overline{\alpha}(v) v \nabla u) - \sigma v \left(1 - \frac{v}{u}\right). \tag{2.14}$$

Note that $0 \leq v_0$, so $v = 0$ is a lower solution of the equation. Moreover,

$$LN = -\sigma N \left(1 - \frac{N}{u}\right). \tag{2.15}$$

Because $N > m_2$, choosing sufficiently large N gives

$$LN \geq 0. \tag{2.16}$$

Thus $v = N$ is an upper solution of the v equation by (2.16). The comparison principle [22] gives

$$0 \leq v(x,t) \leq N. \tag{2.17}$$

$\square$

3. Effect of prey-taxis on the dynamics

In this section, we investigate the effect of prey-taxis on the dynamics. From Theorems 2.3 and 2.4, and Sobolev embedding Theorem, the solution $(u(x,t), v(x,t))$ of (2.4) becomes classical solution. In the following, we consider the local stability and global stability of (u^*, v^*) in

$$Y = \left\{ \omega \in C^2(\Omega) : \frac{\partial \omega}{\partial \nu} = 0, x \in \partial \Omega \right\}.$$

3.1. Global stability of positive constant steady state. First we study the global stability of positive constant solution (u^*, v^*) in (1.1). For this purpose, we impose the following additional hypothesis as pointed in [14]:

(A3) there exists some constant $\widehat{p} > 0$, such that $-\widehat{p} \leq \frac{d}{du}\left(\frac{p(u)}{u}\right) \leq 0$ for $u > 0$.

(A4) $g'(u) \leq -\widehat{g}$, where $\widehat{g} > 0$ is a positive constant.

It is easy to find that system (1.1) has a unique positive equilibrium (u^*, v^*), where $u^* = \frac{g(u^*)}{p(u^*)} = v^*$.

Theorem 3.1. *Assume that (A1)–(A4) hold and $\frac{k}{2} < u^* \leq \frac{\widehat{g}}{\widehat{p}}$, then the positive equilibrium (u^*, v^*) of (1.1) is globally asymptotically stable if $\chi^2 < \frac{d_1 d_2 \sigma}{u^* p(u^*) v^*}$.*

Proof. It follows from Theorems 2.3 and 2.4 that $\mathcal{Y} := \{(u,v) \in R^2 | 0 < u \leq k, 0 \leq v \leq N\}$ is positive invariant for (1.1). Let $(u(x,t), v(x,t))$ be a positive solution of (1.1). Define a Lyapunov function

$$E(t) = \int_\Omega \left(\int_{u^*}^u \frac{\xi - u^*}{\xi p(\xi)} d\xi + A \int_{v^*}^v \frac{\eta - v^*}{\eta} d\eta \right) dx$$

for some positive constant A, which will be chosen later. Then

$$\dot{E} = \int_\Omega \frac{u - u^*}{up(u)} \frac{\partial u}{\partial t} dx + \int_\Omega A \frac{v - v^*}{v} \frac{\partial v}{\partial t} dx$$

$$= \int_\Omega \frac{u - u^*}{up(u)} [d_1 \Delta u + ug(u) - p(u)v] dx$$

$$+ \int_\Omega A \frac{v - v^*}{v} \big[d_2 \Delta v + \sigma v \big(1 - \frac{v}{u} \big) - \chi \nabla \cdot (\alpha(v) v \nabla u) \big] dx$$
$$:= E_1(t) + E_2(t),$$

where

$$E_1(t) = \int_\Omega \big[\frac{u - u^*}{up(u)} d_1 \Delta u + A \frac{v - v^*}{v} (d_2 \Delta v - \chi \nabla(\alpha(v) v \nabla u)) \big] dx$$
$$= - \int_\Omega \big[\frac{d_1}{u^2 p^2(u)} \left(up(u) - (u - u^*)(p(u) + up_u(u)) \right) |\nabla u|^2 \big] dx$$
$$- \int_\Omega \big[A d_2 \frac{v^*}{v^2} |\nabla v|^2 + A \chi v^* \int_\Omega \frac{\alpha(v)}{v} |\nabla u||\nabla v| \big] dx,$$

and

$$E_2(t) = \int_\Omega \big[\frac{u - u^*}{up(u)} \left(ug(u) - p(u)v \right) + A \frac{v - v^*}{v} \sigma v (1 - \frac{v}{u}) \big] dx$$
$$= \int_\Omega \big[\frac{u - u^*}{up(u)} f_1(u, v) + A \frac{v - v^*}{v} f_2(u, v) \big] dx.$$

According to (2.8), we can find a large T such that $u(x, t) \le k + \varepsilon$ in $[T, \infty) \times \Omega$ for any positive constant ε with $\varepsilon \le 2u^* - K$.

Rewrite $E_1(t)$ as

$$E_1(t) = - \int_\Omega Z^T B Z,$$

where

$$Z = \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix}, \quad B = \begin{pmatrix} \frac{d_1}{u^2 p^2(u)} \left(up(u) - (u - u^*)(p(u) + up_u(u)) \right) & \frac{A \chi v^* \alpha(v)}{2v} \\ \frac{A \chi v^* \alpha(v)}{2v} & A d_2 \frac{v^*}{v^2} \end{pmatrix}.$$

Since $\frac{d}{du} \big(\frac{p(u)}{u} \big) = \frac{up_u(u) - p(u)}{u^2} < 0$ for $u > 0$ from $(A3)$, thus $up_u(u) < p(u)$ for $u > 0$. Using this fact and the assumption $\frac{K}{2} < u^*$, we have

$$up(u) - (u - u^*)(p(u) + up_u(u)) = -u^2 p_u(u) + u^*(p(u) + up_u(u))$$
$$\ge -u^2 p_u(u) + u^*[2up_u(u)]$$
$$= up_u(u)[2u^* - u] \ge up_u(u)[2u^* - K - \varepsilon] \ge 0$$

for $t \ge T$, which implies the result.

It is clear that $\text{trace}(B) > 0$, and the determinant of B is

$$\det B = \frac{A d_1 d_2 v^*}{u^2 p^2(u) v^2} \left(up(u) - (u - u^*)(p(u) + up_u(u)) \right) - \frac{A^2 \chi^2 (v^*)^2 \alpha^2(v)}{4v^2}. \quad (3.1)$$

Thus $\det(B) > 0$ is equivalent to

$$d_1 d_2 [up(u) - (u - u^*)(p(u) + up_u(u))] \ge u^2 p^2(u) A \chi^2 v^* \alpha(v). \quad (3.2)$$

Because $0 < u \le u^*, 0 \le v \le N$, a sufficient condition for (3.2) to hold is

$$d_1 d_2 > u^* p(u^*) A \chi^2 v^*. \quad (3.3)$$

Therefore, if (3.3) holds, then

$$E_1(t) = - \int_\Omega Z^T B Z \le 0. \quad (3.4)$$

Moreover,

$$\begin{aligned}
E_2(t) &= \int_\Omega \left[\frac{u - u^*}{p(u)}\big(g(u) - g(u^*)\big)\right]dx \\
&\quad - \int_\Omega \left\{\frac{u - u^*}{p(u)}\left[\Big(\frac{p(u)}{u}v - \frac{p(u)}{u}v^*\Big) + \Big(\frac{p(u)}{u}v^* - \frac{p(u^*)}{u^*}v^*\Big)\right]\right\}dx \\
&\quad + \int_\Omega \left[A\sigma(v - v^*)\Big(-\frac{v}{u} + \frac{v^*}{u} - \frac{v^*}{u} + \frac{v^*}{u^*}\Big)\right]dx \\
&= \int_\Omega \left[\frac{1}{p(u)}\big(g_u(\xi) - v^*\frac{d}{du}\big(\frac{p(u)}{u}\big)|_{u=\eta}\big)(u - u^*)^2\right]dx \\
&\quad + \int_\Omega \left[\frac{(u - u^*)(v - v^*)}{u}\Big(-1 + A\sigma\frac{v^*}{u^*}\Big) + \frac{(v - v^*)^2}{u}(-\sigma A)\right]dx
\end{aligned}$$

for some ξ and η. If we choose $A = \frac{1}{\sigma}$, then $-1 + A\sigma\frac{v^*}{u^*} = 0$. Thus $E_2(t) \leq 0$ since

$$g_u(\xi) - v^*\frac{d}{du}\big(\frac{p(u)}{u}\big)|_{u=\eta} \leq -\widehat{g} + \widehat{p}v^* = -\widehat{g} + \widehat{p}u^* \leq 0. \tag{3.5}$$

from the hypotheses (A1), (A3), and (A4).

Thus $\dot{E} \leq 0$ for all $t \geq 0$, which implies the desired result since the equality holds if and only if when $(u, v) = (u^*, v^*)$. That is

$$\lim_{t\to+\infty} \|u(x,t) - u^*\|_Y = 0, \quad \lim_{t\to+\infty} \|v(x,t) - v^*\|_Y = 0.$$

$\square$

Remark 3.2. If $\frac{d}{du}\big(\frac{p(u)}{u}\big) \equiv 0$, then (3.5) is always satisfied since $g_u(\xi) < 0$, and the same result holds when $\frac{k}{2} < u^*$. It points out that the predator-prey models with Leslie-Gower functional response [13] satisfy the condition $\frac{d}{du}\big(\frac{p(u)}{u}\big) \equiv 0$.

3.2. **Effect of prey-taxis on the stability of (u^*, v^*).** The linearization of (1.1) at $e^* = (u^*, v^*)$ can be expressed as

$$\Psi_t = L(\chi)\Psi := G\Delta\Psi + J\Psi$$

with domain Y, where

$$\Psi(x) = \begin{pmatrix}\phi(x) \\ \psi(x)\end{pmatrix} \in Y, \quad G = \begin{pmatrix} d_1 & 0 \\ -\chi\alpha(v^*)v^* & d_2 \end{pmatrix}, \quad J = \begin{pmatrix} M & -p(u^*) \\ \sigma & -\sigma \end{pmatrix}$$

and $M = g(u^*) + u^*g_u(u^*) - p_u(u^*)v^*$.

$$\operatorname{tr}(G) = d_1 + d_2 > 0, \quad \det(G) = d_1d_2 > 0, \tag{3.6}$$
$$\operatorname{tr}(J) = M - \sigma, \quad \det(J) = \sigma(p(u^*) - M). \tag{3.7}$$

Then the eigenvalue λ of $L(\chi)\Psi = \lambda\Psi$ can be obtained from the Fourier decomposition of the matrix $L_i(\chi)$, where

$$L_i(\chi) = \begin{pmatrix} -d_1\mu_i + M & -p(u^*) \\ C + \chi\alpha(v^*)v^*\mu_i & -d_2\mu_i - \sigma \end{pmatrix}, \quad (i = 0, 1, 2, \dots)$$

and μ_i is the eigenvalue of operator $-\Delta$ under Neumann boundary conditions, which implies that eigenvalues $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots$ and $\lim_{i\to\infty}\mu_i = \infty$.

$$\operatorname{tr}(L_i) = -(d_1 + d_2)\mu_i + M - \sigma = -\operatorname{tr}(G)\mu_i + \operatorname{tr}(J), \tag{3.8}$$

$$\det(L_i) = d_1 d_2 \mu_i^2 + (d_1\sigma - d_2 M + p(u^*)\chi\alpha(v^*)v^*)\mu_i + (p(u^*) - M)\sigma$$
$$= \det(G)\mu_i^2 + F(J,G)\mu_i + \det(J), \tag{3.9}$$

where

$$F(J,G) = -(d_2 A + d_1 D + B\chi\alpha(v^*)v^*) = -(d_2 M + d_1\sigma - p(u^*)\chi\alpha(v^*)v^*).$$

The expression of $\det(L_i)$ lead to its minimum $\min_{\mu\in R^+}\det(L_i)$:

$$\min_{\mu\in R^+}\det(L_i) = \frac{4d_1 d_2(p(u^*) - M)\sigma - (d_1\sigma - d_2 M + p(u^*)\chi\alpha(v^*)v^*)^2}{4d_1 d_2}, \tag{3.10}$$

at

$$\mu = \mu^* = -\frac{d_1\sigma - d_2 M + p(u^*)\chi\alpha(v^*)v^*}{2d_1 d_2} = \frac{d_2 M - d_1\sigma - p(u^*)\chi\alpha(v^*)v^*}{2d_1 d_2}. \tag{3.11}$$

After direct calculations, we obtain the stability/instability of (u^*, v^*) for $\chi = 0$.

Theorem 3.3. *Let $d_1, d_2 > 0$ and Ω is a bounded domain with smooth boundary. Assume $\sigma > M > 0$ and $p(u^*) > M$.*

(1) *If $\sigma \geq \frac{d_2}{d_1}M$ for any $d_1 > 0, d_2 > 0$, (u^*, v^*) is locally asymptotically stable in Y.*

(2) *If $\max\{M, \sigma_1\} < \sigma < \min\{\frac{d_2 M}{d_1}, \sigma_2\}$, (u^*, v^*) is locally asymptotically stable in Y.*

(3) *If $M < \sigma < \sigma_1$ or $\sigma_2 < \sigma < \frac{d_2 M}{d_1}$, $e^* = (u^*, v^*)$ is unstable in Y, where*
$$\sigma_1 = \frac{d_2}{d_1}\left(\sqrt{p(u^*)} - \sqrt{p(u^*) - M}\right)^2 \text{ and } \sigma_2 = \frac{d_2}{d_1}\left(\sqrt{p(u^*)} + \sqrt{p(u^*) - M}\right)^2.$$

Next, we study the effect that the prey-taxis χ has on the stability of (u^*, v^*) for different parameter ranges.

Theorem 3.4. *Let $d_1, d_2 > 0$ and Ω be a bounded domain with smooth boundary. Assume $\sigma > M > 0$ and $p(u^*) > M$. If $\sigma \geq \frac{d_2}{d_1}M$, then (u^*, v^*) is locally asymptotically stable for any $\chi > 0$.*

Proof. From (3.7), $\operatorname{tr}(J) < 0$ as $\sigma > M$. From (3.8), it is noticed that $\operatorname{tr}(L_i) = -(d_1 + d_2)\mu_i + \operatorname{tr}(J) < 0$ for $i = 0, 1, 2, 3, \ldots$. From (3.9), $\det(L_0) = \det(J) > 0$ as $p(u^*) > M$. According to (3.9), by $\sigma \geq \frac{d_2}{d_1}M$ and $\chi > 0$, we have $F(J,G) > 0$, which gives rise to $\det(L_i) = \det(G)\mu_i^2 + F(J,G)\mu_i + \det(J) > 0$, $i = 0, 1, 2, 3, \ldots$. Thus each eigenvalue of $\det(L_i)$ has a negative real part, and (u^*, v^*) is locally asymptotically stable for any $\chi > 0$ from [10]. $\square$

Moreover, the case of $\sigma < d_2 M/d_1$ can be stated as follows.

Theorem 3.5. *Let σ_1 and σ_2 be the smaller and larger roots of $\min_{\mu\in R^+}\det(L_i) = 0$. Assume $d_1, d_2 > 0$ and $p(u^*) > M$. If*

$$\max\{M, \sigma_1\} < \sigma < \min\left\{\frac{d_2 M}{d_1}, \sigma_2\right\},$$

then $e^* = (u^*, v^*)$ is locally asymptotically stable for system (1.1) for any $\chi > 0$.

Proof. According to (3.7), by $\sigma > M$, we have $\operatorname{tr}(J) < 0$, which gives rise to $\operatorname{tr}(L_i) = -(d_1 + d_2)\mu_i + \operatorname{tr}(J) < 0$, $i = 0, 1, 2, 3, \ldots$.

If $d_2 M - d_1\sigma - p(u^*)\chi\alpha(v^*)v^* > 0$, i.e. $\chi < \frac{d_2 M - d_1\sigma}{p(u^*)\alpha(v^*)v^*}$, then

$$\frac{4d_1 d_2(p(u^*) - M)\sigma - (d_1\sigma - d_2 M)^2}{4d_1 d_2} > 0$$

for $\sigma_1 < \sigma < \sigma_2$. Thus we get $\det(L_i) > 0$ for $i = 0, 1, 2, 3, \ldots$, which implies that the constant positive equilibrium solution (u^*, v^*) is locally asymptotically stable. If $d_2 M - d_1 \sigma - p(u^*)\chi\alpha(v^*)v^* \leq 0$, then $\det(L_0) = \det(J) > 0$ from (3.9). It is easy to find that $\det(L_i) > 0$ for $i = 0, 1, 2, 3, \ldots$, thus (u^*, v^*) is still locally asymptotically stable. $\qquad\square$

Similarly, we have the following stability change for different $\chi > 0$. For convenience, we denote

$$D_1 := \frac{d_2 M - d_1 \sigma - 2\sqrt{d_1 d_2 (p(u^*) - M)\sigma}}{p(u^*)\alpha(v^*)v^*},$$

$$D_2 := \frac{d_2 M - d_1 \sigma + 2\sqrt{d_1 d_2 (p(u^*) - M)\sigma}}{p(u^*)\alpha(v^*)v^*}$$

Theorem 3.6. *Let σ_1 and σ_2 be the smaller and larger roots of $\min_{\mu \in R^+} \det(L_i) = 0$. Assume $d_1, d_2 > 0$ and $p(u^*) > M$. If*

$$M < \sigma < \sigma_1 \quad or \quad \sigma_2 < \sigma < \frac{d_2 M}{d_1},$$

then

(1) *$e^* = (u^*, v^*)$ is locally asymptotically stable for $\chi > D_1$;*
(2) *$e^* = (u^*, v^*)$ is unstable for $\chi \leq D_1$.*

Comparing Theorems 3.6 and 3.3, the positive prey tactic coefficient χ makes $e^* = (u^*, v^*)$ from being unstable to being stable. Furthermore, we can find the negative prey tactic $\chi < 0$ also has significant influence on the stability/instability of (u^*, v^*).

Theorem 3.7. *Assume $0 < M < \min\{p(u^*), \frac{\sigma d_1}{d_2}\}$. Then (u^*, v^*) is unstable for (1.1) if*

$$\chi \leq D_1 < 0, \tag{3.12}$$

or

$$D_2 \leq \chi < 0. \tag{3.13}$$

Proof. It is easy to find that (3.10) is smaller than zero when (3.12) or (3.13) holds. $\qquad\square$

Remark 3.8. Note that $e^* = (u^*, v^*)$ is globally asymptotically stable when $p(u^*) > M > 0$ and $\sigma \geq \frac{d_2}{d_1} M$ for $\chi = 0$ from Theorem 3.3. While the negative prey tactic coefficient χ satisfying (3.12) or (3.13) changes the stability of $e^* = (u^*, v^*)$ completely.

Furthermore, we have the stability change of e^* for $M > \frac{\sigma d_1}{d_2}$ if $\chi < 0$.

Theorem 3.9. *Assume $p(u^*) > M > 0$.*

(1) *If*

$$\max\{M, \sigma_1\} < \sigma < \min\left\{\frac{d_2 M}{d_1}, \sigma_2\right\},$$

then $e^* = (u^*, v^*)$ becomes unstable for $\chi < D_1 < 0$.
(2) *If*

$$M < \sigma < \sigma_1 \quad or \quad \sigma_2 < \sigma < \frac{d_2 M}{d_1},$$

then $e^* = (u^*, v^*)$ remains unstable for any $\chi < 0$.

TABLE 1. The stability/instability of (u^*, v^*) for different χ and other parameters.

χ	$p(u^*) < M$	$\sigma = \frac{d_2}{d_1}M > M, d_2 > d_1 > 0$	$\sigma > \frac{d_2}{d_1}M, d_2, d_1 > 0$	$\frac{d_2}{d_1}M > \sigma > M, d_2, d_1 > 0$	
$\chi = 0$	(u^*, v^*) is stable.	(u^*, v^*) is stable.	(u^*, v^*) is stable.	when $\sigma_* < \sigma < \sigma^*$, (u^*, v^*) is stable.	when $M < \sigma < \sigma_1$ or $\sigma_2 < \sigma < \frac{d_2 M}{d_1}$, (u^*, v^*) is unstable.
$\chi > 0$	(u^*, v^*) is stable.	(u^*, v^*) is stable.	(u^*, v^*) is stable.	when $\sigma_* < \sigma < \sigma^*$, (u^*, v^*) is stable for any $\chi > 0$.	when $M < \sigma < \sigma_1$ or $\sigma_2 < \sigma < \frac{d_2 M}{d_1}$, (u^*, v^*) is stable, for $\chi > D_1$.
$\chi < 0$	(u^*, v^*) is unstable.	(u^*, v^*) is unstable for $\chi < D_3 < 0$. (u^*, v^*) is stable for $0 > \chi > D_3$.	$D_2 \leq \chi < 0$, (u^*, v^*) is unstable. $\chi \leq D_1 < 0$, (u^*, v^*) is unstable.	when $\sigma_* < \sigma < \sigma^*$, $\chi \leq D_1 < 0$, (u^*, v^*) is unstable.	when $M < \sigma < \sigma_1$ or $\sigma_2 < \sigma < \frac{d_2 M}{d_1}$, (u^*, v^*) is unstable for any $\chi < 0$.

Comparing the conditions in Theorems 3.3 and 3.9, the negative prey tactic coefficient χ can exacerbate the instability of $e^* = (u^*, v^*)$. Summarizing the above analysis, the stability/instability of (u^*, v^*) can be listed in Table 1 for different

χ and other parameters. For convenience, we denote $\sigma_* = \max\{M, \sigma_1\}$, $\sigma^* = \min\left\{\frac{d_2 M}{d_1}, \sigma_2\right\}$, $D_3 := \frac{-2\sqrt{d_1 d_2 (p(u^*) - M)\sigma}}{p(u^*)\alpha(v^*)v^*}$ in Table 1.

The results for system (1.1) satisfies the following conditions.

Theorem 3.10. *Let $d_1, d_2 > 0$ and Ω is a bounded domain with smooth boundary.*

(1) *If $M \le 0$, (u^*, v^*) is locally asymptotically stable for any χ.*
(2) *If $M > 0$ and $\sigma \le M$, (u^*, v^*) is unstable for any χ.*
(3) *If $\sigma > M > 0$ and $p(u^*) < M$, (u^*, v^*) is unstable for any χ.*
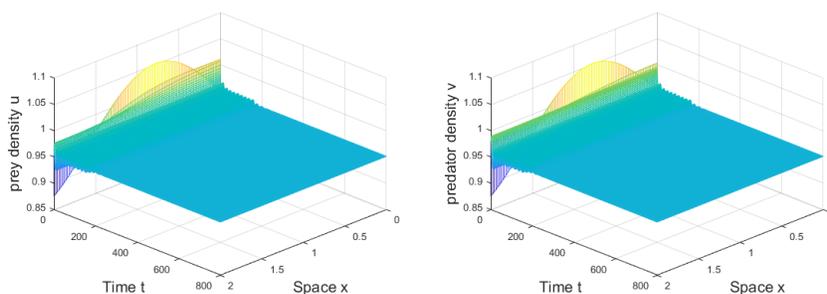
FIGURE 1. u and v components of system (1.1) with $\chi = 0$ and initial value $(u_0(x), v_0(x)) = (0.9512 + 0.1 * rand; 0.9512 + 0.1 * rand)$. The positive equilibrium $(u^*, v^*) = (0.9512, 0.9512)$ is asymptotically stable.
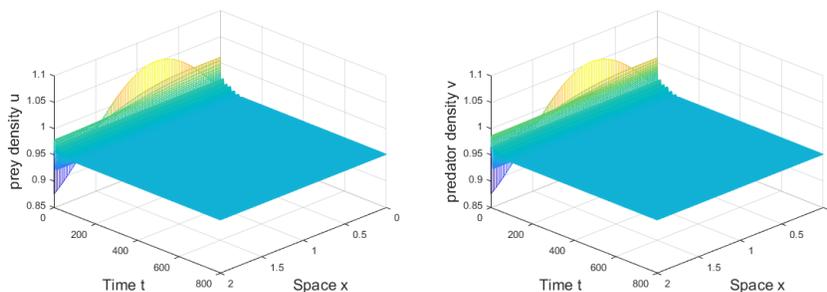
FIGURE 2. u and v components of system (1.1) with $\chi = 1.5579$ and initial value $(u_0(x), v_0(x)) = (0.9512 + 0.1 * rand; 0.9512 + 0.1 * rand)$. The positive equilibrium $(u^*, v^*) = (0.9512, 0.9512)$ is asymptotically stable.

4. Conclusion and simulation

In this paper, we discuss a diffusive predator-prey model with prey-taxis subject to homogeneous Neumann boundary conditions. The global existence and boundedness of nonnegative solution of (1.1) is obtained for every prey tactic. The global stability of positive constant equilibrium remains for small prey tactic, see Theorem
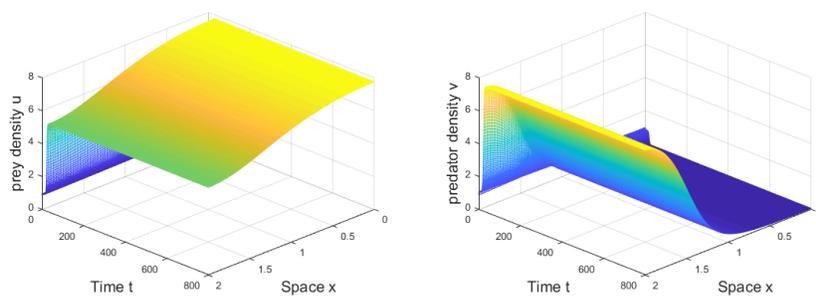
FIGURE 3. u and v components of system (1.1) with $\chi = -8$ and initial value $(u_0(x), v_0(x)) = (0.9512 + 0.1 * rand; 0.9512 + 0.1 * rand)$. The positive equilibrium (u^*, v^*) becomes unstable.
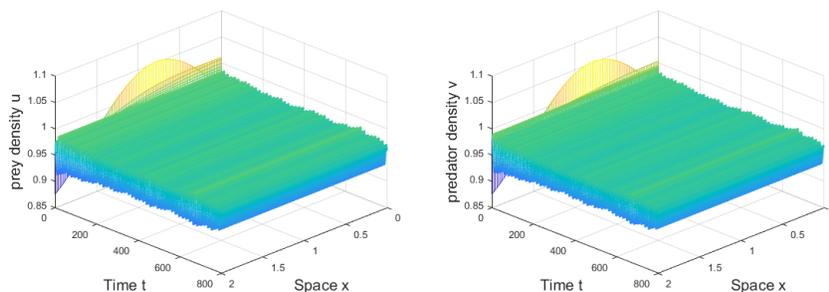
FIGURE 4. u and v components of system (1.1) with $\chi = 0$ and initial value $(u_0(x), v_0(x)) = (0.9512 + 0.1 * rand; 0.9512 + 0.1 * rand)$. The positive equilibrium $(u^*, v^*) = (0.9512, 0.9512)$ is unstable.
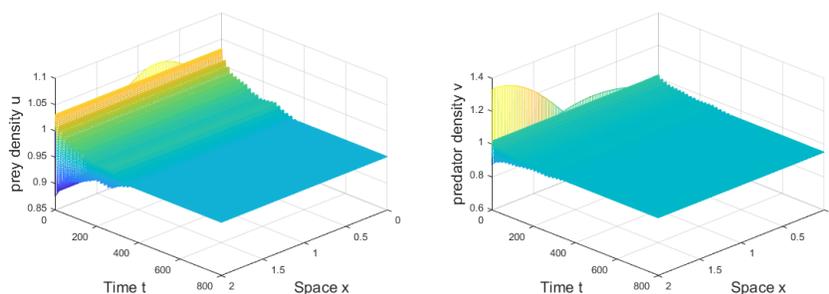
FIGURE 5. u and v components of system (1.1) with $\chi > D_1 = 298.99725$ and initial value $(u_0(x), v_0(x)) = (0.9512 + 0.1 * rand; 0.9512 + 0.1 * rand)$. The positive equilibrium $(u^*, v^*) = (0.9512, 0.9512)$ is asymptotically stable.
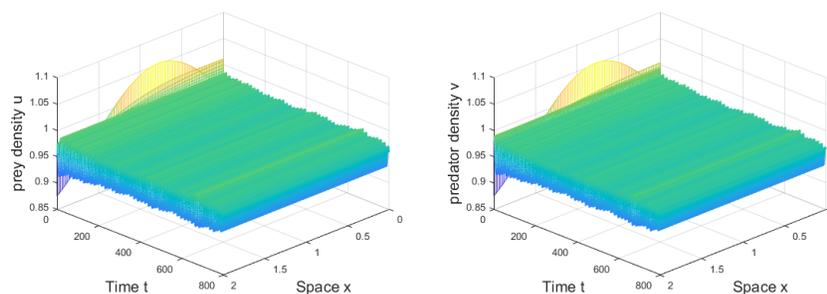
FIGURE 6. u and v components of system (1.1) with $\chi = -0.00001 < 0$ and initial value $(u_0(x), v_0(x)) = (0.9512 + 0.1 * rand; 0.9512 + 0.1 * rand)$. The positive equilibrium $(u^*, v^*) = (0.9512, 0.9512)$ is unstable.

3.1. From Theorems 3.3, 3.4, 3.5, 3.6, 3.7 and 3.9, we find the positive prey tactic coefficient can maintain the stability of the positive constant equilibrium, while negative prey tactic coefficient can lead to the instability of the positive constant equilibrium. The results are applicable in the case of linear functional response $g(u) = p - bu$ and Holling-Tanner type $p(u) = \frac{u}{u+a}$, which satisfy (A1) and (A2). Taking $p = 1$, $a = 0.1$, $b = 0.1$, $d_1 = 1$, $d_2 = 3$, $M = 0.7237$, we can illustrate the above results for different value of σ:

(1) $\sigma = 0.83$. Then $\frac{d_2 M}{d_1} = 2.1710, \sigma_1 = 0.8287, \sigma_2 = 5.6878$. In this case, $\max\{M, \sigma_1\} < \sigma < \min\{\frac{d_2 M}{d_1}, \sigma_2\}$, then the constant positive equilibrium solution (u^*, v^*) is locally asymptotically stable for $\chi = 0$ by Theorem 3.3, see Figure 1. According to Theorem 3.5, the positive equilibrium (u^*, v^*) remains stable for positive prey tactic, see Figure 2; while (u^*, v^*) becomes unstable for negative prey tactic from Theorem 3.9, see Figure 3.

(2) $\sigma = 0.76$. In this case, $M < \sigma < \sigma_1$, then the constant positive equilibrium $e^* = (u^*, v^*)$ is unstable for $\chi = 0$, by Theorem 3.3, see Figure 4. While (u^*, v^*) becomes stable for positive prey tactic from Theorem 3.6, see Figure 5. We observe that (u^*, v^*) is unstable for any $\chi < 0$ from Theorem 3.9, see Figure 6.

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