PERIODIC SOLUTIONS OF STOCHASTIC VOLTERRA EQUATIONS

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ABSTRACT. This article concerns the dynamical behavior of solutions to stochastic Volterra equations. We prove the existence of periodic solutions in distribution of stochastic Volterra equations. We use the Banach fixed point theorem and a Krasnoselski-Schaefer type fixed point theorem.

1. INTRODUCTION

The dynamical behavior of solutions to stochastic differential equations (SDEs for short), such as long time behavior, ergodicity, and periodicity, has been studied in [11, 17, 31]. This paper concerns the existence of periodic solutions in the distribution of stochastic Volterra equations. Stochastic Volterra integral equations arise in many scientific areas such as mathematical finance, biology, etc, see [15, 18]. There are many well-known results concerning the dynamical behavior of the solutions with non-singular kernels and the Lipschitz coefficients, see [4, 29, 30]. The solutions of stochastic Volterra equations with singular kernel in finite-dimensional space have been studied in [10, 13, 34]. For the infinite-dimensional case, the existence, uniqueness, and large deviation estimate for stochastic Volterra equations in Banach spaces were studied in [37]. For more details about these topics and recent developments, we refer to [28] and references therein.

It is well-known that periodic solutions are one of the fundamental problems in the qualitative theory of differential equations [24]. For the stochastic system, the existence of periodic solutions is also worth studying and discussing. Because of the effects of diffusion, we should study periodic solutions for SDEs in the distribution sense. In other words, obtaining periodic solutions in the probability or moment for SDEs is impossible, see [25]. Khasminskii [23] defined periodic solutions in the sense of periodic Markov processes and obtained the existence results via the Lyapunov method. Chen et al. [8] established Halanay's criterion of SDEs and obtained periodic solutions in distribution. Ji et al. [20, 21] studied periodic probability solutions to the Fokker-Planck equations. The existence of periodic solutions for semilinear SDEs has been studied in [27, 16] and references therein. Cheban and Liu [9] studied the problem of Poisson stability (in particular, stationarity, periodicity, quasi-periodicity, Bohr almost periodicity, Bohr almost automorphy, Birkhoff recurrence, Poisson stability) of solutions for semi-linear SDEs. Zhao and

Zheng [38] and Feng et al. [14] studied the existence of random periodic solutions of random dynamical systems. Based on the technique of upper and lower solutions and comparison principle, Ji et al. [19] obtained the existence of periodic solutions in distribution for stochastic differential equations. Via Wong-Zakai approximations, Jiang and Li [22] obtained the existence of periodic solutions in distribution for stochastic dissipative systems. Lv et al. [26] obtained the existence of periodic solutions of the probability density function corresponding to the stochastic process by using the technique of deterministic partial differential equations.

In this article, we consider the stochastic Volterra integral equation

$$x(t) = a(t, x(t)) + \int_{-\infty}^{t} K(t, s) f(s, x(s)) ds + \int_{-\infty}^{t} K(t, s) g(s, x(s)) dW(s), \quad (1.1)$$

where $K : L_{\text{loc}}^2(\mathbb{R} \times \mathbb{R}, \mathbb{R}^{d \times d})$ is a given kernel, $a : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ are continuous functions with additional properties which would be specified below, and W is m-dimensional Brownian motion.

Our first goal is to use the Banach fixed point theorem to obtain the existence and uniqueness of periodic solutions in distribution for stochastic Volterra integral equations (1.1) under the Lipschitz conditions and integrable conditions. Banach fixed point theorem is an effective method to study periodic solutions in distribution for semilinear SDEs under Lipschitz conditions, for example, see [9, 27, 16].

Our second goal is to study the existence of periodic solutions in distribution for stochastic Volterra integral equations (1.1) under the Carathéodory condition, which is a non-Lipschitz condition. We prove the existence of such system by the Krasnoselski-Schaefer type fixed point theorem. Carathéodory condition is widely used in the field of mathematics physics. Taniguchi [33] proved the existence and uniqueness of SDEs under a relatively weak non-Lipschitz assumptions for drift term and diffusion term by successive approximation, which includes the well-known result of [36]. Many authors have studied the property of solutions for SDEs under Carathéodory conditions, see Abouagwa and Li [2], Xu, Pei, and Wu [35], Shao, Wang, and Yuan [32], for more details about these topics and recent developments.

This article is organized as follows. In Section 2, we introduce some notation and definitions. In Section 3, we state and prove our main results by using the Banach fixed point theorem and Krasnoselski-Schaefer fixed point theorem. In Section 4, we provide an example to illustrate the applicability of our results.

2. Preliminaries

Throughout this article, let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be the complete probability space, with the filtration $\mathcal{F}_t$. Let us denote by $\mathbb{R}^{d \times d}$ the set of all $d \times d$-dimensional matrices, equipped with the norm $\| \cdot \|$ defined by $\|M\|^2 = \text{Trace}(MM^T)$ for all $M \in \mathbb{R}^{d \times d}$. The space $\mathbb{R}^d$ is equipped with the Euclidean norm, denoted by $| \cdot |$. Let $L^2(\mathbb{P}, \mathbb{R}^d)$ stand for the space of all $\mathbb{R}^d$-valued random variables X such that $\mathbf{E}|X|^2 = \int_{\Omega} |X|^2 d\mathbb{P} < \infty$. For $X \in L^2(\mathbb{P}, \mathbb{R}^d)$, let $\|X\|_2 = (\int_{\Omega} |X|^2 d\mathbb{P})^{1/2}$. Denote by $C_b(\mathbb{R}, L^2(\mathbb{P}, \mathbb{R}^d))$ the Banach space of all bounded L^2-continuous mappings from $\mathbb{R}$ to $L^2(\mathbb{P}, \mathbb{R}^d))$ endowed with the norm $\| \cdot \|_\infty$ defined by $\|Y\|_\infty := \sup_{t \in \mathbb{R}} \|Y(t)\|_2$ for all $Y \in C_b(\mathbb{R}, L^2(\mathbb{P}, \mathbb{R}^d))$. Let $\mathcal{P}(\mathbb{R}^d)$ be the set of Borel probability measures on $\mathbb{R}^d$. For a given $\mathbb{R}^d$-valued random variable X, let p_X be the distribution of X on $\mathbb{R}^d$. For a given stochastic process, the law of X on $\mathbb{R}^d$ is denoted by $\mu : \mathbb{R} \to \mathcal{P}(\mathbb{R}^d)$.

Following [8], we define

$$\|h\|_\infty = \sup_{x\in\mathbb{R}^d} |h(x)|,$$

$$\|h\|_L = \sup \big\{ \frac{|h(x)-h(y)|}{|x-y|}; x, y\in\mathbb{R}^d, x\neq y \big\},$$

$$\|h\|_{BL} = \max\{\|h\|_\infty, \|h\|_L\},$$

$$d_{BL}(\mu,\nu) = \sup_{\|h\|_{BL}\leq 1} |\int h\,d(\mu-\nu)|$$

for all $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and all Lipschitz continuous real-valued functions h on $\mathbb{R}^d$. It is known that $(d_{BL}, \mathcal{P}(\mathbb{R}^d))$ is a complete metric space. Now we show the definition of the stochastic periodic solution in the distribution sense.

Definition 2.1 ([8]). Let $x(t)$ be a solution of (1.1). Suppose that a, K, f, and g are $\mathcal{T}$-periodic with respect to t and $x(t)$ satisfies the following conditions

 (i) $p_{x(t)} = p_{x(t+\mathcal{T})}$ for all $t \in \mathbb{R}_+$;
 (ii) There exists $\tilde{W}$, which has the same distribution as W, such that $x(t+\mathcal{T})$ is a solution of the equation

$$x(t) = a(t, x(t)) + \int_{-\infty}^{t} K(t,s)f(s,x(s))ds + \int_{-\infty}^{t} K(t,s)g(s,x(s))d\tilde{W}(t).$$

Then $x(t)$ is said to be a stochastic periodic solution in distribution.

We present the Krasnoselskii-Schaefer's fixed point theorem, which is one of the key tools in our paper.

Lemma 2.2 ([5]). *Let S be a Banach space, and $\Xi_1, \Xi_2 : S \to S$ be two operators satisfying: Ξ_1 is a contraction, and Ξ_2 is completely continuous. Then, either the operator equation $x = \Xi_1 x + \Xi_2 x$ has a solution, or the set $\Upsilon = \{x \in S : \lambda\Xi_1(\frac{x}{\lambda}) + \lambda\Xi_2 x = x\}$ is unbounded for some $\lambda \in (0,1)$.*

The modified Bihari-type integral inequality was given in [7, 12] as follows. Suppose the $l \geq 0$ and $u \in BC(\mathbb{R}, \mathbb{R}^+)$ satisfies

$$u(t) \leq l + \int_{\mathbb{R}} K_1(t,s)\gamma_1(s)\gamma_3(u(s))ds + \int_{-\infty}^{t} K_2(t,s)\gamma_2(s)\gamma_3(u(s))ds \qquad (2.1)$$

for all $t \in \mathbb{R}$. Here $K_1, K_2 \in C(\mathbb{R}\times\mathbb{R}, \mathbb{R}^+)$, $\gamma_1, \gamma_2 \in C(\mathbb{R}, \mathbb{R}^+)$. Let $\gamma_3 \in C(\mathbb{R}^+, \mathbb{R}^+)$ be a nondecreasing function such that

$$\int_a^b \gamma_3(r^{1/q})^{-q}dr < +\infty, \qquad \int_a^{+\infty} \gamma_3(r^{1/q})^{-q}dr = +\infty,$$

where $q \geq 1, 0 < a < b < +\infty$. Moreover, suppose that there exists a continuous function $\sigma : \mathbb{R} \to (0,\infty)$ such that

$$N_1 = \sup_{t\in\mathbb{R}} \int_{\mathbb{R}} K_1(t,s)^p \sigma(s)^p ds < +\infty, \qquad (2.2)$$

$$N_2 = \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} K_2(t,s)^p \sigma(s)^p ds < +\infty, \qquad (2.3)$$

$$N_3 = \int_{\mathbb{R}} \sigma(s)^{-q}(\gamma_1(s)^q + \gamma_2(s)^q)ds < +\infty, \qquad (2.4)$$

where $1 < p \leq \infty$, $1 \leq q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For a given $\epsilon > 0$, let $0 < u_0 < 3^{q/p}\epsilon^q$ be fixed and we define

$$G(x) = \int_{u_0}^{x} \gamma_3(r^{1/q})^{-q} dr, x \geq u_0,$$

$$H(x) = G(2x - 3^{q/p}\epsilon^q) - G(x) - \int_{\mathbb{R}} \sigma(s)^{-q} P(s) ds, x \geq \frac{u_0 + 3^{q/p}\epsilon^q}{2},$$

where

$$P(x) = (3N_1 + 3N_2)^{q/p}(\gamma_1(x)^q + \gamma_2(x)^q).$$

If $l > 0$, define G, H with $\epsilon = l$ and assume that H is nondecreasing for $x \geq 3^{q/p}l^q$ and $H(c_0) = 0$ for some $c_0 > 3^{q/p}l^q$. If $l = 0$, assume that there exists $\epsilon_0 > 0$ such that G and H are defined with $\epsilon = \epsilon_0$ and H is nondecreasing for $x \geq 3^{q/p}\epsilon_0^q$ and $H(c_0) = 0$ for some $c_0 > 3^{q/p}\epsilon_0^q$.

Lemma 2.3 ([7, Theorem 2.3]). *Let $1 < p \leq \infty$, $1 \leq q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, and $l \geq 0$. Suppose that $u \in BC(\mathbb{R}, \mathbb{R}^+)$ satisfies (2.1) for all $t \in \mathbb{R}$, where $\gamma_1, \gamma_2, \gamma_3, K_1, K_2$ are defined as above. Then*

$$\|u^q\|_{\infty} \leq G^{-1}[G(c_0) + (3N_1 + 3N_2)^{q/p}N_3].$$

3. EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTION OF (1.1)

In this section, we employ the Banach fixed point theorem on (1.1), and obtain a unique periodic solution. In addition to the continuous conditions on functions a, f, g, and K, we will need to impose some of the following assumptions.

(H1) There exists $\mathcal{T} > 0$, such that $a(t + \mathcal{T}) = a(t)$, $f(t + \mathcal{T}, x) = f(t, x)$, $g(t + \mathcal{T}, x) = g(t, x)$, and $K(t + \mathcal{T}, s + \mathcal{T}) = K(t, s)$ for all $t, s \in \mathbb{R}, x \in \mathbb{R}^d$.

(H2) There exists a constant $L_a > 0$, such that

$$\|a(t, x) - a(t, y)\| \leq L_a \|x - y\|,$$

where $t \in \mathbb{R}, x, y \in \mathbb{R}^d$.

(H3) There exist constants $L_f > 0$ and $L_g > 0$, such that

$$\|f(t, x) - f(t, y)\| \leq L_f \|x - y\|,$$
$$\|g(t, x) - g(t, y)\| \leq L_g \|x - y\|,$$

where $t \in \mathbb{R}$, $x, y \in \mathbb{R}^d$.

(H4) There exists a constant $C^* > 0$, such that

$$\sup_{t \in \mathbb{R}} \int_{-\infty}^{t} \|K(t, s)\|^p ds \leq C^*,$$

where $p \geq 1$.

(H5) There exists a constant $\tilde{C} > 0$, such that for all $0 \leq u \leq v \leq \mathcal{T}$

$$\int_{-\infty}^{u} \|K(u, s) - K(v, s)\|^p ds \leq \tilde{C}|u - v|,$$

where $p \geq 1$.

Theorem 3.1. *Assume* (H1)–(H5) *hold , and that*

$$3L_a + 3C^*(C^* + 1)(L_f^2 + L_g^2) < 1.$$

Then (1.1) *admits a unique continuous $\mathcal{T}$-periodic solution in distribution.*

Proof. Firstly, we show that $(\mathcal{S}\varphi)(t)$ is continuous with respect to t. Consider the nonlinear operator $\mathcal{S}$ defined on $C_b(\mathbb{R}, L^2(\mathbb{P}, \mathbb{R}^d))$:

$$(\mathcal{S}\varphi)(t) = a(t, \varphi(t)) + \int_{-\infty}^{t} K(t, s) f(s, \varphi(s)) ds + \int_{-\infty}^{t} K(t, s) g(s, \varphi(s)) dW(s).$$

If $\varphi(t)$ is an L^2 bounded solution of (1.1), one obtains that

$$\mathbf{E}\|(\mathcal{S}\varphi)(u) - (\mathcal{S}\varphi)(v)\|^2$$

$$\leq 5\mathbf{E}\|a(u, \varphi(u)) - a(v, \varphi(v))\|^2 + 5\mathbf{E}\Big\| \int_{-\infty}^{u} (K(u, s) - K(v, s)) f(s, \varphi(s)) ds \Big\|^2$$

$$+ 5\mathbf{E}\Big\| \int_{u}^{v} K(v, s) f(s, \varphi(s)) ds \Big\|^2$$

$$+ 5\mathbf{E}\Big\| \int_{-\infty}^{u} (K(u, s) - K(v, s)) g(s, \varphi(s)) dW(s) \Big\|^2$$

$$+ 5\mathbf{E}\Big\| \int_{u}^{v} K(v, s) g(s, \varphi(s)) dW(s) \Big\|^2$$

$$:= 5I_1 + 5I_1 + 5I_2 + 5I_3 + 5I_4 + 5I_5,$$

where $-\infty < u \leq v \leq \mathcal{T}$. Since $a(t)$ is continuous and $\mathcal{T}$-periodic with respect to t, we have

$$I_1 \leq C\|u - v\|^2,$$

where C is a constant.

Since $f(t, \varphi(t))$ is Lipschitz in φ and periodic in t and is L^2 bounded, we have

$$\sup_{t \in \mathbb{R}} \|f(t, \varphi(t))\| \leq \sup_{t \in \mathbb{R}} \|f(t, \varphi(t)) - f(t, 0)\| + \sup_{t \in \mathbb{R}} \|f(t, 0)\|$$

$$\leq L_f \|\varphi\| + \sup_{t \in \mathbb{R}} \|f(t, 0)\| < +\infty.$$

Since $\sup_{t \in \mathbb{R}} \|f(t, \varphi(t))\| = \sup_{t \in \mathbb{R}} (\mathbf{E}\|f(t, \varphi(t))\|^2)^{1/2}$, there exists M_1 such that

$$\sup_{s \in \mathbb{R}} \mathbf{E}\|f(s, \varphi(s))\|^2 < M_1,$$

$$\sup_{s \in \mathbb{R}} \mathbf{E}\|g(s, \varphi(s))\|^2 < M_2,$$

where M_1 and M_2 are constants. By Cauchy-Schwarz's inequality, we have

$$I_2 \leq \int_{-\infty}^{u} \|K(u, s) - K(v, s)\| ds \mathbf{E} \int_{-\infty}^{u} \|K(u, s) - K(v, s)\| \|f(s, \varphi(s))\|^2 ds$$

$$\leq \sup_{s \in \mathbb{R}} \mathbf{E}\|f(s, \varphi(s))\|^2 \Big(\int_{-\infty}^{u} \|K(u, s) - K(v, s)\| ds \Big)^2$$

$$\leq M_1 \tilde{C}^2 |u - v|^2.$$

Since K is continuous, there exists a constant $\bar{C}$ such that if $t, s \in \mathbb{R}$, then $\|K(t, s)\| \leq \bar{C}$, a.e.

$$I_3 \leq M_1 \bar{C}^2 |u - v|^2.$$

By Itô's isometry formula, we have

$$I_4 \leq \int_{-\infty}^{u} \|K(u, s) - K(v, s)\|^2 \mathbf{E}\|g(s, \varphi(s))\|^2 ds$$

$$\leq \sup_{s \in \mathbb{R}} \mathbf{E} \|g(s, \varphi(s))\|^2 \int_{-\infty}^{u} \|K(u, s) - K(v, s)\|^2 ds$$

$$\leq M_2 \tilde{C} |u - v|.$$

Similarly,

$$I_5 \leq M_2 \bar{C}^2 |u - v|.$$

Then

$$\mathbf{E} \|(\mathcal{S}\varphi)(u) - (\mathcal{S}\varphi)(v)\|^2 \leq C(|u - v|^2 + |u - v|),$$

where C is a constant. This shows that $(\mathcal{S}\varphi)(t)$ is continuous in t.

Secondly, we show that $\mathcal{S}$ is a contraction mapping on $C_b(\mathbb{R}, L^2(\mathbb{P}, \mathbb{R}^d))$. For $\varphi, \psi \in C_b(\mathbb{R}, L^2(\mathbb{P}, \mathbb{R}^d))$ and each $t \in \mathbb{R}$, we have

$$\mathbf{E} \|(\mathcal{S}\varphi)(t) - (\mathcal{S}\psi)(t)\|^2$$

$$= 3\mathbf{E} \|a(t, \varphi) - a(t, \psi)\|^2 + 3\mathbf{E} \Big\| \int_{-\infty}^{t} K(t, s)(f(s, \varphi(s)) - f(s, \psi(s))) ds \Big\|^2$$

$$+ 3\mathbf{E} \Big\| \int_{-\infty}^{t} K(t, s)(g(s, \varphi(s)) - g(s, \psi(s))) dW(s) \Big\|^2$$

$$\leq 3L_a^2 \|\varphi - \psi\|^2 + 3 \int_{-\infty}^{t} \|K(t, s)\| ds$$

$$\times \mathbf{E} \int_{-\infty}^{t} \|K(t, s)\| \|(f(s, \varphi(s)) - f(s, \psi(s))\|^2 ds$$

$$+ 3\mathbf{E} \int_{-\infty}^{t} \|K(t, s)(g(s, \varphi(s)) - g(s, \psi(s)))\|^2 ds$$

$$\leq 3L_a^2 \|\varphi - \psi\|^2 + 3 \Big(\int_{-\infty}^{t} \|K(t, s)\| ds \Big)^2 L_f^2 \sup_{s \in \mathbb{R}} \mathbf{E} \|\varphi(s) - \psi(s)\|^2$$

$$+ 3 \int_{-\infty}^{t} \|K(t, s)\|^2 ds L_g^2 \sup_{s \in \mathbb{R}} \mathbf{E} \|\varphi(s) - \psi(s)\|^2$$

$$\leq (3L_a + 3C^*(C^* + 1)(L_f^2 + L_g^2)) \sup_{s \in \mathbb{R}} \mathbf{E} \|\varphi(s) - \psi(s)\|^2.$$

Since $3L_a + 3C^*(C^* + 1)(L_f^2 + L_g^2) < 1$, $\mathcal{S}$ is a contraction on $C_b(\mathbb{R}, L^2(\mathbb{P}, \mathbb{R}^d))$. Then there exists a unique $v \in C_b(\mathbb{R}, L^2(\mathbb{P}, \mathbb{R}^d))$ such that $\mathcal{S}v = v$, which is the unique solution to (1.1).

Thirdly, we show that $v(t)$ is a $\mathcal{T}$-periodic solution in distribution to (1.1).

$$\begin{aligned}
v(t + \mathcal{T}) &= a(t + \mathcal{T}, v(t + \mathcal{T})) + \int_{-\infty}^{t+\mathcal{T}} K(t + \mathcal{T}, s) f(s, v(s)) ds \\
&\quad + \int_{-\infty}^{t+\mathcal{T}} K(t + \mathcal{T}, s) g(s, v(s)) dW(s) \\
&= a(t + \mathcal{T}, v(t + \mathcal{T})) + \int_{-\infty}^{t} K(t + \mathcal{T}, s + \mathcal{T}) f(s + \mathcal{T}, v(s + \mathcal{T})) ds \\
&\quad + \int_{-\infty}^{t} K(t + \mathcal{T}, s + \mathcal{T}) g(s + \mathcal{T}, v(s + \mathcal{T})) d\tilde{W}(s),
\end{aligned} \tag{3.1}$$

where $\tilde{W}(t) = W(t + \mathcal{T}) - W(\mathcal{T})$.

We consider the process $\hat{v}(t)$, which satisfies the equation

$$\hat{v}(t) = a(t + \mathcal{T}, \hat{v}(t)) + \int_{-\infty}^{t} K(t + \mathcal{T}, s + \mathcal{T})f(s + \mathcal{T}, \hat{v}(s))ds$$
$$+ \int_{-\infty}^{t} K(t + \mathcal{T}, s + \mathcal{T})g(s + \mathcal{T}, \hat{v}(s))dW(s). \tag{3.2}$$

Note that $v(t + \mathcal{T})$ has the same distribution as $\hat{v}(t)$. By (H1), we have

$$\mathbf{E}\|\hat{v}(t) - v(t)\|^2$$
$$\leq 3\mathbf{E}\|a(t + \mathcal{T}, \hat{v}(t)) - a(t, v(t))\|^2$$
$$+ 3\mathbf{E}\Big\| \int_{-\infty}^{t} K(t + \mathcal{T}, s + \mathcal{T})f(s + \mathcal{T}, \hat{v}(s))ds - \int_{-\infty}^{t} K(t, s)f(s, v(s))ds\Big\|^2$$
$$+ 3\mathbf{E}\Big\| \int_{-\infty}^{t} K(t + \mathcal{T}, s + \mathcal{T})g(s + \mathcal{T}, \hat{v}(s))dW(s)$$
$$- \int_{-\infty}^{t} K(t, s)g(s, v(s))dW(s)\Big\|^2$$
$$\leq 3\mathbf{E}\|a(t + \mathcal{T}, \hat{v}(t)) - a(t, v(t))\|^2$$
$$+ 3\mathbf{E}\Big\| \int_{-\infty}^{t} K(t, s)(f(s + \mathcal{T}, \hat{v}(s)) - f(s, v(s)))ds\Big\|^2$$
$$+ 3\mathbf{E}\Big\| \int_{-\infty}^{t} K(t, s)(g(s + \mathcal{T}, \hat{v}(s)) - g(s, v(s)))dW(s)\Big\|^2$$
$$\leq 3\mathbf{E}\|a(t + \mathcal{T}, \hat{v}(t)) - a(t, v(t))\|^2$$
$$+ 3\Big(\int_{-\infty}^{t} K(t, s)ds\Big)^2 \sup_{s \in \mathbb{R}} \mathbf{E}\|f(s + \mathcal{T}, \hat{v}(s)) - f(s, v(s))\|^2$$
$$+ 3\int_{-\infty}^{t} \|K(t, s)\|^2 ds \sup_{s \in \mathbb{R}} \mathbf{E}\|g(s + \mathcal{T}, \hat{v}(s)) - g(s, v(s))\|^2 = 0.$$

Since $v(t + \mathcal{T})$ has the same distribution with $\hat{v}(t)$, $v(t)$ is a $\mathcal{T}$-periodic solution in distribution to (1.1). This completes the proof. $\qquad\square$

The second result is established by using Krasnoselskii-Schaefer type fixed point theorem with the following Carathéodory conditions on f and g.

(H6) There exist functions $\alpha(t) : \mathbb{R} \to \mathbb{R}^+$ and $\Gamma(v) : \mathbb{R}^+ \to \mathbb{R}^+$ such that
    (1) $\Gamma(v)$ is continuous monotone non-decreasing concave and satisfy

$$\int_a^b \Gamma(r^{1/q})^{-q}dr < +\infty, \int_a^{+\infty} \Gamma(r^{1/q})^{-q}dr = +\infty,$$

       where $p \geq 1, 0 < a < b < +\infty$.
    (2) For any positive $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, the functions $f(t, x)$ and $g(t, x)$ are continuous in x and satisfy

$$\|f(t, x)\|^2 \leq \alpha(t)\Gamma(\|x\|^2), \quad \|g(t, x)\|^2 \leq \alpha(t)\Gamma(\|x\|^2).$$

    (3) There exists a constant $M^* > 0$, such that

$$\sup_{t \in \mathbb{R}} \int_{-\infty}^{t} \|K(t, s)\|^p \alpha(s)ds \leq M^*,$$

where $p \geq 1$.

(4) There exists a constant $\tilde{M} > 0$, such that for all $0 \leq u \leq v \leq \mathcal{T}$,

$$\int_{-\infty}^{u} |(K(u,s) - K(v,s))\alpha(s)|^p ds \leq \tilde{M}|u-v|.$$

Theorem 3.2. *Suppose* (H1), (H2), (H4)–(H6) *hold, and suppose* $3L_a^2 < 1$. *Then* (1.1) *has at least one continuous* $\mathcal{T}$-*periodic solution in distribution.*

Proof. We split the operator $\mathcal{S}$ in Theorem 3.1 into two parts:

$$(\mathcal{S}_1\varphi)(t) = a(t, \varphi(t)), \tag{3.3}$$

$$(\mathcal{S}_2\varphi)(t) = \int_{-\infty}^{t} K(t,s)f(s,\varphi(s))ds + \int_{-\infty}^{t} K(t,s)g(s,\varphi(s))dW(s). \tag{3.4}$$

Step 1. We show that $\mathcal{S}_1$ is a contraction mapping. For $u, v \in C_b(\mathbb{R}, L^2(\mathbb{P}, \mathbb{R}^d))$, we have

$$\mathbf{E}\|(\mathcal{S}_1 u)(t) - (\mathcal{S}_1 v)(t)\|^2 = \mathbf{E}\|a(t,u) - a(t,v)\|^2 \leq L_a^2 \mathbf{E}\|u-v\|^2. \tag{3.5}$$

Since $3L_a^2 < 1$, $\mathcal{S}_1$ is a contraction mapping.

Step 2. We prove that the operator $\mathcal{S}_2$ is completely continuous. The proof is divided into four steps.

Step 2.1. We show that $\mathcal{S}_2$ maps bounded sets $\mathfrak{B}_r = \{v : \|v(t)\|_2^2 \leq r\}$ into bounded sets. Indeed, we will show that there exists a positive constant q such that for each $w \in \mathfrak{B}_r$, one has that $\mathbf{E}\|(\mathcal{S}_2 w)(t)\|^2 \leq q$. For each $w \in \mathfrak{B}_r$, we have

$$\mathbf{E}\|(\mathcal{S}_2 w)(t)\|^2$$

$$\leq 2\mathbf{E}\Big\| \int_{-\infty}^{t} K(t,s)f(s,w(s))ds \Big\|^2 + 2\mathbf{E}\Big\| \int_{-\infty}^{t} K(t,s)g(s,w(s))dW(s) \Big\|^2$$

$$\leq 2 \int_{-\infty}^{t} \|K(t,s)\|ds\mathbf{E}\int_{-\infty}^{t} \|K(t,s)\|\|f(s,w(s))\|^2 ds$$

$$\quad + 2\mathbf{E}\int_{-\infty}^{t} \|K(t,s)\|^2\|g(s,w(s))\|^2 ds$$

$$\leq \Gamma(r)\Big(\int_{-\infty}^{t} \|K(t,s)\|ds \int_{-\infty}^{t} \|K(t,s)\|\alpha(s)ds + \int_{-\infty}^{t} \|K(t,s)\|^2\alpha(s)ds \Big)$$

$$\leq \Gamma(r)M^*(C^* + 1) := q.$$

Then we have for every $w \in \mathfrak{B}_r$, we have $\mathbf{E}\|(\mathcal{S}_2 w)(t)\|^2 \leq q$.

Step 2.2. We prove that $\mathcal{S}_2$ maps bounded sets $w \in \mathfrak{B}_r$ into equicontinuous sets. For $t, t+h \in \mathbb{R}, w \in \mathfrak{B}_r$, we have

$$\mathbf{E}\|(\mathcal{S}_2 w)(t+h) - (\mathcal{S}_2 w)(t)\|^2$$

$$\leq 4\mathbf{E}\Big\| \int_{-\infty}^{t} (K(t+h,s) - K(t,s))f(s,w(s))ds \Big\|^2$$

$$\quad + 4\mathbf{E}\Big\| \int_{t}^{t+h} K(t+h,s)f(s,w(s))ds \Big\|^2$$

$$\quad + 4\mathbf{E}\Big\| \int_{-\infty}^{t} (K(t+h,s) - K(t,s))g(s,w(s))dW(s) \Big\|^2$$

$$+ 4\mathbf{E}\big\| \int_t^{t+h} K(t+h,s)g(s,w(s))dW(s)\big\|^2$$

$$\leq 4\Gamma(r) \int_{-\infty}^t \|K(t+h,s) - K(t,s)\|ds \int_{-\infty}^t \|K(t+h,s) - K(t,s)\|\alpha(s)ds$$

$$+ 4\Gamma(r) \int_t^{t+h} \|K(t+h,s)\|ds \int_t^{t+h} \|K(t+h,s)\|\alpha(s)ds$$

$$+ 4\Gamma(r) \int_{-\infty}^t \|(K(t+h,s) - K(t,s))\alpha(s)\|^2 ds$$

$$+ 4\Gamma(r) \int_t^{t+h} \|K(t+h,s)\alpha(s)\|^2 ds$$

$$\leq 4\Gamma(r)(\tilde{C}\tilde{M}h^2 + \bar{C}\bar{M}h^2 + \tilde{M}h + \bar{M}h).$$

We deduce that the right-hand side of the above inequality is independent of $w \in \mathfrak{B}_r$ and tends to zero as $h \to 0$. Thus, the set $\{\mathcal{S}_2 w : w \in \mathfrak{B}_r\}$ is equicontinuous.

Step 2.3. We show that $\mathcal{S}_2$ is continuous. Let $\{w_n\}_{n=0}^\infty \subset \mathfrak{B}_r$, with $w_n \to w$ in $\mathfrak{B}_r$. By assumption (H6), we have $f(t, w_n(t)) \to f(t, w(t))$, $g(t, w_n(t)) \to g(t, w(t))$ $(n \to \infty)$, for each $t \in \mathbb{R}$, and since

$$\mathbf{E}\|f(t, w_n(t)) - f(t, w(t))\|^2 \leq 2\alpha(t)\Gamma(r),$$

$$\mathbf{E}\|g(t, w_n(t)) - g(t, w(t))\|^2 \leq 2\alpha(t)\Gamma(r).$$

Then, by the dominated convergence theorem, we obtain

$$\mathbf{E}\|\mathcal{S}_2 w_n - \mathcal{S}_2 w\|^2 \leq 2\mathbf{E}\big\| \int_{-\infty}^t K(t,s)(f(t,w_n(t)) - f(t,w(t))ds\big\|^2$$

$$+ 2\mathbf{E}\big\| \int_{-\infty}^t K(t,s)(g(t,w_n(t)) - g(t,w(t))dW(s)\big\|^2$$

$$\leq 2\Big(\int_{-\infty}^t \|K(t,s)\|ds\Big)^2 \int_{-\infty}^t \mathbf{E}\|f(t,w_n(t)) - f(t,w(t))\|^2 ds$$

$$+ 2 \int_{-\infty}^t \|K(t,s)\|^2 ds \int_{-\infty}^t \mathbf{E}\|g(t,w_n(t)) - g(t,w(t))\|^2 ds$$

$$\to 0, \quad \text{as } n \to \infty.$$

Thus, $\mathcal{S}_2$ is continuous.

Step 2.4. $\mathcal{S}_2$ maps $\mathfrak{B}_r$ into a precompact set in $\mathfrak{B}_r$. Let $t \in \mathbb{R}$ be fixed and be a real number $0 < \varepsilon < t$. For $w \in \mathfrak{B}_r$, define the operator

$$(\mathcal{S}_2^* w)(t)$$

$$= K(\varepsilon) \int_{-\infty}^{t-\varepsilon} K(t-\varepsilon,s)f(s,w(s))ds + K(\varepsilon) \int_{-\infty}^{t-\varepsilon} K(t-\varepsilon,s)g(s,w(s))dW(s)$$

$$= \int_{-\infty}^{t-\varepsilon} K(t,s)f(s,w(s))ds + \int_{-\infty}^{t-\varepsilon} K(t,s)g(s,w(s))dW(s).$$

Since $K(t)$ is a compact operator, the set $\{(\mathcal{S}_2^* w)(t), w \in \mathfrak{B}_r\}$ is relatively compact for every $\varepsilon, 0 < \varepsilon < t$. Moreover, for every $w \in \mathfrak{B}_r$, we have

$$\mathbf{E}\|(\mathcal{S}_2 w)(t) - (\mathcal{S}_2^* w)(t)\|^2$$

$$\leq 2\mathbf{E}\Big\| \int_{t-\varepsilon}^t K(t,s)f(s,w(s))ds \Big\|^2 + 2\mathbf{E}\Big\| \int_{t-\varepsilon}^t K(t,s)g(s,w(s))dW(s) \Big\|^2$$

$$\leq 2\varepsilon \int_{t-\varepsilon}^t \|K(t,s)\|^2 \mathbf{E}\|f(s,w(s))\|^2 ds + 2 \int_{t-\varepsilon}^t \|K(t,s)\|^2 \mathbf{E}\|g(s,w(s))\|^2 ds$$

$$\leq 2\varepsilon\Gamma(r) \int_{t-\varepsilon}^t \|K(t,s)\|^2 \alpha(s)ds + 2\Gamma(r) \int_{t-\varepsilon}^t \|K(t,s)\|^2 \alpha(s)ds.$$

Thus, $\mathbf{E}\|(\mathcal{S}_2 w)(t) - (\mathcal{S}_2^* w)(t)\|^2 \to 0$ as $\varepsilon \to 0$. Hence the set $\{\mathcal{S}_2^* w)(t), w \in \mathfrak{B}_r\}$ is precompact, and by Arzela-Ascoli theorem, the operator $\mathcal{S}_2$ is completely continuous.

Step 3. We show that the set $\Upsilon = \{u \in S : \lambda S_1(\frac{u}{\lambda}) + \lambda S_2 u = u, \lambda \in (0,1)\}$ is bounded. Indeed, for $u \in \Upsilon$ and $\lambda \in (0,1)$, one has

$$\mathbf{E}\|u(t)\|^2 \leq 3\mathbf{E}\|\lambda a(t, \frac{u}{\lambda})\|^2 + 3\mathbf{E}\Big\| \lambda \int_{-\infty}^t K(t,s)f(s,\varphi(s))ds \Big\|^2$$

$$+ 3\mathbf{E}\Big\| \lambda \int_{-\infty}^t K(t,s)g(s,\varphi(s))dW(s) \Big\|^2$$

$$\leq 3L_a^2 \mathbf{E}\|u(t)\|^2 + 3 \int_{-\infty}^t \|K(t,s)\|ds \int_{-\infty}^t \|K(t,s)\|\alpha(s)\Gamma(\mathbf{E}\|u(s)\|^2)ds$$

$$+ 3 \int_{-\infty}^t \|K(t,s)\|^2 \alpha(s)\Gamma(\mathbf{E}\|u(s)\|^2)ds.$$

Hence,

$$\mathbf{E}\|u(t)\|^2 \leq \frac{3C^*}{1 - 3L_a^2} \int_{-\infty}^t \|K(t,s)\|\alpha(s)\Gamma(\mathbf{E}\|u(s)\|^2)ds$$

$$+ \frac{3}{1 - 3L_a^2} \int_{-\infty}^t \|K(t,s)\|^2 \alpha(s)\Gamma(\mathbf{E}\|u(s)\|^2)ds.$$

By Lemma 2.3, it follows that Υ is bounded.

In concluding, by Krasnoselskii-Schaefer's fixed point theorem, $\mathcal{S}$ has a fixed point. Then (1.1) has a continuous solution.

Step 4. We show that $v(t)$ is a $\mathcal{T}$-periodic solution in distribution to (1.1). Using the same argument of (3.1)-(3.2), and (H1) in Theorem 3.1, we have

$$\mathbf{E}\|\hat{v}(t) - v(t)\|^2 = 0.$$

Since $v(t + \mathcal{T})$ has the same distribution with $\hat{v}(t)$, $v(t)$ is a $\mathcal{T}$-periodic solution of (1.1) in distribution. This completes the proof. $\square$

4. Application

Consider the following stochastic Volterra equation

$$x(t) = ax(t)\sin t + \int_{-\infty}^t K(t,s)\sigma(s,x(s))dW(s), \tag{4.1}$$

where $0 < a < \frac{1}{2}, K(t,s) = e^{-w(t-s)}, w > 0, \sigma(t,x) = \cos t \sum_{k=1}^\infty \sigma_k(t,x),$ $\sigma_{2k+1}(t,x) = b_{2k+1}\sin k^p x, \sigma_{2k}(t,x) = b_{2k}\cos k^p x, p > 0, b_{2k} = O(k^{-(p+\frac{1}{2})}),$ $b_{2k+1} = O(k^{-(p+\frac{1}{2})})$. Then by [3, 6] we have

$$\|\sigma(t,x) - \sigma(t,y)\|^2$$

$$= |\cos t|^2 \sum_{k=1}^{\infty} (b_{2k}^2 (\cos k^p x - \cos k^p y)^2 + b_{2k+1}^2 (\sin k^p x - \sin k^p y)^2)$$

$$\leq C \sum_{k=1}^{\infty} k^{-(2p+1)} ((\cos k^p x - \cos k^p y)^2 + (\sin k^p x - \sin k^p y)^2)$$

$$\leq C \sum_{k=1}^{\infty} k^{-(2p+1)} \sin^2 \frac{k^p(x-y)}{2}$$

$$\leq \Gamma(|x-y|^2)$$

for all $|x - y|$ is sufficiently small, where

$$\Gamma(x) = \begin{cases} 0, & x = 0, \\ Cx(\log \frac{1}{x})^{1/2}, & 0 < x \leq \delta, \\ C\delta(\log \frac{1}{x})^{1/2}, & x > \delta \end{cases}$$

is a concave non-decreasing continuous function on $\mathbb{R}^+$ satisfying $\int_{0+}^{+\infty} \frac{x}{\Gamma(x)} dx = +\infty$ and $\delta \in (0, 1)$ is sufficiently small. It is easy to get that $(H1) - (H2), (H4) - (H6)$ hold, and $3L_a^2 < 1$. By Theorem 3.2, (4.1) admits a continuous 2π-periodic solution in distribution.

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