HIGHER DIFFERENTIABILITY FOR SOLUTIONS TO NONHOMOGENEOUS OBSTACLE PROBLEMS WITH $1 < p < 2$

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ABSTRACT. In this article, we establish integer and fractional higher-order differentiability of weak solutions to non-homogeneous obstacle problems that satisfy the variational inequality

$$\int_\Omega \langle A(x, Du), D(\varphi - u) \rangle \, dx \geq \int_\Omega \langle |F|^{p-2}F, D(\varphi - u) \rangle \, dx,$$

where $1 < p < 2$, $\varphi \in \mathcal{K}_\psi(\Omega) = \{v \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}) : v \geq \psi \text{ a.e. in } \Omega\}$, $u_0 \in W^{1,p}(\Omega)$ is a fixed boundary datum. We show that the higher differentiability of integer or fractional order of the gradient of the obstacle ψ and the nonhomogeneous term F can transfer to the gradient of the weak solution, provided the partial map $x \mapsto A(x, \xi)$ belongs to a suitable Sobolev or Besov-Lipschitz space.

1. INTRODUCTION

This article is devoted to studying the higher differentiability properties of the gradient of the solutions $u \in W^{1,p}(\Omega)$ to the variational inequality

$$\int_\Omega \langle A(x, Du), D(\varphi - u) \rangle \, dx \geq \int_\Omega \langle |F|^{p-2}F, D(\varphi - u) \rangle \, dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ ($n > 2$) is a bounded domain, the function $\psi : \Omega \mapsto [-\infty, +\infty)$, called obstacle, belongs to the Sobolev space $W_{\text{loc}}^{1,p}(\Omega)$, and

$$\mathcal{K}_\psi(\Omega) := \{v \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}) : v \geq \psi \text{ a.e. in } \Omega\}$$

is the class of the admissible functions, with $u_0 \in W^{1,p}(\Omega)$ is a fixed boundary datum. Moreover, $\varphi \in \mathcal{K}_\psi(\Omega)$, $F \in L^p(\Omega, \mathbb{R}^n)$ is a given exterior force and the vector field $A(x, \xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-Carathéodory function, namely, A is measurable in x for all $\xi \in \mathbb{R}^n$ and differentiable in ξ for almost all $x \in \Omega$. Meanwhile, we assume that A is a p-harmonic operator, that is it satisfies the following p-ellipticity and p-growth conditions with respect to the ξ-variable. There exist positive constants ν, γ, Γ and an exponent $p \in (1, 2)$ and a parameter $\mu \in [0, 1]$, such that

$$\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \nu |\xi - \eta|^2 (\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}, \tag{1.2}$$

$$|A(x, \xi) - A(x, \eta)| \leq \Gamma |\xi - \eta| (\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}, \tag{1.3}$$

$$|A(x,\xi)| \le \gamma(\mu^2 + |\xi|^2)^{\frac{p-1}{2}} \tag{1.4}$$

for all $\xi, \eta \in \mathbb{R}^n$ and for almost all $x \in \Omega$.

In the previous decades, the study of the regularity theory for obstacle problems has been developing rapidly as a popular topic in calculus of variations and partial differential equations. The obstacle problems can be dated back to the work of Stampacchia and Lions [9, 24]. Stampacchia discussed the special case $\psi = \chi_E$ firstly, and then Lions and Stampacchia proposed the theory of variational inequalities in order to solve the regularity of obstacle problems. In an earlier work, Fichera [10] solved the elastostatic problems with unilateral constraints, namely, the Signorini problem with ambiguous boundary conditions. These problems can be solved by applying methods of functional analysis, and then the regularity of the solutions were connected to the obstacle problems.

It is often observed that the regularity of the solutions to the obstacle problems is affected by the obstacle itself. For linear obstacle problems, obstacle and solutions have the same regularity [3, 5, 16], but nonlinear is not like this. Therefore, people pay more attention to the nonlinear case in recent years [19, 20]. A first result was about the Hölder continuity of the weak solutions to the obstacle problems by Michael and Ziemer [23], it was related to the Hölder continuity of obstacle itself. Choe [6] established the Hölder continuity of the gradient of the weak solutions when the gradient of the obstacle is Hölder continuous. Many recent works focus on the regularity of solutions to variational problem. Most of papers give that the regularity of the solution depend on the regularity of the obstacle itself and the nonhomogeneous term F, provided an advisable assumption is given on the map $x \mapsto A(x,\xi)$. It is worth noting that there is not higher differentiability for the solution of obstacle problems (1.1) even though the obstacle and nonhomogeneous term are smooth.

The aim of this article is to extend some higher differentiability results in [11] to non-homogeneous elliptic obstacle problems under the suitable conditions on the x-dependence of A. We make a suitable estimate of the nonhomogeneous term F by using the crucial Lemma 2.1 and other assumptions on $A(x,\xi)$, thus obtaining new conclusions in Sobolev or Besov-Lipschitz space. First, we show that a higher differentiability property of integer order, provided the partial map $x \mapsto A(x,\xi)$ belongs to a suitable Sobolev class. More specifically, we assume that the partial map $x \mapsto A(x,\xi)$ belongs to $W_{\text{loc}}^{1,n}(\Omega)$ for any $\xi \in \mathbb{R}^n$. Equivalently, there exits a function $\iota \in L_{\text{loc}}^n(\Omega)$ such that

$$|D_x A(x,\xi)| \le \iota(x)(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}, \tag{1.5}$$

see [16]. Since $A(x,\xi)$ is a C^1 function with respect to ξ, (1.3) implies

$$|D_\xi A(x,\xi)| \le c(\mu^2 + |\xi|^2)^{\frac{p-2}{2}},$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and for almost every $x \in \Omega$.

Now we are in position to state our main results of this paper. For convenience, we introduce a special function $V_p : \mathbb{R}^n \to \mathbb{R}^n$, defined as $V_p(\xi) := (\mu^2 + |\xi|^2)^{\frac{p-2}{4}} \xi$ for $\xi \in \mathbb{R}^n$. The first result we prove reads as follows.

Theorem 1.1. *Let $u \in W_{\text{loc}}^{1,p}(\Omega)$ be a solution to the obstacle problem* (1.1) *under assumptions* (1.2)–(1.5) *for $1 < p < 2$. Let $V_p(D\psi) \in W_{\text{loc}}^{1,2}(\Omega)$, $Du \in W_{\text{loc}}^{1,\max(\frac{4n}{n-2},n)}(\Omega)$, $D\psi \in W_{\text{loc}}^{1,\max(\frac{2n}{n-2},n)}(\Omega)$, $F \in W_{\text{loc}}^{1,2}(\Omega)$, $|\tau_h D\psi| < \theta|\tau_h Du|$ for*

any $\theta > 0$ small enough, $|\tau_h(u - \psi)| > 1$, $|\tau_h F| > 1$, then $V_p(Du) \in W_{\text{loc}}^{1,2}(\Omega)$. Moreover, for any $B_R \Subset \Omega$, we have the following estimate

$$\|DV_p(Du)\|_{L^2(B_{R/2})} \leq C\Big[1 + \|Du\|_{W^{1,\max(\frac{4n}{n-2},n)}(B_R)} + \|DV_p(D\psi)\|_{L^2(B_R)}$$

$$+ \|D\psi\|_{W^{1,\max(\frac{2n}{n-2},n)}(B_R)} + \|DF\|_{L^2(B_R)} + \|\iota\|_{L^n(B_R)}\Big]^{\sigma},$$

where C, and σ are positive constants depending on n, p, R, θ, ν, γ and Γ.

Next we plan to prove that an analogous conclusion holds true in case the obstacle belongs to a Besov-Lipschitz space, provided the operator A is related to x-variable. Specifically, given $\alpha \in (0,1)$ and $q \in [1,\infty)$, we assume that there is a sequence of non-negative measurable functions $\iota_k \in L^{\frac{n}{\alpha}}(\Omega)$ such that

$$\sum_{k=1}^{\infty} \|\iota_k\|_{L^{n/\alpha}(\Omega)}^q < \infty.$$

Simultaneously, we have

$$|A(x,\xi) - A(y,\xi)| \leq (\iota_k(x) + \iota_k(y))|x - y|^{\alpha}(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}, \qquad (1.6)$$

for each $\xi \in \mathbb{R}^n$ and almost every $x, y \in \Omega$ such that $2^{-k}\text{diam}(\Omega) \leq |x - y| < 2^{-k+1}\text{diam}(\Omega)$. For ease of statement, we will shortly write then that $(\iota_k)_k \in \ell^q(L^{n/\alpha}(\Omega))$.

Now we state the Besov regularity of the obstacle.

Theorem 1.2. *Let $u \in W_{\text{loc}}^{1,p}(\Omega)$ be a solution to the obstacle problem (1.1) under assumptions (1.2)–(1.4) and (1.6) for $1 < p < 2$. Let $V_p(D\psi) \in B_{2,q,\text{loc}}^{\alpha}(\Omega)$, $D^2\psi \in L_{\text{loc}}^{\frac{np}{n-2\alpha}}(\Omega)$, $F \in W_{\text{loc}}^{1,\frac{np}{n-2\alpha}}(\Omega)$. Then $V_p(Du) \in B_{2,q,\text{loc}}^{\alpha\beta}(\Omega)$, for any $q \leq \frac{2n}{n-2\alpha}$ and $\beta \in (0,1)$. Moreover, for any $B_{4R} \Subset \Omega$, we have the estimate*

$$\Big\|\frac{\tau_h V_p(Du)}{|h|^{\alpha\beta}}\Big\|_{L^q(\frac{dh}{|h|^n}; L^2(B_{R/2}))}$$

$$\leq C\Big[1 + \|Du\|_{L^p(B_{4R})} + \|V_p(D\psi)\|_{B_{2,q}^{\alpha}(B_{4R})} + \|D^2\psi\|_{L^{\frac{np}{n-2\alpha}}(B_{4R})}$$

$$+ \|F\|_{W^{1,\frac{np}{n-2\alpha}}(B_{4R})} + \|(\iota_k)_k\|_{\ell^q(L^{n/\alpha}(B_{2R}))}\Big]^{\sigma},$$

where C and σ are positive constants depending on $n, p, q, R, \alpha, \nu, \gamma$ and Γ.

In the Besov-Lipschitz space we discussed above, if $q = \infty$, we still have a fractional differentiability property of the obstacle. More specifically, we prove the following result.

Theorem 1.3. *Let $u \in W_{\text{loc}}^{1,p}(\Omega)$ be a solution to the obstacle problem (1.1) under assumptions (1.2)–(1.4) for $1 < p < 2$. If for any $\xi \in \mathbb{R}^n$ and almost every $x, y \in \Omega$, there exists $\alpha \in (0,1)$ and a function $\iota \in L_{\text{loc}}^{\frac{n}{\alpha}}(\Omega)$ such that*

$$|A(x,\xi) - A(y,\xi)| \leq (\iota(x) + \iota(y))|x - y|^{\alpha}(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}, \qquad (1.7)$$

then, provided $0 < \alpha < \delta < 1$, $V_p(D\psi) \in B_{2,\infty,\text{loc}}^{\delta}(\Omega)$, $D^2\psi \in L_{\text{loc}}^{\frac{np}{n-2\alpha}}(\Omega)$, $F \in W_{\text{loc}}^{1,\frac{np}{n-2\alpha}}(\Omega)$, we have $V_p(Du) \in B_{2,\infty,\text{loc}}^{\alpha\beta}(\Omega)$, for any $\beta \in (0,1)$. Moreover, for any $B_{4R} \Subset \Omega$, we have the estimate

$$[V_p(Du)]_{\dot{B}_{2,\infty}^{\alpha\beta}(B_{\frac{R}{2}})} \leq C\Big[1 + \|Du\|_{L^p(B_{4R})} + \|V_p(D\psi)\|_{B_{2,\infty}^{\delta}(B_{4R})}$$

$$+ \|D^2\psi\|_{L^{\frac{np}{n-2\alpha}}(B_{4R})} + \|F\|_{W^{1,\frac{np}{n-2\alpha}}(B_{4R})} + \|\iota\|_{L^{n/\alpha}(B_{2R})}\Big]^{\sigma},$$

where C and σ are positive constants depending on $n, p, q, R, \alpha, \beta, \delta, \nu, \gamma$ and Γ.

It is worth mentioning that in integer and fractional order cases, the fundamental tools are the difference quotient method and Calderón-Zygmund type estimates proved in [4].

This article is organized as follows. In Section 2, we give notation and preliminary results. Section 3 is devoted to the proof of Theorem 1.1, while Section 4 is devoted to the proofs of Theorems 1.2 and 1.3.

2. Notation and preliminary results

In this section we will list some definitions and recall a few of fundamental tools for the proof of our main results in the following content. We shall use C or c to denote a general constant that may depend on different parameters, even within the same line of estimates. Relevant dependencies will be appropriately emphasized using parentheses. In the following, $B(x,r) = B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\} \Subset \Omega$ will denote the open ball centered at x of radius $r > 0$. For a function $u \in L^1(B_r(x_0))$, the symbol

$$\fint_{B_r(x_0)} u(x)\, dx := \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u(x)\, dx$$

will denote the integral mean of the function $u(x)$ over the open ball $B_r(x_0)$. Next we recall a crucial result for the function V_p, see [1, 13].

Lemma 2.1. Let $p \in (1, \infty)$, $\mu \in [0,1]$. There is a constant $c = c(n,p) > 0$ such that

$$c^{-1}(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \le \frac{|V_p(\xi) - V_p(\eta)|^2}{|\xi - \eta|^2} \le c(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}},$$

for any $\xi, \eta \in \mathbb{R}^n$.

Particularly, there is a constant $c = c(n,p) > 0$ such that

$$\left| |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \right| \le c|\xi - \eta|^{p-1}.$$

In addition, for any $\Phi \in C^2(\mathbb{R}^n)$, there exists a constant $C = C(p) > 0$ such that

$$C^{-1}|D^2\Phi|^2(\mu^2 + |D\Phi|^2)^{\frac{p-2}{2}} \le |DV_p(D\Phi)|^2 \le C|D^2\Phi|^2(\mu^2 + |D\Phi|^2)^{\frac{p-2}{2}}.$$

2.1. **Difference quotients.** To obtain the higher differentiability of the gradient of the weak solution, let us recall some results of the finite difference operator.

Given $h \in \mathbb{R}^n$, for every function $v : \mathbb{R}^n \to \mathbb{R}$, the finite difference operator is defined by $\tau_h v(x) := v(x + h) - v(x)$. We start with the description of some basic properties that can be founded in [14].

Proposition 2.2. Let F and G be two functions such that $F, G \in W^{1,p}(\Omega)$, with $p \in [1, \infty)$, and we consider the set

$$\Omega_{|h|} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > |h|\}.$$

Then

(a) $\tau_h F \in W^{1,p}(\Omega_{|h|})$ and $D_i(\tau_h F(x)) = \tau_h(D_i F(x))$.

(b) *If at least one of the functions F or G has support contained in $\Omega_{|h|}$, then*

$$\int_\Omega F(x)\tau_h G(x)\,dx = \int_\Omega G(x)\tau_{-h}F(x)\,dx.$$

(c) *We have*

$$\tau_h(FG)(x) = F(x+h)\tau_h G(x) + G(x)\tau_h F(x).$$

The next result about finite difference operator is a kind of integral version of the Lagrange Theorem.

Lemma 2.3. *If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $p \in (1,\infty)$, and F, $DF \in L^p(B_R)$, then*

$$\int_{B_\rho} |\tau_h F(x)|^p dx \le c(n,p)|h|^p \int_{B_R} |DF(x)|^p dx.$$

Moreover

$$\int_{B_\rho} |F(x+h)|^p dx \le \int_{B_R} |F(x)|^p dx.$$

For each function $v : \mathbb{R}^n \to \mathbb{R}^N$ and $h \in \mathbb{R}$, we denote

$$\tau_{s,h}v(x) := v(x + he_s) - v(x),$$

where e_s is the unit vector in the x_s direction for any $s \in \{1,\ldots,n\}$. Now we recall the essential Sobolev embedding property that is proved in [14].

Lemma 2.4. *Let $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^p(B_R)$ with $p \in (1,\infty)$. Suppose that there exists $\rho \in (0,R)$ and $M > 0$ such that*

$$\sum_{s=1}^n \int_{B_\rho} |\tau_{s,h}F(x)|^p dx \le M^p |h|^p,$$

for all h with $|h| < \frac{R-\rho}{2}$. Then $F \in W^{1,p}(B_\rho) \cap L^{\frac{np}{n-p}}(B_\rho)$. Moreover

$$\|DF\|_{L^p(B_\rho)} \le M,$$
$$\|F\|_{L^{\frac{np}{n-p}}(B_\rho)} \le c(M + \|F\|_{L^p(B_R)}),$$

with $c = c(n,N,p,R,\rho)$.

Now we introduce a fractional version of Lemma 2.4, whose proof can be found in [18].

Lemma 2.5. *Let $F \in L^2(B_R)$. Suppose that there exist $\rho \in (0,R)$, $\alpha \in (0,1)$ and $M > 0$ such that*

$$\sum_{s=1}^n \int_{B_\rho} |\tau_{s,h}F(x)|^2 dx \le M^2 |h|^{2\alpha},$$

for all h with $|h| < \frac{R-\rho}{2}$. Then $F \in L^{\frac{2n}{n-2\beta}}(B_\rho)$ for all $\beta \in (0,\alpha)$. Moreover

$$\|F\|_{L^{\frac{2n}{n-2\beta}}(B_\rho)} \le c(M + \|F\|_{L^2(B_R)}),$$

with $c = c(n,N,R,\rho,\alpha,\beta)$.

Next we give a Sobolev embedding theorem, which includes the fractional version, see [8, 12].

Lemma 2.6. *Assume that $\Omega \subset \mathbb{R}^n$ has the extension property.*

(a) *Let $p \in [1, n)$. Then there exists a constant $c = c(n, p, \Omega) > 0$ such that*

$$\|f\|_{L^q(\Omega)} \leq c\|f\|_{W^{1,p}(\Omega)}$$

for all $f \in W^{1,p}(\Omega)$ and $p \leq q \leq \frac{np}{n-p}$.

(b) *Let $\alpha \in (0, 1)$ and $p \in [1, \frac{n}{\alpha})$. Then there exists a constant $c = c(n, p, \alpha, \Omega) > 0$ such that*

$$\|f\|_{L^q(\Omega)} \leq c\|f\|_{W^{\alpha,p}(\Omega)}$$

for all $f \in W^{\alpha,p}(\Omega)$ and $p \leq q \leq \frac{np}{n-\alpha p}$.

2.2. Besov-Lipschitz spaces.

Definition 2.7. For a given $\alpha \in (0, 1)$, and $p, q \in [1, \infty)$, we say that v belongs to the Besov-Lipschitz space $B_{p,q}^{\alpha}(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$\|v\|_{B_{p,q}^{\alpha}(\mathbb{R}^n)} = \|v\|_{L^p(\mathbb{R}^n)} + [v]_{\dot{B}_{p,q}^{\alpha}(\mathbb{R}^n)} < \infty, \tag{2.1}$$

where

$$[v]_{\dot{B}_{p,q}^{\alpha}(\mathbb{R}^n)} := \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \frac{|\tau_h v(x)|^p}{|h|^{\alpha p}} dx\Big)^{q/p} \frac{dh}{|h|^n}\Big)^{1/q} < \infty. \tag{2.2}$$

It is worth noticing that $B_{p,q}^{\alpha}(\mathbb{R}^n)$ is a Banach space. We can say that a function $v \in L^p(\mathbb{R}^n)$ belongs to $B_{p,q}^{\alpha}(\mathbb{R}^n)$ if and only if $\frac{\tau_h v}{|h|^{\alpha}} \in L^q\big(\frac{dh}{|h|^n}; L^p(\mathbb{R}^n)\big)$. In addition, for $h \in B_{\delta}(0)$ where $\delta > 0$ is a fixed constant, we can simply (2.2) to obtain an equivalent norm of (2.1), that is

$$\|v\|_{B_{p,q}^{\alpha}(\mathbb{R}^n)} \simeq \|v\|_{L^p(\mathbb{R}^n)} + \Big(\int_{\{|h| \leq \delta\}} \Big(\int_{\mathbb{R}^n} \frac{|\tau_h v(x)|^p}{|h|^{\alpha p}} dx\Big)^{q/p} \frac{dh}{|h|^n}\Big)^{1/q} < \infty,$$

this is so because

$$\Big(\int_{\{|h| \geq \delta\}} \Big(\int_{\mathbb{R}^n} \frac{|\tau_h v(x)|^p}{|h|^{\alpha p}} dx\Big)^{q/p} \frac{dh}{|h|^n}\Big)^{1/q} \leq c(n, p, q, \alpha, \delta)\|v\|_{L^p(\mathbb{R}^n)} < \infty.$$

Definition 2.8. For a given $\alpha \in (0, 1)$, and $p \in [1, \infty)$, $q = \infty$, we say that v belongs to the Besov-Lipschitz space $B_{p,\infty}^{\alpha}(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$[v]_{\dot{B}_{p,\infty}^{\alpha}(\mathbb{R}^n)} := \sup_{h \in \mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \frac{|\tau_h v(x)|^p}{|h|^{\alpha p}} dx\Big)^{1/p} < \infty. \tag{2.3}$$

Similarly, we can take the supremum over $|h| \leq \delta$ in (2.3) and obtain an equivalent norm. According to the construction of the norm of Besov-Lipschitz space, it is easy to see that $B_{p,q}^{\alpha}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. Now we give the Sobolev-type embeddings for Besov-Lipschitz spaces, see [15].

Lemma 2.9. *Suppose that $\alpha \in (0, 1)$.*

(a) *If $1 < p < \frac{n}{\alpha}$ and $1 \leq q \leq p_{\alpha}^* := \frac{np}{n-\alpha p}$, then there is a continuous embedding $B_{p,q}^{\alpha}(\mathbb{R}^n) \hookrightarrow L^{p_{\alpha}^*}(\mathbb{R}^n)$.*

(b) *If $p = \frac{n}{\alpha}$ and $1 \leq q \leq \infty$, then there is a continuous embedding $B_{p,q}^{\alpha}(\mathbb{R}^n) \hookrightarrow BMO(\mathbb{R}^n)$, where BMO denotes the space of functions with bounded mean oscillations [14].*

The following lemma describes the relationship between Besov-Lipschitz spaces, see [15].

Lemma 2.10. *Suppose that $0 < \beta < \alpha < 1$.*

(a) *If $1 < p < +\infty$ and $1 \le q \le r \le +\infty$, then $B_{p,q}^{\alpha}(\mathbb{R}^n) \subset B_{p,r}^{\alpha}(\mathbb{R}^n)$.*
(b) *If $1 < p < +\infty$ and $1 \le q, r \le +\infty$, then $B_{p,q}^{\alpha}(\mathbb{R}^n) \subset B_{p,r}^{\beta}(\mathbb{R}^n)$.*
(c) *If $1 \le q \le \infty$, then $B_{\frac{n}{\alpha},q}^{\alpha}(\mathbb{R}^n) \subset B_{\frac{n}{\beta},q}^{\beta}(\mathbb{R}^n)$.*

The following lemma is follows from the definition of the local Besov-Lipschitz spaces, and its proof can be found in [2].

Lemma 2.11. *A function $v \in L_{\mathrm{loc}}^p(\Omega)$ belongs to the local Besov space $B_{p,q,\mathrm{loc}}^{\alpha}(\Omega)$ if and only if*

$$\|\frac{\tau_h v}{|h|^{\alpha}}\|_{L^q(\frac{dh}{|h|^n};L^p(B))} < \infty$$

for any open ball $B \subset 2B \subset \Omega$ with radius r_B. Here the measure $\frac{dh}{|h|^n}$ is restricted to the ball $B_{r_B}(0)$ on the h-space.

As we know, the Besov-Lipschitz spaces of fractional order $\alpha \in (0,1)$ can be characterized in pointwise terms. Given a measurable function $v(x) : \mathbb{R}^n \to \mathbb{R}$, a fractional α-Hajłasz gradient for v is a sequence $(\iota_k)_k$ of measurable, non-negative functions $\iota_k(x) : \mathbb{R}^n \to \mathbb{R}$, together with a null set $N \subset \mathbb{R}^n$, such that

$$|v(x) - v(y)| \le (\iota_k(x) + \iota_k(y))|x - y|^{\alpha}$$

whenever $k \in \mathbb{Z}$ and $x, y \in \mathbb{R}^n \backslash N$ are such that $2^{-k} \le |x - y| < 2^{-k+1}$. We say that $(\iota_k) \in \ell^q(\mathbb{Z}; L^p(\mathbb{R}^n))$ if

$$\|(\iota_k)_k\|_{\ell^q(L^p)} = \left(\sum_{k \in \mathbb{Z}} \|\iota_k\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} < \infty.$$

Now we give a necessary and sufficient condition of a function v to belong to the Besov-Lipschitz space $B_{p,q}^{\alpha}(\mathbb{R}^n)$, which was proved in [17].

Theorem 2.12. *Let $0 < \alpha < 1$, $1 \le p < \infty$ and $1 \le q \le \infty$. Let $v \in L^p(\mathbb{R}^n)$. One has $v \in B_{p,q}^{\alpha}(\mathbb{R}^n)$ if and only if there exists a fractional α-Hajłasz gradient $(\iota_k)_k \in \ell^q(\mathbb{Z}; L^p(\mathbb{R}^n))$ for v. Moreover*

$$\|v\|_{B_{p,q}^{\alpha}(\mathbb{R}^n)} \simeq \inf \|(\iota_k)_k\|_{\ell^q(L^p)},$$

where the infimum runs over all possible fractional α-Hajłasz gradients for v.

Now we give a crucial lemma, which proof can be found in [11].

Lemma 2.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $1 < p < 2$, $0 < \alpha < 1$, and $1 \le q \le \infty$. Then*

$$V_p(D\psi) \in B_{2,q,\mathrm{loc}}^{\alpha}(\Omega) \Rightarrow D\psi \in B_{p,q,\mathrm{loc}}^{\alpha}(\Omega).$$

Moreover, for any all $B_R \Subset \Omega$ and $0 < \rho < R$, we have

$$[D\psi]_{\dot{B}_{p,q}^{\alpha}(B_{\rho})} \le C\big[1 + \|D\psi\|_{L^p(B_R)} + \|V_p(D\psi)\|_{B_{2,q}^{\alpha}(B_R)}\big]^{\sigma},$$

where C and σ are positive constants depending on n, p, q and α.

2.3. **VMO coefficients.** To prove our main results, we shall introduce the related content of VMO coefficients. For convenience, given a ball $B \subset \Omega$, let us introduce the operator

$$A_B(\xi) = \fint_B A(x, \xi)\, dx.$$

Definition 2.14. Suppose that $B \subset \Omega$ is an open ball, $1 < p < \infty$, setting

$$V(x, B) := \sup_{\xi \neq 0} \frac{|A(x, \xi) - A_B(\xi)|}{(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}},$$

we say that $x \mapsto A(x, \xi)$ is locally uniformly in VMO if for each set $K \Subset \Omega$ we have that

$$\lim_{R \to 0} \sup_{r < R} \sup_{x_0 \in K} \fint_{B_r(x_0)} V(x, B)\, dx = 0. \tag{2.4}$$

The next lemma is an important tool, which proof for $p \geq 2$ can be found in [7], but it holds exactly in the same way for $1 < p < 2$.

Lemma 2.15. *Let A be such that* (1.2), (1.3), (1.4) *and* (1.5) *or* (1.6) *hold. Then A is locally uniformly in VMO, that is* (2.4) *holds.*

Next we shall give a Calderón-Zygmund type estimate for solutions to the obstacle problem with VMO coefficients. Its proof can be found in [4].

Theorem 2.16. *Let $p > 1$ and $q > p$. Assume that* (1.2), (1.3), (1.4) *hold, and that $x \mapsto A(x, \xi)$ is locally uniformly in VMO. Let $u \in \mathcal{K}_\psi(\Omega)$ be the weak solution of the variational inequality* (1.1)*. Then*

$$D\psi, F \in L^q_{\mathrm{loc}}(\Omega) \Longrightarrow Du \in L^q_{\mathrm{loc}}(\Omega).$$

Moreover, there exists a constant $C = C(n, p, q, v, \gamma, \Gamma)$ such that

$$\fint_{B_R} |Du|^q dx \leq C[1 + \fint_{B_{2R}} |F|^q dx + \fint_{B_{2R}} |D\psi|^q dx + (\fint_{B_{2R}} |Du|^p dx)^{q/p}]$$

for all ball B_R such that $B_{2R} \Subset \Omega$.

3. HIGHER ORDER INTEGER DIFFERENTIABILITY

Existence of weak solutions to the variational inequality (1.1) can be easily proved through classical theories, so in the paper we will concentrate more on the proof of the regularity results. The key point is to choose an appropriate test function φ in (1.1) such that φ turns to be admissible for the obstacle class $\mathcal{K}_\psi(\Omega)$. It is worth noticing that for the higher order integer differentiability, unfortunately, we cannot get a similar result as [11, Theorem 2.2] so that we must require a higher regularity of u and ψ.

Proof of Theorem 1.1. Let us fix a ball B_R such that $B_R \subset B_{4R} \Subset \Omega$ and arbitrary radii $\frac{R}{2} < r < l_1 < l_2 < \lambda r < R$, with $1 < \lambda < 2$. Let us consider a cut off function $\eta \in C_0^\infty(B_{l_2})$ such that $0 \leq \eta \leq 1, \eta \equiv 1$ on B_{l_1} and $|D\eta| \leq \frac{c}{R}$. Because of the local nature of our results, with no loss of generality, we suppose $R \leq 1$. Let us consider $\varphi := u + \theta v$ for any $\theta \in [0, 1]$ and a suitable $v \in W_0^{1,p}(\Omega)$ such that

$$u - \psi + \theta v \geq 0. \tag{3.1}$$

It is easy to see $\varphi \in \mathcal{K}_\psi(\Omega)$ since $\varphi = u + \theta v \geq \psi$. Now, for $|h| < \frac{R}{4}$, we consider

$$v_1(x) = \eta^2(x)\tau_h(u - \psi)(x).$$

From the regularity of u and ψ, we have $v_1 \in W_0^{1,p}(\Omega)$. Moreover, v_1 satisfies (3.1). Indeed, for almost every $x \in \Omega$ and for all $\theta \in [0,1]$

$$
\begin{aligned}
u(x) - \psi(x) + \theta v_1(x) &= u(x) - \psi(x) + \theta \eta^2(x) \tau_h(u - \psi)(x) \\
&= u(x) - \psi(x) + \theta \eta^2(x)[(u - \psi)(x + h) - (u - \psi)(x)] \\
&= \theta \eta^2(x)(u - \psi)(x + h) + (1 - \theta \eta^2(x))(u - \psi)(x) \geq 0,
\end{aligned}
$$

since $u \in \mathcal{K}_\psi(\Omega)$ and $\eta \in [0,1]$.

Therefore, we can use $\varphi = u + \theta v_1$ as an admissible test function in variational inequality (1.1), thus we have

$$
\begin{aligned}
&\int_\Omega \langle A(x, Du(x)), D[\eta^2(x)[(u - \psi)(x + h) - (u - \psi)(x)]]\rangle dx \\
&\geq \int_\Omega \langle |F(x)|^{p-2} F(x), D[\eta^2(x)[(u - \psi)(x + h) - (u - \psi)(x)]]\rangle dx.
\end{aligned}
\tag{3.2}
$$

Similarly, if we define

$$
v_2(x) = \eta^2(x - h) \tau_{-h}(u - \psi)(x),
$$

we have $v_2 \in W_0^{1,p}(\Omega)$, and v_2 satisfies (3.1).

So we can also use $\varphi = u + \theta v_2$ as an admissible test function in (1.1), thus we obtain

$$
\begin{aligned}
&\int_\Omega \langle A(x, Du(x)), D[\eta^2(x - h)[(u - \psi)(x - h) - (u - \psi)(x)]]\rangle dx \\
&\geq \int_\Omega \langle |F(x)|^{p-2} F(x), D[\eta^2(x - h)[(u - \psi)(x - h) - (u - \psi)(x)]]\rangle dx.
\end{aligned}
\tag{3.3}
$$

By means of a simple change of variable in (3.3), we have

$$
\begin{aligned}
&\int_\Omega \langle A(x + h, Du(x + h)), D[\eta^2(x)[(u - \psi)(x) - (u - \psi)(x + h)]]\rangle dx \\
&\geq \int_\Omega \langle |F(x + h)|^{p-2} F(x + h), D[\eta^2(x)[(u - \psi)(x) - (u - \psi)(x + h)]]\rangle dx.
\end{aligned}
\tag{3.4}
$$

We can add (3.2) and (3.4), thus obtaining

$$
\begin{aligned}
&\int_\Omega \langle A(x + h, Du(x + h)) - A(x, Du(x)), \\
&\quad D[\eta^2(x)[(u - \psi)(x + h) - (u - \psi)(x)]]\rangle dx \\
&\leq \int_\Omega \langle |F(x + h)|^{p-2} F(x + h) - |F(x)|^{p-2} F(x), \\
&\quad D[\eta^2(x)[(u - \psi)(x + h) - (u - \psi)(x)]]\rangle dx,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_\Omega \langle A(x + h, Du(x + h)) - A(x, Du(x)), \eta^2(x)[Du(x + h) - Du(x)]\rangle dx \\
&- \int_\Omega \langle A(x + h, Du(x + h)) - A(x, Du(x)), \eta^2(x)[D\psi(x + h) - D\psi(x)]\rangle dx \\
&+ \int_\Omega \langle A(x + h, Du(x + h)) - A(x, Du(x)), 2\eta(x) D\eta(x) \tau_h(u - \psi)(x)\rangle dx
\end{aligned}
$$

$$\leq \int_\Omega \langle |F(x+h)|^{p-2}F(x+h) - |F(x)|^{p-2}F(x), \eta^2(x)[Du(x+h) - Du(x)]\rangle dx$$

$$- \int_\Omega \langle |F(x+h)|^{p-2}F(x+h) - |F(x)|^{p-2}F(x), \eta^2(x)[D\psi(x+h) - D\psi(x)]\rangle dx$$

$$+ \int_\Omega \langle |F(x+h)|^{p-2}F(x+h) - |F(x)|^{p-2}F(x), 2\eta(x)D\eta(x)\tau_h(u-\psi)(x)\rangle dx.$$

We can write the previous inequality in the form

$$I := \int_\Omega \langle A(x+h, Du(x+h)) - A(x+h, Du(x)), \eta^2[Du(x+h) - Du(x)]\rangle dx$$

$$\leq \int_\Omega \langle A(x+h, Du(x+h)) - A(x+h, Du(x)), \eta^2[D\psi(x+h) - D\psi(x)]\rangle dx$$

$$- \int_\Omega \langle A(x+h, Du(x+h)) - A(x+h, Du(x)), 2\eta D\eta \tau_h(u-\psi)(x)\rangle dx$$

$$- \int_\Omega \langle A(x+h, Du(x)) - A(x, Du(x)), \eta^2[Du(x+h) - Du(x)]\rangle dx$$

$$+ \int_\Omega \langle A(x+h, Du(x)) - A(x, Du(x)), \eta^2[D\psi(x+h) - D\psi(x)]\rangle dx$$

$$- \int_\Omega \langle A(x+h, Du(x)) - A(x, Du(x)), 2\eta D\eta \tau_h(u-\psi)(x)\rangle dx$$

$$+ \int_\Omega \langle |F(x+h)|^{p-2}F(x+h) - |F(x)|^{p-2}F(x), \eta^2[Du(x+h) - Du(x)]\rangle dx$$

$$- \int_\Omega \langle |F(x+h)|^{p-2}F(x+h) - |F(x)|^{p-2}F(x), \eta^2[D\psi(x+h) - D\psi(x)]\rangle dx$$

$$+ \int_\Omega \langle |F(x+h)|^{p-2}F(x+h) - |F(x)|^{p-2}F(x), 2\eta D\eta \tau_h(u-\psi)(x)\rangle dx$$

$$=: II + III + IV + V + VI + VII + VIII + IX.$$

so we have

$$I \leq |II| + |III| + |IV| + |V| + |VI| + |VII| + |VIII| + |IX|. \qquad (3.5)$$

Next, we estimate $I, II, III, IV, V, VI, VII, VIII$ and IX.

By ellipticity assumption (1.2) we have

$$I \geq \nu \int_\Omega \eta^2(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2 dx. \qquad (3.6)$$

For the term II, by assumption (1.3) and using the fact that $|\tau_h D\psi| < \theta|\tau_h Du|$, we have

$$|II| \leq \Gamma \int_\Omega \eta^2(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}}|\tau_h Du||\tau_h D\psi|dx$$

$$\leq \varepsilon \int_\Omega \eta^2(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2 dx. \qquad (3.7)$$

For the term III, by Young's Inequality with exponents $(p, \frac{p}{p-1})$ and $|\tau_h(u-\psi)| > 1$, we obtain

$$|III| \leq 2\Gamma \int_\Omega \eta|D\eta|(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}}|\tau_h Du|\,|\tau_h(u-\psi)|dx$$

$$\leq \varepsilon \int_\Omega \eta^{\frac{p}{p-1}} (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx$$

$$+ \frac{C(\varepsilon)}{R^p} |h|^2 \int_{B_{\lambda r}} |D(u-\psi)|^2 dx. \tag{3.8}$$

To estimate IV, we use assumption (1.5), Young's Inequality with exponents $(2,2)$ and Hölder's Inequality with exponents $(\frac{n}{2}, \frac{n}{n-2})$ to obtain

$$|IV| \leq \int_\Omega |h| \iota (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-1}{2}} \eta^2 |\tau_h Du| dx$$

$$\leq \varepsilon \int_\Omega \eta^2 (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx$$

$$+ C(\varepsilon) |h|^2 \Big(\int_{B_{l_2}} (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{np}{2(n-2)}} dx \Big)^{\frac{n-2}{n}} \tag{3.9}$$

$$\times \Big(\int_{B_{l_2}} \iota^n dx \Big)^{2/n}.$$

Assumption (1.5) also implies

$$|V| \leq \int_\Omega |h| \iota (\mu^2 + |Du|^2)^{\frac{p-1}{2}} \eta^2 |\tau_h D\psi| dx$$

$$\leq \int_{B_{l_2}} |h| \iota (\mu^2 + |Du|^2)^{\frac{p-1}{2}} (\mu^2 + |D\psi(x)|^2 + |D\psi(x+h)|^2)^{\frac{2-p}{4}}$$

$$\times (\mu^2 + |D\psi(x)|^2 + |D\psi(x+h)|^2)^{\frac{p-2}{4}} |\tau_h D\psi| dx$$

$$\leq c|h|^2 \Big(\int_{B_{l_2}} (\mu^2 + |Du|^2)^{\frac{n(p-1)}{n-2}} (\mu^2 + |D\psi(x)|^2 + |D\psi(x+h)|^2)^{\frac{n(2-p)}{2(n-2)}} dx \Big)^{\frac{n-2}{n}}$$

$$\times \Big(\int_{B_{l_2}} \iota^n dx \Big)^{2/n} + c \int_{B_{l_2}} |\tau_h V_p(D\psi)|^2 dx$$

$$\leq c|h|^2 \Big(\int_{B_{l_2}} \iota^n dx \Big)^{2/n} \Big[\Big(\int_{B_{l_2}} (\mu^2 + |Du|^2)^{\frac{2n(p-1)}{n-2}} dx \Big)^{\frac{n-2}{n}}$$

$$+ \Big(\int_{B_{l_2}} (\mu^2 + |D\psi(x)|^2 + |D\psi(x+h)|^2)^{\frac{n(2-p)}{(n-2)}} dx \Big)^{\frac{n-2}{n}} \Big]$$

$$+ c|h|^2 \int_{B_{\lambda r}} |DV_p(D\psi)|^2 dx, \tag{3.10}$$

where we also used Young's Inequality with exponents $(2,2)$, Hölder's Inequality with exponents $(\frac{n}{2}, \frac{n}{n-2})$, Lemma 2.1 and Lemma 2.3.

For the term VI, arguing as we did for the estimate of V, we obtain

$$|VI| \leq 2 \int_\Omega |h| \iota (\mu^2 + |Du|^2)^{\frac{p-1}{2}} \eta |D\eta| |\tau_h(u-\psi)| dx$$

$$\leq \frac{c}{R} \int_\Omega |h| \iota (\mu^2 + |Du|^2)^{\frac{p-1}{2}} \eta |\tau_h(u-\psi)| dx$$

$$\leq \frac{c}{R} |h|^2 \Big(\int_{B_{\lambda r}} |D(u-\psi)|^n dx \Big)^{2/n} \Big(\int_{B_{\lambda r}} (\mu^2 + |Du|^2)^{\frac{n(p-1)}{2(n-2)}} dx \Big)^{\frac{n-2}{n}}$$

$$+ \frac{c}{R}|h|^2 \Big(\int_{B_{l_2}} \iota^n dx \Big)^{2/n} \Big(\int_{B_{l_2}} (\mu^2 + |Du|^2)^{\frac{n(p-1)}{2(n-2)}} dx \Big)^{\frac{n-2}{n}}. \qquad (3.11)$$

Using Lemma 2.1, Lemma 2.3, Young's Inequality with exponents $(2,2)$ and $|\tau_h F| > 1$ for the term VII, we obtain

$$\begin{aligned}
|VII| &\le \int_\Omega \eta^2 ||F(x+h)|^{p-2}F(x+h) - |F(x)|^{p-2}F(x)||\tau_h Du| dx \\
&\le c(n,p)|h|^2 \big(\int_{B_{\lambda r}} |DF|^2 dx + \int_{B_{\lambda r}} |D^2 u|^2 dx \big).
\end{aligned} \qquad (3.12)$$

Arguing analogously, for the terms $VIII$ and IX, we have

$$\begin{aligned}
|VIII| &\le \int_\Omega \eta^2 ||F(x+h)|^{p-2}F(x+h) - |F(x)|^{p-2}F(x)||\tau_h D\psi| dx \\
&\le c(n,p)|h|^2 \Big(\int_{B_{\lambda r}} |DF|^2 dx + \int_{B_{\lambda r}} |D^2 \psi|^2 dx \Big),
\end{aligned} \qquad (3.13)$$

and

$$\begin{aligned}
|IX| &\le 2 \int_\Omega \eta ||F(x+h)|^{p-2}F(x+h) - |F(x)|^{p-2}F(x)|\, |D\eta||\tau_h(u-\psi)| dx \\
&\le \frac{c(n,p)}{R}|h|^2 \Big(\int_{B_{\lambda r}} |DF|^2 dx + \int_{B_{\lambda r}} |D(u-\psi)|^2 dx \Big).
\end{aligned} \qquad (3.14)$$

Now, plugging (3.6)-(3.14) into (3.5), choosing a sufficiently small value of ε, we obtain

$$\begin{aligned}
&\int_{B_R} \eta^2 (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx \\
&\le \frac{c}{R^p}|h|^2 \int_{B_{\lambda r}} |D(u-\psi)|^2 dx \\
&\quad + c|h|^2 \Big(\int_{B_{l_2}} (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{np}{2(n-2)}} dx \Big)^{\frac{n-2}{n}} \Big(\int_{B_{l_2}} \iota^n dx \Big)^{2/n} \\
&\quad + c|h|^2 (\int_{B_{l_2}} \iota^n dx)^{2/n} \cdot \Big[\big(\int_{B_{l_2}} (\mu^2 + |Du|^2)^{\frac{2n(p-1)}{n-2}} dx \big)^{\frac{n-2}{n}} \\
&\quad + \Big(\int_{B_{l_2}} (\mu^2 + |D\psi(x)|^2 + |D\psi(x+h)|^2)^{\frac{n(2-p)}{(n-2)}} dx \Big)^{\frac{n-2}{n}} \Big] \\
&\quad + c|h|^2 \int_{B_{\lambda r}} |DV_p(D\psi)|^2 dx \\
&\quad + \frac{c}{R}|h|^2 \Big(\int_{B_{\lambda r}} |D(u-\psi)|^n dx \Big)^{2/n} \Big(\int_{B_{\lambda r}} (\mu^2 + |Du|^2)^{\frac{n(p-1)}{2(n-2)}} dx \Big)^{\frac{n-2}{n}} \\
&\quad + \frac{c}{R}|h|^2 (\int_{B_{l_2}} \iota^n dx)^{2/n} \Big(\int_{B_{l_2}} (\mu^2 + |Du|^2)^{\frac{np}{2(n-2)}} dx \Big)^{\frac{n-2}{n}} \\
&\quad + c|h|^2 \Big(\int_{B_{\lambda r}} |DF|^2 dx + \int_{B_{\lambda r}} |D^2 u|^2 dx \Big) \\
&\quad + c|h|^2 \Big(\int_{B_{\lambda r}} |DF|^2 dx + \int_{B_{\lambda r}} |D^2 \psi|^2 dx \Big)
\end{aligned}$$

$$+ \frac{c}{R}|h|^2 \Big(\int_{B_{\lambda r}} |DF|^2 dx + \int_{B_{\lambda r}} |D(u - \psi)|^2 dx \Big). \tag{3.15}$$

By Lemma 2.1, the left-hand side of (3.5) can be bounded from below as follows

$$\int_{B_R} \eta^2 (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx \geq \int_{B_{R/2}} |\tau_h V_p(Du)|^2 dx.$$

By Lemma 2.3 and Young's Inequality with exponents $(\frac{n}{2}, \frac{n}{n-2})$ to (3.15), we obtain

$$\begin{aligned}
\int_{B_{R/2}} |\tau_h V_p(Du)|^2 dx \leq c|h|^2 \Big[&\int_{B_R} (\mu^2 + |Du|^2)^{\frac{2n}{n-2}} dx + \int_{B_R} \iota^n \, dx \\
&+ \int_{B_R} (\mu^2 + |D\psi|^2)^{\frac{n}{n-2}} dx + \int_{B_R} |DF|^2 dx \\
&+ \int_{B_R} |D^2 u|^2 dx + \int_{B_R} |D^2 \psi|^2 dx + \int_{B_R} |Du|^n dx \\
&+ \int_{B_R} |D\psi|^n dx + \int_{B_R} |DV_p(D\psi)|^2 dx \Big] \\
=: &\, M|h|^2,
\end{aligned}$$

for a suitable $c = c(n, p, R, \theta, \nu, \gamma, \Gamma)$.

So applying Lemma 2.4 for a suitable choice of C and σ, we obtain

$$\begin{aligned}
\|DV_p(Du)\|_{L^2(B_{R/2})} \leq C \Big[1 + \|Du\|_{W^{1,\max(\frac{4n}{n-2}, n)}(B_R)} &+ \|DV_p(D\psi)\|_{L^2(B_R)} \\
+ \|D\psi\|_{W^{1,\max(\frac{2n}{n-2}, n)}(B_R)} &+ \|DF\|_{L^2(B_R)} + \|\iota\|_{L^n(B_R)} \Big]^\sigma,
\end{aligned}$$

so the conclusion is established. $\qquad\square$

4. Higher order fractional differentiability

This section is devoted to the proofs of Theorems 1.2 and 1.3. The proof of Theorem 1.2 is the same as the proof of the one presented in the previous section until the estimate (3.6). Differences come when starting estimate II-IX, in which the different assumptions on the partial map $x \mapsto A(x, \xi)$ and on the obstacle come into play.

4.1. Proof of Theorem 1.2. Our starting point is the estimate

$$\begin{aligned}
\nu \int_\Omega \eta^2 (\mu^2 + |Du(x)|^2 &+ |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx \\
&\leq |II| + |III| + |IV| + |V| + |VI| + |VII| + |VIII| + |IX|.
\end{aligned} \tag{4.1}$$

Let us observe that, since $V_p(D\psi) \in B^\alpha_{2,q,\mathrm{loc}}(\Omega)$ for $q \leq \frac{2n}{n-2\alpha}$, then by Lemmas 2.9 and 2.13, $V_p(D\psi) \in L^{\frac{2n}{n-2\alpha}}_{\mathrm{loc}}(\Omega)$ and $D\psi \in L^{\frac{np}{n-2\alpha}}_{\mathrm{loc}}(\Omega)$. Moreover, because $F \in L^{\frac{np}{n-2\alpha}}_{\mathrm{loc}}(\Omega)$, by Theorem 2.16, we have $Du \in L^{\frac{np}{n-2\alpha}}_{\mathrm{loc}}(\Omega)$.

Now we consider the term II, by assumption (1.3), we obtain

$$|II| \leq \Gamma \int_\Omega \eta^2 (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du| |\tau_h D\psi| dx. \tag{4.2}$$

Setting $E_1 := \{x \in \Omega : |Du(x)|^2 + |Du(x+h)|^2 > |D\psi(x)|^2 + |D\psi(x+h)|^2\}$ and $E_2 := \Omega \setminus E_1$, so (4.2) becomes

$$
\begin{aligned}
|II| \le{}& \Gamma \int_{E_1} \eta^2 (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du||\tau_h D\psi| dx \\
&+ \Gamma \int_{E_2} \eta^2 (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du||\tau_h D\psi| dx \\
=:{}& II_1 + II_2.
\end{aligned} \tag{4.3}
$$

Using Young's Inequality with exponents (2,2) and Lemma 2.1, we have

$$
\begin{aligned}
|II_1| \le{}& \Gamma \int_{E_1} \eta^2 (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{4}} |\tau_h Du| \\
&\times \big(\mu^2 + |D\psi(x)|^2 + |D\psi(x+h)|^2\big)^{\frac{p-2}{4}} |\tau_h D\psi| dx \\
\le{}& \varepsilon \int_\Omega \eta^2 (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx \\
&+ C(n,p,\Gamma,\varepsilon) \int_{B_{l_2}} |\tau_h V_p(D\psi)|^2 dx.
\end{aligned} \tag{4.4}
$$

Similarly, for the term II_2, we obtain

$$
|II_2| \le C(n,p,\Gamma) \int_{B_{l_2}} |\tau_h V_p(D\psi)|^2 dx + C(n,p,\Gamma) \int_{B_{\lambda r}} (\mu^p + |D\psi|^p) dx.
$$

Noticing that $D\psi \in L_{\mathrm{loc}}^{\frac{np}{n-2\alpha}}(\Omega)$ and $p < \frac{np}{n-2\alpha}$, then by Hölder's Inequality with exponents $(\frac{n}{2\alpha}, \frac{n}{n-2\alpha})$, we obtain

$$
\begin{aligned}
\int_{B_R} |D\psi|^p dx &\le \big(\omega_n R^n\big)^{2\alpha/n} \Big(\int_{B_R} |D\psi|^{\frac{np}{n-2\alpha}} dx\Big)^{\frac{n-2\alpha}{n}} \\
&\le C(n,p) R^{2\alpha} \Big(\int_{B_R} |D\psi|^{\frac{np}{n-2\alpha}} dx\Big)^{\frac{n-2\alpha}{n}},
\end{aligned}
$$

where ω_n is the measure of the ball of radius 1 in $\mathbb{R}^n$. Then

$$
\begin{aligned}
|II_2| \le{}& C(n,p,\Gamma) \int_{B_{l_2}} |\tau_h V_p(D\psi)|^2 dx \\
&+ C(n,p,\Gamma) R^{2\alpha} \Big[\int_{B_R} (1 + |D\psi|^{\frac{np}{n-2\alpha}}) dx\Big]^{\frac{n-2\alpha}{n}}.
\end{aligned} \tag{4.5}
$$

Plugging (4.4) and (4.5) into (4.3), we obtain

$$
\begin{aligned}
|II| \le{}& \varepsilon \int_\Omega \eta^2 (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx \\
&+ C(n,p,\Gamma,\varepsilon) \int_{B_{l_2}} |\tau_h V_p(D\psi)|^2 dx \\
&+ C(n,p,\Gamma) R^{2\alpha} \Big[\int_{B_R} (1 + |D\psi|^{\frac{np}{n-2\alpha}}) dx\Big]^{\frac{n-2\alpha}{n}}.
\end{aligned} \tag{4.6}
$$

For the term III, by assumption 1.3, Young's Inequality with exponents $(\frac{p}{p-1}, p)$, Lemma 2.3 and $D(u - \psi) \in L_{\mathrm{loc}}^{\frac{np}{n-2\alpha}}(\Omega)$, arguing like in (4.5), thus obtaining

$$
\begin{aligned}
|III| &\leq 2\Gamma \int_{\Omega} \eta |D\eta| (\mu^2 + |Du|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du| |\tau_h(u-\psi)| dx \\
&\leq \varepsilon \int_{\Omega} \eta^{\frac{p}{p-1}} (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx \\
&\quad + \frac{C(n,p,\Gamma,\varepsilon)}{R^{p-2\alpha}} |h|^p \Big(\int_{B_{2R}} |D(u-\psi)|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}}.
\end{aligned} \tag{4.7}
$$

For the term IV, by assumption (1.6) and Young's Inequality with exponents $(2,2)$, we have

$$
\begin{aligned}
|IV| &\leq |h|^\alpha \int_{\Omega} \eta^2 (\iota_k(x) + \iota_k(x+h))(\mu^2 + |Du|^2)^{\frac{p-1}{2}} |\tau_h Du| dx \\
&\leq \varepsilon \int_{\Omega} \eta^2 (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx \\
&\quad + C(\varepsilon)|h|^{2\alpha} \int_{B_{l_2}} (\iota_k(x) + \iota_k(x+h))^2 (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{p/2} dx,
\end{aligned}
$$

where $2^{-k}\frac{R}{4} \leq |h| \leq 2^{-k+1}\frac{R}{4}$ for $k \in \mathbb{N}$. Then by using Hölder's Inequality with exponents $(\frac{n}{2\alpha}, \frac{n}{n-2\alpha})$ and Lemma 2.3 we obtain

$$
\begin{aligned}
|IV| &\leq \varepsilon \int_{\Omega} \eta^2 (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx \\
&\quad + C(n,p,\alpha,\varepsilon)|h|^{2\alpha} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{2\alpha/n} \\
&\quad \times \Big(\int_{B_R} (\mu^{\frac{np}{n-2\alpha}} + |Du|^{\frac{np}{n-2\alpha}}) dx \Big)^{\frac{n-2\alpha}{n}}.
\end{aligned}
$$

Now we apply Theorem 2.16 with $q = \frac{np}{n-2\alpha}$, thus obtaining

$$
\begin{aligned}
&\fint_{B_R} |Du|^{\frac{np}{n-2\alpha}} dx \\
&\leq C\Big[1 + \fint_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \fint_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\fint_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big].
\end{aligned}
$$

Then for the term IV, we obtain

$$
\begin{aligned}
|IV| &\leq \varepsilon \int_{\Omega} \eta^2 (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx \\
&\quad + C(n,p,\alpha,\varepsilon)|h|^{2\alpha} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{2\alpha/n} \\
&\quad \times \Big[1 + \fint_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \fint_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx \\
&\quad + \Big(\fint_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}.
\end{aligned} \tag{4.8}
$$

To estimate the term V, we consider $2^{-k}\frac{R}{4} \leq |h| \leq 2^{-k+1}\frac{R}{4}$ for $k \in \mathbb{N}$, using (1.6), Young's Inequality with exponents $(2,2)$ and Lemma 2.1, thus obtaining

$$
\begin{aligned}
|V| &\leq |h|^\alpha \int_\Omega \eta^2(\iota_k(x) + \iota_k(x+h))(\mu^2 + |Du|^2)^{\frac{p-1}{2}} |\tau_h D\psi| dx \\
&\leq |h|^\alpha \int_{B_{l_2}} (\iota_k(x) + \iota_k(x+h))(\mu^2 + |Du|^2)^{\frac{p-1}{2}} |\tau_h D\psi| \\
&\quad \times (\mu^2 + |D\psi(x)|^2 + |D\psi(x+h)|^2)^{\frac{2-p}{4}} (\mu^2 + |D\psi(x)|^2 + |D\psi(x+h)|^2)^{\frac{p-2}{4}} dx \\
&\leq c|h|^{2\alpha} \int_{B_{l_2}} (\iota_k(x) + \iota_k(x+h))^2(\mu^2 + |Du|^2)^{p-1} \\
&\quad \times (\mu^2 + |D\psi(x)|^2 + |D\psi(x+h)|^2)^{\frac{2-p}{2}} dx \\
&\quad + c \int_{B_R} |\tau_h V_p(D\psi)|^2 dx.
\end{aligned}
$$

By using Hölder's Inequality with exponents $(\frac{n}{2\alpha}, \frac{n}{n-2\alpha})$, $(\frac{p}{2(p-1)}, \frac{p}{2-p})$, and Lemma 2.3, the previous estimate becomes

$$
\begin{aligned}
|V| &\leq c|h|^{2\alpha} \Big(\int_{B_{l_2}} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{2\alpha/n} \\
&\quad \times \Big(\int_{B_{l_2}} (\mu^2 + |Du|^2)^{\frac{n(p-1)}{n-2\alpha}} \big(\mu^2 + |D\psi(x)|^2 + |D\psi(x+h)|^2\big)^{\frac{n(2-p)}{2(n-2\alpha)}} dx \Big)^{\frac{n-2\alpha}{n}} \\
&\quad + c \int_{B_R} |\tau_h V_p(D\psi)|^2 dx \\
&\leq c|h|^{2\alpha} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{2\alpha/n} \\
&\quad \times \Big(\int_{B_R} (\mu^{\frac{np}{n-2\alpha}} + |Du|^{\frac{np}{n-2\alpha}}) dx \Big)^{\frac{2(p-1)}{p}\frac{n-2\alpha}{n}} \\
&\quad \times \Big(\int_{B_R} (\mu^{\frac{np}{n-2\alpha}} + |D\psi|^{\frac{np}{n-2\alpha}}) dx \Big)^{\frac{2-p}{p}\frac{n-2\alpha}{n}} + c \int_{B_R} |\tau_h V_p(D\psi)|^2 dx.
\end{aligned}
$$

Using Young's Inequality with exponents $(\frac{p}{2(p-1)}, \frac{p}{2-p})$, we obtain

$$
\begin{aligned}
|V| &\leq c \int_{B_R} |\tau_h V_p(D\psi)|^2 dx + c|h|^{2\alpha} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{2\alpha/n} \\
&\quad \times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx \\
&\quad + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}.
\end{aligned} \tag{4.9}
$$

Now we consider the term VI, taking $2^{-k}\frac{R}{4} \leq |h| \leq 2^{-k+1}\frac{R}{4}$ for $k \in \mathbb{N}$, using assumption (1.6), Hölder's Inequality with exponents $(\frac{n}{2\alpha}, \frac{n}{n-2\alpha})$, $(\frac{p}{p-1}, p)$ and Lemma 2.3, we obtain

$$
\begin{aligned}
|VI| &\leq |h|^\alpha \int_\Omega \eta |D\eta|(\iota_k(x) + \iota_k(x+h))(\mu^2 + |Du|^2)^{\frac{p-1}{2}} |\tau_h(u-\psi)| dx \\
&\leq \frac{c}{R} |h|^\alpha \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{\frac{n}{2\alpha}} dx \Big)^{2\alpha/n}
\end{aligned}
$$

$$\times \Big(\int_{B_R} (\mu^2 + |Du|^2)^{\frac{n(p-1)}{2(n-2\alpha)}} |\tau_h(u-\psi)|^{\frac{n}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}}$$

$$\leq \frac{c}{R} |h|^\alpha \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{\frac{n}{2\alpha}} dx \Big)^{\frac{2\alpha}{n}}$$

$$\times \Big[\Big(\int_{B_R} (\mu^2 + |Du|^2)^{\frac{np}{2(n-2\alpha)}} dx \Big)^{\frac{p-1}{p} \frac{n-2\alpha}{n}} \Big(\int_{B_R} |\tau_h(u-\psi)|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{np}} \Big]$$

$$\leq \frac{C(n,p,\alpha)}{R} |h|^{\alpha+1} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{\frac{n}{2\alpha}} dx \Big)^{2\alpha/n}$$

$$\times \Big[\Big(\int_{B_R} (\mu^{\frac{np}{n-2\alpha}} + |Du|^{\frac{np}{n-2\alpha}}) dx \Big)^{\frac{p-1}{p} \frac{n-2\alpha}{n}} \Big(\int_{B_{\lambda R}} |D(u-\psi)|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{np}} \Big].$$

Notice that $\{\iota_k\}_k \subset L^{n/\alpha}(\Omega) \subset L^{\frac{n}{2\alpha}}(\Omega)$ with the estimate

$$\|\iota_k\|_{L^{\frac{n}{2\alpha}}(B_R)} \leq \Big(\int_{B_R} 1^2 dx \Big)^{\alpha/n} \Big(\int_{B_R} |\iota_k|^{n/\alpha} dx \Big)^{\alpha/n} \leq cR^\alpha \|\iota_k\|_{L^{n/\alpha}(B_R)}. \quad (4.10)$$

By Young's Inequality with exponents $(\frac{p}{p-1}, p)$ and (4.10), the previous estimate becomes

$$
\begin{aligned}
|VI| &\leq \frac{C(n,p,\alpha)}{R^{1-\alpha}} |h|^{\alpha+1} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{\alpha/n} \\
&\quad \times \Big[\Big(\int_{B_R} (\mu^{\frac{np}{n-2\alpha}} + |Du|^{\frac{np}{n-2\alpha}}) dx \Big)^{\frac{n-2\alpha}{n}} \\
&\quad + \Big(\int_{B_{\lambda R}} |D(u-\psi)|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}} \Big] \\
&\leq \frac{C(n,p,\alpha)}{R^{1-\alpha}} |h|^{\alpha+1} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{\alpha/n} \\
&\quad \times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx \\
&\quad + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}.
\end{aligned}
\quad (4.11)
$$

It remains to estimate VII, $VIII$, and IX. For the term VII, using Young's Inequality with exponents $(\frac{p}{p-1}, p)$, Lemma 2.1, Lemma 2.3, and Hölder's Inequality with exponents $(\frac{n}{2\alpha}, \frac{n}{n-2\alpha})$, we have

$$
\begin{aligned}
|VII| &\leq \int_\Omega \eta^2 ||F(x+h)|^{p-2} F(x+h) - |F(x)|^{p-2} F(x)| |\tau_h Du| dx \\
&\leq C_0(n,p)\varepsilon \int_\Omega \eta^2 (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx \\
&\quad + C(n,p,\alpha,\varepsilon) |h|^p R^{2\alpha} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}}.
\end{aligned}
\quad (4.12)
$$

For $VIII$, using Hölder's Inequality with exponents $(\frac{p}{p-1}, p)$ and $(\frac{n}{2\alpha}, \frac{n}{n-2\alpha})$, Lemma 2.1, and Lemma 2.3, we have

$$
|VIII|
$$

$$
\leq \int_\Omega \eta^2 ||F(x+h)|^{p-2}F(x+h) - |F(x)|^{p-2}F(x)||\tau_h D\psi|dx
$$

$$
\leq C(n,p)|h|^p \Big(\int_{B_R} |DF|^p dx \Big)^{\frac{p-1}{p}} \Big(\int_{B_R} |D^2\psi|^p dx \Big)^{1/p} \tag{4.13}
$$

$$
\leq C(n,p)|h|^p R^{2\alpha} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}\frac{p-1}{p}} \Big(\int_{B_R} |D^2\psi|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{np}}.
$$

Arguing as we did for the estimate of $VIII$, for the term IX, we obtain

$$
|IX| \leq 2\int_\Omega \eta|D\eta|||F(x+h)|^{p-2}F(x+h) - |F(x)|^{p-2}F(x)|\,|\tau_h(u-\psi)|dx
$$

$$
\leq \frac{C(n,p)}{R}|h|^p \Big(\int_{B_R} |DF|^p dx \Big)^{\frac{p-1}{p}} \big(\int_{B_R} (|Du|^p + |D\psi|^p) dx \big)^{1/p}
$$

$$
\leq \frac{C(n,p)}{R}|h|^p R^{2\alpha} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}\frac{p-1}{p}} \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx \tag{4.14}
$$

$$
+ \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{np}}.
$$

Plugging (4.6), (4.7), (4.8), (4.9), (4.11), (4.12), (4.13), and (4.14) into (4.1), and then choosing $\varepsilon = \frac{\nu}{2(3+C_0)}$, we obtain

$$
\int_{B_{R/2}} |\tau_h V_p(Du)|^2 dx
$$

$$
\leq cR^{2\alpha} \Big[\int_{B_R} (1 + |D\psi|^{\frac{np}{n-2\alpha}}) dx \Big]^{\frac{n-2\alpha}{n}}
$$

$$
+ \frac{c}{R^{p-2\alpha}}|h|^p \Big(\int_{B_{2R}} |D(u-\psi)|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}}
$$

$$
+ c|h|^{2\alpha} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{2\alpha/n}
$$

$$
\times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}
$$

$$
+ c\int_{B_R} |\tau_h V_p(D\psi)|^2 dx + \frac{c}{R^{1-\alpha}}|h|^{\alpha+1} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{\alpha/n}
$$

$$
\times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}
$$

$$
+ c|h|^p R^{2\alpha} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}}
$$

$$
+ c|h|^p R^{2\alpha} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}\frac{p-1}{p}} \Big(\int_{B_R} |D^2\psi|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{np}}
$$

$$
+ c|h|^p R^{2\alpha-1} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}\frac{p-1}{p}} \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx
$$

$$+ \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{np}}. \tag{4.15}$$

Now, we use a covering argument [5] to the balls of the integrals in (4.15), whose radii are proportional to R, we have $R \propto |h|^\beta$ for $\beta \in (0,1)$ and a sufficiently small value of $|h|$, then (4.15) becomes

$$\int_{B_{R/2}} |\tau_h V_p(Du)|^2 dx$$

$$\leq c|h|^{2\alpha\beta} \Big[\int_{B_R} (1 + |D\psi|^{\frac{np}{n-2\alpha}}) dx \Big]^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^{p(1-\beta)+2\alpha\beta} \Big(\int_{B_{2R}} |D(u-\psi)|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^{2\alpha} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{2\alpha/n}$$

$$\times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}$$

$$+ c \int_{B_R} |\tau_h V_p(D\psi)|^2 dx + c|h|^{\alpha-\beta+\alpha\beta+1} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{\alpha/n}$$

$$\times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^{p+2\alpha\beta} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^{p+2\alpha\beta} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n} \frac{p-1}{p}} \Big(\int_{B_R} |D^2\psi|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{np}}$$

$$+ c|h|^{p-\beta+2\alpha\beta} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n} \frac{p-1}{p}} \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx$$

$$+ \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{np}}. \tag{4.16}$$

Since $\alpha, \beta \in (0,1)$, by setting

$$p_1 = 2\alpha\beta \in (0,2), \quad p_2 = p(1-\beta) + 2\alpha\beta \in (0,4),$$

$$p_3 = 2\alpha \in (0,2), \quad p_4 = \alpha - \beta + \alpha\beta + 1 = (\alpha+1)(1-\beta) + 2\alpha\beta \in (0,3),$$

$$p_5 = p + 2\alpha\beta \in (1,4), \quad p_6 = p - \beta + 2\alpha\beta \in (0,3),$$

we have

$$\min_{1 \leq i \leq 6} p_i = p_1 = 2\alpha\beta.$$

We divide both sides of (4.16) by $|h|^{2\alpha\beta}$, and notice that $|h|^{-2\alpha\beta} \leq |h|^{-2\alpha}$ for $|h| < 1$, $0 < \alpha, \beta < 1$. Then we have

$$\int_{B_{R/2}} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha\beta}} dx$$

$$\leq c \Big[\int_{B_R} (1 + |D\psi|^{\frac{np}{n-2\alpha}}) dx \Big]^{\frac{n-2\alpha}{n}} + c|h|^{p(1-\beta)} \Big(\int_{B_{2R}} |D(u-\psi)|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^{2\alpha(1-\beta)} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{2\alpha/n}$$

$$\times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \big(\int_{B_{2R}} |Du|^p dx \big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}$$

$$+ c \int_{B_R} \frac{|\tau_h V_p(D\psi)|^2}{|h|^{2\alpha}} dx + c|h|^{(\alpha+1)(1-\beta)} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{\alpha/n}$$

$$\times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^p \big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \big)^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^p \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n} \frac{p-1}{p}} \Big(\int_{B_R} |D^2\psi|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{np}}$$

$$+ c|h|^{p-\beta} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n} \frac{p-1}{p}} \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx$$

$$+ \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{np}}. \tag{4.17}$$

Next we take the L^q norm with the measure $\frac{dh}{|h|^n}$ restricted to the ball $B(0, \frac{R}{4})$. For any $k \in \mathbb{N}$, the integral in the third and fifth lines of (4.17) are taken for $2^{-k}\frac{R}{4} \leq |h| \leq 2^{-k+1}\frac{R}{4}$, so it is essential to notice that

$$B(0, \frac{R}{4}) = \cup_{k=1}^{\infty} \big(B(0, 2^{-k+1}\frac{R}{4}) \backslash B(0, 2^{-k}\frac{R}{4}) \big) =: \cup_{k=1}^{\infty} E_k.$$

It is also worth noticing that the choice of the radius $R = |h|^\beta$ is possible for small values of $|h|$. This is because $2^{-k}\frac{R}{4} \leq |h| \leq 2^{-k+1}\frac{R}{4}$ if and only if $2^{-\frac{k+2}{1-\beta}} \leq |h| \leq 2^{-\frac{k+1}{1-\beta}}$, for any $k \in \mathbb{N}$. Thus we have the estimate

$$\int_{B_{\frac{R}{4}}(0)} \Big(\int_{B_{\frac{R}{2}}} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha\beta}} dx \Big)^{q/2} \frac{dh}{|h|^n}$$

$$\leq c \int_{B_{\frac{R}{4}}(0)} \Big[\int_{B_R} (1 + |D\psi|^{\frac{np}{n-2\alpha}}) dx \Big]^{\frac{q(n-2\alpha)}{2n}} \frac{dh}{|h|^n}$$

$$+ c \int_{B_{\frac{R}{4}}(0)} |h|^{\frac{qp(1-\beta)}{2}} \frac{dh}{|h|^n} \Big(\int_{B_R} |D(u-\psi)|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{q(n-2\alpha)}{2n}}$$

$$+ c \sum_{k=1}^{\infty} \int_{E_k} |h|^{q\alpha(1-\beta)} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{\frac{q\alpha}{n}} \frac{dh}{|h|^n}$$

$$\times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{q(n-2\alpha)}{2n}}$$

$$+ c \int_{B_{\frac{R}{4}}(0)} \Big(\int_{B_R} \frac{|\tau_h V_p(D\psi)|^2}{|h|^{2\alpha}} dx \Big)^{q/2} \frac{dh}{|h|^n}$$

$$+ c \sum_{k=1}^{\infty} \int_{E_k} |h|^{\frac{q(\alpha+1)(1-\beta)}{2}} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{\frac{q\alpha}{2n}} \frac{dh}{|h|^n}$$

$$\times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{q(n-2\alpha)}{2n}}$$

$$+ c \int_{B_{\frac{R}{4}}(0)} |h|^{qp/2} \frac{dh}{|h|^n} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{(n-2\alpha)q}{2n}}$$

$$+ c \int_{B_{\frac{R}{4}}(0)} |h|^{qp/2} \frac{dh}{|h|^n} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}\frac{q(p-1)}{2p}} \Big(\int_{B_R} |D^2\psi|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{q(n-2\alpha)}{2np}}$$

$$+ c \int_{B_{\frac{R}{4}}(0)} |h|^{\frac{q(p-\beta)}{2}} \frac{dh}{|h|^n} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}\frac{q(p-1)}{2p}}$$

$$\times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{q(n-2\alpha)}{2np}}.$$
$$(4.18)$$

To simplify notation, we set

$$S^* = \int_{B_{2R}} (1 + |Du|^p + |F|^{\hat{p}} + |DF|^{\hat{p}} + |Du|^{\hat{p}} + |D\psi|^{\hat{p}} + |D^2\psi|^{\hat{p}}) dx. \qquad (4.19)$$

where $\hat{p} = \frac{np}{n-2\alpha}, 0 < \alpha < 1$. Then (4.18) can be written as

$$\int_{B_{\frac{R}{4}}(0)} \Big(\int_{B_{\frac{R}{2}}} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha\beta}} dx \Big)^{q/2} \frac{dh}{|h|^n}$$

$$\leq C \int_{B_{\frac{R}{4}}(0)} |h|^{\frac{qp(1-\beta)}{2}} \frac{dh}{|h|^n}$$

$$+ C \sum_{k=1}^{\infty} \int_{E_k} |h|^{q\alpha(1-\beta)} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{\frac{q\alpha}{n}} \frac{dh}{|h|^n}$$

$$+ C \int_{B_{\frac{R}{4}}(0)} \Big(\int_{B_R} \frac{|\tau_h V_p(D\psi)|^2}{|h|^{2\alpha}} dx \Big)^{q/2} \frac{dh}{|h|^n}$$

$$+ C \sum_{k=1}^{\infty} \int_{E_k} |h|^{\frac{q(\alpha+1)(1-\beta)}{2}} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{\frac{q\alpha}{2n}} \frac{dh}{|h|^n}$$

$$+ C \int_{B_{\frac{R}{4}}(0)} |h|^{qp/2} \frac{dh}{|h|^n} + C \int_{B_{\frac{R}{4}}(0)} |h|^{\frac{q(p-\beta)}{2}} \frac{dh}{|h|^n}, \qquad (4.20)$$

where the constant $C = C(n, p, q, R, \alpha, \nu, \gamma, \Gamma, S^*)$.

Now we apply Young's Inequality with exponents $(2, 2)$ to the second and the fourth integrals of the right-hand side of (4.20), thus obtaining

$$\int_{B_{\frac{R}{4}}(0)} \Big(\int_{B_{\frac{R}{2}}} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha\beta}} dx \Big)^{q/2} \frac{dh}{|h|^n}$$

$$\leq C \int_{B_{\frac{R}{4}}(0)} |h|^{\frac{qp(1-\beta)}{2}} \frac{dh}{|h|^n} + C \int_{B_{\frac{R}{4}}(0)} |h|^{2q\alpha(1-\beta)} \frac{dh}{|h|^n}$$

$$+ C \sum_{k=1}^{\infty} \int_{E_k} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{\frac{2q\alpha}{n}} \frac{dh}{|h|^n}$$

$$+ C \int_{B_{\frac{R}{4}}(0)} |h|^{q(\alpha+1)(1-\beta)} \frac{dh}{|h|^n}$$

$$+ C \sum_{k=1}^{\infty} \int_{E_k} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{\frac{q\alpha}{n}} \frac{dh}{|h|^n}$$

$$+ C \int_{B_{\frac{R}{4}}(0)} |h|^{qp/2} \frac{dh}{|h|^n} + C \int_{B_{\frac{R}{4}}(0)} |h|^{\frac{q(p-\beta)}{2}} \frac{dh}{|h|^n}$$

$$+ C \int_{B_{\frac{R}{4}}(0)} \Big(\int_{B_R} \frac{|\tau_h V_p(D\psi)|^2}{|h|^{2\alpha}} dx \Big)^{q/2} \frac{dh}{|h|^n}. \tag{4.21}$$

Noticing that $\alpha, \beta \in (0,1)$ and $p \in (1,2)$, if we set

$$q_1 = \frac{qp(1-\beta)}{2}, \quad q_2 = 2q\alpha(1-\beta), \quad q_3 = q(\alpha+1)(1-\beta),$$

$$q_4 = \frac{qp}{2}, \quad q_5 = \frac{q(p-\beta)}{2},$$

then we have

$$\vartheta := \min_{1 \le i \le 5} q_i > 0.$$

Since $|h| < 1$, we can write (4.21) as follows

$$\int_{B_{\frac{R}{4}}(0)} \Big(\int_{B_{\frac{R}{2}}} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha\beta}} dx \Big)^{q/2} \frac{dh}{|h|^n}$$

$$\le C \sum_{k=1}^{\infty} \int_{E_k} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{\frac{2q\alpha}{n}} \frac{dh}{|h|^n}$$

$$+ C \sum_{k=1}^{\infty} \int_{E_k} \Big(\int_{B_R} (\iota_k(x) + \iota_k(x+h))^{n/\alpha} dx \Big)^{\frac{q\alpha}{n}} \frac{dh}{|h|^n} \tag{4.22}$$

$$+ C \int_{B_{\frac{R}{4}}(0)} \Big(\int_{B_R} \frac{|\tau_h V_p(D\psi)|^2}{|h|^{2\alpha}} dx \Big)^{q/2} \frac{dh}{|h|^n} + C \int_{B_{\frac{R}{4}}(0)} |h|^{\vartheta} \frac{dh}{|h|^n}$$

$$=: I_1 + I_2 + I_3 + I_4.$$

By assumption, we have

$$I_3 \le \|V_p(D\psi)\|_{B_{2,q}^{\alpha}(B_R)} < \infty. \tag{4.23}$$

For the term I_4, by polar coordinates transformation, we obtain

$$I_4 = C \int_{B_{\frac{R}{4}}(0)} |h|^{\vartheta} \frac{dh}{|h|^n} = C \int_0^{R/4} \rho^{\vartheta-1} d\rho = C(n, p, q, R, \alpha, \beta) < \infty, \tag{4.24}$$

since $\vartheta > 0$, i.e. $\vartheta - 1 > -1$, which implies the last integral is finite.

We now write the integral I_1 in polar coordinates, so $h \in E_k$ if and only if $h = \rho\xi$ for $2^{-k}\frac{R}{4} \le \rho < 2^{-k+1}\frac{R}{4}$ and some ξ in the unit sphere S^{n-1} on $\mathbb{R}^n$. We denote by

$d\sigma(\xi)$ the surface measure on S^{n-1}, then we have

$$
\begin{aligned}
I_1 &= C\sum_{k=1}^{\infty}\int_{r_k}^{r_{k-1}}\int_{S^{n-1}}\Big(\int_{B_R}(\iota_k(x)+\iota_k(x+\rho\xi))^{n/\alpha}dx\Big)^{\frac{2q\alpha}{n}}d\sigma(\xi)\frac{d\rho}{\rho}\\
&\le C\sum_{k=1}^{\infty}\int_{r_k}^{r_{k-1}}\int_{S^{n-1}}\|\iota_k(x)+\iota_k(x+\rho\xi)\|_{L^{n/\alpha}(B_R)}^{2q}d\sigma(\xi)\frac{d\rho}{\rho}\\
&\le C\Big(\sum_{k=1}^{\infty}\|\iota_k\|_{L^{n/\alpha}(B_{2R})}^{2q}\Big)\int_{r_k}^{r_{k-1}}\frac{d\rho}{\rho}\\
&= C\ln 2\sum_{k=1}^{\infty}\|\iota_k\|_{L^{n/\alpha}(B_{2R})}^{2q}\\
&= C\|(\iota_k)_k\|_{\ell^{2q}(L^{n/\alpha}(B_{2R}))}^{2q}\\
&\le C\|(\iota_k)_k\|_{\ell^q(L^{n/\alpha}(B_{2R}))}^{2q} < \infty,
\end{aligned}
\tag{4.25}
$$

where we set $r_k = 2^{-k}\frac{R}{4}$. Moreover, for any $\xi \in S^{n-1}$ and $r_k \le \rho < r_{k-1}$, by Lemma 2.3, we have

$$
\|(\iota_k(x)+\iota_k(x+\rho\xi))\|_{L^{n/\alpha}(B_R)} \le 2\|\iota_k\|_{L^{n/\alpha}(B_{R+\frac{R}{4}})} \le 2\|\iota_k\|_{L^{n/\alpha}(B_{2R})}.
$$

Recalling the continuous embedding $\ell^q(L^{n/\alpha}(B_{2R})) \hookrightarrow \ell^{2q}(L^{n/\alpha}(B_{2R}))$, so the estimate is easily established.

Arguing as the estimate to I_1, for the term I_2, we obtain

$$
I_2 \le C\|(\iota_k)_k\|_{\ell^q(L^{n/\alpha}(B_{2R}))}^q < \infty.
\tag{4.26}
$$

Inserting (4.23)-(4.26) in (4.22), we have

$$
\|\frac{\tau_h V_p(Du)}{|h|^{\alpha\beta}}\|_{L^q(\frac{dh}{|h|^n};L^2(B_{R/2}))} \le C\Big(1+\|V_p(D\psi)\|_{B_{2,q}^\alpha(B_R)}+\|(\iota_k)_k\|_{\ell^q(L^{n/\alpha}(B_{2R}))}^{2q}\Big).
$$

Noticing that the constant C depends on S^* given by (4.19), then for a suitable exponent $\sigma = \sigma(n,p,q,\alpha) > 0$, utilizing that $p < \frac{np}{n-2\alpha}$, Theorem 2.16, Lemma 2.9, and Lemma 2.11, we obtain

$$
\begin{aligned}
&\|\frac{\tau_h V_p(Du)}{|h|^{\alpha\beta}}\|_{L^q(\frac{dh}{|h|^n};L^2(B_{R/2}))}\\
&\le C\Big[1+\|Du\|_{L^p(B_{4R})}+\|V_p(D\psi)\|_{B_{2,q}^\alpha(B_{4R})}+\|D^2\psi\|_{L^{\frac{np}{n-2\alpha}}(B_{4R})}\\
&\quad +\|F\|_{W^{1,\frac{np}{n-2\alpha}}(B_{4R})}+\|(\iota_k)_k\|_{\ell^q(L^{n/\alpha}(B_{2R}))}\Big]^\sigma,
\end{aligned}
$$

that is the conclusion. $\qquad\square$

4.2. Proof of Theorem 1.3.
To prove the theorem we use the arguments of the previous section. Differences come when we estimate IV, V and VI.

Proof. By hypothesis, since $V_p(D\psi) \in B_{2,\infty,\mathrm{loc}}^\delta(\Omega)$ with $0 < \alpha < \delta < 1$, we have $V_p(D\psi) \in L_{\mathrm{loc}}^{\frac{2n}{n-2\alpha}}(\Omega)$, and so $D\psi \in L_{\mathrm{loc}}^{\frac{np}{n-2\alpha}}(\Omega)$. Similar to the proof of Theorem 1.2, the estimate to $I, II, III, VII, VIII$ and IX can be treated in the same way. We only need to use assumption (1.7) instead of (1.6) so as to estimate the term IV, V and VI.

For the term IV, using the assumption (1.7), Young's Inequality with exponents $(2,2)$, Hölder's Inequality with exponents $(\frac{n}{2\alpha}, \frac{n}{n-2\alpha})$ and Lemma 2.3, for $|h| < \frac{R}{4}$, we have

$|IV|$

$$\leq |h|^\alpha \int_\Omega \eta^2(\iota(x) + \iota(x+h))(\mu^2 + |Du|^2)^{\frac{p-1}{2}} |\tau_h Du| dx$$

$$\leq \varepsilon \int_\Omega \eta^2(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx$$

$$+ C(\varepsilon)|h|^{2\alpha} \int_{B_{l_2}} (\iota(x) + \iota(x+h))^2(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{p/2} dx \tag{4.27}$$

$$\leq \varepsilon \int_\Omega \eta^2(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h Du|^2 dx$$

$$+ C(n,p,\alpha,\varepsilon)|h|^{2\alpha} \Big(\int_{B_R} (\iota(x) + \iota(x+h))^{n/\alpha} dx \Big)^{2\alpha/n} \Big[1$$

$$+ \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}.$$

For the term V, by the same arguments that we used in the previous section, we have

$$|V| \leq c|h|^{2\alpha} \Big(\int_{B_R} (\iota(x) + \iota(x+h))^{n/\alpha} dx \Big)^{2\alpha/n} \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx$$

$$+ \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}} \tag{4.28}$$

$$+ c \int_{B_R} |\tau_h V_p(D\psi)|^2 dx.$$

For the term VI, by assumption (1.7), for $|h| < \frac{R}{4}$, we obtain

$$|VI| \leq \frac{C|h|^{\alpha+1}}{R^{1-\alpha}} \Big(\int_{B_R} (\iota(x) + \iota(x+h))^{n/\alpha} dx \Big)^{\alpha/n} \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx$$

$$+ \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}} \tag{4.29}$$

Now we plug (4.6), (4.7), (4.12),(4.13), (4.14), (4.27), (4.28), and (4.29) into (4.1), then choosing $\varepsilon = \frac{\nu}{2(3+C_0)}$, we obtain

$$\int_{B_{R/2}} |\tau_h V_p(Du)|^2 dx$$

$$\leq cR^{2\alpha} \Big[\int_{B_R} (1 + |D\psi|^{\frac{np}{n-2\alpha}}) dx \Big]^{\frac{n-2\alpha}{n}}$$

$$+ \frac{c}{R^{p-2\alpha}} |h|^p \Big(\int_{B_{2R}} |D(u-\psi)|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^{2\alpha} \Big(\int_{B_R} (\iota(x) + \iota(x+h))^{n/\alpha} dx \Big)^{2\alpha/n}$$

$$\times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}$$

$$+ c \int_{B_R} |\tau_h V_p(D\psi)|^2 dx + \frac{c}{R^{1-\alpha}} |h|^{\alpha+1} \Big(\int_{B_R} (\iota(x) + \iota(x+h))^{n/\alpha} dx \Big)^{\alpha/n}$$

$$\times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^p R^{2\alpha} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^p R^{2\alpha} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n} \frac{p-1}{p}} \Big(\int_{B_R} |D^2\psi|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{np}}$$

$$+ c|h|^p R^{2\alpha-1} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n} \frac{p-1}{p}} \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx$$

$$+ \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{np}}. \tag{4.30}$$

Now let us notice that $|h| \le \frac{|h|^\beta}{4}$ if and only if $|h| \le 2^{-\frac{2}{1-\beta}}$ for any $\beta \in (0,1)$. Recalling the covering argument that we used in the previous section, so we have $R \propto |h|^\beta$. Dividing both sides of (4.30) by $|h|^{2\alpha\beta}$, using Lemma 2.3 and that for $|h| < \frac{R}{4} < R \le 1$, since $0 < \alpha, \beta < 1$, $|h|^{-2\alpha\beta} < |h|^{-2\alpha}$, we obtain

$$\int_{B_{R/2}} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha\beta}} dx$$

$$\le c \Big[\int_{B_R} (1 + |D\psi|^{\frac{np}{n-2\alpha}}) dx \Big]^{\frac{n-2\alpha}{n}} + c|h|^{p(1-\beta)} \Big(\int_{B_{2R}} |D(u-\psi)|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^{2\alpha(1-\beta)} \Big(\int_{B_{2R}} \iota^{n/\alpha} dx \Big)^{2\alpha/n}$$

$$\times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}$$

$$+ c \int_{B_R} \frac{|\tau_h V_p(D\psi)|^2}{|h|^{2\alpha}} dx + c|h|^{(\alpha+1)(1-\beta)} \Big(\int_{B_{2R}} \iota^{n/\alpha} dx \Big)^{\alpha/n}$$

$$\times \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^p \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^p \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n} \frac{p-1}{p}} \Big(\int_{B_R} |D^2\psi|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{np}}$$

$$+ c|h|^{p-\beta} \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n} \frac{p-1}{p}} \Big[1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx$$

$$+ \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{np}}. \tag{4.31}$$

Using Young's Inequality with exponents $(\frac{n}{2\alpha}, \frac{n}{n-2\alpha})$ and $(\frac{p}{p-1}, p)$, then (4.31) becomes

$$\int_{B_{R/2}} \frac{|\tau_h V_p(Du)|^2}{|h|^{2\alpha\beta}} dx$$

$$\leq c \Big[\int_{B_R} (1 + |D\psi|^{\frac{np}{n-2\alpha}}) dx \Big]^{\frac{n-2\alpha}{n}} + c|h|^{p(1-\beta)} \Big(\int_{B_{2R}} |D(u - \psi)|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^{2\alpha(1-\beta)} \Big[\int_{B_{2R}} \iota^{n/\alpha} dx + 1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx$$

$$+ \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]$$

$$+ c \int_{B_R} \frac{|\tau_h V_p(D\psi)|^2}{|h|^{2\alpha}} dx + c|h|^{(\alpha+1)(1-\beta)} \cdot \Big[(\int_{B_{2R}} \iota^{n/\alpha} dx)^{\frac{1}{2}}$$

$$+ 1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]$$

$$+ c|h|^p \Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}}$$

$$+ c|h|^p \Big[\Big(\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}} + \Big(\int_{B_R} |D^2\psi|^{\frac{np}{n-2\alpha}} dx \Big)^{\frac{n-2\alpha}{n}} \Big]$$

$$+ c|h|^{p-\beta} \Big[\int_{B_R} |DF|^{\frac{np}{n-2\alpha}} dx + 1 + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx$$

$$+ \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\frac{n-2\alpha}{n}}. \tag{4.32}$$

By Lemma 2.13, $V_p(D\psi) \in B_{2,\infty,\text{loc}}^{\delta}(\Omega)$ implies that $D\psi \in B_{p,\infty,\text{loc}}^{\delta}(\Omega)$. Using that $0 < \alpha < \delta < 1$ and Lemma 2.10, we have $V_p(D\psi) \in B_{2,\infty,\text{loc}}^{\alpha}(\Omega)$ and $D\psi \in B_{p,\infty,\text{loc}}^{\alpha}(\Omega)$. Taking the supremum for $|h| < \frac{R}{4}$ at both sides of (4.32), we obtain

$$[V_p(Du)]_{\dot{B}_{2,\infty}^{\alpha\beta}(B_{R/2})}$$

$$\leq C[V_p(D\psi)]_{\dot{B}_{2,\infty}^{\alpha}(B_R)} + C \Big[1 + \int_{B_{2R}} \iota^{n/\alpha} dx + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |DF|^{\frac{np}{n-2\alpha}} dx$$

$$+ \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D^2\psi|^{\frac{np}{n-2\alpha}} dx + \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\sigma},$$

where the exponent $\sigma = \sigma(n, p, \alpha)$ and the constant $C = C(n, p, R, \alpha, \nu, \Gamma)$. By the definition of the norm in Besov-Lipschitz spaces and using Lemma 2.10, we have

$$[V_p(Du)]_{\dot{B}_{2,\infty}^{\alpha\beta}(B_{R/2})}$$

$$\leq C\|V_p(D\psi)\|_{B_{2,\infty}^{\delta}(B_R)} + C \Big[1 + \int_{B_{2R}} \iota^{n/\alpha} dx + \int_{B_{2R}} |F|^{\frac{np}{n-2\alpha}} dx$$

$$+ \int_{B_{2R}} |DF|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D\psi|^{\frac{np}{n-2\alpha}} dx + \int_{B_{2R}} |D^2\psi|^{\frac{np}{n-2\alpha}} dx$$

$$+ \Big(\int_{B_{2R}} |Du|^p dx \Big)^{\frac{n}{n-2\alpha}} \Big]^{\sigma}.$$

Noticing for $0 < \alpha < \delta < 1$, we have $p < \frac{np}{n-2\alpha} < \frac{np}{n-2\delta}$, then we obtain

$$[V_p(Du)]_{\dot{B}_{2,\infty}^{\alpha\beta}(B_{\frac{R}{2}})} \leq C \Big[1 + \|Du\|_{L^p(B_{2R})} + \|V_p(D\psi)\|_{B_{2,\infty}^{\delta}(B_R)}$$

$$+ \|D\psi\|_{L^{\frac{np}{n-2\delta}}(B_{2R})} + \|D^2\psi\|_{L^{\frac{np}{n-2\alpha}}(B_{2R})} + \|F\|_{W^{1,\frac{np}{n-2\alpha}}(B_{2R})}$$

$$+ \|\iota\|_{L^{n/\alpha}(B_{2R})} \Big]^{\sigma}.$$

Applying Lemma 2.5 to the function $V_p(D\psi)$, we obtain

$$[V_p(Du)]_{\dot{B}^{\alpha\beta}_{2,\infty}(B_{\frac{R}{2}})} \leq C \Big[1 + \|Du\|_{L^p(B_{4R})} + \|V_p(D\psi)\|_{B^{\delta}_{2,\infty}(B_{4R})}$$

$$+ \|D^2\psi\|_{L^{\frac{np}{n-2\alpha}}(B_{4R})} + \|F\|_{W^{1,\frac{np}{n-2\alpha}}(B_{4R})} + \|\iota\|_{L^{n/\alpha}(B_{2R})} \Big]^{\sigma}.$$

The proof of Theorem 1.3 is complete. $\qquad\square$

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