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INTEGRABLE NONLINEAR PERTURBED HIERARCHIES OF NLS-MKDV EQUATION AND SOLITON SOLUTIONS

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Abstract. We propose three spectral problems for NLS-mKdV equation by combining three integrable coupling ways. Then we obtain three nonlinear perturbation terms to derive three integrable nonlinear perturbed hierarchies of the NLS-mKdV equation. We proved the Lax integrability of the integrable nonlinear perturbed hierarchies. On the basis of a special orthogonal group, we prove the Liouville integrability of a third-order integrable nonlinear perturbed hierarchy of NLS-mKdV equation by deriving its bi-Hamiltonian structures. We build three Darboux matrices for constructing the Darboux transformations of the first two equations. As applications of the Darboux transformation, we present explicit solutions of these equations, three-dimensional plots, and density profiles the evolution of solitary waves.

1. Introduction

The nonlinear Schrödinger (NLS) equation has been derived in fields such as plasma physics, deep water waves, and nonlinear fiber optics; see [15, 20]. The modified Korteweg-de Vries (mKdV) equation appears in the description of van Alfvén waves in collisionless plasma, cosmic plasma, water waves, and so on; see [8, 25]. Both mKdV equation and the NLS equation are well-known for their physical and mathematical significance in nonlinear evolution models. The study of NLS-mKdV hierarchy is a significant topic in soliton theory. We consider the effect of perturbations so that their applicability can be extended to higher order nonlinearity or to larger amplitude waves. During the past few decades, there has been an increasing interest in the study of NLS-mKdV hierarchy, which was proposed as subalgebras of the loop algebra $\widetilde{A}_1$ in [6],

$$\begin{aligned} q_t &= \beta r_{xx} + 4\beta r(r^2 - q^2), \\ r_t &= \beta q_{xx} + 4\beta q(r^2 - q^2). \end{aligned} \tag{1.1}$$

A system of integrable coupling system is a larger system include the original integrable system as its sub-system; see [13]. A few ways to construct integrable coupling systems includes perturbations [11], creating new loop algebras [12], and enlarging spectral problem [22]. Integrable coupling makes integrable system more abundant and complex. Based on integrable coupling, the multi-component integrable couplings of the NLS-mKdV hierarchy was proposed in [27]. The super

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Hamiltonian structure of a super NLS-mKdV hierarchy is obtained by using super trace identity [26]. To further study NLS-mKdV hierarchy, researchers construct the completion of the NLS-mKdV integrable coupling systems and binary nonlinearization [18, 24]. There are many methods to obtain explicit solutions of integrable equations, for instance, Darboux transformation method [4, 14, 23], inverse scattering transformation [1, 2], Hirota method [7, 21], Bäcklund transformation [16, 9], and so on. Darboux transformation is an efficient method for solving nonlinear partial differential equations. The Darboux transformation studies the explicit solution of an integrable system from a seed solution. We choose different seed solutions and analyze the relations among them. In 2019, the generalized super-NLS-mKdV equation was solved with Darboux transformation in [5]. Then they gave analytic solutions by using symbolic computations, and plot their graphs.

It is important to derive a soliton hierarchy from its corresponding spectral problem. In 2009, Dong et al [3] studied the spectral problem

$$\Phi_x = U\Phi, \quad U = \begin{bmatrix} \lambda & p+q \\ -p+q & -\lambda \end{bmatrix}. \tag{1.2}$$

Starting from this equation, they constructed integrable couplings of NLS-mKdV hierarchy, and established their Hamiltonian structures and Super-Hamiltonian structures. We enlarge the spectral problem (1.2) as follows

$$\Phi_x = \overline{U}\Phi, \quad \overline{U} = \begin{bmatrix} U & U_0 \\ 0 & U \end{bmatrix} = \left[\begin{array}{cc|cc} \lambda & p+q & \lambda & r+s \\ -p+q & -\lambda & -r+s & -\lambda \\ \hline 0 & 0 & \lambda & p+q \\ 0 & 0 & -p+q & -\lambda \end{array}\right]. \tag{1.3}$$

By using the perturbation technique, we would like to generalize the spectral problem (1.3)

$$\Phi_x = U_1\Phi, \quad U_1 = \left[\begin{array}{cc|cc} \lambda+h & p+q & \lambda & r+s \\ -p+q & -\lambda-h & -r+s & -\lambda \\ \hline 0 & 0 & \lambda+h & p+q \\ 0 & 0 & -p+q & -\lambda-h \end{array}\right], \tag{1.4}$$

where $h = \epsilon(qs - pr)$, ϵ is an arbitrary constant. This way of adding the nonlinear term h is a "completion process of integrable couplings" [17]. We construct a fourth-order spectral problem (1.4) by enlarging spectral problem and adding a perturbed term.

In 2013, Ma [10] considered the spectral problem

$$\Phi_x = \hat{U}\Phi, \quad \hat{U} = \begin{bmatrix} 0 & q & \lambda \\ -q & 0 & -p \\ -\lambda & p & 0 \end{bmatrix}, \tag{1.5}$$

established a soliton hierarchy from zero curvature equation associated with so(3,$\mathbb{R}$) and their Hamiltonian structures. Motivated by the spectral problem (1.5) , we construct the spectral problem

$$\Phi_x = U_2\Phi, \quad U_2 = \begin{bmatrix} 0 & -p+q & \lambda+h \\ p-q & 0 & -p-q \\ -\lambda-h & p+q & 0 \end{bmatrix}, \tag{1.6}$$

where $h = \epsilon(p^2 + q^2)$. We construct the third-order spectral problem (1.6) by enlarging the Lie algebra and adding a perturbed term. By enlarging (1.6), we obtain the spectral problem $\Phi_x = U_3\Phi$ with

$$U_3 = \left[\begin{array}{ccc|ccc} 0 & -p+q & \lambda+h & 0 & -r+s & \lambda \\ p-q & 0 & -p-q & r-s & 0 & -r-s \\ -\lambda-h & p+q & 0 & -\lambda & r+s & 0 \\ \hline 0 & 0 & 0 & 0 & -p+q & \lambda+h \\ 0 & 0 & 0 & p-q & 0 & -p-q \\ 0 & 0 & 0 & -\lambda-h & p+q & 0 \end{array}\right], \tag{1.7}$$

where $h = \epsilon(qs+pr)$. In this article, starting from (1.4), (1.6) and (1.7) we propose three nonlinear integrable perturbed hierarchies of the NLS-mKdV equation. Our aim is to study explicit solutions of three nonlinear integrable perturbed hierarchies of NLS-mKdV equation by Darboux transformation method. Actually, it is difficult to select a Darboux matrix in integrable system and it is more difficult to obtain the explicit solution by Darboux transformation method in perturbed system than in unperturbed system. Moreover, three generalized forms of the NLS-mKdV equation are discussed together facilitates classification and comparisons. In addition, three-dimensional plots and density profiles of explicit solutions are visually presented to show their properties.

This article is outlined as follows. In Section 2, we obtain a fourth-order integrable perturbed hierarchy of NLS-mKdV equation by zero curvature equation, and explicit solutions by using Darboux transformation method. Also we present plots of these explicit solutions. In Section 3, we prove the Liouville integrability of a third-order integrable perturbed hierarchy of NLS-mKdV equation. We study explicit solutions of the first two nontrivial equations by using Darboux transformation, and plot their graphs. In Section 4, we obtain a sixth-order integrable perturbed hierarchy of NLS-mKdV equation on the basis of a sixth-order spectral problem. Also we obtain explicit solutions by using Darboux transformations, and present three-dimensional plots, and density profiles of explicit solutions. In Section 5, we give some conclusions.

2. Fourth-order nonlinear integrable perturbed hierarchy of NLS-mKdV equation and their explicit solutions

2.1. Fourth-order nonlinear integrable perturbed hierarchy of NLS-mKdV equation. To obtain a fourth-order nonlinear integrable perturbed hierarchy of NLS-mKdV equation, we consider the stationary zero curvature equation [19] associated with spectral problem (1.4),

$$V_{1,x} = [U_1, V_1] = U_1V_1 - V_1U_1, \tag{2.1}$$

with

$$\Phi_t = V_1\Phi, \quad V_1 = \left[\begin{array}{cc|cc} a & b+c & d & f+g \\ b-c & -a & f-g & -d \\ \hline 0 & 0 & a & b+c \\ 0 & 0 & b-c & -a \end{array}\right]. \tag{2.2}$$

Obviously, the above equation becomes

$$\begin{gathered} a_x = 2pb - 2qc, \\ b_x = 2\lambda c - 2pa + 2hc, \\ c_x = 2\lambda b - 2qa + 2hb, \\ d_x = 2pf - 2qg + 2rb - 2sc, \\ f_x = 2\lambda g + 2\lambda c + 2hg - 2pd - 2ra, \\ g_x = 2\lambda f + 2\lambda b + 2hf - 2qd - 2sa. \end{gathered} \tag{2.3}$$

By assuming the Laurent series expansions

$$V_1 = \sum_{i=0}^{\infty} \left[\begin{array}{cc|cc} a_i & b_i + c_i & d_i & f_i + g_i \\ b_i - c_i & -a_i & f_i - g_i & -d_i \\ \hline 0 & 0 & a_i & b_i + c_i \\ 0 & 0 & b_i - c_i & -a_i \end{array}\right] \lambda^{-i}. \tag{2.4}$$

Substituting (2.4) into (2.3), and comparing the powers of coefficients of λ, we arrive at

$$\begin{gathered} a_{m+1,x} = 2pb_{m+1} - 2qc_{m+1}, \\ b_{m+1} = \frac{1}{2}c_{m,x} + qa_m - hb_m, \\ c_{m+1} = \frac{1}{2}b_{m,x} + pa_m - hc_m, \\ d_{m+1,x} = 2pf_{m+1} - 2qg_{m+1} + 2rb_{m+1} - 2sc_{m+1}, \\ f_{m+1} = \frac{1}{2}g_{m,x} - \frac{1}{2}c_{m,x} + (s-p)a_m + hc_m + qd_m - hf_m, \\ g_{m+1} = \frac{1}{2}f_{m,x} - \frac{1}{2}b_{m,x} + (r-q)a_m + hb_m + pd_m - hg_m, \end{gathered} \tag{2.5}$$

and

$$a_{0x} = 0, \quad b_0 = 0, \quad c_0 = 0, \quad d_{0x} = 0, \quad f_0 = 0, \quad g_0 = 0.$$

We take the initial values $a_0 = \alpha$ and $d_0 = \beta$, where α, β are arbitrary constants. The values of first few terms are calculated as follows

$$\begin{gathered} a_1 = 0, \quad b_1 = \alpha q, \quad c_1 = \alpha p, \quad d_1 = 0, \quad f_1 = \alpha(s-q) + \beta q, \\ g_1 = \alpha(r-p) + \beta p, \quad a_2 = \frac{1}{2}\alpha(p^2 - q^2), \quad b_2 = \frac{1}{2}\alpha p_x - \alpha hq, \\ c_2 = \frac{1}{2}\alpha q_x - \alpha hp, \quad d_2 = \alpha pr - \alpha qs + \alpha(q^2 - p^2) + \frac{1}{2}\beta(p^2 - q^2), \\ f_2 = \frac{1}{2}\alpha r_x - \alpha p_x + \frac{1}{2}\beta p_x + 2\alpha hq - \alpha hs - \beta hq, \\ g_2 = \frac{1}{2}\alpha s_x - \alpha q_x + \frac{1}{2}\beta q_x + 2\alpha hp - \alpha hr - \beta hp, \\ b_3 = \frac{1}{4}\alpha q_x x - \alpha hp_x + \alpha h^2 q + \frac{1}{2}\alpha q(p^2 - q^2 - \frac{1}{2}\alpha ph_x), \\ c_3 = \frac{1}{4}\alpha p_x x - \alpha hq_x + \alpha h^2 p + \frac{1}{2}\alpha p(p^2 - q^2 - \frac{1}{2}\alpha qh_x), \\ a_3 = \frac{1}{2}\alpha(pq_x - qp_x) + \alpha h(q^2 - p^2), \end{gathered}$$

$$
\begin{aligned}
f_3 &= \frac{1}{4}\alpha s_x x - \frac{3}{4}\alpha q_x x + \frac{1}{4}\beta q_x x + \frac{3}{2}\alpha h_x p + 3\alpha h p_x - \frac{1}{2}\alpha h_x r - \frac{1}{2}\alpha h r_x - \frac{1}{2}\beta h_x p \\
&\quad - \beta h p_x - \frac{1}{2}\alpha h r_x - 3\alpha h^2 q + \alpha h^2 s + \beta h^2 q + \alpha pqr - \alpha q^2 s + \frac{3}{2}\alpha q(q^2 - p^2) \\
&\quad + \frac{1}{2}\beta q(p^2 - q^2) + \frac{1}{2}\alpha s(p^2 - q^2), \\
g_3 &= \frac{1}{4}\alpha r_x x - \frac{3}{4}\alpha p_x x + \frac{1}{4}\beta p_x x + \frac{3}{2}\alpha h_x q + 3\alpha h q_x - \frac{1}{2}\alpha h_x s - \frac{1}{2}\alpha h s_x - \frac{1}{2}\beta h_x q \\
&\quad - \beta h q_x - \frac{1}{2}\alpha h s_x - 3\alpha h^2 p + \alpha h^2 r + \beta h^2 p + \alpha p^2 r - \alpha pqr + \frac{3}{2}\alpha p(q^2 - p^2) \\
&\quad + \frac{1}{2}\beta p(p^2 - q^2) + \frac{1}{2}\alpha r(p^2 - q^2), \\
d_3 &= \frac{1}{2}\alpha(ps_x - sp_x) + \frac{3}{2}\alpha(qp_x - qp_x) + \frac{1}{2}\beta(pq_x - qp_x) + \frac{1}{2}\alpha(rq_x - qr_x) \\
&\quad + 3\alpha h(p^2 - q^2) + \beta h(q^2 - p^2) + 2h^2, \dots
\end{aligned}
$$

Now, taking

$$
V_1 = \sum_{i=0}^{m} \left[\begin{array}{cc|cc} a_i & b_i + c_i & d_i & f_i + g_i \\ b_i - c_i & -a_i & f_i - g_i & -d_i \\ \hline 0 & 0 & a_i & b_i + c_i \\ 0 & 0 & b_i - c_i & -a_i \end{array}\right] \lambda^{m-i} + \left[\begin{array}{cc|cc} \delta_m & 0 & 0 & 0 \\ 0 & -\delta_m & 0 & 0 \\ \hline 0 & 0 & \delta_m & 0 \\ 0 & 0 & 0 & -\delta_m \end{array}\right].
$$

Then the corresponding zero curvature equation

$$
U_{1,t} - V_{1,x} + [U_1, V_1] = 0 \tag{2.6}
$$

gives

$$
\begin{gathered}
p_t = 2b_{m+1} + 2q\delta_m, \\
q_t = 2c_{m+1} + 2p\delta_m, \\
r_t = 2b_{m+1} + 2f_{m+1} + 2s\delta_m, \\
s_t = 2c_{m+1} + 2g_{m+1} + 2r\delta_m, \\
h_t = \delta_{mx}.
\end{gathered} \tag{2.7}
$$

From this equation, we can obtain

$$
\begin{aligned}
\delta_{mx} &= h_t \\
&= \epsilon(q_t s + q s_t - p_t r - p r_t) \\
&= \epsilon[(2c_{m+1} + 2p\delta_m)s + q(2c_{m+1} + 2g_{m+1} + 2r\delta_m) \\
&\quad - (2b_{m+1} + 2q\delta_m)r - p(2b_{m+1} + 2f_{m+1} + 2s\delta_m)] \\
&= \epsilon(2c_{m+1}s + 2g_{m+1}q - 2b_{m+1}r - 2f_{m+1}p + 2c_{m+1}q - 2b_{m+1}p) \\
&= -\epsilon(a_{m+1,x} + d_{m+1,x}).
\end{aligned}
$$

Thus we introduce

$$
\delta_m = -\epsilon(a_{m+1} + d_{m+1}), \tag{2.8}
$$

and then we have generated a fourth-order nonlinear integrable perturbed hierarchy of NLS-mKdV equation

$$\begin{aligned} p_t &= 2b_{m+1} - 2\epsilon q(a_{m+1} + e_{m+1}), \\ q_t &= 2c_{m+1} - 2\epsilon p(a_{m+1} + e_{m+1}), \\ r_t &= 2b_{m+1} + 2f_{m+1} - 2\epsilon s(a_{m+1} + e_{m+1}), \\ s_t &= 2c_{m+1} + 2g_{m+1} - 2\epsilon r(a_{m+1} + e_{m+1}). \end{aligned} \tag{2.9}$$

When $m = 1$, by setting $\epsilon = 1$, (2.9) becomes

$$\begin{aligned} p_t &= \alpha p_x - (2\alpha - 2\beta)q(p^2 - q^2), \\ q_t &= \alpha q_x - (2\alpha - 2\beta)p(p^2 - q^2), \\ r_t &= \alpha r_x - \alpha p_x + \beta p_x + (2\alpha - 2\beta)(p^2 s - qqr), \\ s_t &= \alpha s_x - \alpha q_x + \beta q_x + (2\alpha - 2\beta)(pqs - q^2 r). \end{aligned} \tag{2.10}$$

When $m = 2$, (2.9) becomes

$$\begin{aligned} p_t = \;& \frac{1}{2}\alpha q_{xx} - 2\alpha h p_x + 2\alpha h^2 q + \alpha h^2 q + \alpha q(p^2 - q^2) - \alpha p h_x \\ & - 4q(\alpha(qp_x - pq_x) + \frac{1}{2}\alpha(ps_x - sp_x) + \frac{1}{2}\beta(pq_x - qp_x) \\ & + \frac{1}{2}\alpha(rq_x - qr_x) + 2\alpha h(p^2 - q^2 + \beta h(q^2 - p^2) + 2h^2)), \\ q_t = \;& \frac{1}{2}\alpha p_{xx} - 2\alpha h q_x + 2\alpha h^2 p + \alpha h^2 p + \alpha p(p^2 - q^2) - \alpha q h_x \\ & - 4p(\alpha(qp_x - pq_x) + \frac{1}{2}\alpha(ps_x - sp_x) + \frac{1}{2}\beta(pq_x - qp_x) \\ & + \frac{1}{2}\alpha(rq_x - qr_x) + 2\alpha h(p^2 - q^2 + \beta h(q^2 - p^2) + 2h^2)), \\ r_t = \;& \frac{1}{2}\alpha s_{xx} - 2\alpha q_{xx} + \frac{1}{2}\beta q_{xx} + 2\alpha p h_x + 4\alpha h p_x - \alpha r h_x \\ & - \alpha h r - \beta p h_x - 2\beta h p_x - \alpha h r_x - 4\alpha h^2 q + 2\alpha h^2 s + 2\beta h^2 q \\ & + 2\alpha pqr - 2\alpha q^2 s + 2\alpha q(p^2 - q^2) + \beta q(p^2 - q^2) + \alpha s(p^2 - q^2) \\ & - 4s(\alpha(qp_x - pq_x) + \frac{1}{2}\alpha(ps_x - sp_x) + \frac{1}{2}\beta(pq_x - qp_x) \\ & + \frac{1}{2}\alpha(rq_x - qr_x) + 2\alpha h(p^2 - q^2 + \beta h(q^2 - p^2) + 2h^2)), \\ s_t = \;& \frac{1}{2}\alpha r_{xx} - 2\alpha p_{xx} + \frac{1}{2}\beta p_{xx} + 2\alpha q h_x + 4\alpha h q_x - \alpha s h_x - \alpha h s \\ & - q h_x - 2\beta h q_x - \alpha h s_x - 4\alpha h^2 p + 2\alpha h^2 r + 2\beta h^2 p + 2\alpha p^2 r - 2\alpha pqs \\ & + 2\alpha p(p^2 - q^2) + \beta p(p^2 - q^2) + \alpha r(p^2 - q^2) - 4r(\alpha(qp_x - pq_x) \\ & + \frac{1}{2}\alpha(ps_x - sp_x) + \frac{1}{2}\beta(pq_x - qp_x) + frac12\alpha(rq_x - qr_x) \\ & + 2\alpha h(p^2 - q^2 + \beta h(q^2 - p^2) + 2h^2)), \end{aligned} \tag{2.11}$$

by setting $\epsilon = 1$. Therefore, we obtain (2.9) which is a Lax integrable. In the next two subsections, we try to obtain explicit solutions of (2.10) and (2.11) by the Darboux transformation method.

2.2. **N-fold Darboux transformation of** (2.10) **and its explicit solutions.** We will construct the Darboux transformation of (2.10). Firstly, we introduce the gauge transformation

$$\widetilde{\Phi} = T_1\Phi. \tag{2.12}$$

The original Lax pair under Darboux transformation are transformed into the new Lax pair

$$\widetilde{\Phi}_x = \widetilde{U_1}(\widetilde{p},\widetilde{q},\widetilde{r},\widetilde{s},\lambda)\widetilde{\phi}, \quad \widetilde{\Phi}_t = \widetilde{V_1}(\widetilde{p},\widetilde{q},\widetilde{r},\widetilde{s},\lambda)\widetilde{\phi}, \quad \widetilde{\phi} = \begin{bmatrix}\widetilde{\phi_1}\\ \widetilde{\phi_2}\\ \widetilde{\phi_3}\\ \widetilde{\phi_4}\end{bmatrix}, \tag{2.13}$$

where

$$\widetilde{U}_1 = (T_{1,x} + T_1U_1)T_1^{-1}, \quad \widetilde{V}_1 = (T_{1,t} + T_1V_1)T_1^{-1}. \tag{2.14}$$

For this, we consider the Darboux matrix

$$T_1 = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14}\\ A_{21} & A_{22} & A_{23} & A_{24}\\ 0 & 0 & A_{11} & A_{12}\\ 0 & 0 & A_{21} & A_{22}\end{bmatrix}, \tag{2.15}$$

where

$$\begin{gathered}
A_{11} = \lambda^N + \sum_{i=0}^{N-1} A_{11}^{(i)}\lambda^i, \quad A_{12} = \sum_{i=0}^{N-1} A_{12}^{(i)}\lambda^i, \quad A_{13} = \sum_{i=0}^{N-1} A_{13}^{(i)}\lambda^i,\\
A_{14} = \sum_{i=0}^{N-1} A_{14}^{(i)}\lambda^i, \quad A_{21} = \sum_{i=0}^{N-1} A_{21}^{(i)}\lambda^i, \quad A_{22} = \lambda^N + \sum_{i=0}^{N-1} A_{22}^{(i)}\lambda^i,\\
A_{23} = \sum_{i=0}^{N-1} A_{23}^{(i)}\lambda^i, \quad A_{24} = \sum_{i=0}^{N-1} A_{24}^{(i)}\lambda^i\,.
\end{gathered}$$

We define

$$\Phi = \begin{bmatrix}\varphi_1 & \psi_1\\ \varphi_2 & \psi_2\\ \varphi_3 & \psi_3\\ \varphi_4 & \psi_4\end{bmatrix}. \tag{2.16}$$

From (2.15) and (2.16), we have

$$\begin{aligned}
\widetilde{\Phi} = T_1\Phi &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14}\\ A_{21} & A_{22} & A_{23} & A_{24}\\ 0 & 0 & A_{11} & A_{12}\\ 0 & 0 & A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}\varphi_1 & \psi_1\\ \varphi_2 & \psi_2\\ \varphi_3 & \psi_3\\ \varphi_4 & \psi_4\end{bmatrix}\\
&= \begin{bmatrix} A_{11}\varphi_1 + A_{12}\varphi_2 + A_{13}\varphi_3 + A_{14}\varphi_4 & A_{11}\psi_1 + A_{12}\psi_2 + A_{13}\psi_3 + A_{14}\psi_4\\ A_{21}\varphi_1 + A_{22}\varphi_2 + A_{23}\varphi_3 + A_{24}\varphi_4 & A_{21}\psi_1 + A_{22}\psi_2 + A_{23}\psi_3 + A_{24}\psi_4\\ A_{11}\varphi_3 + A_{12}\varphi_4 & A_{11}\psi_3 + A_{12}\psi_4\\ A_{21}\varphi_3 + A_{22}\varphi_4 & A_{21}\psi_3 + A_{22}\psi_4\end{bmatrix}.
\end{aligned} \tag{2.17}$$

So there exists γ_j $(j = 1, 2)$ satisfying

$$\gamma_j^{(1)}\widetilde{\varphi} + \gamma_j^{(2)}\widetilde{\psi} = 0. \tag{2.18}$$

Substituting (2.18) into (2.17), we obtain

$$
\begin{aligned}
&\gamma_j^{(1)}(A_{11}\varphi_1+A_{12}\varphi_2+A_{13}\varphi_3+A_{14}\varphi_4)\\
&\quad+\gamma_j^{(2)}(A_{11}\psi_1+A_{12}\psi_2+A_{13}\psi_3+A_{14}\psi_4)=0,\\
&\gamma_j^{(1)}(A_{21}\varphi_1+A_{22}\varphi_2+A_{23}\varphi_3+A_{24}\varphi_4)\\
&\quad+\gamma_j^{(2)}(A_{21}\psi_1+A_{22}\psi_2+A_{23}\psi_3+A_{24}\psi_4)=0,\\
&\gamma_j^{(1)}(A_{11}\varphi_3+A_{12}\varphi_4)+\gamma_j^{(2)}(A_{11}\psi_3+A_{12}\psi_4)=0,\\
&\gamma_j^{(1)}(A_{21}\varphi_3+A_{22}\varphi_4)+\gamma_j^{(2)}(A_{23}\psi_3+A_{24}\psi_4)=0.
\end{aligned}
$$

From the above equalities, we obtain

$$
\begin{gathered}
A_{11}+A_{12}\omega_j^{(1)}+A_{13}\omega_j^{(2)}+A_{14}\omega_j^{(3)}=0,\\
A_{21}+A_{22}\omega_j^{(1)}+A_{23}\omega_j^{(2)}+A_{24}\omega_j^{(3)}=0,\\
A_{11}\omega_j^{(2)}+A_{12}\omega_j^{(3)}=0,\\
A_{21}\omega_j^{(2)}+A_{22}\omega_j^{(3)}=0,
\end{gathered}
\tag{2.19}
$$

where

$$
\omega_j^{(1)}=\frac{\gamma_j^{(1)}\varphi_2+\gamma_j^{(2)}\psi_2}{\gamma_j^{(1)}\varphi_1+\gamma_j^{(2)}\psi_1},\quad \omega_j^{(2)}=\frac{\gamma_j^{(1)}\varphi_3+\gamma_j^{(2)}\psi_3}{\gamma_j^{(1)}\varphi_1+\gamma_j^{(2)}\psi_1},\quad \omega_j^{(3)}=\frac{\gamma_j^{(1)}\varphi_4+\gamma_j^{(2)}\psi_4}{\gamma_j^{(1)}\varphi_1+\gamma_j^{(2)}\psi_1}.
$$

Furthermore, the following equations are obtained by (2.19)

$$
\begin{gathered}
\sum_{i=0}^{N-1}A_{11}^{(i)}\lambda_j^i+\omega_j^{(1)}\sum_{i=0}^{N-1}A_{12}^{(i)}\lambda_j^i+\omega_j^{(2)}\sum_{i=0}^{N-1}A_{13}^{(i)}\lambda_j^i+\omega_j^{(3)}\sum_{i=0}^{N-1}A_{14}^{(i)}\lambda_j^i=-\lambda_j^N,\\
\sum_{i=0}^{N-1}A_{21}^{(i)}\lambda_j^i+\omega_j^{(1)}\sum_{i=0}^{N-1}A_{22}^{(i)}\lambda_j^i+\omega_j^{(2)}\sum_{i=0}^{N-1}A_{23}^{(i)}\lambda_j^i+\omega_j^{(3)}\sum_{i=0}^{N-1}A_{24}^{(i)}\lambda_j^i=-\omega_j^{(1)}\lambda_j^N,\\
\omega_j^{(2)}\sum_{i=0}^{N-1}A_{11}^{(i)}\lambda_j^i+\omega_j^{(3)}\sum_{i=0}^{N-1}A_{12}^{(i)}\lambda_j^i=-\omega_j^{(2)}\lambda_j^N,\\
\omega_j^{(2)}\sum_{i=0}^{N-1}A_{21}^{(i)}\lambda_j^i+\omega_j^{(3)}\sum_{i=0}^{N-1}A_{22}^{(i)}\lambda_j^i=-\omega_j^{(3)}\lambda_j^N.
\end{gathered}
$$

Proposition 2.1. *Matrix $\widetilde{U}_1$ is of the same type as U_1 defined by (1.4), i.e, $\widetilde{U}_1$ can be written as*

$$
\widetilde{U}_1=\begin{bmatrix}
\lambda+\widetilde{q}\widetilde{s}-\widetilde{p}\widetilde{r} & \widetilde{p}+\widetilde{q} & \lambda & \widetilde{r}+\widetilde{s}\\
-\widetilde{p}+\widetilde{q} & -\lambda-\widetilde{q}\widetilde{s}-\widetilde{p}\widetilde{r} & -\widetilde{r}+\widetilde{s} & -\lambda\\
0 & 0 & \lambda+\widetilde{q}\widetilde{s}+\widetilde{p}\widetilde{r} & \widetilde{p}+\widetilde{q}\\
0 & 0 & \widetilde{p}+\widetilde{q} & -\lambda-\widetilde{q}\widetilde{s}+\widetilde{p}\widetilde{r}
\end{bmatrix},
\tag{2.20}
$$

in which the transformation formulae between old and new potentials are defined by

$$
\begin{aligned}
\widetilde{p} &= p - A_{12}^{(N-1)} - A_{21}^{(N-1)},\\
\widetilde{q} &= q - A_{21}^{(N-1)} + A_{21}^{(N-1)},\\
\widetilde{r} &= r - A_{12}^{(N-1)} - A_{14}^{(N-1)} - A_{21}^{(N-1)} - A_{23}^{(N-1)},\\
\widetilde{s} &= s - A_{12}^{(N-1)} - A_{14}^{(N-1)} + A_{21}^{(N-1)} + A_{23}^{(N-1)}.
\end{aligned} \tag{2.21}
$$

Proof. Setting

$$
(T_{1,x} + T_1 U_1) T_1^* = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ 0 & 0 & \Gamma_{11} & \Gamma_{12} \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \end{bmatrix},
$$

where

$$
T_1^{-1} = \frac{T_1^*}{\det(T_1)}, \tag{2.22}
$$

we deduce

$$
(T_{1,x} + T_1 U_1) T_1^* = (det T_1) O(\lambda), \tag{2.23}
$$

where

$$
O(\lambda) = \begin{bmatrix} O_{11}^{(1)}\lambda + O_{11}^{(0)} & O_{12}^{(0)} & O_{13}^{(1)}\lambda & O_{14}^{(0)} \\ O_{21}^{(0)} & O_{22}^{(1)}\lambda + O_{22}^{(0)} & O_{23}^{(0)} & O_{24}^{(1)}\lambda \\ 0 & 0 & O_{11}^{(1)}\lambda + O_{11}^{(0)} & O_{12}^{(0)} \\ 0 & 0 & O_{21}^{(0)} & O_{22}^{(1)}\lambda + O_{22}^{(0)} \end{bmatrix}. \tag{2.24}
$$

Substituting (2.23) into (2.22), we obtain

$$
T_{1,x} + T_1 U_1 = O(\lambda) T_1. \tag{2.25}
$$

From this equation, by equating the coefficients of λ^N and λ^{N+1}, we find

$$
\begin{gathered}
O_{11}^{(1)} = 1, \quad O_{12}^{(0)} = p + q - 2A_{12}^{(N-1)} = \widetilde{p} + \widetilde{q},\\
O_{14}^{(0)} = r + s - 2A_{12}^{(N-1)} - 2A_{14}^{(N-1)} = \widetilde{r} + \widetilde{s},\\
O_{21}^{(0)} = -p + q + 2A_{21}^{(N-1)} = -\widetilde{p} + \widetilde{q},\\
O_{23}^{(0)} = -r + s + 2A_{21}^{(N-1)} + 2A_{23}^{(N-1)} = -\widetilde{r} + \widetilde{s},\\
O_{24}^{(1)} = -1, \quad O_{13}^{(1)} = 1, \quad O_{22}^{(1)} = -1,\\
O_{11}^{(0)} = qs - pr = \widetilde{q}\widetilde{s} - \widetilde{p}\widetilde{r}, \quad O_{22}^{(0)} = -qs + pr = -\widetilde{q}\widetilde{s} + \widetilde{p}\widetilde{r}.
\end{gathered}
$$

We see that $\widetilde{U}_1 = O(\lambda)$. The proof is complete. □

Proposition 2.2. *The matrix $\widetilde{V}_1^{(1)}$ is of the same type as $V_1^{(1)}$ defined by (2.2); therefore, $\widetilde{V}_1^{(1)}$ can be rewritten as*

$$
\widetilde{V}_1^{(1)} = \begin{bmatrix} \alpha\lambda + \delta_1 & \alpha\widetilde{p} + \alpha\widetilde{q} & \beta\lambda & \alpha(\widetilde{s} - \widetilde{r} + \widetilde{p} + \widetilde{q}) + \beta(\widetilde{p} + \widetilde{q}) \\ \alpha\widetilde{q} - \alpha\widetilde{p} & -\alpha\lambda - \delta_1 & \alpha(\widetilde{s} - \widetilde{q} - \widetilde{r} + \widetilde{p}) + \beta(\widetilde{q} - \widetilde{p}) & -\beta\lambda \\ 0 & 0 & \alpha\lambda + \delta_1 & \alpha\widetilde{p} + \alpha\widetilde{q} \\ 0 & 0 & \alpha\widetilde{q} - \alpha\widetilde{p} & -\alpha\lambda - \delta_1 \end{bmatrix}.
$$

Proof. Setting

$$(T_{1,t}+T_1V_1^{(1)})T_1^*=\begin{bmatrix}\Xi_{11}&\Xi_{12}&\Xi_{13}&\Xi_{14}\\ \Xi_{21}&\Xi_{22}&\Xi_{23}&\Xi_{24}\\ 0&0&\Xi_{11}&\Xi_{12}\\ 0&0&\Xi_{21}&\Xi_{22}\end{bmatrix},$$

where

$$T_1^{-1}=\frac{T_1^*}{\det(T_1)}. \tag{2.26}$$

Now we define

$$P=\begin{pmatrix}P_{11}^{(1)}\lambda+P_{11}^{(0)}&P_{12}^{(0)}&P_{13}^{(1)}\lambda&P_{14}^{(0)}\\ P_{21}^{(0)}&P_{22}^{(1)}\lambda+P_{22}^{(0)}&P_{23}^{(0)}&P_{24}^{(1)}\lambda\\ 0&0&P_{11}^{(1)}\lambda+P_{11}(0)&P_{12}^{(0)}\\ 0&0&P_{21}^{(0)}&P_{22}^{(1)}\lambda+P_{22}^{(0)}\end{pmatrix}. \tag{2.27}$$

Noting that $(T_{1,t}+T_1V_1)T_1^*=(\det T_1)P(\lambda)$, we have

$$T_{1,t}+T_1V_1^{(1)}=P(\lambda)T_1. \tag{2.28}$$

By comparing the coefficients of λ in (2.28), we obtain

$$\begin{gathered}P_{11}^{(1)}=\alpha,\quad P_{13}^{(1)}=\beta,\quad P_{22}^{(1)}=-\alpha,\quad P_{24}^{(1)}=-\beta,\\ P_{11}^{(0)}=\delta=\widetilde{\delta},\quad P_{22}^{(0)}=-\delta=-\widetilde{\delta},\\ \begin{aligned}P_{14}^{(0)}&=\alpha(s+r-p-q)+\beta(p+q)-2\beta A_{12}^{(N-1)}-2\alpha A_{14}^{(N-1)}\\ &=\alpha(\widetilde{s}+\widetilde{r}-\widetilde{p}-\widetilde{q})+\beta(\widetilde{p}+\widetilde{q}),\\ P_{23}^{(0)}&=\alpha(s-q-r+p)+\beta(q-p)+2\alpha A_{23}^{(N-1)}+2\beta A_{21}^{(N-1)}\\ &=\alpha(\widetilde{s}-\widetilde{q}-\widetilde{r}+\widetilde{p})+\beta(\widetilde{q}+\widetilde{q}),\end{aligned}\\ P_{21}^{(0)}=\alpha(q-p)+2\alpha A_{21}^{(N-1)}=\alpha(\widetilde{q}-\widetilde{p}),\\ P_{12}^{(0)}=\alpha(q+p)+2\alpha A_{12}^{(N-1)}=\alpha(\widetilde{q}+\widetilde{p}).\end{gathered}$$

The proof is complete. □

To obtain the explicit solutions of (2.10), we choose the seed solution $p=q=0$, $r=s=1$. The spectral problems become

$$\Phi_x=U_1\Phi,\quad U_1=\begin{bmatrix}\lambda&0&\lambda&2\\ 0&-\lambda&0&-\lambda\\ 0&0&\lambda&0\\ 0&0&0&-\lambda\end{bmatrix}, \tag{2.29}$$

and

$$\Phi_t=V_1^{(1)}\Phi,\quad V_1^{(1)}=\begin{bmatrix}\alpha\lambda&0&\beta\lambda&2\alpha\\ 0&-\alpha\lambda&0&-\beta\lambda\\ 0&0&\alpha\lambda&0\\ 0&0&0&-\alpha\lambda\end{bmatrix}. \tag{2.30}$$

Solving these spectral problems, we obtain

$$
\begin{gathered}
\varphi=\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{bmatrix}=\begin{bmatrix}-\frac{1}{\lambda}e^{-\lambda x-\lambda\alpha t}\\ -((\beta t+x)\lambda-1)e^{-\alpha\lambda t-\lambda x}\\ 0\\ e^{-\lambda x-\alpha\lambda t}\end{bmatrix},\\
\psi=\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix}=\begin{bmatrix}((\beta t+x)\lambda+1)e^{\lambda x+\alpha\lambda t}\\ e^{-\lambda x-\lambda\alpha t}\\ e^{\alpha\lambda t+\lambda x}\\ 0\end{bmatrix}.
\end{gathered}
\tag{2.31}
$$

In particular, when $N=1$,

$$
T_1=\begin{bmatrix}\lambda+A_{11}^{(0)} & A_{12}^{(0)} & A_{13}^{(0)} & A_{14}^{(0)}\\ A_{21}^{(0)} & \lambda+A_{22}^{(0)} & A_{23}^{(0)} & A_{24}^{(0)}\\ 0 & 0 & \lambda+A_{11}^{(0)} & A_{12}^{(0)}\\ 0 & 0 & A_{21}^{(0)} & \lambda+A_{22}^{(0)}\end{bmatrix}.
\tag{2.32}
$$

If we set

$$
\omega_j^{(1)}=\frac{\varphi_2+\gamma_j\psi_2}{\varphi_1+\gamma_j\psi_1},\quad \omega_j^{(2)}=\frac{\varphi_3+\gamma_j\psi_3}{\varphi_1+\gamma_j\psi_1},\quad \omega_j^{(3)}=\frac{\varphi_4+\gamma_j\psi_4}{\varphi_1+\gamma_j\psi_1},
\tag{2.33}
$$

from $\widetilde{\Phi}=T_1\Phi$, we obtain

$$
\begin{gathered}
A_{11}^{(0)}+A_{12}^{(0)}\omega_j^{(1)}+A_{13}^{(0)}\omega_j^{(2)}+A_{14}^{(0)}\omega_j^{(3)}=-\lambda_j,\\
A_{21}^{(0)}+A_{22}^{(0)}\omega_j^{(1)}+A_{23}^{(0)}\omega_j^{(2)}+A_{24}^{(0)}\omega_j^{(3)}=-\lambda_j\omega_j^{(1)},\\
A_{11}^{(0)}\omega_j^{(2)}+A_{12}^{(0)}\omega_j^{(3)}=-\lambda_j\omega_j^{(2)},\\
A_{21}^{(0)}\omega_j^{(2)}+A_{22}^{(0)}\omega_j^{(3)}=-\lambda_j\omega_j^{(3)}.
\end{gathered}
\tag{2.34}
$$

The solution to this equations is

$$
A_{12}^{(0)}=\frac{\Delta A_{12}^{(0)}}{\Delta'},\quad A_{14}^{(0)}=\frac{\Delta A_{14}^{(0)}}{\Delta},\quad A_{21}^{(0)}=\frac{\Delta A_{21}^{(0)}}{\Delta},\quad A_{23}^{(0)}=\frac{\Delta A_{23}^{(0)}}{\Delta},
\tag{2.35}
$$

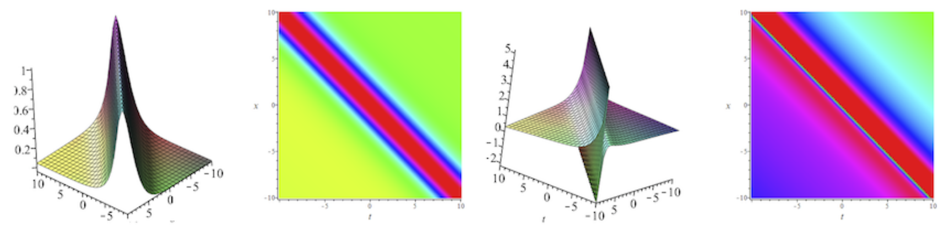
here,

$$
\begin{gathered}
\Delta=\begin{bmatrix}1 & \omega_1^{(1)} & \omega_1^{(2)} & \omega_1^{(3)}\\ 1 & \omega_2^{(1)} & \omega_2^{(2)} & \omega_2^{(3)}\\ 1 & \omega_3^{(1)} & \omega_3^{(2)} & \omega_3^{(3)}\\ 1 & \omega_4^{(1)} & \omega_4^{(2)} & \omega_4^{(3)}\end{bmatrix},\quad \Delta A_{21}^{(0)}=\begin{bmatrix}-\lambda_1\omega_1^{(1)} & \omega_1^{(1)} & \omega_1^{(2)} & \omega_1^{(3)}\\ -\lambda_2\omega_2^{(1)} & \omega_2^{(1)} & \omega_2^{(2)} & \omega_2^{(3)}\\ -\lambda_3\omega_3^{(1)} & \omega_3^{(1)} & \omega_3^{(2)} & \omega_3^{(3)}\\ -\lambda_4\omega_4^{(1)} & \omega_4^{(1)} & \omega_4^{(2)} & \omega_4^{(3)}\end{bmatrix},\\
\Delta A_{14}^{(0)}=\begin{bmatrix}1 & \omega_1^{(1)} & \omega_1^{(2)} & -\lambda_1\\ 1 & \omega_2^{(1)} & \omega_2^{(2)} & -\lambda_2\\ 1 & \omega_3^{(1)} & \omega_3^{(2)} & -\lambda_3\\ 1 & \omega_4^{(1)} & \omega_4^{(2)} & -\lambda_4\end{bmatrix},\quad \Delta A_{23}^{(0)}=\begin{bmatrix}1 & \omega_1^{(1)} & -\lambda_1\omega_1^{(1)} & \omega_1^{(3)}\\ 1 & \omega_2^{(1)} & -\lambda_2\omega_2^{(1)} & \omega_2^{(3)}\\ 1 & \omega_3^{(1)} & -\lambda_3\omega_3^{(1)} & \omega_3^{(3)}\\ 1 & \omega_4^{(1)} & -\lambda_4\omega_4^{(1)} & \omega_4^{(3)}\end{bmatrix},\\
\Delta'=\begin{bmatrix}\omega_1^{(2)} & \omega_1^{(3)}\\ \omega_2^{(2)} & \omega_2^{(3)}\end{bmatrix},\quad \Delta A_{12}^{(0)}=\begin{bmatrix}\omega_1^{(2)} & -\lambda_1\omega_1^{(2)}\\ \omega_2^{(2)} & -\lambda_2\omega_2^{(2)}\end{bmatrix}.
\end{gathered}
\tag{2.36}
$$

Accordingly we gain the explicit solutions of (2.10) as

$$\begin{aligned}
\widetilde{p} &= -A_{12}^{(0)} - A_{21}^{(0)},\\
\widetilde{q} &= -A_{21}^{(0)} + A_{21}^{(0)},\\
\widetilde{r} &= 1 - A_{12}^{(0)} - A_{14}^{(0)} - A_{21}^{(0)} - A_{23}^{(0)},\\
\widetilde{s} &= 1 - A_{12}^{(0)} - A_{14}^{(0)} + A_{21}^{(0)} + A_{23}^{(0)}.
\end{aligned} \tag{2.37}$$

Remark 2.3. By choosing suitable parameters, the left column displays the space-time distributions and the right column displays the density profiles different time for components p, q, r, s. Figure 1 shows that the bell soliton (a, b) and kink soliton (c, d) formed, respectively, by the one-soliton solutions and propagate along the negative direction of x-axis in the process of evolution.



(a) $p = q$ (b) $p = q$ (c) $r = s$ (d) $r = s$

FIGURE 1. Soliton solutions of (2.10) and density plots with $\alpha = 1$, $\beta = 1$, $\lambda_1 = -0.3$, $\lambda_2 = 0.3$, $\lambda_3 = -0.01$, $\lambda_4 = 0.01$, $\gamma_1 = -0.3$, $\gamma_2 = 0.3$, $\gamma_3 = -0.1$, $\gamma_4 = 0.1$.

2.3. **Darboux transformation of** (2.11) **and its explicit solutions.** We use the same Darboux matrix (2.15) to solve the second nonlinear equation (2.11).

Proposition 2.4. *The matrix* $\widetilde{V}_1^{(2)}$ *has the same form as* V_1 *defined by* (2.2)

$$\widetilde{V}_1^{(2)} = \begin{bmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ 0 & 0 & V_{11} & V_{12} \\ 0 & 0 & V_{21} & V_{22} \end{bmatrix}, \tag{2.38}$$

where

$$V_{11} = \alpha\lambda^2 + \delta_1\lambda + \frac{1}{2}\alpha(\widetilde{p}^2 - \widetilde{q}^2), \quad V_{21} = \alpha\widetilde{q}\lambda - \alpha\widetilde{p}\lambda + \frac{1}{2}\alpha\widetilde{p}_x - \alpha h\widetilde{q} - \frac{1}{2}\alpha\widetilde{p}_x + \alpha h\widetilde{p},$$

$$V_{12} = \alpha\widetilde{q}\lambda + \alpha\widetilde{p}\lambda + \frac{1}{2}\alpha\widetilde{p}_x - \alpha h\widetilde{q} + \frac{1}{2}\alpha\widetilde{p}_x - \alpha h\widetilde{p},$$

$$V_{22} = -\alpha\lambda^2 - \delta_1\lambda - \frac{1}{2}\alpha(\widetilde{p}^2 - \widetilde{q}^2),$$

$$V_{13} = \beta\lambda^2 + \alpha\widetilde{p}\widetilde{r} - \alpha\widetilde{q}\widetilde{s} + \alpha(\widetilde{q}^2 - \widetilde{p}^2) + \frac{1}{2}\beta(\widetilde{q}^2 - \widetilde{p}^2),$$

$$V_{24} = -\beta\lambda^2 - \alpha\widetilde{p}\widetilde{r} + \alpha\widetilde{q}\widetilde{s} - \alpha(\widetilde{q}^2 - \widetilde{p}^2) - \frac{1}{2}\beta(\widetilde{q}^2 - \widetilde{p}^2),$$

$$
\begin{aligned}
V_{14} &= \alpha(\widetilde{s}+\widetilde{r}-\widetilde{q}-\widetilde{p})\lambda+\beta(\widetilde{p}+\widetilde{q})\lambda+\frac{1}{2}\alpha\widetilde{r}_x-\alpha\widetilde{p}_x+\frac{1}{2}\beta\widetilde{p}_x+2\alpha h\widetilde{q}-\alpha h\widetilde{s}-\beta h\widetilde{q}\\
&\quad+\frac{1}{2}\alpha\widetilde{s}_x-\alpha\widetilde{q}_x+\frac{1}{2}\beta\widetilde{q}_x+2\alpha h\widetilde{p}-\alpha h\widetilde{r}-\beta h\widetilde{p},\\
V_{23} &= \alpha(\widetilde{s}-\widetilde{q}-\widetilde{r}+\widetilde{p})\lambda+\beta(\widetilde{q}-\widetilde{p})\lambda+\frac{1}{2}\alpha\widetilde{r}_x-\alpha\widetilde{p}_x+\frac{1}{2}\beta\widetilde{p}p_x+2\alpha h\widetilde{q}-\alpha h\widetilde{s}-\beta h\widetilde{q}\\
&\quad-\frac{1}{2}\alpha\widetilde{s}_x+\alpha\widetilde{q}_x-\frac{1}{2}\beta\widetilde{q}_x-2\alpha h\widetilde{p}+\alpha h\widetilde{r}+\beta h\widetilde{p}.
\end{aligned}
$$

Proof. Setting

$$
(T_{1,t}+T_1V_1^{(2)})T_1^*=\begin{bmatrix}\Theta_{11}&\Theta_{12}&\Theta_{13}&\Theta_{14}\\ \Theta_{21}&\Theta_{22}&\Theta_{23}&\Theta_{24}\\ 0&0&\Theta_{11}&\Theta_{12}\\ 0&0&\Theta_{21}&\Theta_{22}\end{bmatrix}, \tag{2.39}
$$

we have $T_1^{-1}=T_1^*/\det(T_1)$. Therefore,

$$
T_{1,t}+T_1V_1^{(2)}=Q(\lambda)T_1, \tag{2.40}
$$

where $Q(\lambda)$ has columns 1 and 2:

$$
\begin{pmatrix}
Q_{11}^{(2)}\lambda^2+Q_{11}^{(1)}\lambda+Q_{11}^{(0)} & Q_{12}^{(1)}\lambda+Q_{12}^{(0)}\\
Q_{21}^{(1)}\lambda+Q_{21}^{(0)} & Q_{22}^{(2)}\lambda^2+Q_{22}^{(1)}\lambda+Q_{22}^{(0)}\\
0&0\\
0&0
\end{pmatrix}
$$

and columns 3 and 4:

$$
\begin{pmatrix}
Q_{13}^{(2)}\lambda^2+Q_{13}^{(1)}\lambda+Q_{13}^{(0)} & Q_{14}^{(1)}\lambda+Q_{14}^{(0)}\\
Q_{23}^{(1)}\lambda+Q_{23}^{(0)} & Q_{24}^{(2)}\lambda^2+Q_{24}^{(1)}\lambda+Q_{24}^{(0)}\\
Q_{11}^{(2)}\lambda^2+Q_{11}^{(1)}\lambda+Q_{11}^{(0)} & Q_{12}^{(1)}\lambda+Q_{12}^{(0)}\\
Q_{21}^{(1)}\lambda+Q_{21}^{(0)} & Q_{22}^{(2)}\lambda^2+Q_{22}^{(1)}\lambda+Q_{22}^{(0)}
\end{pmatrix}.
$$

Comparing the coefficients of λ in (2.40), we have

$$
\begin{gathered}
Q_{11}^{(2)}=\alpha,\quad Q_{13}^{(2)}=\beta,\quad Q_{22}^{(2)}=-\alpha,\quad Q_{24}^{(2)}=-\beta,\\
Q_{11}^{(1)}=\delta=\widetilde{\delta},\quad Q_{22}^{(1)}=-\delta=-\widetilde{\delta},\quad Q_{13}^{(1)}=0,\\
\begin{aligned}
Q_{14}^{(1)}&=\alpha(s+r-p-q)+\beta(p+q)-2\beta A_{12}^{(N-1)}-2\alpha A_{14}^{(N-1)}\\
&=\alpha(\widetilde{s}+\widetilde{r}-\widetilde{p}-\widetilde{q})+\beta(\widetilde{p}+\widetilde{q}),\\
Q_{23}^{(1)}&=\alpha(s-q-r+p)+\beta(q-p)+2\alpha A_{23}^{(N-1)}+2\beta A_{21}^{(N-1)}\\
&=\alpha(\widetilde{s}-\widetilde{q}-\widetilde{r}+\widetilde{p})+\beta(\widetilde{q}+\widetilde{q}),
\end{aligned}\\
Q_{21}^{(1)}=\alpha(q-p)+2\alpha A_{21}^{(N-1)}=\alpha(\widetilde{q}-\widetilde{p}),\\
Q_{12}^{(1)}=\alpha(q+p)+2\alpha A_{12}^{(N-1)}=\alpha(\widetilde{q}+\widetilde{p}),\quad Q_{24}^{(1)}=0,\\
\begin{aligned}
Q_{11}^{(0)}&=\frac{1}{2}\alpha(p^2-q^2)+\alpha(q-p)A_{12}^{(N-1)}-\alpha(p+q)A_{21}^{(N-1)}+2\alpha A_{12}^{(N-1)}A_{21}^{(N-1)}\\
&=\frac{1}{2}\alpha(\widetilde{p}^2-\widetilde{q}^2),
\end{aligned}
\end{gathered}
$$

$$
\begin{aligned}
Q_{22}^{(0)} &= -\frac{1}{2}\alpha(p^2-q^2) - \alpha(q-p)A_{12}^{(N-1)} + \alpha(p+q)A_{21}^{(N-1)} - 2\alpha A_{12}^{(N-1)}A_{21}^{(N-1)} \\
&= \frac{1}{2}\alpha(\widetilde{q}^2 - \widetilde{p}^2), \\
Q_{12}^{(0)} &= \frac{1}{2}\alpha(p_x - A_{21,x}^{(N-1)} - A_{21,x}^{(N-1)}) - \alpha h(q - A_{12}^{(N-1)} - A_{21}^{(N-1)}) \\
&\quad + \frac{1}{2}\alpha(q_x - A_{21,x}^{(N-1)} + A_{21,x}^{(N-1)}) - \alpha h(p - A_{12}^{(N-1)} + A_{21}^{(N-1)}) \\
&= \frac{1}{2}\alpha\widetilde{p}_x - \alpha h\widetilde{q} + \frac{1}{2}\alpha\widetilde{q}_x - \alpha h\widetilde{p}, \\
Q_{21}^{(0)} &= \frac{1}{2}\alpha(p_x - A_{21,x}^{(N-1)} - A_{21,x}^{(N-1)}) - \alpha h(q - A_{12}^{(N-1)} - A_{21}^{(N-1)}) \\
&\quad - \frac{1}{2}\alpha(q_x - A_{21,x}^{(N-1)} + A_{21,x}^{(N-1)}) + \alpha h(p - A_{12}^{(N-1)} + A_{21}^{(N-1)}) \\
&= \frac{1}{2}\alpha\widetilde{p}_x - \alpha h\widetilde{q} - \frac{1}{2}\alpha\widetilde{q}_x + \alpha h\widetilde{p}.
\end{aligned}
$$

It is easy to see that $\widetilde{V}_1^{(2)} = Q(\lambda)$. The proof is complete. $\square$

To obtain the explicit solutions of (2.11), we choose the seed solution $p = q = 0, r = s = 1$. Then, the spectral problems become

$$
\Phi_x = U_1\Phi, \quad U_1 = \begin{bmatrix} \lambda & 0 & \lambda & 2 \\ 0 & -\lambda & 0 & -\lambda \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}, \tag{2.41}
$$

and

$$
\Phi_t = V_1^{(2)}\Phi, \quad V_1^{(2)} = \begin{bmatrix} \alpha\lambda^2 & 0 & \beta\lambda^2 & 2\alpha\lambda \\ 0 & -\alpha\lambda^2 & 0 & -\beta\lambda^2 \\ 0 & 0 & \alpha\lambda^2 & 0 \\ 0 & 0 & 0 & -\alpha\lambda^2 \end{bmatrix}. \tag{2.42}
$$

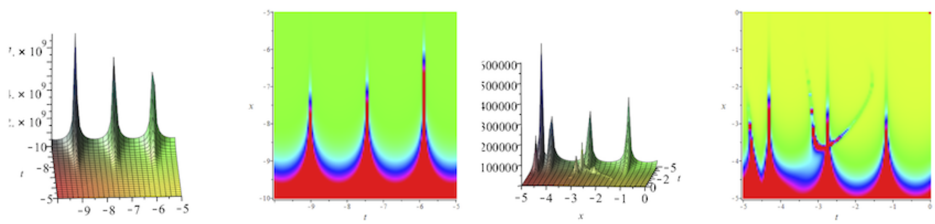
Solving these spectral problems, we obtain

$$
\begin{gathered}
\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} = \begin{bmatrix} e^{-\lambda x - \lambda^2\alpha t}(t\lambda + x)\lambda \\ e^{-\alpha\lambda^2 t - \lambda x} \\ 0 \\ e^{-\lambda x - \alpha\lambda^2 t} \end{bmatrix}, \\
\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} ((\beta t\lambda^2 + x)\lambda)e^{\lambda x + \alpha\lambda t} \\ e^{-\lambda x - \lambda^2\alpha t} \\ e^{\alpha\lambda^2 t + \lambda x} \\ 0 \end{bmatrix}.
\end{gathered} \tag{2.43}
$$

When $N = 1$, the explicit solutions of (2.11) is

$$
\begin{aligned}
\widetilde{p} &= -A_{12}^{(0)} - A_{21}^{(0)}, \\
\widetilde{q} &= -A_{21}^{(0)} + A_{21}^{(0)}, \\
\widetilde{r} &= -A_{12}^{(0)} - A_{14}^{(0)} - A_{21}^{(0)} - A_{23}^{(0)}, \\
\widetilde{s} &= -A_{12}^{(0)} - A_{14}^{(0)} + A_{21}^{(0)} + A_{23}^{(0)}.
\end{aligned} \tag{2.44}
$$

Remark 2.5. By choosing suitable parameters in solution (2.11), we provide four figures to analyze the transmission form of soliton solutions. In Figure 2(a, b), we can find that three sets of the parallel solitons, but they have different heights. Three single solitons keep at a certain interval which stably propagate and have no interaction. In Figure 2(c, d), solitons do not stably propagate. That is to say, the difference between Figure 2(a, b) and Figure 2(c, d) under the influence of the same parameter.



(a) $p = q$ (b) $p = q$ (c) $r = s$ (d) $r = s$

FIGURE 2. Soliton solutions of (2.11) and density plots with $\alpha = 1$, $\beta = 1$, $\lambda_1 = 0.05$, $\lambda_2 = -0.05$, $\lambda_3 = 0.1$, $\lambda_4 = -0.1$, $\gamma_1 = 0.2$, $\gamma_2 = -0.2$, $\gamma_3 = 2$, $\gamma_4 = -2$.

3. Third-order nonlinear integrable perturbed hierarchy of NLS-mKdV equation and their explicit solutions

3.1. **Third-order nonlinear integrable perturbed hierarchy of NLS-mKdV equation.** To obtain a third-order nonlinear integrable perturbed hierarchy of NLS-mKdV equation, we consider the auxiliary problem of the spectral problem (1.6) as follows

$$\Phi_t = V_2\Phi, \quad V_2 = \begin{bmatrix} 0 & b-c & a \\ -b+c & 0 & -b-c \\ -a & b+c & 0 \end{bmatrix}. \tag{3.1}$$

Solving the stationary zero curvature equation (2.1), we obtain

$$\begin{aligned} a_x &= 2pb - 2qc, \\ b_x &= \lambda c + hc - pa, \\ c_x &= -\lambda b - hb + qa. \end{aligned} \tag{3.2}$$

By substituting expansions

$$V_2 = \sum_{i=0}^{\infty} \begin{bmatrix} 0 & b_i - c_i & a_i \\ -b_i + c_i & 0 & -b_i - c_i \\ -a_i & b_i + c_i & 0 \end{bmatrix} \lambda^{-i} \tag{3.3}$$

into (3.2), we obtain

$$\begin{aligned} a_{m+1,x} &= 2pb_{m+1} - 2qc_{m+1}, \\ b_{m+1} &= -c_{m,x} + qa_m - hb_m, \\ c_{m+1} &= b_{m,x} + pa_m - hc_m, \end{aligned} \tag{3.4}$$

and the initial conditions $a_{0x} = 2pb_0 - 2qc_0$, $b_0 = 0$, $c_0 = 0$. Now we choose $a_0 = \alpha$ an arbitrary constant. Starting from the above initial values, we obtain $a_1 = 0$, $b_1 = \alpha q$, $c_1 = \alpha p$, $a_2 = \alpha(-p^2 - q^2)$, $b_2 = -p_x - \alpha hq$, $c_2 = \alpha q_x - \alpha hp$, $\dots$. Setting

$$\mathrm{V_{2,m}} = \sum_{i=0}^{m} \begin{bmatrix} 0 & b_i - c_i & a_i \\ -b_i + c_i & 0 & -b_i - c_i \\ -a_i & b_i + c_i & 0 \end{bmatrix} \lambda^{m-i}.$$

We have

$$V_{2,mx} - [U, V_{2,m}] = \begin{bmatrix} 0 & b_{m+1} + c_{m+1} & 0 \\ -b_{m+1} - c_{m+1} & 0 & b_{m+1} - c_{m+1} \\ 0 & -b_{m+1} + c_{m+1} & 0 \end{bmatrix}.$$

By considering $\Phi_t = V_2 \Phi$ with

$$V_2 = V_{2,m} + \Delta_{2,m}.$$

We take the correction term

$$\Delta_{2,m} = \begin{bmatrix} 0 & 0 & \delta_m \\ 0 & 0 & 0 \\ -\delta_m & 0 & 0 \end{bmatrix},$$

where $\delta_m = -2\epsilon(a_{m+1} + d_{m+1})$. Then the corresponding zero curvature equation (2.6) give rise to the following a third-order nonlinear integrable perturbed hierarchy of NLS-mKdV equation

$$\begin{aligned} p_t &= -b_{m+1} + \frac{1}{2}\epsilon q a_{m+1}, \\ q_t &= c_{m+1} - \frac{1}{2}\epsilon p a_{m+1}. \end{aligned} \tag{3.5}$$

When $m = 1$, setting $\epsilon = 1$, we obtain

$$\begin{aligned} p_t &= \alpha p_x + \alpha hq - \frac{1}{2}\alpha q(p^2 + q^2), \\ q_t &= \alpha q_x - \alpha hp + \frac{1}{2}\alpha p(p^2 + q^2). \end{aligned} \tag{3.6}$$

When $m = 2$, setting $\epsilon = 1$, we obtain

$$\begin{aligned} p_t &= \alpha q_{xx} - \alpha h_x q - 2\alpha h p_x - \alpha h^2 q + \alpha q(p^2 + q^2) \\ &\quad + \alpha q((qp_x - pq_x) + 2\alpha h(p^2 + q^2)), \\ q_t &= -\alpha p_{xx} - \alpha h_x p - 2\alpha h p_x + \alpha h^2 p - \alpha p(p^2 + q^2) \\ &\quad - \alpha p((qp_x - pq_x) + 2\alpha h(p^2 + q^2)). \end{aligned} \tag{3.7}$$

We can construct bi-Hamiltonian structures for (3.5) by using the variational-trace identity

$$\frac{\delta}{\delta \bar{u}} \int \langle V, \frac{\partial U(\bar{u}, \lambda)}{\partial \lambda} \rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \langle V, \frac{\partial U(\bar{u}, \lambda)}{\partial \bar{u}} \rangle, \tag{3.8}$$

where $\bar{u}$ stands for $(p, q)^T$, and $\langle a, b \rangle$ stands for the trace of the matrix $a \cdot b$; here $a \cdot b$ stands for the matrix a times the matrix b. Computations yield

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & -1 & 2\epsilon p \\ 1 & 0 & -1 \\ -2\epsilon p & 1 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & 1 & 2\epsilon q \\ -1 & 0 & -1 \\ -2\epsilon q & 1 & 0 \end{bmatrix},$$

and so we can obtain

$$\big\langle V, \frac{\partial U}{\partial \lambda}\big\rangle = -2a, \quad \big\langle V, \frac{\partial U}{\partial p}\big\rangle = -4c - 4\epsilon pa, \quad \big\langle V, \frac{\partial U}{\partial q}\big\rangle = -4b - 4\epsilon qa. \tag{3.9}$$

Now the corresponding trace identity (3.8) becomes

$$\frac{\delta}{\delta \bar{u}} \int -2a\,dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \begin{bmatrix} -4c - 4\epsilon pa \\ -4b - 4\epsilon qa \end{bmatrix}. \tag{3.10}$$

Balancing coefficients of each power of λ in the above equality, we have

$$\frac{\delta}{\delta \bar{u}} \int -2a_{m+1}\,dx = (\gamma - m) \begin{bmatrix} -4c_m - 4\epsilon pa_m \\ -4b_m - 4\epsilon qa_m \end{bmatrix}. \tag{3.11}$$

The case of $m = 1$ tells $\gamma = 0$, and thus we have

$$\frac{\delta}{\delta \bar{u}} \int \frac{2a_{m+2}}{m+1}\,dx = \begin{bmatrix} -4c_{m+1} - 4\epsilon pa_{m+1} \\ -4b_{m+1} - 4\epsilon qa_{m+1} \end{bmatrix}. \tag{3.12}$$

Consequently, we obtain the following Hamiltonian structures for (3.5),

$$u_t = \begin{bmatrix} -4b_{m+1} + \epsilon qa_{m+1} \\ -4c_{m+1} - \epsilon pa_{m+1} \end{bmatrix} = J\frac{\delta H_m}{\delta \overline{u}}, \tag{3.13}$$

with the Hamiltonian operator

$$J = \begin{bmatrix} \epsilon q\partial^{-1}q & \frac{1}{4} - \epsilon q\partial^{-1}p \\ -\frac{1}{4} - \epsilon p\partial^{-1}q & \epsilon p\partial^{-1}p \end{bmatrix}, \tag{3.14}$$

and the Hamilton functionals $H_m = \int -\frac{2a_{m+2}}{m+1}dx$.

Thus (3.5) has the following bi-Hamiltonian structures

$$U_t = J\frac{\delta H_m}{\delta \overline{u}} = JL\frac{\delta H_{m-1}}{\delta \overline{u}}, \quad m \geq 0. \tag{3.15}$$

Through a series of calculations, we obtain

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \tag{3.16}$$

where

$$\begin{aligned} L_{11} &= -2p\partial^{-1}q - h - 2\epsilon p\partial^{-1}p\partial + 2\epsilon q\partial^{-1}q - 8\epsilon\partial p\partial^{-1}hq, \\ L_{12} &= \partial + 2p\partial^{-1}p - 2\epsilon p\partial^{-1}q\partial - 2\epsilon q\partial^{-1}p + 8\epsilon\partial p\partial^{-1}hp, \\ L_{21} &= -\partial - 2q\partial^{-1}q - 2\epsilon q\partial^{-1}p\partial - 2\epsilon p\partial^{-1}q + 8\epsilon\partial q\partial^{-1}hq, \\ L_{22} &= 2q\partial^{-1}p - h - 2\epsilon q\partial^{-1}q\partial + 2\epsilon p\partial^{-1}p - 8\epsilon\partial q\partial^{-1}hp. \end{aligned} \tag{3.17}$$

So far, we are ready to see that (3.5) is intregrable in the sense of Liouville.

3.2. **N-fold Darboux transformation of** (3.6) **and its explicit solution.** In what follows, we search for a Darboux transformation of (3.6), which is the first equation in the hierarchy (3.5). The Darboux transformation is a special gauge transformation

$$\widetilde{\Phi} = T_2\Phi, \tag{3.18}$$

with

$$T_2 = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \tag{3.19}$$

where

$$A_{11}=A_{33}=0,\quad A_{12}=-A_{21}=\sum_{i=0}^{N-1}A_{12}^{(i)}\lambda^i,\quad A_{13}=-A_{31}=\lambda^N+\sum_{i=0}^{N-1}A_{13}^{(i)}\lambda^i,$$
$$A_{22}=\lambda^N+\sum_{i=0}^{N-1}A_{22}^{(i)}\lambda^i,\quad A_{23}=-A_{32}=\lambda^N+\sum_{i=0}^{N-1}A_{23}^{(i)}\lambda^i.$$

Then the Lax pair becomes

$$\widetilde{\Phi}_x=\widetilde{U}_2\widetilde{\Phi},\quad \widetilde{\Phi}_t=\widetilde{V}_2\widetilde{\Phi}, \tag{3.20}$$

and $\widetilde{U}_2$, $\widetilde{V}_2$ satisfy

$$T_{2,x}+T_2U_2=\widetilde{U}_2T_2,\quad T_{2,t}+T_2V_2=\widetilde{V}_2T_2. \tag{3.21}$$

Now we define

$$\Phi=\begin{bmatrix}\varphi_1 & \psi_1\\ \varphi_2 & \psi_2\\ \varphi_3 & \psi_3\end{bmatrix}. \tag{3.22}$$

Using (3.22) we define the linear algebraic system

$$\begin{gathered}\sum_{i=0}^{N-1}A_{11}^{(i)}\lambda_j^i+\omega_j^{(1)}\sum_{i=0}^{N-1}A_{12}^{(i)}\lambda_j^i+\omega_j^{(2)}\sum_{i=0}^{N-1}A_{13}^{(i)}\lambda_j^i=-\omega_j^{(2)}\lambda_j^N,\\ -\sum_{i=0}^{N-1}A_{12}^{(i)}\lambda_j^i+\omega_j^{(1)}\sum_{i=0}^{N-1}A_{22}^{(i)}\lambda_j^i-\omega_j^{(2)}\sum_{i=0}^{N-1}A_{32}^{(i)}\lambda_j^i=-\omega_j^{(1)}\lambda_j^N,\\ -\sum_{i=0}^{N-1}A_{13}^{(i)}\lambda_j^i+\omega_j^{(1)}\sum_{i=0}^{N-1}A_{32}^{(i)}\lambda_j^i+\omega_j^{(2)}\sum_{i=0}^{N-1}A_{33}^{(i)}\lambda_j^i=\lambda_j^N,\end{gathered} \tag{3.23}$$

with

$$\omega_j^{(1)}=\frac{\gamma_j^{(1)}\varphi_2+\gamma_j^{(2)}\psi_2}{\gamma_j^{(1)}\varphi_1+\gamma_j^{(2)}\psi_1},\quad \omega_j^{(2)}=\frac{\gamma_j^{(1)}\varphi_3+\gamma_j^{(2)}\psi_3}{\gamma_j^{(1)}\varphi_1+\gamma_j^{(2)}\psi_1}.$$

Proposition 3.1. *Noting $T_2^{-1}=T_2^*/\det T_2$, we have*

$$T_{2,t}+T_2U_2=R(\lambda)T_2, \tag{3.24}$$

with

$$R(\lambda)=\begin{pmatrix}0 & R_{12}^{(0)} & R_{13}^{(1)}\lambda+R_{13}^{(0)}\\ -R_{12}^{(0)} & 0 & -R_{32}^{(0)}\\ -R_{13}^{(1)}\lambda-R_{13}(0) & R_{32}^{(0)} & 0\end{pmatrix}. \tag{3.25}$$

Proof. By (3.24), equating the coefficients of λ^{N+i} $(i=1,2)$, we obtain

$$\begin{gathered}R_{13}^{(1)}=1,\quad R_{32}^{(0)}=p-q+A_{32}^{(N-1)}=\widetilde{p}+\widetilde{q},\quad R_{12}^{(0)}=p+q+A_{12}^{(N-1)}=-\widetilde{p}+\widetilde{q},\\ R_{13}^{(0)}=(-q+\frac{1}{2}A_{32}^{(N-1)}-\frac{1}{2}A_{12}^{(N-1)})^2+(p+\frac{1}{2}A_{32}^{(N-1)}+\frac{1}{2}A_{12}^{(N-1)})^2=\widetilde{p}^2+\widetilde{q}^2.\end{gathered}$$

We see that $\widetilde{U}_2=R(\lambda)$, which completes the proof. $\square$

Proposition 3.2. *Noting $T_2^{-1} = T_2^*/\det T_2$, we have*

$$T_{2,t} + T_2 V_2^{(1)} = S(\lambda) T_2, \tag{3.26}$$

with

$$S(\lambda) = \begin{pmatrix} 0 & S_{12}^{(0)} & S_{13}^{(1)}\lambda + S_{13}^{(0)} \\ -S_{12}^{(0)} & 0 & -S_{32}^{(0)} \\ -S_{13}^{(1)}\lambda - S_{13}(0) & S_{32}^{(0)} & 0 \end{pmatrix}. \tag{3.27}$$

Proof. By (3.27), comparing the coefficients of λ on both sides, we have

$$S_{13}^{(1)} = \alpha, \quad S_{13}^{(0)} = \delta_1, \quad S_{32}^{(0)} = \alpha p - \alpha q + \alpha A_{32}^{(N-1)} = \alpha\widetilde{p} + \alpha\widetilde{q},$$
$$S_{12}^{(0)} = \alpha p + \alpha q + \alpha A_{12}^{(N-1)} = -\alpha\widetilde{p} + \alpha\widetilde{q}.$$

We see that $\widetilde{V}_2^{(1)} = S(\lambda)$. The Proof is complete. □

Next we discuss the explicit solutions of (3.6). Firstly, we give a seed solutions $p = q = 0$ of (3.6), the spectral problems are

$$\Phi_x = U_2\Phi = \begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ -\lambda & 0 & 0 \end{bmatrix} \Phi, \tag{3.28}$$

and

$$\Phi_t = V_2^{(1)}\Phi = \begin{bmatrix} 0 & 0 & \alpha\lambda \\ 0 & 0 & 0 \\ -\alpha\lambda & 0 & 0 \end{bmatrix} \Phi. \tag{3.29}$$

Solving the above two equations, we have

$$\begin{gathered} \varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} \sin(\lambda t) sin(\lambda x) - \cos(\lambda t)\cos(\lambda x) \\ 1 \\ \sin(\lambda x)\cos(\lambda t) + \sin(\lambda t)\cos(\lambda x) \end{bmatrix}, \\ \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} \cos(\lambda x)\sin(\lambda t) + \cos(\lambda t)\sin(\lambda x) \\ 1 \\ \cos(\lambda t)\cos(\lambda x) - \sin(\lambda t)\sin(\lambda x) \end{bmatrix}. \end{gathered} \tag{3.30}$$

In particular, when $N = 1$,

$$T_2 = \begin{bmatrix} 0 & A_{12}^{(0)} & \lambda + A_{13}^{(0)} \\ -A_{12}^{(0)} & \lambda + A_{22}^{(0)} & -A_{32}^{(0)} \\ -\lambda - A_{13}^{(0)} & A_{32}^{(0)} & 0 \end{bmatrix}, \tag{3.31}$$

and

$$\widetilde{p} = \frac{1}{2}A_{12}^{(0)} - \frac{1}{2}A_{32}^{(0)}, \quad \widetilde{q} = \frac{1}{2}A_{12}^{(0)} + \frac{1}{2}A_{32}^{(0)}. \tag{3.32}$$

If we set

$$\omega_j^{(1)} = \frac{\varphi_2 + \gamma_j\psi_2}{\varphi_1 + \gamma_j\psi_1}, \quad \omega_j^{(2)} = \frac{\varphi_3 + \gamma_j\psi_3}{\varphi_1 + \gamma_j\psi_1},$$

we have

$$\begin{aligned} \omega_j^{(1)} &= \frac{1 + \gamma_j}{\sin(\lambda t)\sin(\lambda x) - \cos(\lambda t)\cos(\lambda x) + \gamma_j(\cos(\lambda x) sin(\lambda t) + \cos(\lambda t)\sin(\lambda x))}, \\ \omega_j^{(1)} &= \frac{\sin(\lambda x)\cos(\lambda t) + \sin(\lambda t)\cos(\lambda x) + \gamma_j(\cos(\lambda t)\cos(\lambda x) - \sin(\lambda t)\sin(\lambda x))}{\sin(\lambda t)\sin(\lambda x) - \cos(\lambda t)\cos(\lambda x) + \gamma_j(\cos(\lambda x)\sin(\lambda t) + \cos(\lambda t)\sin(\lambda x))}. \end{aligned}$$

By $\widetilde{\Phi} = T_2\Phi$, we obtain

$$\begin{gathered}
A_{11}^{(0)} + A_{12}^{(0)}\omega_j^{(1)} + A_{13}^{(0)}\omega_j^{(2)} = -\lambda_j\omega_j^{(2)}, \\
-A_{12}^{(0)} + A_{22}^{(0)}\omega_j^{(1)} - A_{32}^{(0)}\omega_j^{(2)} = -\lambda_j\omega_j^{(1)}, \\
-A_{13}^{(0)} + A_{32}^{(0)}\omega_j^{(1)} + A_{33}^{(0)}\omega_j^{(2)} = \lambda_j.
\end{gathered} \tag{3.33}$$

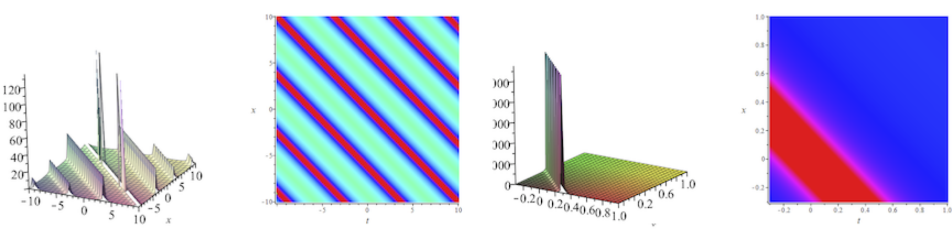
Solving the above equations by using of Cramer's rule, we can obtain

$$A_{12}^{(0)} = \frac{\Delta A_{12}^{(0)}}{\Delta A_{12}}, \quad A_{32}^{(0)} = \frac{\Delta A_{32}^{(0)}}{\Delta A_{32}}, \tag{3.34}$$

and

$$\begin{gathered}
\Delta A_{12} = \begin{bmatrix} 1 & \omega_1^{(1)} & \omega_1^{(2)} \\ 1 & \omega_2^{(1)} & \omega_2^{(2)} \\ 1 & \omega_3^{(1)} & \omega_3^{(2)} \end{bmatrix}, \quad \Delta A_{12}^{(0)} = \begin{bmatrix} 1 & -\lambda_1\omega_2^{(1)} & \omega_1^{(2)} \\ 1 & -\lambda_2\omega_2^{(2)} & \omega_2^{(2)} \\ 1 & -\lambda_3\omega_2^{(3)} & \omega_3^{(2)} \end{bmatrix}, \\
\Delta A_{32} = \begin{bmatrix} -1 & \omega_1^{(1)} & \omega_1^{(2)} \\ -1 & \omega_2^{(1)} & \omega_2^{(2)} \\ -1 & \omega_3^{(1)} & \omega_3^{(2)} \end{bmatrix}, \quad \Delta A_{32}^{(0)} = \begin{bmatrix} -1 & \lambda_1 & \omega_1^{(2)} \\ -1 & \lambda_2 & \omega_2^{(2)} \\ -1 & \lambda_3 & \omega_3^{(2)} \end{bmatrix}.
\end{gathered} \tag{3.35}$$

Remark 3.3. In Figure 3, we can see the different pulse propagation patterns. Figure 3(a, b), the solitons keep at a certain interval and have no interaction, we can find that they have certain symmetry. Figure 3(c, d) shows that the bright soliton structures of solutions. Therefore, the difference between Figure 3(a, b) and Figure 3(c, d) is obvious.



(a) p (b) p (c) q (d) q

FIGURE 3. Three-dimensional structure figures of explicit solutions of (3.6) and the density plots with $\alpha = 1$; $\lambda_1 = -0.5$, $\lambda_2 = 0.2$, $\lambda_3 = -0.01$, $\gamma_1 = -0.2$, $\gamma_2 = 0.2$, $\gamma_3 = -0.1$.

3.3. **N-fold Darboux transformation of** (3.7) **and its explicit solutions.** In this subsection, we apply the same Darboux matrix (3.19) to construct the Darboux transformation of (3.7).

Proposition 3.4. *Noting* $T_2^{-1} = T_2^*/\det T_2$, *we have*

$$T_{2,t} + T_2V_2^{(2)} = W(\lambda)T_2, \tag{3.36}$$

with

$$W(\lambda)=\begin{bmatrix} 0 & W_{12}^{(1)}+W_{12}^{(0)} & W_{13}^{(2)}\lambda^2+W_{13}^{(1)}\lambda+W_{13}^{(0)} \\ -W_{12}^{(1)}-W_{12}^{(0)} & 0 & -W_{32}^{(1)}-W_{32}^{(0)} \\ -W_{13}^{(2)}\lambda^2-W_{13}^{(1)}\lambda-W_{13}(0) & W_{32}^{(1)}+W_{32}^{(0)} & 0 \end{bmatrix}. \tag{3.37}$$

Proof. By (3.37), comparing the coefficients of λ on both sides, we have

$$W_{32}^{(1)}=\alpha p-\alpha q+\alpha A_{32}^{(N-1)}=\alpha\widetilde{p}+\alpha\widetilde{q},$$

$$W_{12}^{(1)}=\alpha p+\alpha q+\alpha A_{12}^{(N-1)}=-\alpha\widetilde{p}+\alpha\widetilde{q},\quad W_{13}^{(2)}=\alpha,$$

$$\begin{aligned} W_{13}^{(0)}&=-\alpha[(-q+\frac{1}{2}A_{32}^{(N-1)}-\frac{1}{2}A_{12}^{(N-1)})^2+(p+\frac{1}{2}A_{32}^{(N-1)}+\frac{1}{2}A_{12}^{(N-1)})^2] \\ &=-\alpha(\widetilde{p}^2+\widetilde{q}^2), \end{aligned}$$

$$W_{13}^{(1)}=\delta_1,$$

$$\begin{aligned} W_{12}^{(0)}&=-\alpha(-q_x+\frac{1}{2}A_{32,x}^{(N-1)}-\frac{1}{2}A_{21,x}^{(N-1)})-\alpha h(p+\frac{1}{2}A_{32}^{(N-1)}+\frac{1}{2}A_{12}^{(N-1)}) \\ &=-\alpha\widetilde{p}_x-\alpha h\widetilde{q}. \end{aligned}$$

We see that $\widetilde{V}_2^{(2)}=S(\lambda)$. The Proof is complete. □

Next we will apply Darboux transformation (3.19) to give explicit solution of (3.7). We choose the seed solution $p=q=0$, then the spectral problems become

$$\Phi_x=U_2\Phi,\quad U_2=\begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ -\lambda & 0 & 0 \end{bmatrix}, \tag{3.38}$$

and

$$\Phi_t=V_2^{(2)}\Phi,\quad \mathrm{V}_2^{(2)}=\begin{bmatrix} 0 & 0 & \alpha\lambda^2 \\ 0 & 0 & 0 \\ -\alpha\lambda^2 & 0 & 0 \end{bmatrix}. \tag{3.39}$$

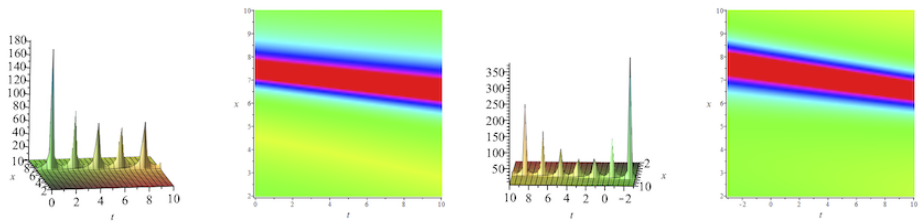
Solving the above two equations, we have

$$\begin{gathered} \varphi=\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix}=\begin{bmatrix} \sin(\lambda^2 t)\sin(\lambda x)-\cos(\lambda^2 t)\cos(\lambda x) \\ 1 \\ \sin(\lambda x)\cos(\lambda^2 t)+sin(\lambda^2 t)\cos(\lambda x) \end{bmatrix}, \\ \psi=\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}=\begin{bmatrix} \cos(\lambda x)\sin(\lambda t)+cos(\lambda t)\sin(\lambda x) \\ 1 \\ \cos(\lambda t)\cos(\lambda x)-\sin(\lambda t)\sin(\lambda x) \end{bmatrix}. \end{gathered} \tag{3.40}$$

Ultimately, we obtain explicit solution of (3.7),

$$\widetilde{p}=\frac{1}{2}A_{32}^{(0)}-\frac{1}{2}A_{12}^{(0)},\quad \widetilde{q}=\frac{1}{2}A_{32}^{(0)}+\frac{1}{2}A_{12}^{(0)}. \tag{3.41}$$

Remark 3.5. In Figure 4, one can see that all of these solitary waves move from right to left. During the propagation, their shapes nearly keep invariant but amplitudes changes. It is worth pointing out that these one solitary waves display oscillating nonlinear waves with multiple peaks. Compared with Figure 4(a, b), the amplitude change of Figure 4(c, d) are more obvious.



(a) p (b) p (c) q (d) q

Figure 4. Three-dimensional structure figures of explicit solutions of (3.7) and the density plots with $\alpha = 1$, $\lambda_1 = -0.2$, $\lambda_2 = -0.1$, $\lambda_3 = 0.3$, $\gamma_1 = -0.5$, $\gamma_2 = 0.2$, $\gamma_3 = 0.3$.

4. Sixth-order nonlinear integrable perturbed hierarchy of NLS-mKdV equation and their explicit solutions

4.1. Sixth-order nonlinear integrable perturbed hierarchy of NLS-mKdV equation.

To obtain a sixth-order nonlinear integrable perturbed hierarchy of NLS-mKdV equation, we assume that the V_3 has the form: $\Phi_t = V_3\Phi$ and

$$V_3 = \begin{bmatrix} V_2 & V_4 \\ 0 & V_2 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 0 & b-c & a & 0 & f-g & d \\ -b+c & 0 & -b-c & -f+g & 0 & -f-g \\ -a & b+c & 0 & -d & f+g & 0 \\ \hline 0 & 0 & 0 & 0 & b-c & a \\ 0 & 0 & 0 & -b+c & 0 & -b-c \\ 0 & 0 & 0 & -a & b+c & 0 \end{array}\right]. \tag{4.1}$$

Now, we set

$$V_3 = \sum_{i=0}^{\infty} \left[\begin{array}{ccc|ccc} 0 & b_i-c_i & a_i & 0 & f_i-g_i & d_i \\ -b_i+c_i & 0 & -b_i-c_i & -f_i+g_i & 0 & -f_i-g_i \\ -a_i & b_i+c_i & 0 & -d_i & b_i+c_i & 0 \\ \hline 0 & 0 & 0 & 0 & b_i-c_i & a_i \\ 0 & 0 & 0 & -b_i+c_i & 0 & -b_i-c_i \\ 0 & 0 & 0 & -a_i & b_i+c_i & 0 \end{array}\right] \lambda^{-i}. \tag{4.2}$$

Solving the stationary zero curvature equation (2.1) we obtain

$$\begin{gathered} a_{m+1,x} = 2pb_{m+1} - 2qc_{m+1}, \\ b_{m+1} = -c_{m,x} + qa_m - hb_m, \\ c_{m+1} = b_{m,x} + pa_m - hc_m, \\ , d_{m+1,x} = 2pf_{m+1} - 2qg_{m+1} + 2rb_{m+1} - 2sc_{m+1}, \\ f_{m+1} = -g_{m,x} - \frac{1}{2}c_{m,x} + (s-p)a_m + hc_m + qd_m - hf_m, \\ g_{m+1} = f_{m,x} - \frac{1}{2}b_{m,x} + (r-q)a_m + hb_m + pd_m - hg_m. \end{gathered} \tag{4.3}$$

Substituting (1.7) and (4.2) into (2.1), and comparing the powers of coefficient of λ, we obtain initial values $a_0 = \alpha, b_0 = 0, c_0 = 0, e_0 = \beta, f_0 = 0, g_0 = 0$, with α, β are arbitrary constants. The values of first few terms are calculated as follows

$$a_1 = 0, \quad b_1 = \alpha q, \quad c_1 = \alpha p, \quad d_1 = 0, \quad f_1 = \alpha(s-q) + \beta q,$$

$$
\begin{gathered}
g_1=\alpha(r-p)+\beta p, \quad a_2=\alpha(-p^2-q^2), \\
b_2=-p_x-\alpha hq, \quad c_2=\alpha q_x-\alpha hp, \\
d_2=-2\alpha pr-2\alpha qs+2\alpha(q^2+p^2)+\beta(-p^2-q^2), \\
f_2=-\alpha r_x+2\alpha p_x-\beta p_x+2\alpha hq-\alpha hs-\beta hq, \\
g_2=\alpha s_x-2\alpha q_x+\beta q_x+2\alpha hp-\alpha hr-\beta hp, \dots
\end{gathered}
$$

We choose the auxiliary problem $\Phi_t=V_3\Phi$. Noting that

$$
V_3=V_{3,m}+\Delta_{3,m},
$$

where the correction term is

$$
\Delta\Delta_{3,m}=\left[\begin{array}{ccc|ccc}
0 & 0 & \delta_m & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\delta_m & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \delta_m \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\delta_m & 0 & 0
\end{array}\right],
$$

where $\delta_m=-2\epsilon(a_{m+1}+d_{m+1})$. Next, we set

$$
V_{3,m}=\sum_{i=0}^{m}\left[\begin{array}{ccc|ccc}
0 & b_i-c_i & a_i & 0 & f_i-g_i & d_i \\
-b_i+c_i & 0 & -b_i-c_i & -f_i+g_i & 0 & -f_i-g_i \\
-a_i & b_i+c_i & 0 & -d_i & b_i+c_i & 0 \\
\hline
0 & 0 & 0 & 0 & b_i-c_i & a_i \\
0 & 0 & 0 & -b_i+c_i & 0 & -b_i-c_i \\
0 & 0 & 0 & -a_i & b_i+c_i & 0
\end{array}\right]\lambda^{m-i}.
$$

Then the corresponding zero curvature equation (2.6) give rise to the integrable coupling of NLS-mKdV system

$$
\begin{gathered}
p_t=-b_{m+1}+\frac{1}{2}\epsilon q(a_{m+1}+e_{m+1}), \\
q_t=c_{m+1}-\frac{1}{2}\epsilon p(a_{m+1}+e_{m+1}), \\
r_t=-b_{m+1}-f_{m+1}+\frac{1}{2}\epsilon s(a_{m+1}+e_{m+1}), \\
s_t=c_{m+1}+g_{m+1}-\frac{1}{2}\epsilon r(a_{m+1}+e_{m+1}).
\end{gathered} \tag{4.4}
$$

When $m=1$, setting $\epsilon=1$ in (4.4) gives rise to

$$
\begin{gathered}
p_t=\alpha p_x+\alpha hq-\frac{1}{2}\alpha q^2 s-\frac{1}{2}\alpha pqr-\frac{1}{2}(\alpha+\beta)q(p^2+q^2), \\
q_t=\alpha q_x+\alpha hp+\frac{1}{2}\alpha qs+\frac{1}{2}\alpha p^2 r+\frac{1}{2}(\alpha+\beta)p(p^2+q^2), \\
r_t=\alpha r_x-\alpha p_x+\beta p_x-\alpha hq+\alpha hs+\beta hq-\frac{1}{2}\alpha qs^2-2\alpha psr-\frac{1}{2}(\alpha+\beta)s(p^2+q^2), \\
r_t=\alpha s_x-\alpha q_x+\beta q_x+\alpha hq-\alpha hs-\beta hq+\frac{1}{2}\alpha qsr-2\alpha pr^2+\frac{1}{2}(\alpha+\beta)r(p^2+q^2).
\end{gathered} \tag{4.5}
$$

When $m=2$, setting $\epsilon=1$ in (4.4) yields

$$
\begin{aligned}
p_t &= \alpha q_{xx} - 2\alpha h p_x + \alpha p h_x + \frac{1}{2} q(6\alpha(qp_x - pq_x) + 2\alpha(sp_x - ps_x) \\
&\quad + \beta(pq_x - qp_x) + 2\alpha(qr_x - rq_x) + 4\alpha h(p^2+q^2) - \beta h(q^2-p^2) + 2h^2), \\
q_t &= -\alpha p_{xx} - 2\alpha h q_x + \alpha h^2 p - \alpha h^2 p + \alpha p(p^2+q^2) - \alpha q h_x \\
&\quad - \frac{1}{2} p(6\alpha(qp_x - pq_x) + 2\alpha(sp_x - ps_x) + \beta(pq_x - qp_x) \\
&\quad + 2\alpha(qr_x - rq_x) + 4\alpha h(p^2+q^2) - \beta h(q^2-p^2) + 2h^2), \\
r_t &= \alpha s_{xx} - \alpha q_{xx} + \beta q_{xx} + 2\alpha p h_x + 4\alpha h p_x - \alpha r h_x - \alpha h r_\beta p h_x - 2\beta h p_x \\
&\quad - \alpha h r_x - 4\alpha h^2 q + 2\alpha h^2 s + 2\beta h^2 q + 2\alpha pqr - 2\alpha q^2 s + 2\alpha q(p^2 - q^2) \\
&\quad + \beta q(p^2-q^2) + \alpha s(p^2-q^2) + \frac{1}{2} s(6\alpha(qp_x - pq_x) + 2\alpha(sp_x - ps_x) \\
&\quad + \beta(pq_x - qp_x) + 2\alpha(qr_x - rq_x) + 4\alpha h(p^2+q^2) - \beta h(q^2-p^2) + 2h^2), \\
s_t &= -\alpha r_{xx} - \alpha p_{xx} + \beta p_{xx} + 2\alpha q h_x + 4\alpha h q_x - \alpha s h_x - \alpha h s_\beta q h_x - 2\beta h q_x \\
&\quad - \alpha h s_x - 4\alpha h^2 p + 2\alpha h^2 r + 2\beta h^2 p + 2\alpha p^2 r - 2\alpha pqs + 2\alpha p(p^2-q^2) \\
&\quad + \beta p(p^2-q^2) + \alpha r(p^2-q^2) - \frac{1}{2} r(6\alpha(qp_x - pq_x) 2\alpha(sp_x - ps_x) \\
&\quad + \beta(pq_x - qp_x) + 2\alpha(qr_x - rq_x) + 4\alpha h(p^2+q^2) - \beta h(q^2-p^2) + 2h^2).
\end{aligned}
\tag{4.6}
$$

We are ready to show that a sixth-order nonlinear integrable perturbed hierarchy of NLS-mKdV equation (4.4) is Lax integrable.

4.2. **N-fold Darboux transformation of** (4.5) **and its explicit solutions.** In this section, we investigate the Darboux transformation of (4.5). To construct the Darboux transformation of (4.5), we consider a gauge transformation

$$
\widetilde{\Phi} = T_3 \Phi, \tag{4.7}
$$

where T_3 satisfies

$$
\widetilde{U}_3 = (T_{3,x} + T_3 U_3) T_3^{-1}, \quad \widetilde{V}_3 = (T_{3,x} + T_3 V_3) T_3^{-1}. \tag{4.8}
$$

Suppose the Darboux matrix T_3 is of the form

$$
T_3 = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\
A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\
A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\
0 & 0 & 0 & A_{11} & A_{12} & A_{13} \\
0 & 0 & 0 & A_{21} & A_{22} & A_{23} \\
0 & 0 & 0 & A_{31} & A_{32} & A_{33}
\end{bmatrix}, \tag{4.9}
$$

with

$$
A_{11} = A_{33} = A_{16} = A_{34} = 0, \quad A_{12} = -A_{21} = \sum_{i=0}^{N-1} A_{12}^{(i)} \lambda^i,
$$

$$
A_{13} = -A_{31} = \lambda^N + \sum_{i=0}^{N-1} A_{13}^{(i)} \lambda^i,
$$

$$
A_{16} = \lambda^N + \sum_{i=0}^{N-1} A_{16}^{(i)} \lambda^i, \quad A_{22} = \lambda^N + \sum_{i=0}^{N-1} A_{22}^{(i)} \lambda^i,
$$

$$
\begin{gathered}
A_{23}=-A_{32}=\lambda^N+\sum_{i=0}^{N-1}A_{23}^{(i)}\lambda^i,\\
A_{25}=\lambda^N+\sum_{i=0}^{N-1}A_{25}^{(i)}\lambda^i,\quad A_{35}=-A_{26}=\sum_{i=0}^{N-1}A_{26}^{(i)}\lambda^i,\\
A_{15}=-A_{24}=\sum_{i=0}^{N-1}A_{15}^{(i)}\lambda^i.
\end{gathered}
$$

We define

$$
\Phi=\begin{bmatrix}\varphi_1 & \psi_1\\ \varphi_2 & \psi_2\\ \varphi_3 & \psi_3\\ \varphi_4 & \psi_4\\ \varphi_5 & \psi_5\\ \varphi_6 & \psi_6\end{bmatrix}. \tag{4.10}
$$

By (4.9) and (4.10), we have

$$
\widetilde{\Phi}=T_3\Phi=\begin{bmatrix}A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16}\\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26}\\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36}\\ 0 & 0 & 0 & A_{11} & A_{12} & A_{13}\\ 0 & 0 & 0 & A_{21} & A_{22} & A_{23}\\ 0 & 0 & 0 & A_{31} & A_{32} & A_{33}\end{bmatrix}\begin{bmatrix}\varphi_1 & \psi_1\\ \varphi_2 & \psi_2\\ \varphi_3 & \psi_3\\ \varphi_4 & \psi_4\\ \varphi_5 & \psi_5\\ \varphi_6 & \psi_6\end{bmatrix}, \tag{4.11}
$$

which equals a matrix or two Column 1 is

$$
\begin{pmatrix}A_{11}\varphi_1+A_{12}\varphi_2+A_{13}\varphi_3+A_{14}\varphi_4+A_{15}\varphi_5+A_{16}\varphi_6\\ A_{21}\varphi_1+A_{22}\varphi_2+A_{23}\varphi_3+A_{24}\varphi_4+A_{25}\varphi_5+A_{26}\varphi_6\\ A_{31}\varphi_1+A_{32}\varphi_2+A_{33}\varphi_3+A_{34}\varphi_4+A_{35}\varphi_5+A_{36}\varphi_6\\ A_{11}\varphi_4+A_{12}\varphi_5+A_{13}\varphi_6\\ A_{21}\varphi_4+A_{22}\varphi_5+A_{23}\varphi_6\\ A_{31}\varphi_4+A_{32}\varphi_5+A_{33}\varphi_6\end{pmatrix}
$$

and the column 2 is

$$
\begin{pmatrix}A_{11}\psi_1+A_{12}\psi_2+A_{13}\psi_3+A_{14}\psi_4+A_{15}\varphi_5+A_{16}\varphi_6\\ A_{21}\psi_1+A_{22}\psi_2+A_{23}\psi_3+A_{24}\psi_4+A_{25}\varphi_5+A_{26}\varphi_6\\ A_{31}\psi_1+A_{32}\psi_2+A_{33}\psi_3+A_{34}\psi_4+A_{35}\varphi_5+A_{36}\varphi_6\\ A_{11}\psi_4+A_{12}\psi_5+A_{13}\varphi_6\\ A_{21}\psi_4+A_{22}\psi_5+A_{23}\psi_5\\ A_{31}\psi_4+A_{32}\psi_5+A_{33}\psi_5\end{pmatrix}.
$$

So there exists γ_j $(j=1,2)$ satisfying

$$
\gamma_j^{(1)}\widetilde{\varphi}+\gamma_j^{(2)}\widetilde{\psi}=0. \tag{4.12}
$$

Substituting (4.12) into (4.11), we obtain

$$
\begin{aligned}
&\gamma_j^{(1)}(A_{11}\varphi_1+A_{12}\varphi_2+A_{13}\varphi_3+A_{14}\varphi_4)\\
&+\gamma_j^{(2)}(A_{11}\psi_1+A_{12}\psi_2+A_{13}\psi_3+A_{14}\psi_4)=0,\\
&\gamma_j^{(1)}(A_{21}\varphi_1+A_{22}\varphi_2+A_{23}\varphi_3+A_{24}\varphi_4)\\
&+\gamma_j^{(2)}(A_{21}\psi_1+A_{22}\psi_2+A_{23}\psi_3+A_{24}\psi_4)=0,
\end{aligned}
$$

$$
\begin{gathered}
\gamma_j^{(1)}(A_{11}\varphi_3 + A_{12}\varphi_4) + \gamma_j^{(2)}(A_{11}\psi_3 + A_{12}\psi_4) = 0, \\
\gamma_j^{(1)}(A_{21}\varphi_3 + A_{22}\varphi_4) + \gamma_j^{(2)}(A_{23}\psi_3 + A_{24}\psi_4) = 0.
\end{gathered}
$$

Proposition 4.1. *The matrix $\widetilde{U}_3$ has the same form as U_3 determined by* (1.7)*, that is*

$$
U_3 = \left[\begin{array}{ccc|ccc}
0 & -\widetilde{p}+\widetilde{q} & \lambda+\widetilde{q}\widetilde{s}+\widetilde{p}\widetilde{r} & 0 & -\widetilde{r}+\widetilde{s} & \lambda \\
\widetilde{p}-\widetilde{q} & 0 & -\widetilde{p}-\widetilde{q} & \widetilde{r}-\widetilde{s} & 0 & -\widetilde{r}-\widetilde{s} \\
-\lambda-\widetilde{q}\widetilde{s}-\widetilde{p}\widetilde{r} & \widetilde{p}+\widetilde{q} & 0 & -\lambda & \widetilde{r}+\widetilde{s} & 0 \\
\hline
0 & 0 & 0 & 0 & -\widetilde{p}+\widetilde{q} & \lambda+\widetilde{q}\widetilde{s}+\widetilde{p}\widetilde{r} \\
0 & 0 & 0 & \widetilde{p}-\widetilde{q} & 0 & -\widetilde{p}-\widetilde{q} \\
0 & 0 & 0 & -\lambda-\widetilde{q}\widetilde{s}-\widetilde{p}\widetilde{r} & \widetilde{p}+\widetilde{q} & 0
\end{array}\right],
$$

in which the relations between old potentials and new ones are

$$
\begin{aligned}
\widetilde{p} &= -q + \frac{1}{2}A_{32}^{(N-1)} - \frac{1}{2}A_{12}^{(N-1)}, \\
\widetilde{q} &= p + \frac{1}{2}A_{32}^{(N-1)} + \frac{1}{2}A_{12}^{(N-1)}, \\
\widetilde{r} &= -s + \frac{1}{2}A_{35}^{(N-1)} - \frac{1}{2}A_{15}^{(N-1)}, \\
\widetilde{s} &= r + \frac{1}{2}A_{35}^{(N-1)} + \frac{1}{2}A_{15}^{(N-1)}.
\end{aligned} \tag{4.13}
$$

Proof. Setting

$$
(T_{3,t} + T_3V_3)T_3^* = \begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} & \Gamma_{16} \\
\Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & \Gamma_{25} & \Gamma_{26} \\
\Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{34} & \Gamma_{35} & \Gamma_{36} \\
0 & 0 & 0 & \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\
0 & 0 & 0 & \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\
0 & 0 & 0 & \Gamma_{31} & \Gamma_{32} & \Gamma_{33}
\end{bmatrix},
$$

where $T_3^{-1} = T_3^*/\det T_3$. Then, we deduce that

$$
T_{3,t} + T_3U_3 = X(\lambda)T_3, \tag{4.14}
$$

with

$$
X(\lambda) = \begin{bmatrix}
0 & X_{12}^{(0)} & X_{13}^{(1)}\lambda + X_{13}^{(0)} & 0 & X_{15}^{(0)} & X_{16}^{(1)}\lambda \\
-X_{12}^{(0)} & 0 & -X_{32}^{(0)} & -X_{15}^{(0)} & 0 & -X_{35}^{(1)}\lambda \\
-X_{13}^{(1)}\lambda - X_{13}^{(0)} & -X_{32}^{(0)} & 0 & -X_{16}^{(0)}\lambda & -X_{35}^{(0)} & 0 \\
0 & 0 & 0 & 0 & X_{12}^{(0)} & X_{13}^{(1)}\lambda + X_{13}^{(0)} \\
0 & 0 & 0 & -X_{12}^{(0)} & 0 & -X_{32}^{(0)} \\
0 & 0 & 0 & -X_{13}^{(1)}\lambda - X_{13}^{(0)} & -X_{32}^{(0)} & 0
\end{bmatrix}.
$$

Equating the coefficients of λ^{N+i} $(i = 0, 1)$ in the above expression, we obtain

$$
\begin{gathered}
X_{13}^{(1)} = 1, \quad X_{12}^{(0)} = p + q + A_{12}^{(N-1)} = -\widetilde{p} + \widetilde{q}, \\
X_{15}^{(0)} = r + s + A_{15}^{(N-1)} = -\widetilde{r} + \widetilde{s}, \quad X_{16}^{(1)} = 1, \\
X_{13}^{(0)} = qs + pr = \widetilde{q}\widetilde{s} + \widetilde{p}\widetilde{r}, \quad X_{32}^{(0)} = p - q + 2A_{32}^{(N-1)} = \widetilde{p} + \widetilde{q}, \\
X_{35}^{(0)} = r - s + A_{35}^{(N-1)} = \widetilde{r} + \widetilde{s}.
\end{gathered}
$$

The proof is complete. □

Proposition 4.2. *We have*

$$
(T_{3,t} + T_3V_3^{(1)})T_3^* = (detT_3)Y(\lambda), \tag{4.15}
$$

where

$$Y(\lambda)=\begin{bmatrix} 0 & Y_{12}^{(0)} & Y_{13}^{(1)}\lambda+Y_{13}^{(0)} & 0 & Y_{15}^{(0)} & Y_{16}^{(1)}\lambda \\ -Y_{12}^{(0)} & 0 & -Y_{32}^{(0)} & -Y_{15}^{(0)} & 0 & -Y_{35}^{(1)}\lambda \\ -Y_{13}^{(1)}\lambda-Y_{13}^{(0)} & -Y_{32}^{(0)} & 0 & -Y_{16}^{(0)}\lambda & -Y_{35}^{(0)} & 0 \\ 0 & 0 & 0 & 0 & Y_{12}^{(0)} & Y_{13}^{(1)}\lambda+Y_{13}^{(0)} \\ 0 & 0 & 0 & -Y_{12}^{(0)} & 0 & -Y_{32}^{(0)} \\ 0 & 0 & 0 & -Y_{13}^{(1)}\lambda-Y_{13}^{(0)} & -Y_{32}^{(0)} & 0 \end{bmatrix}.$$

Proof. Equation (4.15) can be rewritten as

$$T_{3,t}+T_3V_3^{(1)}=Y(\lambda)T_3, \tag{4.16}$$

by means of $T_3^{-1}=T_3^*/\det T_3$. According to (4.16), equating the coefficients of λ^{N+i} $(i=1,2)$, we obtain

$$\begin{aligned} &Y_{13}^{(1)}=\alpha,\quad Y_{12}^{(0)}=\alpha p+\alpha q+\alpha A_{12}^{(N-1)}=-\alpha\widetilde{p}+\alpha\widetilde{q},\\ &Y_{13}^{(0)}=qs+pr,\quad Y_{32}^{(0)}=p-q+2A_{32}^{(N-1)}=\widetilde{p}+\widetilde{q},\quad Y_{16}^{(1)}=\beta,\\ &Y_{15}^{(0)}=\alpha(s+r-q-p)+\beta(p+q)+\alpha A_{15}^{(N-1)}+(\beta-\alpha)A_{12}^{(N-1)}\\ &\quad\;\; =\alpha(-\widetilde{s}-\widetilde{q}-\widetilde{r}+\widetilde{p})+\beta(\widetilde{q}-\widetilde{p}),\\ &Y_{35}^{(0)}=\alpha(r-p-s+q)+\beta(p-q)+\alpha A_{35}^{(N-1)}+(\beta-\alpha)A_{32}^{(N-1)}\\ &\quad\;\; =\alpha(\widetilde{s}+\widetilde{r}-\widetilde{q}-\widetilde{p})+\beta(\widetilde{q}+\widetilde{p}). \end{aligned}$$

The proof is complete. □

To obtain the explicit solutions of (4.5), we choose the seed solution $p=q=0, r=s=1$. The spectral problems become

$$\Phi_x=U_3\Phi,\quad U_3=\begin{bmatrix} 0 & 0 & \lambda & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & -2 \\ -\lambda & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 0 \end{bmatrix}, \tag{4.17}$$

and

$$\Phi_t=V_3^{(1)}\Phi,\quad V_3^{(1)}=\begin{bmatrix} 0 & 0 & \alpha\lambda & 0 & 0 & \beta\lambda \\ 0 & 0 & 0 & 0 & 0 & -2\alpha \\ -\alpha\lambda & 0 & 0 & -\beta\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha\lambda \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha\lambda & 0 & 0 \end{bmatrix}. \tag{4.18}$$

In particular, when $N=1$,

$$T_3=\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ 0 & 0 & 0 & A_{11} & A_{12} & A_{13} \\ 0 & 0 & 0 & A_{21} & A_{22} & A_{23} \\ 0 & 0 & 0 & A_{31} & A_{32} & A_{33} \end{bmatrix},$$

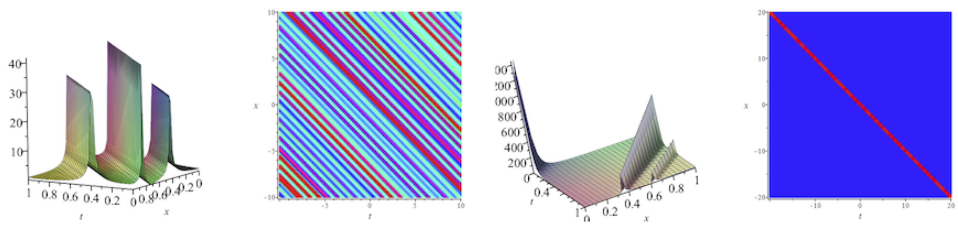
where

$$A_{11}=A_{33}=A_{16}=A_{34}=0,\quad A_{12}=-A_{21}=A_{12}^{(0)},\quad A_{13}=-A_{31}=\lambda+A_{31}^{(0)},$$

$$A_{15} = -A_{24} = A_{15}^{(0)}, \quad A_{16} = \lambda + A_{16}^{(0)}, \quad A_{22} = \lambda + A_{22}^{(0)},$$
$$A_{23} = -A_{32} = \lambda + A_{32}^{(0)}, \quad A_{25} = \lambda + A_{25}^{(0)}, \quad A_{35} = -A_{26} = A_{35}^{(0)}.$$

From the above results, we obtain the explicit solutions of (4.5) as follows

$$\begin{aligned} \widetilde{p} &= \frac{1}{2}A_{32}^{(0)} - \frac{1}{2}A_{12}^{(0)}, \\ \widetilde{q} &= \frac{1}{2}A_{32}^{(0)} + \frac{1}{2}A_{12}^{(0)}, \\ \widetilde{r} &= -1 + \frac{1}{2}A_{35}^{(0)} - \frac{1}{2}A_{15}^{(0)}, \\ \widetilde{s} &= 1 + \frac{1}{2}A_{35}^{(0)} + \frac{1}{2}A_{15}^{(0)}. \end{aligned} \tag{4.19}$$



(a) $p = q$ (b) $p = q$ (c) $r = s$ (d) $r = s$

FIGURE 5. Soliton solutions of (4.5) and the density plots with $\alpha = 1$, $\beta = 0$, $\lambda_1 = -0.6$, $\lambda_2 = -0.6$, $\lambda_3 = -0.2$, $\lambda_4 = 0.2$, $\lambda_5 = -0.3$, $\lambda_6 = -0.3$, $\gamma_1 = -0.8$, $\gamma_2 = 0.8$, $\gamma_3 = -0.3$, $\gamma_4 = 0.3$, $\gamma_5 = -0.2$, $\gamma_6 = 0.2$.

Remark 4.3. From Figure 5(a, b), it can be observed that the shapes and amplitudes of three single solitons hardly change during propagation, which means that these propagation process is elastic. Compared to Figure 5(a, b), Figure 5(c, d) shows that both the shapes and amplitudes of two single solitons change during propagation, thus the propagation is inelastic. In addition, the existence of elastic and inelastic in the same system is an interesting physical phenomenon.

4.3. **N-fold Darboux transformation of** (4.6) **and its explicit solutions.** Next we use the same Darboux matrix (4.9) to solve the (4.6).

Proposition 4.4. *Let*

$$(T_{3,t} + T_3 V_3^{(2)})T_3^* = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & \Theta_{15} & \Theta_{16} \\ \Theta_{21} & \Theta_{22} & \Theta_{23} & \Theta_{24} & \Theta_{25} & \Theta_{26} \\ \Theta_{31} & \Theta_{32} & \Theta_{33} & \Theta_{34} & \Theta_{35} & \Theta_{36} \\ 0 & 0 & 0 & \Theta_{11} & \Theta_{12} & \Theta_{13} \\ 0 & 0 & 0 & \Theta_{21} & \Theta_{22} & \Theta_{23} \\ 0 & 0 & 0 & \Theta_{31} & \Theta_{32} & \Theta_{33} \end{bmatrix}.$$

Then

$$(T_{3,t} + T_3 V_3^{(2)})T_3^* = (detT_3)Z(\lambda), \tag{4.20}$$

with

$$Z(\lambda)=\begin{bmatrix} 0 & Z_{12} & Z_{13} & 0 & Z_{15} & Z_{16}\\ Z_{21} & 0 & Z_{23} & Z_{24} & 0 & Z_{26}\\ Z_{31} & Z_{32} & 0 & Z_{34} & Z_{35} & 0\\ 0 & 0 & 0 & 0 & Z_{12} & Z_{13}\\ 0 & 0 & 0 & Z_{21} & 0 & Z_{23}\\ 0 & 0 & 0 & Z_{31} & Z_{32} & 0 \end{bmatrix}, \tag{4.21}$$

where

$$Z_{12}=Z_{12}^{(1)}+Z_{12}^{(0)},\quad Z_{13}=Z_{13}\lambda^{(2)}+Z_{13}^{(1)}\lambda+Z_{13}^{(0)},\quad Z_{21}=-Z_{12}^{(1)}\lambda-Z_{12}^{(0)},$$
$$Z_{23}=-Z_{32}^{(1)}\lambda-Z_{32}^{(0)},\quad Z_{31}=-Z_{13}^{(1)}\lambda^2-Z_{13}^{(1)}\lambda-Z_{13}^{(0)},\quad Z_{15}=Z_{15}^{(1)}\lambda+Z_{15}^{(0)},$$
$$Z_{16}=Z_{16}^{(1)}\lambda^2+Z_{16}^{(0)},\quad Z_{24}=Z_{15}^{(1)}\lambda-Z_{15}^{(0)},\quad Z_{26}=-Z_{35}^{(1)}\lambda-Z_{35}^{(0)},$$
$$Z_{35}=Z_{35}^{(1)}\lambda+Z_{35}^{(0)},\quad Z_{32}=Z_{32}^{(1)}\lambda+Z_{32}^{(0)}.$$

Proof. From (4.20), we obtain

$$T_{3,t}+T_3V_3^{(2)}=Z(\lambda)T_3. \tag{4.22}$$

Comparing the coefficients of λ in (4.22), we have

$$\begin{gathered}
Z_{13}^{(2)}=\alpha,\quad Z_{12}^{(1)}=\alpha p+\alpha q+\alpha A_{12}^{(N-1)}=-\alpha\widetilde{p}+\alpha\widetilde{q},\\
Z_{13}^{(1)}=qs+pr, Z_{32}^{(1)}=p-q+2A_{32}^{(N-1)}=\widetilde{p}+\widetilde{q},
\end{gathered}$$
$$\begin{aligned}
Z_{16}^{(2)}&=\beta, Z_{15}^{(1)}=\alpha(s+r-q-p)+\beta(p+q)+\alpha A_{15}^{(N-1)}+(\beta-\alpha)A_{12}^{(N-1)}\\
&=\alpha(-\widetilde{s}-\widetilde{q}-\widetilde{r}+\widetilde{p})+\beta(\widetilde{q}-\widetilde{p}),\\
Z_{35}^{(1)}&=\alpha(r-p-s+q)+\beta(p-q)+\alpha A_{35}^{(N-1)}+(\beta-\alpha)A_{32}^{(N-1)}\\
&=\alpha(\widetilde{s}+\widetilde{r}-\widetilde{q}-\widetilde{p})+\beta(\widetilde{q}+\widetilde{p}),\\
Z_{13}^{(0)}&=-\alpha[(-q+\frac{1}{2}A_{32}^{(N-1)}-\frac{1}{2}A_{12}^{(N-1)})^2+(p+\frac{1}{2}A_{32}^{(N-1)}+\frac{1}{2}A_{12}^{(N-1)})^2]\\
&=-\alpha(\widetilde{p}^2+\widetilde{q}^2),\\
Z_{12}^{(0)}&=-\alpha(-q_x+\frac{1}{2}A_{32,x}^{(N-1)}-\frac{1}{2}A_{21,x}^{(N-1)})-\alpha h(p+\frac{1}{2}A_{32}^{(N-1)}+\frac{1}{2}A_{12}^{(N-1)})\\
&\quad -\alpha(p_x-A_{32,x}^{(N-1)}+A_{12,x}^{(N-1)})+\alpha h(-q_x+\frac{1}{2}A_{32,x}^{(N-1)}+A_{21}^{(N-1)})\\
&=-\alpha\widetilde{p}_x-\alpha h\widetilde{q}-\alpha\widetilde{q}_x+\alpha h\widetilde{p}.
\end{aligned}$$

The proof is complete. □

Next we apply Darboux transformation (4.9) to give explicit solution of (4.6). We choose the seed solution $p=q=0, r=s=1$. Then the spectral problems become

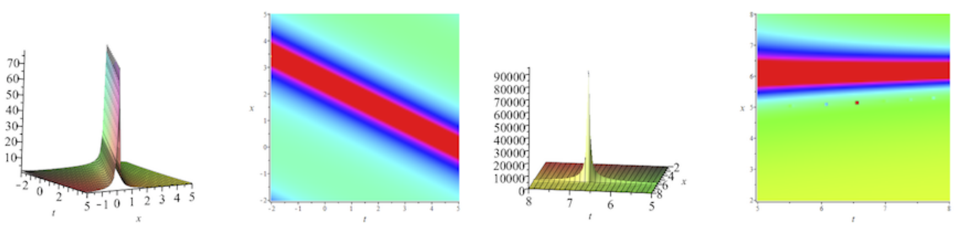
$$\Phi_x=U_3\Phi,\quad U_3=\begin{bmatrix} 0 & 0 & \lambda & 0 & 0 & \lambda\\ 0 & 0 & 0 & 0 & 0 & -2\\ -\lambda & 0 & 0 & -\lambda & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \lambda\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\lambda & 0 & 0 \end{bmatrix}, \tag{4.23}$$

and

$$\Phi_t = V_3^{(2)}\Phi, \quad V_3^{(2)} = \begin{bmatrix} 0 & 0 & \alpha\lambda^2 & 0 & 0 & \beta\lambda^2 \\ 0 & 0 & 0 & 0 & 0 & -2\alpha\lambda \\ -\alpha\lambda^2 & 0 & 0 & -\beta\lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha\lambda^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha\lambda^2 & 0 & 0 \end{bmatrix}. \tag{4.24}$$

The exact solution of (4.6) is

$$\begin{aligned} \widetilde{p} &= \frac{1}{2}A_{32}^{(0)} - \frac{1}{2}A_{12}^{(0)}, \\ \widetilde{q} &= \frac{1}{2}A_{32}^{(0)} + \frac{1}{2}A_{12}^{(0)}, \\ \widetilde{r} &= -1 + \frac{1}{2}A_{35}^{(0)} - \frac{1}{2}A_{15}^{(0)}, \\ \widetilde{s} &= 1 + \frac{1}{2}A_{35}^{(0)} + \frac{1}{2}A_{15}^{(0)}. \end{aligned} \tag{4.25}$$



(a) $p = q$ (b) $p = q$ (c) $r = s$ (d) $r = s$

FIGURE 6. Soliton solutions of (4.6) and the density plots with $\alpha = 1$, $\beta = 0$, $\lambda_1 = -0.2$, $\lambda_2 = 0.2$, $\lambda_3 = -0.3$, $\lambda_4 = 0.3$, $\lambda_5 = -0.3$, $\lambda_6 = 0.3$, $\gamma_1 = -0.2$, $\gamma_2 = 0.2$, $\gamma_3 = -0.5$, $\gamma_4 = 0.3$, $\gamma_5 = -0.2$, $\gamma_6 = 0.2$.

Remark 4.5. Based on single soliton solution (4.6) under the suitable parameters, Figure 6 describes the soliton pulse propagation in (x, t) plane. In Figure 6, we find that the components p q and r s are composed of bright solitons and the amplitude of r s are significantly higher than that of p q. It should be pointed out that the image shapes of components in the same system are similar but the amplitudes are different, which is an interesting phenomenon that may explain some physical significance and worthy of further investigation.

5. Conclusions

In this article, three integrable nonlinear perturbed hierarchies of NLS-mKdV equation associated with their corresponding spectral problem are proposed and their Lax integrability is proved. Although the three problems are integrable coupled in different ways and have different perturbation terms, they all have Lax integrability, which is the interesting part of this paper. The purpose of studying the three problems together is that they all belong to an integrable coupled system of NLS-mKdV equations, which are promising for classification and comparison.

We found three different Darboux matrices to construct their Darboux transformations and derive the relationship between old and new potentials. Explicit solutions of these equations are investigated by Darboux transformation. Three-dimensional plots and density profiles of these explicit solution behaviors are presented. These will help us understand the role of higher order nonlinearity or larger amplitude waves.

The propagation of pulsed soliton in the integrable nonlinear perturbed hierarchies of the NLS-mKdV equation is systematically analyzed, three-dimensional plots and density profiles of soliton solutions are obtained by balancing nonlinear term and dispersion term by adjusting the parameters. In the perturbed system, the adjustment of parameter is more complicated, we can find the desired image by adjusting the parameters to achieve a balance between nonlinear term and dispersion term. For unselected images, these images maybe irregular because the balance of nonlinear term and dispersion term is not achieved.

If we take $h = 0$, the perturbed system becomes unperturbed. In other words, unperturbed system is a special case of perturbed system, and the image of the perturbed system we show already includes the image of the unperturbed system. Therefore, there is no need to discuss unperturbed systems separately.

In fact, soliton images of unperturbed systems have been widely studied [28], but soliton images of perturbed systems have not been widely studied due to their complexity. It seems that perturbed systems are important to study than unperturbed systems.

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