Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 73, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO COUPLED POROUS MEDIUM SYSTEMS WITH BOUNDARY DEGENERACY

XUTONG ZHAO, MINGJUN ZHOU, QIAN ZHOU

Abstract. This article concerns the asymptotic behavior of solutions of one-dimensional porous medium systems with boundary degeneracy in bounded and unbounded intervals. It is shown that the degree of the boundary degeneracy and the exponent of the nonlinear diffusion determine asymptotic behaviors of solutions. For the problem in a bounded interval, if the degeneracy is not strong, the problem admits both nontrivial global and blowing-up solutions, while if the degeneracy is strong enough, any nontrivial solution to the problem must blow up in a finite time. For the problem in an unbounded interval, the Fujita type blowing-up theorems are established and the critical Fujita exponent is formulated by the degree of the boundary degeneracy and the exponent of nonlinear diffusion.

1. Introduction

The semilinear parabolic equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial u}{\partial x}\Big) = f(x,t,u), \quad x \in \mathbb{R},\ t > 0,\ (\lambda > 0) \tag{1.1}$$

is a typical parabolic equation degenerating on the boundary. It is clear that (1.1) becomes degenerate at $x = 0$, a portion of the lateral boundary. Equations with such degeneracy as (1.1) can be used to describe some models, such as the Budyko-Sellers climate model [23], the Black-Scholes model coming from the option pricing problem [3], and a simplified Crocco-type equation coming from the velocity field of a laminar flow on a flat plate [5]. In recent decades, semilinear equations with boundary degeneracy have received a lot of attentions from mathematicians. Among them, it was proved that for the control system

$$\begin{gathered}
\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial u}{\partial x}\Big) = h(x,t)\chi_\omega, \quad (x,t) \in (0,1)\times(0,T),\\
u(0,t) = 0, \quad t \in (0,T) \quad \text{if } 0 < \lambda < 1,\\
\Big(x^\lambda \frac{\partial u}{\partial x}\Big)(0,t) = 0, \quad t \in (0,T) \quad \text{if } \lambda \ge 1,\\
u(1,t) = 0, \quad t \in (0,T),\\
u(x,T) = u_0(x), \quad x \in (0,1),
\end{gathered}$$

2020 *Mathematics Subject Classification.* 35K59, 35B33, 35K65.
Key words and phrases. Porous medium equation; boundary degeneracy; asymptotic behavior.

Submitted April 4, 2022. Published October 28, 2022.

there is a threshold $\lambda = 2$ in the sense that the system is null controllable if $0 < \lambda < 2$ [1, 5, 6, 20], while not if $\lambda \geq 2$ [4], where $u_0 \in L^2((0,1))$, h is the control function, ω is a subinterval of $(0,1)$, and χ_ω is the characteristic function of ω. In addition, the null controllability of other control systems with boundary degeneracy were studied, see, e.g., [7, 10, 11, 28, 31, 32, 9, 26]. For another instance, the quenching phenomenon of solutions to the problem

$$\begin{gathered}
\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial u}{\partial x}\Big) = f(u), \quad (x,t) \in (0,a) \times (0,T), \\
\Big(x^\lambda \frac{\partial u}{\partial x}\Big)(0,t) = u(a,t) = 0, \quad t \in (0,T), \\
u(x,0) = 0, \quad x \in (0,a)
\end{gathered}$$

was studied in [37], where $a > 0$ and $f \in C^2([0,c))$ with a constant $c > 0$ satisfies

$$f(0) > 0, \quad f'(0) > 0, \quad f''(s) \geq 0 \text{ for } 0 < s < c, \quad \lim_{s \to c^-} f(s) = +\infty.$$

It was shown that $\lambda = 2$ is also a threshold in the sense that the critical length satisfies

$$a_* \begin{cases} > 0, & \text{if } 0 < \lambda < 2, \\ = 0, & \text{if } \lambda \geq 2. \end{cases}$$

That is to say, in the case that $0 < \lambda < 2$, there is a critical length $a_* > 0$ such that the solution exists globally in time if $a < a_*$, while quenches in a finite time if $a > a_*$. As to the case that $\lambda \geq 2$, the solution must quench in a finite time for each $a > 0$. In [27] and [33], the asymptotic behavior of solutions to the following problem in a bounded interval was studied

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial u^m}{\partial x}\Big) = u^p, \quad 0 < x < 1, \ t > 0, \tag{1.2}$$

$$\Big(x^\lambda \frac{\partial u^m}{\partial x}\Big)(0,t) = u(1,t) = 0, \quad t > 0, \tag{1.3}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1 \tag{1.4}$$

where $\lambda > 0$, $p > m \geq 1$ and $u_0 \in L^\infty((0,1))$ is a nonnegative function. For the problem (1.2)–(1.4), it was proved that there exist both nontrivial global and blowing-up solutions if the degeneracy is not strong such that $\lambda < 2$, while any nontrivial solution must blow up in a finite time if the degeneracy is so strong that $\lambda \geq 2$. Furthermore, the blowing-up theorems of Fujita type were also established in [27] and [33] for the following problem in an unbounded interval

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial u^m}{\partial x}\Big) = u^p, \quad x > 0, \ t > 0, \tag{1.5}$$

$$\Big(x^\lambda \frac{\partial u^m}{\partial x}\Big)(0,t) = 0, \quad t > 0, \tag{1.6}$$

$$u(x,0) = u_0(x), \quad x > 0, \tag{1.7}$$

where $\lambda > 0$, $p > m \geq 1$ and and $u_0 \in L^\infty((0,1))$ is a nonnegative function. It was proved that the critical Fujita exponent can be formulated as

$$p_c = \begin{cases} m + 2 - \lambda, & \text{if } 0 < \lambda < 2, \\ +\infty, & \text{if } \lambda \geq 2. \end{cases}$$

In 1966, Fujita [13] showed that for the Cauchy problem of

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \quad x \in \mathbb{R}^N,\ t > 0,$$

any nontrivial solution must blow up in a finite time if $1 < p < 1 + 2/N$, whereas there exist both nontrivial global and blowing-up solutions if $p > 1 + 2/N$. Then, the critical case $p = 1+2/N$, which is well known as the Fujita exponent, was proved to be the blowing-up case in [14, 17]. Such results revealed the relationship between the asymptotic behavior of the solutions to nonlinear partial differential equations and the exponents of nonlinear internal sources. Different extension directions, such as different types of parabolic equations and systems in various of geometries with or without degeneracies or singularities, have been obtained since then, see the survey papers [8, 18] and also the recent papers [2, 19, 22, 29, 30, 34, 35, 36, 38]. Among them, the Cauchy problem of the following coupled semilinear parabolic system was studied

$$\frac{\partial u}{\partial t} - \Delta u = t^{\mu_1}|x|^{\nu_1} v^p, \quad \frac{\partial v}{\partial t} - \Delta v = t^{\mu_2}|x|^{\nu_2} u^q, \quad x \in \mathbb{R}^N,\ t > 0,$$

where $\mu_1, \mu_2, \nu_1, \nu_2 \geq 0$, and $p, q \geq 1$. Escobedo and Herrero in [12] considered this Cauchy problem with $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, and they proved that the critical Fujita curve is

$$(pq)_c = 1 + \frac{2}{N} \max\{p+1,\ q+1\}.$$

In [25] and [21], it was proved that the critical Fujita curve is

$$(pq)_c = 1 + \frac{2}{N} \max\{(\mu_2+1)p + \mu_1 + 1,\ (\mu_1+1)q + \mu_2 + 1\}$$

if $\nu_1 = \nu_2 = 0$, while is

$$(pq)_c = 1 + \frac{1}{N} \max\{(\nu_2+2)p + \nu_1 + 2,\ (\nu_1+2)q + \nu_2 + 2\}$$

if $\mu_1 = \mu_2 = 0$. There are also some studies on the Cauchy problem of the coupled porous medium systems with fast diffusion

$$\frac{\partial u}{\partial t} - \Delta u^m = v^p, \quad \frac{\partial v}{\partial t} - \Delta v^n = u^q, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.8}$$

where $0 < m, n < 1$, $p, q \geq 1$ and $pq > 1$. Qi et al [24] proved that the critical Fujita curve of the Cauchy problem of (1.8) is

$$(pq)_c = mn + \frac{2}{N} \max\{p+n, q+m\},$$

and proved that any nontrivial solution to the Cauchy problem of (1.8) must blow up in a finite time if $pq < (pq)_c$, whereas there exist both nonnegative nontrivial global and blowing-up solutions if $pq > (pq)_c$ with $m = n$. They pointed out that the method of constructing the global supersolutions fails for the case that $m \neq n$ because of the different propagating rates for the two kinds of diffusions. Later in [15], it was shown that for the "very fast diffusions" case that $0 < m, n < (N-2)_+/N$, the Cauchy problem of (1.8) admits nontrivial global solutions if the initial data is small enough even though m is not equal to n.

In this article, we study the asymptotic behavior of the solutions to the following two coupled porous medium systems with the boundary degeneracy

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial u^m}{\partial x}\Big) + v^p, \quad 0 < x < 1,\ t > 0, \tag{1.9}$$
$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial v^m}{\partial x}\Big) + u^q, \quad 0 < x < 1,\ t > 0, \tag{1.10}$$
$$\Big(x^\lambda \frac{\partial u^m}{\partial x}\Big)(0,t) = \Big(x^\lambda \frac{\partial v^m}{\partial x}\Big)(0,t) = 0, \quad t > 0, \tag{1.11}$$
$$u(1,t) = v(1,t) = 0, \quad t > 0, \tag{1.12}$$
$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad 0 < x < 1, \tag{1.13}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial u^m}{\partial x}\Big) + v^p, \quad x > 0,\ t > 0, \tag{1.14}$$
$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial v^m}{\partial x}\Big) + u^q, \quad x > 0,\ t > 0, \tag{1.15}$$
$$\Big(x^\lambda \frac{\partial u^m}{\partial x}\Big)(0,t) = \Big(x^\lambda \frac{\partial v^m}{\partial x}\Big)(0,t) = 0, \quad t > 0, \tag{1.16}$$
$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x > 0, \tag{1.17}$$

where $0 < m < 1$, $\lambda > 0$ and $p, q > 1$. In [16], the semilinear case with $m = 1$ was considered. It was shown that for the problem (1.9)–(1.13) with $m = 1$, there exist both nontrivial global and blowing-up solutions if $\lambda < 2$, while any nontrivial solution must blow up in a finite time if $\lambda \geq 2$; for the problem (1.14)–(1.17) with $m = 1$, the critical Fujita curve is

$$(pq)_c = \begin{cases} 1 + (2-\lambda)\max\{p+1,\, q+1\}, & \text{if } 0 < \lambda < 2, \\ +\infty, & \text{if } \lambda \geq 2. \end{cases}$$

Compared with the semilinear case with $m = 1$, the quasilinear case with $0 < m < 1$ processes both the degeneracy and the singularity. Indeed, (1.9) and (1.14) are degenerate at $x = 0$ and are singular at points where $u = 0$, while (1.10) and (1.15) are degenerate at $x = 0$ and are singular at points where $v = 0$. For the problem (1.9)–(1.13) in a bounded interval, we prove that $\lambda = 2$ is a threshold in the sense that there exist both nontrivial global and blowing-up solutions to problem (1.9)–(1.13) if $\lambda < 2$, while any nontrivial solution to problem (1.9)–(1.13) must blow up in a finite time if $\lambda \geq 2$. For problem (1.14)–(1.17) in an unbounded interval, it is proved that $\lambda = 2$ is also a threshold in the sense that the critical Fujita exponent is finite if $\lambda < 2$, while infinite if $\lambda \geq 2$. More precisely, it is proved that the critical Fujita exponent is

$$(pq)_c = \begin{cases} m^2 + (2-\lambda)\max\{p+m,\, q+m\}, & \text{if } 0 < \lambda < 2, \\ +\infty, & \text{if } \lambda \geq 2. \end{cases}$$

That is to say, in the case $0 < \lambda < 2$, any nontrivial solution to the problem (1.14)–(1.17) must blow up in a finite time if $pq < (pq)_c$, while the problem (1.14)–(1.17) admits both nontrivial global and blowing-up solutions if $pq > (pq)_c$. Whereas in the case $\lambda \geq 2$, any nontrivial solution to the problem (1.14)–(1.17) must blow up in a finite time. The methods used in this paper are mainly inspired by [27, 33, 16, 24]. But the discussions and estimates are much more complicated than the semilinear case with $m = 1$ in [16]. To prove the blowing-up of the solution to the problem (1.9)–(1.13) with $\lambda > 0$ and (1.14)–(1.17) with $\lambda \geq 2$, we analyze the interactions between the nonlinear degenerate diffusions and the sources through estimating energy integrals constructed by choosing appropriate weight functions, and show the blowing-up of the energy integrals instead of constructing blowing-up subsolutions. As for the blowing-up of the solutions to the problem (1.14)–(1.17) with $0 < \lambda < 2$, we establish the ordinary differential inequalities satisfied by the two energy integrals, and prove the blowing-up of the energy integrals by using the theory of invariant region for ordinary differential systems. For the global existence of the solutions to the problem (1.14)–(1.17) with $0 < \lambda < 2$, we construct the self-similar global supersolutions for the case that $1 + m < \lambda < 2$, while use the nontrivial explicit solution to

$$\frac{\partial \omega}{\partial t} - \frac{\partial}{\partial x}\Big(x^\lambda \frac{\partial \omega^m}{\partial x}\Big) = 0, \quad x > 0,\ t > 0,$$
$$\Big(x^\lambda \frac{\partial \omega^m}{\partial x}\Big)(0,t) = 0, \quad t > 0$$

to construct global supersolutions for the case that $0 < \lambda \leq 1 + m$.

This article is organized as follows. Some preliminaries, including the definition of solutions, comparison principles and well-posedness, are stated in Section 2. The asymptotic behavior of solutions to the problem (1.9)–(1.13) and (1.14)–(1.17) are studied in Section 3 and Section 4, respectively.

2. Preliminaries

The subsolutions, supersolutions, and solutions to problems (1.9)–(1.13) and (1.14)–(1.17) are defined as follows.

Definition 2.1. Let $0 < T \leq +\infty$. A pair of nonnegative functions (u, v) in $L^\infty((0,1)\times(0,T))$ is called a subsolution (supersolution, solution) to problem (1.9)–(1.13) in $(0,T)$, if

(i) For any $0 < \tau < T$, $x^{\lambda/2}\frac{\partial u^m}{\partial x}$, $x^{\lambda/2}\frac{\partial v^m}{\partial x} \in L^2((0,1)\times(0,\tau))$.

(ii) For any $0 < \tau < T$ and any nonnegative functions $\varphi, \psi \in C^1([0,1]\times[0,\tau])$ vanishing at $t = \tau$ and $x = 1$, it hold

$$\int_0^\tau \int_0^1 \Big(-u(x,t)\frac{\partial \varphi}{\partial t}(x,t) + x^\lambda \frac{\partial u^m}{\partial x}(x,t)\frac{\partial \varphi}{\partial x}(x,t)\Big)\mathrm{d}x\mathrm{d}t$$
$$\leq (\geq, =) \int_0^\tau \int_0^1 v^p(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t,$$

and

$$\int_0^\tau \int_0^1 \Big(-v(x,t)\frac{\partial \psi}{\partial t}(x,t) + x^\lambda \frac{\partial v^m}{\partial x}(x,t)\frac{\partial \psi}{\partial x}(x,t)\Big)\mathrm{d}x\mathrm{d}t$$
$$\leq (\geq, =) \int_0^\tau \int_0^1 u^q(x,t)\psi(x,t)\mathrm{d}x\mathrm{d}t.$$

Definition 2.2. Let $0 < T \leq +\infty$. A pair of nonnegative functions (u, v) in $L^\infty((0,+\infty)\times(0,T))$ is called a subsolution (supersolution, solution) to problem (1.14)–(1.17) in $(0,T)$, if

(i) For any $0 < \tau < T$, $x^{\lambda/2}\frac{\partial u^m}{\partial x}$, $x^{\lambda/2}\frac{\partial v^m}{\partial x} \in L^2((0,+\infty)\times(0,\tau))$.

(ii) For any $0 < \tau < T$ and any nonnegative $\varphi, \psi \in C^1([0,+\infty)\times[0,\tau])$ vanishing at $t = \tau$ and for large x, it holds

$$\begin{aligned}
&\int_0^\tau \int_0^{+\infty} \Big(- u(x,t)\frac{\partial \varphi}{\partial t}(x,t) + x^\lambda \frac{\partial u^m}{\partial x}(x,t)\frac{\partial \varphi}{\partial x}(x,t)\Big)\mathrm{d}x\mathrm{d}t \\
&\le (\ge, =) \int_0^\tau \int_0^{+\infty} v^p(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\tau \int_0^{+\infty} \Big(- v(x,t)\frac{\partial \psi}{\partial t}(x,t) + x^\lambda \frac{\partial v^m}{\partial x}(x,t)\frac{\partial \psi}{\partial x}(x,t)\Big)\mathrm{d}x\mathrm{d}t \\
&\le (\ge,=) \int_0^\tau \int_0^{+\infty} u^q(x,t)\phi(x,t)\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

If (u, v) is a solution to (1.9)–(1.13) (or to (1.14)–(1.17)) in $(0,+\infty)$, then (u, v) is called a global solution in time. Otherwise, there exists $T > 0$ such that (u, v) is a solution in $(0, T)$ and

$$\begin{gathered}
\|u(\cdot,t)\|_{L^\infty((0,1))} + \|v(\cdot,t)\|_{L^\infty((0,1))} \to +\infty, \quad \text{as } t \to T^-, \\
(\text{or } \|u(\cdot,t)\|_{L^\infty((0,+\infty))} + \|v(\cdot,t)\|_{L^\infty((0,+\infty))} \to +\infty, \quad \text{as } t \to T^-),
\end{gathered}$$

and we say that (u, v) blows up in a finite time.

Similarly to [27] and [33], one can establish the well-posedness and the comparison principles for problems (1.9)–(1.13) and (1.14)–(1.17).

Proposition 2.3. (i) *For any $0 \le u_0, v_0 \in L^\infty((0,1))$, the problem* (1.9)–(1.13) *admits at least one solution locally in time.*

(ii) *Assume that $(\hat u, \hat v)$ and $(\check u, \check v)$ are a supersolution and a subsolution to the problem* (1.9)–(1.13) *in $(0,T)$, respectively. Then $(\check u, \check v) \le (\hat u, \hat v)$ in $(0,1)\times(0,T)$.*

Proposition 2.4. (i) *For any $0 \le u_0, v_0 \in L^\infty((0,+\infty)) \cap L^1((0,+\infty))$, the problem* (1.14)–(1.17) *admits at least one solution locally in time.*

(ii) *Assume that $(\hat u, \hat v)$ and $(\check u, \check v)$ are a supersolution and a subsolution to the problem* (1.14)–(1.17) *in $(0,T)$, respectively. Then $(\check u, \check v) \le (\hat u, \hat v)$ in $(0,+\infty)\times(0,T)$.*

3. Problem in a bounded domain

In this section, we investigate the blowing-up and global existence of the solution to the problem (1.9)–(1.13).

Theorem 3.1. *Assume that $\lambda \ge 2$, $0 < m < 1$ and $p, q > 1$. Then for any nontrivial $0 \le u_0, v_0 \in L^\infty((0,1))$, the solution to problem* (1.9)–(1.13) *must blow up in a finite time.*

Proof. Without loss of generality, we assume that $p \ge q$. For $0 < \delta < 1$, set

$$\xi_\delta(x) = \begin{cases} \frac{\lambda-1}{\delta}2^{\lambda-1-\delta}x^{-\delta} - \frac{\lambda-1-\delta}{\delta}2^{\lambda-1} - 1, & 0 < x < 1/2, \\ x^{1-\lambda} - 1, & 1/2 \le x \le 1. \end{cases}$$

It is not hard to check that $0 \le \xi_\delta(x) \in C^{1,1}((0,1])$ satisfies

$$(x^\lambda \xi_\delta'(x))' \ge -M_1\delta\xi_\delta(x), \quad 0 < x < 1; \tag{3.1}$$

$$\int_0^1 \xi_\delta(x)\mathrm{d}x \le M_2, \tag{3.2}$$

where $M_1, M_2 > 0$ is a constant depending only on λ but independent of δ.

Assume that (u, v) is a solution to (1.9)–(1.13) in $(0, +\infty)$, and denote

$$w_\delta(t) = \int_0^1 (u(x,t) + v(x,t))\xi_\delta(x)\mathrm{d}x, \quad t \ge 0.$$

By a similar argument as in [33, Theorem 3.1] and Hölder inequality, one can obtain

$$\begin{aligned}
w_\delta'(t) &= \int_0^1 (u^m(x,t) + v^m(x,t))(x^\lambda \xi_\delta'(x))'\mathrm{d}x \\
&\quad + \int_0^1 v^p(x,t)\xi_\delta(x)\mathrm{d}x + \int_0^1 u^q(x,t)\xi_\delta(x)\mathrm{d}x \\
&\ge -M_1\delta \int_0^1 (u^m(x,t) + v^m(x,t))\xi_\delta(x)\mathrm{d}x \\
&\quad + \Big(\int_0^1 \xi_\delta(x)\mathrm{d}x\Big)^{1-q}\Big(\int_0^1 u(x,t)\xi_\delta(x)\mathrm{d}x\Big)^q \\
&\quad + \Big(\int_0^1 \xi_\delta(x)\mathrm{d}x\Big)^{1-p}\Big(\int_0^1 v(x,t)\xi_\delta(x)\mathrm{d}x\Big)^p \\
&\ge -M_1\delta\Big(\int_0^1 \xi_\delta(x)\mathrm{d}x\Big)^{1-m} \\
&\qquad \times \Big[\Big(\int_0^1 u(x,t)\xi_\delta(x)\mathrm{d}x\Big)^m + \Big(\int_0^1 v(x,t)\xi_\delta(x)\mathrm{d}x\Big)^m\Big] \\
&\quad + \min\{M_2^{1-p},\, M_2^{1-q}\} \\
&\qquad \times \Big[\Big(\int_0^1 u(x,t)\xi_\delta(x)\mathrm{d}x\Big)^q + \Big(\int_0^1 v(x,t)\xi_\delta(x)\mathrm{d}x\Big)^p\Big] \\
&\ge \begin{cases} -2M_1M_2^{1-m}\delta w_\delta^m(t) + 2^{-2p}\min\{M_2^{1-p},\, M_2^{1-q}\}w_\delta^p(t), \\ \quad \text{if } w_\delta(t) \le 2,\ t > 0, \\ -2M_1M_2^{1-m}\delta w_\delta^m(t) + 2^{-p}\min\{M_2^{1-p},\, M_2^{1-q}\}w_\delta^q(t), \\ \quad \text{if } w_\delta(t) > 2\ t > 0, \end{cases}
\end{aligned} \tag{3.3}$$

Owing to $\inf_{\delta\in(0,1)} w_\delta(0) > 0$, there exists a sufficient small constant $\delta_0 \in (0,1)$ such that

$$2M_1M_2^{1-m}\delta_0 \le 2^{-2p-1}\min\{M_2^{1-p},\, M_2^{1-q}\}w_{\delta_0}^{p-m}(0), \tag{3.4}$$

$$2M_1M_2^{1-m}\delta_0 \le 2^{-p-1}\min\{M_2^{1-p},\, M_2^{1-q}\}w_{\delta_0}^{q-m}(0). \tag{3.5}$$

If $w_{\delta_0}(0) < 2$, we claim that there exists $\tilde{T} > 0$ such that $w_{\delta_0}(\tilde{T}) \ge 2$. Otherwise,

$$w_{\delta_0}(t) < 2, \quad t \in [0, +\infty). \tag{3.6}$$

It follows from (3.3), (3.4) and (3.6) that

$$w_{\delta_0}'(t) \ge 2^{-2p-1}\min\{M_2^{1-p},\, M_2^{1-q}\}w_\delta^p(t), \quad t > 0.$$

Since $p > 1$, there exists $\hat{T} > 0$ such that $\lim_{t\to\hat{T}^-} w_{\delta_0}(t) = +\infty$, which contradicts (3.6). Therefore, one can assume that $w_{\delta_0}(0) \ge 2$. Then from (3.3) and (3.5) one obtains

$$w'_{\delta_0}(t) \geq 2^{-p-1} \min\{M_2^{1-p},\ M_2^{1-q}\} w_{\delta_0}^q(t), \quad t > 0.$$

Since $q > 1$, there exists some $T > 0$ such that $\lim_{t\to T^-} w_{\delta_0}(t) = +\infty$, which leads to

$$\|u(\cdot,t)\|_{L^\infty((0,1))} + \|v(\cdot,t)\|_{L^\infty((0,1))} \to +\infty, \quad \text{as } t \to T^-.$$

That is to say, (u, v) must blow up in a finite time. $\square$

Theorem 3.2. *Assume that $0 < \lambda < 2$, $0 < m < 1$, $p, q > 1$. Then the solution to problem (1.9)–(1.13) exists globally in time if $(u_0, v_0) \in L^\infty((0,1))$ is suitably small, while blows up in a finite time if $(u_0, v_0) \in L^\infty((0,1))$ is large enough, where $0 \leq u_0,\ v_0 \in L^\infty((0,1))$.*

Proof. First we consider the global existence case. To show the existence of a global solution to (1.9)–(1.13), we try to find a nonnegative nontrivial self-similar supersolutions to the problem (1.9)–(1.13) of the form

$$\overline{u}(x,t) = (t+\tau)^{-\alpha_1} U((t+\tau)^{-\beta} x), \quad 0 \leq x \leq 1,\ t \geq 0, \tag{3.7}$$

$$\overline{v}(x,t) = (t+\tau)^{-\alpha_2} V((t+\tau)^{-\beta} x), \quad 0 \leq x \leq 1,\ t \geq 0, \tag{3.8}$$

where

$$\alpha_1 = \frac{p+1}{pq-1}, \quad \alpha_2 = \frac{q+1}{pq-1}, \quad \beta = \frac{1}{2-\lambda},$$

and $\tau \geq 1$ will be determined below. If $0 \leq U,\ V \in C^{0,1}((0,\tau^{-\beta}))$ with $U^m,\ V^m \in C^{1,1}((0,\tau^{-\beta}))$ satisfies

$$(t+\tau)^{(1-m)\alpha_1+1}(r^\lambda (U^m(r))')' + \alpha_1 U(r) + \beta r U'(r) + V^p(r) \leq 0, \quad 0 < r < \tau^{-\beta},$$

$$(t+\tau)^{(1-m)\alpha_2+1}(r^\lambda (V^m(r))')' + \alpha_2 V(r) + \beta r V'(r) + U^q(r) \leq 0, \quad 0 < r < \tau^{-\beta},$$

then $(\overline{u}, \overline{v})$ given by (3.7) and (3.8) is a supersolution to (1.9) and (1.10). Set

$$U(r) = V(r) = \frac{1}{(2-\lambda)^{1/m}} \Big(\frac{1}{\tau} - r^{2-\lambda}\Big)^{1/m}, \quad 0 \leq r \leq \tau^{-\beta}.$$

For $0 < r < \tau^{-\beta}$, a direct calculation shows that

$$\begin{aligned}
&(t+\tau)^{(1-m)\alpha_1+1}(r^\lambda (U^m(r))')' + \alpha_1 U(r) + \beta r U'(r) + V^p(r) \\
&= -(t+\tau)^{(1-m)\alpha_1+1} + \alpha_1 U(r) - \frac{\beta}{m} r^{1-m} U^{1-m}(r) + U^p(r) \\
&\leq -1 + \alpha_1 U(r) + U^p(r) \\
&\leq -1 + \frac{1}{\tau^{1/m}} \Big(\frac{\alpha_1}{(2-\lambda)^{1/m}} + \frac{1}{(2-\lambda)^{p/m}}\Big),
\end{aligned}$$

and

$$\begin{aligned}
&(t+\tau)^{(1-m)\alpha_2+1}(r^\lambda (V^m(r))')' + \alpha_2 V(r) + \beta r V'(r) + U^q(r) \\
&= -(t+\tau)^{(1-m)\alpha_2+1} + \alpha_2 V(r) - \frac{\beta}{m} r^{1-m} V^{1-m}(r) + V^q(r) \\
&\leq -1 + \alpha_2 V(r) + V^q(r) \\
&\leq -1 + \frac{1}{\tau^{1/m}} \Big(\frac{\alpha_2}{(2-\lambda)^{1/m}} + \frac{1}{(2-\lambda)^{q/m}}\Big).
\end{aligned}$$

Hence $(\overline{u},\overline{v})$ is a supersolution to (1.9) and (1.10) for each

$$\tau \ge \tau_0 = 2^m\Big(\frac{\alpha_1^m+\alpha_2^m}{2-\lambda}+\frac{1}{(2-\lambda)^p}\Big)+1.$$

It is noted that

$$\lim_{r\to 0^+} r^\lambda (U^m(r))' = \lim_{r\to 0^+} r^\lambda (V^m(r))' = 0.$$

Therefore, $(\overline{u},\overline{v})$ is a supersolution to the problem (1.9)–(1.13) if

$$u_0(x)\le \overline{u}(x,0),\quad v_0(x)\le \overline{v}(x,0),\quad 0<x<1 \tag{3.9}$$

for some $\tau\ge\tau_0$. Thanks to Proposition 2.3 (ii), one obtains that the solution to (1.9)–(1.13) exists globally in time if (u_0,v_0) satisfies (3.9).

Now we turn to the blowing-up case. Without loss of generality, it is still assumed that $p\ge q$. Set

$$\eta(x)=\begin{cases}2, & 0\le x\le 1/2,\\ 1+\cos(2x-1)\pi, & 1/2<x\le 1.\end{cases}$$

It is easy to verify that $\eta\in C^{1,1}([0,1])$ satisfies

$$(x^\lambda\eta(x))'\ge -4\pi^2\eta(x),\quad 1/2<x<1. \tag{3.10}$$

Assume that (u,v) is a solution to (1.9)–(1.13) in $(0,+\infty)$, and denote

$$w(t)=\int_0^1 (u(x,t)+v(x,t))\eta(x)\mathrm{d}x,\quad t\ge 0.$$

It follows from Definition 2.1, Hölder's inequality and (3.10) that

$$\begin{aligned}
w'(t) &= \int_0^1 (u^m(x,t)+v^m(x,t))(x^\lambda\eta'(x))'\mathrm{d}x\\
&\quad +\int_0^1 v^p(x,t)\eta(x)\mathrm{d}x+\int_0^1 u^q(x,t)\eta(x)\mathrm{d}x\\
&\ge -4\pi^2\int_0^1 (u^m(x,t)+v^m(x,t))\eta(x)\mathrm{d}x\\
&\quad +\Big(\int_0^1\eta(x)\mathrm{d}x\Big)^{1-q}\Big(\int_0^1 u(x,t)\eta(x)\mathrm{d}x\Big)^q\\
&\quad +\Big(\int_0^1\eta(x)\mathrm{d}x\Big)^{1-p}\Big(\int_0^1 v(x,t)\eta(x)\mathrm{d}x\Big)^p\\
&\ge -4\pi^2\Big(\int_0^1\eta(x)\mathrm{d}x\Big)^{1-m}\\
&\quad\times\Big[\Big(\int_0^1 u(x,t)\eta(x)\mathrm{d}x\Big)^m+\Big(\int_0^1 v(x,t)\eta(x)\mathrm{d}x\Big)^m\Big]\\
&\quad +2^{1-p}\Big[\Big(\int_0^1 u(x,t)\eta(x)\mathrm{d}x\Big)^q+\Big(\int_0^1 v(x,t)\eta(x)\mathrm{d}x\Big)^p\Big]\\
&\ge\begin{cases}-16\pi^2w^m(t)+2^{1-3p}w^p(t), & \text{if } w(t)\le 2,\ t>0\\ -16\pi^2w^m(t)+2^{1-2p}w^q(t), & \text{if } w(t)>2,\ t>0\end{cases}
\end{aligned}\tag{3.11}$$

If (u_0,v_0) is sufficiently large such that

$$w(0)>2,\quad w^{q-m}(0)\ge 2^{4+2p}\pi^2, \tag{3.12}$$

from (3.11) and (3.12) one obtains

$$w'(t) \geq 2^{-2p} w^q(t), \quad t > 0.$$

Since $q > 1$, there exists $T > 0$ such that $\lim_{t\to T^-} w(t) = +\infty$, which leads to

$$\|u(\cdot,t)\|_{L^\infty((0,1))} + \|v(\cdot,t)\|_{L^\infty((0,1))} \to +\infty, \quad \text{as } t \to T^-.$$

That is, if (u_0, v_0) satisfies (3.12), then (u, v) must blow up in a finite time. $\square$

4. Problem in unbounded domains

In this section, we formulate the critical Fujita exponent for problem (1.14)–(1.17) and establish a Fujita-type blowing-up theorem.

Theorem 4.1. *Assume that* $0 < \lambda < 2$, $0 < m < 1$, $p, q > 1$ *and*

$$pq < (pq)_c = m^2 + (2-\lambda)\max\{p+m,\, q+m\}.$$

Then for any nontrivial $0 \leq u_0, v_0 \in L^\infty((0,+\infty)) \cap L^1((0,+\infty))$*, the solution to* (1.14)–(1.17) *must blow up in a finite time.*

Proof. Without loss of generality, we assume that $p \geq q$. For $R > 0$, we set

$$\zeta_R(x) = \begin{cases} 1, & x \in [0,R], \\ \frac{1}{2}\Big(1 + \cos\frac{(x-R)\pi}{R}\Big), & x \in (R, 2R), \\ 0, & x \in [2R, +\infty). \end{cases} \tag{4.1}$$

It follows from the proof of [27, Lemma 2.1] that $\eta_R \in C^{1,1}([0,+\infty))$ satisfies

$$(x^\lambda \zeta_R(x))' \geq -2^\lambda \pi^2 R^{\lambda-2} \zeta_R(x), \quad x > 0. \tag{4.2}$$

Assume that (u, v) is a solution to problem (1.14)–(1.17) in $(0,+\infty)$, and denote

$$F_R(t) = \frac{1}{2R}\int_0^{+\infty} u(x,t)\zeta_R(x)\mathrm{d}x, \quad G_R(t) = \frac{1}{2R}\int_0^{+\infty} v(x,t)\zeta_R(x)\mathrm{d}x, \quad t \geq 0.$$

From Definition 2.2, Hölder's inequality, and (4.2) one gets

$$\begin{aligned} F_R'(t) &= \frac{1}{2R}\int_0^{+\infty} u^m(x,t)(x^\lambda \zeta_R'(x))'\mathrm{d}x + \frac{1}{2R}\int_0^{+\infty} v^p(x,t)\zeta_R(x)\mathrm{d}x \\ &\geq -2^{\lambda-1}\pi^2 R^{\lambda-3}\int_0^{+\infty} u^m(x,t)\zeta_R(x)\mathrm{d}x \\ &\quad + \frac{1}{2R}\Big(\int_0^{+\infty}\zeta_R(x)\mathrm{d}x\Big)^{1-p}\Big(\int_0^{+\infty} v(x,t)\zeta_R(x)\mathrm{d}x\Big)^p \\ &\geq -2^{\lambda-1}\pi^2 R^{\lambda-3}\Big(\int_0^{+\infty}\zeta_R(x)\mathrm{d}x\Big)^{1-m}\Big(\int_0^{+\infty} u(x,t)\zeta_R(x)\mathrm{d}x\Big)^m + G_R^p(t) \\ &\geq -2^\lambda \pi^2 R^{\lambda-2} F_R^m(t) + G_R^p(t), \quad t > 0. \end{aligned} \tag{4.3}$$

Similarly,

$$G_R'(t) \geq -2^\lambda \pi^2 R^{\lambda-2} G_R^m(t) + F_R^q(t), \quad t > 0. \tag{4.4}$$

If $pq < (pq)_c$ then either $(\lambda-2)(p+m)/(pq-m^2) < -1$ or $(\lambda-2)(q+m)/(pq-m^2) < -1$, which, together with the non-negativity and non-triviality of (u_0, v_0), yields either

$$\lim_{R\to+\infty}\frac{F_R(0)}{R^{(\lambda-2)(p+m)/(pq-m^2)}} = \lim_{R\to+\infty}\frac{\frac{1}{2R}\int_0^{+\infty} u_0(x)\zeta_R(x)\mathrm{d}x}{R^{(\lambda-2)(p+m)/(pq-m^2)}} = +\infty$$

or

$$\lim_{R\to+\infty}\frac{G_R(0)}{R^{(\lambda-2)(q+m)/(pq-m^2)}}=\lim_{R\to+\infty}\frac{\frac{1}{2R}\int_0^{+\infty}v_0(x)\zeta_R(x)\mathrm{d}x}{R^{(\lambda-2)(q+m)/(pq-m^2)}}=+\infty.$$

It is known from (4.3), (4.4) and Corollary 2 in [24] that $(F_R(t),G_R(t))$ blows up in a finite time for sufficient large $R>0$. Therefore, (u,v) must blow up in a finite time. $\square$

Theorem 4.2. *Assume that* $0<\lambda<2$, $0<m<1$, $p,q>1$ *and*

$$pq>(pq)_c=m^2+(2-\lambda)\max\{p+m,\ q+m\}.$$

Then the solution to (1.14)–(1.17) *exists globally in time if* (u_0,v_0) *is suitably small, while blows up in a finite time if* (u_0,v_0) *is large enough, where* $0\le u_0,\ v_0\in L^\infty((0,+\infty))\cap L^1((0,+\infty))$.

Proof. Thanks to Theorem 3.2 and Proposition 2.4, the solution to (1.14)–(1.17) blows up in a finite time if (u_0,v_0) is large enough. Below we prove that there exists a nonnegative nontrivial global solution to (1.14)–(1.17) if (u_0,v_0) is suitably small by constructing proper nonnegative nontrivial global supersolution.

Without loss of generality, we still assume that $p\ge q$ in the remaining part of proof. Then

$$p^2\ge pq>(pq)_c=m^2+(2-\lambda)(p+m),$$

which implies that $p>m+2-\lambda$. The discussion will be divided into three cases.

Case 1. $1+m<\lambda<2$. Set

$$\widehat{u}(x,t)=(t+1)^{-\gamma_1}\Phi((t+1)^{-\beta}x),\quad x\ge0,\ t\ge0, \tag{4.5}$$

$$\widehat{v}(x,t)=(t+1)^{-\gamma_2}\Psi((t+1)^{-\beta}x),\quad x\ge0,\ t\ge0, \tag{4.6}$$

where

$$\gamma_1=\frac{p+m}{pq-m^2},\quad \gamma_2=\frac{q+m}{pq-m^2},\quad \beta=\frac{1}{2-\lambda}.$$

If $0\le\Phi,\ \Psi\in C^{0,1}((0,+\infty))$ with $\Phi^m,\ \Psi^m\in C^{1,1}((0,+\infty))$ satisfies

$$(t+1)^{(1-m)\gamma_1}(r^\lambda(\Phi^m(r))')'+\gamma_1\Phi(r)+\beta r\Phi'(r)+(t+1)^{(1-m)\gamma_1}\Psi^p(r)\le0,$$
$$(t+1)^{(1-m)\gamma_2}(r^\lambda(\Psi^m(r))')'+\gamma_2\Psi(r)+\beta r\Psi'(r)+(t+1)^{(1-m)\gamma_2}\Phi^q(r)\le0,$$

for $r>0$, then $(\widehat{u},\widehat{v})$ given by (4.5) and (4.6) is a supersolution to (1.14) and (1.15). We take

$$\Phi(r)=\Psi(r)=(1+Ar^{2-\lambda})^{1/(m-1)},\quad r\ge0, \tag{4.7}$$

where

$$A=\frac{(1-m)^2(\gamma_1+\gamma_2+1)}{m(2-\lambda)(\lambda-1-m)}. \tag{4.8}$$

For $r>0$, by direct calculations we have

$$\begin{aligned}
&(t+1)^{(1-m)\gamma_1}(r^\lambda(\Phi^m(r))')'+\gamma_1\Phi(r)+\beta r\Phi'(r)+(t+1)^{(1-m)\gamma_1}\Psi^p(r)\\
&=\Phi(r)\Big[-\frac{m(2-\lambda)A}{1-m}\Big(1-\frac{(2-\lambda)A}{1-m}r^{2-\lambda}\Phi^{1-m}(r)\Big)(t+1)^{(1-m)\gamma_1}\\
&\quad+\gamma_1-\frac{A}{1-m}r^{2-\lambda}\Phi^{1-m}(r)+(t+1)^{(1-m)\gamma_1}\Phi^{p-1}(r)\Big]\\
&=\Phi(r)\Big[-\frac{m(2-\lambda)A}{1-m}\Big(1-\frac{2-\lambda}{1-m}\Big)(t+1)^{(1-m)\gamma_1}+\gamma_1+(t+1)^{(1-m)\gamma_1}\Phi^{p-1}(r)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le (t+1)^{(1-m)\gamma_1}\varPhi(r)\Big[-\frac{m(2-\lambda)(\lambda-1-m)A}{(1-m)^2}+\gamma_1+1\Big]\\
&= -\gamma_2(t+1)^{(1-m)\gamma_1}\varPhi(r)\le 0,
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&(t+1)^{(1-m)\gamma_2}(r^\lambda(\varPsi^m(r))')'+\gamma_2\varPsi(r)+\beta r\varPsi'(r)+(t+1)^{(1-m)\gamma_2}\varPhi^q(r)\\
&\le -\gamma_1(t+1)^{(1-m)\gamma_2}\varPsi(r)\le 0.
\end{aligned}
$$

Hence $(\widehat{u},\widehat{v})$ given by (4.5)–(4.8) is a supersolution to (1.14) and (1.15). Additionally,

$$
\lim_{r\to 0^+} r^\lambda(\varPhi^m(r))' = \lim_{r\to 0^+} r^\lambda(\varPsi^m(r))' = 0.
$$

Therefore, $(\widehat{u},\widehat{v})$ is a supersolution to the problem (1.14)–(1.17) if

$$
u_0(x)\le \widehat{u}(x,0),\quad v_0(x)\le \widehat{v}(x,0),\quad x>0. \tag{4.9}
$$

Case 2. $\lambda = 1+m$. It is easy to check that

$$
\omega_{1+m}(x,t)=\mathrm{e}^{-t}(1+\mathrm{e}^{-(1-m)t}x^{1-m})^{1/(m-1)},\quad x\ge 0,\, t\ge 0 \tag{4.10}
$$

solves

$$
\frac{\partial\omega}{\partial t}-\frac{\partial}{\partial x}\Big(x^\lambda\frac{\partial\omega^m}{\partial x}\Big)=0,\quad x>0,\ t>0, \tag{4.11}
$$

$$
\Big(x^\lambda\frac{\partial\omega^m}{\partial x}\Big)(0,t)=0,\quad t>0 \tag{4.12}
$$

with $\lambda = 1+m$. Set

$$
\widehat{u}(x,t)=\widehat{v}(x,t)=\varTheta_0^{1/m}(t)\omega_{1+m}\Big(x,\int_0^t\varTheta_0^{(m-1)/m}(s)\mathrm{d}s\Big),\quad x\ge 0,\, t\ge 0. \tag{4.13}
$$

If $0\le\varTheta_0\in C^1([0,+\infty))$ satisfies

$$
\mathrm{e}^{-t}\varTheta_0(t)\in L^\infty([0,+\infty)),\quad \varTheta_0'(t)\ge 0 \text{ for } t>0, \tag{4.14}
$$

and

$$
\varTheta_0'(t)\ge\varTheta_0^{1+(p-1)/m}(t)\exp\Big\{-(p-1)\int_0^t\varTheta_0^{(m-1)/m}(s)\mathrm{d}s\Big\},\quad t>0, \tag{4.15}
$$

$$
\varTheta_0'(t)\ge\varTheta_0^{1+(q-1)/m}(t)\exp\Big\{-(q-1)\int_0^t\varTheta_0^{(m-1)/m}(s)\mathrm{d}s\Big\},\quad t>0, \tag{4.16}
$$

then $(\widehat{u},\widehat{v})$ given by (4.10) and (4.13) is a supersolution to (1.14) and (1.15). We suppose that

$$
0<\frac{\theta}{2}\le\varTheta_0(t)\le\theta<1 \tag{4.17}
$$

holds for some constant $\theta\in(0,1)$ to be determined. Then (4.15) and (4.16) hold provided that

$$
\varTheta_0'(t)\ge\theta^{1+(q-1)/m}\mathrm{e}^{-(q-1)t},\quad t>0. \tag{4.18}
$$

We take

$$
\varTheta_0(t)=\frac{\theta}{2}+\frac{\theta^{1+(q-1)/m}}{q-1}(1-\mathrm{e}^{-(q-1)t}),\quad t\ge 0, \tag{4.19}
$$

where

$$
\theta=\min\Big\{\frac12,\Big(\frac{q-1}{2}\Big)^{m/(q-1)}\Big\}. \tag{4.20}
$$

Then (4.14), (4.17) and (4.18) hold. Hence $(\widehat{u},\widehat{v})$ given by (4.10), (4.13), (4.19) and (4.20) is a supersolution to (1.14)–(1.17) if

$$u_0(x)\le \widehat{u}(x,0),\quad v_0(x)\le \widehat{v}(x,0),\quad x>0. \tag{4.21}$$

Case 3. $0<\lambda<1+m$. One can check that

$$\omega_\lambda(x,t)=(t+1)^{-\mu}\Big(1+\frac{(1-m)\mu}{(2-\lambda)m}(t+1)^{-(2-\lambda)\mu}x^{2-\lambda}\Big)^{1/(m-1)}, \tag{4.22}$$

for $x\ge 0$ and $t\ge 0$, solves (4.11) and (4.12) with $0<\lambda<1+m$, where $\mu=1/(1+m-\lambda)$. Set

$$\widehat{u}(x,t)=\Theta_1^{1/m}(t)\omega_\lambda\Big(x,\int_0^t\Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big),\quad x\ge 0,\ t\ge 0, \tag{4.23}$$

$$\widehat{v}(x,t)=\Theta_2^{1/m}(t)\omega_\lambda\Big(x,\int_0^t\Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big),\quad x\ge 0,\ t\ge 0. \tag{4.24}$$

If $0\le\Theta_1,\,\Theta_2\in C^1([0,+\infty))$ satisfy

$$\begin{gathered}(t+1)^{-\mu}\Theta_1(t),\quad (t+1)^{-\mu}\Theta_2(t)\in L^\infty([0,+\infty)),\\ \Theta_1'(t),\quad \Theta_2'(t)\ge 0\quad\text{for } t>0,\end{gathered} \tag{4.25}$$

and

$$\Theta_1'(t)\ge m\Theta_1^{(m-1)/m}(t)\Theta_2^{p/m}(t)\frac{\omega_\lambda^p\Big(x,\int_0^t\Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)}{\omega_\lambda\Big(x,\int_0^t\Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)},\quad x>0,\ t>0,$$

$$\Theta_2'(t)\ge m\Theta_2^{(m-1)/m}(t)\Theta_1^{q/m}(t)\frac{\omega_\lambda^q\Big(x,\int_0^t\Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)}{\omega_\lambda\Big(x,\int_0^t\Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)},\quad x>0,\ t>0,$$

then $(\widehat{u},\widehat{v})$ given by (4.22)–(4.24) is a supersolution to (1.14) and (1.15).

For $\Theta_1(t)\le\Theta_2(t)$, a directly calculatin shows that

$$\begin{aligned}
&\frac{\omega_\lambda^p\Big(x,\int_0^t\Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)}{\omega_\lambda\Big(x,\int_0^t\Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)}\\
&=\Big(1+\int_0^t\Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)^{-p\mu}\Big(1+\int_0^t\Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{\mu}\\
&\quad\times\Big(1+\frac{(1-m)\mu}{(2-\lambda)m}\Big(1+\int_0^t\Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)^{-(2-\lambda)\mu}x^{2-\lambda}\Big)^{(p-1)/(m-1)}\\
&\quad\times\Big(\frac{1+\frac{(1-m)\mu}{(2-\lambda)m}\Big(1+\int_0^t\Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)^{-(2-\lambda)\mu}x^{2-\lambda}}{1+\frac{(1-m)\mu}{(2-\lambda)m}\Big(1+\int_0^t\Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{-(2-\lambda)\mu}x^{2-\lambda}}\Big)^{1/(m-1)}\\
&\le\Big(1+\int_0^t\Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)^{-p\mu}\Big(1+\int_0^t\Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{\mu},\quad x\ge 0,\ t\ge 0,
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\frac{\omega_\lambda^q\Big(x,\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)}{\omega_\lambda\Big(x,\int_0^t \Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)}\\
&\le \Big(1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{-(q-1)\mu+1/(1-m)}\Big(1+\int_0^t \Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)^{1/(m-1)}
\end{aligned}$$

for $x \ge 0$ and $t \ge 0$. Therefore $(\widehat{u},\widehat{v})$ are a supersolution to (1.14) and (1.15) if

$$\begin{aligned}
\Theta_1'(t) \ge \Theta_1^{(m-1)/m}(t)\Theta_2^{p/m}(t)\Big(1+\int_0^t \Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)^{-p\mu}&\\
\times \Big(1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{\mu}, \quad t>0,&
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
\Theta_2'(t) \ge \Theta_2^{(m-1)/m}(t)\Theta_1^{q/m}(t)\Big(1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{-(q-1)\mu+1/(1-m)}&\\
\times \Big(1+\int_0^t \Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)^{1/(m-1)}, \quad t>0.&
\end{aligned} \tag{4.27}$$

Case 3.1. $p \ge q > m+2-\lambda$. We assume in advance that

$$0<\frac{\theta_0}{2} \le \Theta_1(t) \le \theta_0 \le \Theta_2(t) \le 2\theta_0 <1 \tag{4.28}$$

holds for some constant $\theta_0 \in (0,1/2)$ to be determined. It is not hard to check that

$$\begin{aligned}
&\Big(1+\int_0^t \Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)^{-p\mu}\Big(1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{\mu}\\
&=\Big(1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{-(p-1)\mu}\Big(\frac{1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s}{1+\int_0^t \Theta_2^{(m-1)/m}(s)\mathrm{d}s}\Big)^{p\mu}\\
&\le (1+t)^{-(p-1)\mu}\Big(\frac{1+(\theta_1/2)^{(m-1)/m}t}{1+(2\theta_1)^{(m-1)/m}t}\Big)^{p\mu/m}\\
&\le 4^{p\mu/m}(1+t)^{-(p-1)\mu}, \quad t>0,
\end{aligned}$$

and

$$\begin{aligned}
&\Big(1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{-(q-1)\mu+1/(1-m)}\Big(1+\int_0^t \Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)^{1/(m-1)}\\
&=\Big(1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{-(q-1)\mu}\Big(\frac{1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s}{1+\int_0^t \Theta_2^{(m-1)/m}(s)\mathrm{d}s}\Big)^{1/(1-m)}\\
&\le 4^{1/m}(1+t)^{-(q-1)\mu}, \quad t>0.
\end{aligned}$$

In order that (4.26) and (4.27) hold, it is sufficient that

$$\Theta_1'(t) \ge 4^{p(\mu+2)/m}\theta_0^{1+(p-1)/m}(1+t)^{-(p-1)\mu}, \quad t>0, \tag{4.29}$$

$$\Theta_2'(t) \ge 4^{1/m}\theta_0^{1+(q-1)/m}(1+t)^{-(q-1)\mu}, \quad t>0. \tag{4.30}$$

We take

$$\Theta_1(t)=\frac{\theta_0}{2}+\frac{4^{p(\mu+2)/m}\theta_0^{1+(p-1)/m}}{(p-1)\mu-1}\Big(1-(1+t)^{1-(p-1)\mu}\Big), \quad t\ge 0, \tag{4.31}$$

$$\Theta_2(t)=\theta_0+\frac{4^{1/m}\theta_0^{1+(q-1)/m}}{(q-1)\mu-1}\Big(1-(1+t)^{1-(q-1)\mu}\Big),\quad t\ge 0, \tag{4.32}$$

where

$$\theta_0=\min\Big\{\frac{1}{2},\ \Big(\frac{(p-1)\mu-1}{4^{1+p(\mu+2)/m}}\Big)^{m/(p-1)},\ \Big(\frac{(q-1)\mu-1}{4^{1/m}}\Big)^{m/(q-1)}\Big\}. \tag{4.33}$$

It follows from

$$1-(p-1)\mu\le 1-(q-1)\mu<0$$

that (4.25) and (4.28)–(4.30) hold. Hence $(\widehat{u},\widehat{v})$ given by (4.22)–(4.24) and (4.31)–(4.33) is a supersolution to the problem (1.14)–(1.17) if (4.9) holds.

Case 3.2. $p>m+2-\lambda=q>1$. We assume in advance that

$$0<\frac{\theta_1}{2}\le\Theta_1(t)\le\theta_1\le\Theta_2(t)=\theta_1(1+t)^{m\sigma_1} \tag{4.34}$$

holds for some constant θ_1 and $\sigma_1\in(0,1)$ to be determined. One can show that

$$\begin{aligned}
&\Big(1+\int_0^t\Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)^{-p\mu}\Big(1+\int_0^t\Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{\mu}\\
&=\Big(1+\int_0^t\Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{-(p-1)\mu}\Big(\frac{1+\int_0^t\Theta_1^{(m-1)/m}(s)\mathrm{d}s}{1+\int_0^t\Theta_2^{(m-1)/m}(s)\mathrm{d}s}\Big)^{p\mu}\\
&\le(1+t)^{-(p-1)\mu}\Big(\frac{1+\big(\frac{\theta_1}{2}\big)^{(m-1)/m}t}{1+\frac{\theta_1^{(m-1)/m}}{1-(1-m)\sigma_1}\Big((1+t)^{1-(1-m)\sigma_1}-1\Big)}\Big)^{p\mu}\\
&\le(1+t)^{-(p-1)\mu}\Big(\frac{1+(\theta_1/2)^{(m-1)/m}t}{1+\theta_1^{(m-1)/m}t}\Big)^{p\mu}\\
&\le 2^{p\mu/m}(1+t)^{-(p-1)\mu},\quad t>0,
\end{aligned}$$

and

$$\begin{aligned}
&\Big(1+\int_0^t\Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{-(q-1)\mu+1/(1-m)}\Big(1+\int_0^t\Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)^{1/(m-1)}\\
&\le\Big(1+\int_0^t\Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{-1+\sigma_1}\Big(\frac{1+\int_0^t\Theta_1^{(m-1)/m}(s)\mathrm{d}s}{1+\int_0^t\Theta_2^{(m-1)/m}(s)\mathrm{d}s}\Big)^{1/(1-m)}\\
&\le 2^{1/m}(1+t)^{-1+\sigma_1},\quad t>0.
\end{aligned}$$

In order that (4.26) and (4.27) hold, it is sufficient that

$$\Theta_1'(t)\ge 2^{(p\mu+1)/m}\theta_1^{1+(p-1)/m}(1+t)^{-(p-1)\mu+p\sigma_1},\quad t>0, \tag{4.35}$$

$$\Theta_2'(t)=m\sigma_1\theta_1(1+t)^{m\sigma_1-1}\ge 2^{1/m}\theta_1^{1+(q-1)/m}(1+t)^{m\sigma_1-1},\quad t>0. \tag{4.36}$$

We take

$$\Theta_1(t)=\frac{\theta_1}{2}+\frac{2^{(p\mu+1)/m}\theta_1^{1+(p-1)/m}}{(p-1)\mu-p\sigma_1-1}\Big(1-(1+t)^{1-(p-1)\mu+p\sigma_1}\Big),\quad t\ge 0, \tag{4.37}$$

$$\Theta_2(t)=\theta_1(1+t)^{m\sigma_1},\quad t\ge 0, \tag{4.38}$$

where

$$\sigma_1=\min\Big\{\frac{1}{2},\ \frac{(p-1)\mu-1}{2p}\Big\}, \tag{4.39}$$

$$\theta_1 = \min\Big\{\frac{1}{2},\ \Big(\frac{(p-1)\mu - p\sigma_1 - 1}{2^{1+(p\mu+1)/m}}\Big)^{m/(p-1)},\ \frac{(m\sigma_1)^{m/(q-1)}}{2^{1/(q-1)}}\Big\}. \tag{4.40}$$

It follows from $1-(p-1)\mu+p\sigma_1<0$ that

$$\sigma_1 < \frac{(p-1)\mu - 1}{p} = \mu - \frac{\mu+1}{p} < \mu,$$

and hence (4.25) and (4.34)–(4.36) hold. Therefore, $(\widehat{u},\widehat{v})$ given by (4.22)–(4.24) and (4.37)–(4.40) is a supersolution to the problem (1.14)–(1.17) if (4.9) holds.

Case 3.3. $p > m+2-\lambda > q > 1$. We assume in advance that

$$0 < \frac{\theta_2}{2} \le \Theta_1(t) \le \theta_2 \le \Theta_2(t) = \theta_2(1+t)^{m\sigma_2} \tag{4.41}$$

holds for some constants θ_2 and $\sigma_2 \in (0,1)$ to be determined. One can show that

$$\begin{aligned}
&\Big(1+\int_0^t \Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)^{-p\mu}\Big(1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{\mu}\\
&= \Big(1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{-(p-1)\mu}\Big(\frac{1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s}{1+\int_0^t \Theta_2^{(m-1)/m}(s)\mathrm{d}s}\Big)^{p\mu}\\
&\le (1+t)^{-(p-1)\mu}\Big(\frac{1+\big(\frac{\theta_2}{2}\big)^{(m-1)/m}t}{1+\frac{\theta_2^{(m-1)/m}}{1-(1-m)\sigma_2}\Big((1+t)^{1-(1-m)\sigma_2}-1\Big)}\Big)^{p\mu}\\
&\le (1+t)^{-(p-1)\mu}\Big(\frac{1+(\theta_2/2)^{(m-1)/m}t}{1+\theta_2^{(m-1)/m}t}\Big)^{p\mu}\\
&\le 2^{p\mu/m}(1+t)^{-(p-1)\mu}, \quad t>0,
\end{aligned}$$

and

$$\begin{aligned}
&\Big(1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{-(q-1)\mu+1/(1-m)}\\
&\times\Big(1+\int_0^t \Theta_2^{(m-1)/m}(s)\mathrm{d}s\Big)^{1/(m-1)}\\
&= \Big(1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s\Big)^{-(q-1)\mu}\Big(\frac{1+\int_0^t \Theta_1^{(m-1)/m}(s)\mathrm{d}s}{1+\int_0^t \Theta_2^{(m-1)/m}(s)\mathrm{d}s}\Big)^{1/(1-m)}\\
&\le 2^{1/m}(1+t)^{-(q-1)\mu}, \quad t>0.
\end{aligned}$$

In order that (4.26) and (4.27) hold, it is sufficient that

$$\Theta_1'(t) \ge 2^{(p\mu+1)/m}\theta_2^{1+(p-1)/m}(1+t)^{-(p-1)\mu+p\sigma_2}, \tag{4.42}$$

$$\Theta_2'(t) = m\sigma_2\theta_2(1+t)^{m\sigma_2-1} \ge 2^{1/m}\theta_2^{1+(q-1)/m}(1+t)^{(m-1)\sigma_2-(q-1)\mu}, \tag{4.43}$$

for $t>0$. We take

$$\Theta_1(t) = \frac{\theta_2}{2} + \frac{2^{(p\mu+1)/m}\theta_2^{1+(p-1)/m}}{(p-1)\mu - p\sigma_2 - 1}\Big(1-(1+t)^{1-(p-1)\mu+p\sigma_2}\Big), \quad t\ge 0, \tag{4.44}$$

$$\Theta_2(t) = \theta_2(1+t)^{m\sigma_2}, \quad t\ge 0, \tag{4.45}$$

where

$$\sigma_2 = 1-(q-1)\mu,$$
$$\theta_2 = \min\Big\{\frac{1}{2}, \Big(\frac{(p-1)\mu - p\sigma_2 - 1}{2^{1+(p\mu+1)/m}}\Big)^{m/(p-1)}, \frac{(m\sigma_2)^{m/(q-1)}}{2^{1/(q-1)}}\Big\}. \tag{4.46}$$

From $pq > (pq)_c = m^2 + (2-\lambda)(p+m)$ and $p > m+2-\lambda$ it follows that

$$\begin{aligned}
1-(p-1)\mu + p\sigma_2 &= \frac{(p+1)(m+1-\lambda)+1-pq}{m+1-\lambda}\\
&\le \frac{(p+1)(m+1-\lambda)+1-m^2-(2-\lambda)(p+m)}{m+1-\lambda}\\
&= \frac{(m-1)(p-m-2-\lambda)}{m+1-\lambda} \le 0,
\end{aligned}$$

and

$$\begin{aligned}
\sigma_2 - \mu = 1-q\mu &< 1-\frac{1}{m+1-\lambda}\Big(\frac{m(m+2-\lambda)}{p}+2-\lambda\Big)\\
&< 1-\frac{2-\lambda}{m+1-\lambda} = \frac{m-1}{m+1-\lambda} < 0.
\end{aligned}$$

Then (4.25) and (4.34)–(4.36) hold. Therefore, $(\widehat{u},\widehat{v})$ given by (4.22)–(4.24) and (4.37)–(4.40) is a supersolution to the problem (1.14)–(1.17) if (4.9) holds. $\square$

Acknowledgments. This research was supported by National Key R & D Program of China (No. 2020YFA0714101), and by the National Natural Science Foundation of China (Nos. 11925105 and 12001227). X. Zhao wants thank his supervisor Professor Chunpeng Wang for his helpful guidance.

References

[1] F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli; *Carleman estimates for degenerate parabolic operators with applications to null controllability*, Journal of Evolution Equations, 6 (2) (2006), 161–204.

[2] D. Andreucci, G. Cirmi, S. Leonardi, A. Tedeev; *Large time behavior of solutions to the Neumann problem for a quasilinear second order degenerate parabolic equation in domains with noncompact boundary*, Journal of Differential Equations, 174 (2) (2001), 253–288.

[3] F. Black, M. Scholes; *The pricing of options and corporate liabilities*, Journal of Political Economy, 81 (3) (1973) 637–654.

[4] P. Cannarsa, P. Martinez, J. Vancostenoble; *Persistent regional null controllability for a class of degenerate parabolic equations*, Communications on Pure and Applied Analysis, 3 (4) (2004), 607–635.

[5] P. Cannarsa, P. Martinez, J. Vancostenoble, *Null controllability of degenerate heat equations*, Advances in Differential Equations, 10 (2) (2005), 153–190.

[6] P. Cannarsa, P. Martinez, J. Vancostenoble; *Carleman estimates for a class of degenerate parabolic operators*, SIAM Journal on Control and Optimization, 47 (1) (2008) 1–19.

[7] P. Cannarsa, L. de Teresa; *Controllability of 1-D coupled degenerate parabolic equations*, Electronic Journal of Differential Equations, 2009 (2009), No. 73, 21 pp.

[8] K. Deng, H. Levine; *The role of critical exponents in blow-up theorems: the sequel*, Journal of Mathematical Analysis and Applications, 243 (1) (2000), 85–126.

[9] R. Du; *Null controllability for a class of degenerate parabolic equations with the gradient terms*, Journal of Evolution Equations, 19 (2) (2019), 585–613.

[10] R. Du, J. Eichhorn, Q. Liu, C. Wang, *Carleman estimates and null controllability of a class of singular parabolic equations*, Advances in Nonlinear Analysis, 8 (1) (2019), 1057–1082.

[11] R. Du, C. Wang; *Null controllability of a class of systems governed by coupled degenerate equations*, Applied Mathematics Letters, 26 (1) (2013), 113–119.

[12] M. Escobedo, M. Herrero; *Boundedness and blow up for a semilinear reaction-diffusion system*, Journal of Differential Equations, 89 (1) (1991), 176–202.
[13] H. Fujita; *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\lambda}$, Journal of the Faculty of Science. University of Tokyo. Section I, 13 (1966), 109–124.
[14] K. Hayakawa; *On nonexistence of global solutions of some semilinear parabolic differential equations*, Proceedings of the Japan Academy, 49 (1973), 503–505.
[15] Q. Huang, K. Mochizuki, *A note on the global solutions of a degenerate parabolic system*, Tokyo Journal of Mathematics, 20 (1) (1997), 63–66.
[16] X. Jing, Y. Nie, C. Wang; *Asymptotic behavior of solutions to coupled semilinear parabolic systems with boundary degeneracy*, Electronic Journal of Differential Equations, 2021 (2021), Paper No. 67, 17 pp.
[17] K. Kobayashi, T. Siaro, H. Tanaka; *On the blowing up problem for semilinear heat equations*, Journal of the Mathematical Society of Japan, 29 (1) (1977), 407–424.
[18] H. Levine; *The role of critical exponents in blow-up theorems*, SIAM Review, 32 (2) (1990), 262–288.
[19] H. Li, X. Wang, Y. Nie, H. He; *Asymptotic behavior of solutions to a degenerate quasilinear parabolic equation with a gradient term*, Electronic Journal of Differential Equations, 2015 (2015), No. 298, 12 pp.
[20] P. Martinez, J. Vancostenoble; *Carleman estimates for one-dimensional degenerate heat equations*, Journal of Evolution Equations, 6 (2) (2006), 325–362.
[21] K. Mochizuki, Q. Huang; *Existence and behavior of solutions for a weakly coupled system of reaction-diffusion equations*, Methods and Applications of Analysis, 5 (2) (1998), 109–124.
[22] Y. Na, Y. Nie, X. Zhou; *Asymptotic behavior of solutions to a class of coupled semilinear parabolic systems with gradient terms*, Journal of Nonlinear Sciences and Applications, 10 (11) (2017), 5813–5824.
[23] G. North, L. Howard, D. Pollard, B. Wielicki; *Variational formulation of Budyko-Sellers climate models*, Journal of the Atmospheric Science, 36 (2) (1979) 255–259.
[24] Y. Qi, H. Levine; *The critical exponent of degenerate parabolic systems*, Zeitschrift für Angewandte Mathematik und Physik, 44 (2) (1993), 249–265.
[25] Y. Uda; *The critical exponent for a weakly coupled system of the generalized Fujita type reaction-diffusion equations*, Zeitschrift für Angewandte Mathematik und Physik, 46 (3) (1995), 366–383.
[26] C. Wang; *Approximate controllability of a class of semilinear systems with boundary degeneracy*, Journal of Evolution Equations, 10 (1) (2010), 163–193.
[27] C. Wang; *Asymptotic behavior of solutions to a class of semilinear parabolic equations with boundary degeneracy*, Proceedings of the American Mathematical Society, 141 (9) (2013) 3125–3140.
[28] C. Wang, R. Du; *Carleman estimates and null controllability for a class of degenerate parabolic equations with convection terms*, SIAM Journal on Control and Optimization, 52 (3) (2014), 1457–1480.
[29] C. Wang, S. Zheng; *Critical Fujita exponents of degenerate and singular parabolic equations*, Proceedings of the Royal Society of Edinburgh. Section A, 136 (2) (2006), 415–430.
[30] C. Wang, S. Zheng, Z. Wang; *Critical Fujita exponents for a class of quasilinear equations with homogeneous Neumann boundary data*, Nonlinearity, 20 (6) (2007), 1343–1359.
[31] C. Wang, Y. Zhou, R. Du, Q. Liu, *Carleman estimate for solutions to a degenerate convection-diffusion equation*, Discrete and Continuous Dynamical Systems. Series B, 23 (10) (2018), 4207–4222.
[32] J. Xu, C. Wang, Y. Nie; *Carleman estimate and null controllability of a cascade degenerate parabolic system with general convection terms*, Electronic Journal of Differential Equations, 2018 (2018), Paper No. 195, 20 pp.
[33] X. Zhao, M. Zhou, X. Jing; *Asymptotic behavior of solutions to porous medium equations with boundary degeneracy*, Electronic Journal of Differential Equations, 2021 (2021), Paper No. 96, 19 pp.
[34] S. Zheng, *Global existence and global non-existence of solutions to a reaction-diffusion system*, Nonlinear Analysis. Theory, Methods & Applications, 39 (3) (2000), 327–340.
[35] S. Zheng, X. Song, Z. Jiang; *Critical Fujita exponents for degenerate parabolic equations coupled via nonlinear boundary flux*, Journal of Mathematical Analysis and Applications, 298 (1) (2004), 308–324.

[36] S. Zheng, C. Wang; *Large time behaviour of solutions to a class of quasilinear parabolic equations with convection terms*, Nonlinearity, 21 (9) (2008), 2179–2200.

[37] M. Zhou, C. Wang, Y. Nie; *Quenching of solutions to a class of semilinear parabolic equations with boundary degeneracy*, Journal of Mathematical Analysis and Applications, 421 (1) (2015), 59–74.

[38] Q. Zhou, Y. Nie, X. Han; *Large time behavior of solutions to semilinear parabolic equations with gradient*, Journal of Dynamical and Control Systems, 22 (1) (2016), 191–205.

Xutong Zhao
School of Mathematics, Jilin University, Changchun 130012, China
Email address: `847692570@qq.com`

Mingjun Zhou (corresponding author)
School of Mathematics, Jilin University, Changchun 130012, China
Email address: `zhoumingjun@jlu.edu.cn`

Qian Zhou
School of Mathematics, Jilin University, Changchun 130012, China
Email address: `zhouqian@jlu.edu.cn`