Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 82, pp. 1–16.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu or https://ejde.math.unt.edu

REGULAR TRAVELING WAVES FOR A REACTION-DIFFUSION EQUATION WITH TWO NONLOCAL DELAYS

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Abstract. This article concerns regular traveling waves of a reaction-diffusion equation with two nonlocal delays arising from the study of a single species with immature and mature stages and different ages at reproduction. Establishing a necessary condition on the regular traveling waves, we prove the uniqueness of noncritical regular traveling waves, regardless of being monotone or not. Under a quasi-monotone assumption and among other things, we further show that all noncritical monotone traveling waves are exponentially stable, by establishing two comparison theorems and constructing an auxiliary lower equation.

1. Introduction

It is well known that many species, such as Ixodes ticks, may exhibit totally different ages at successful reproduction [10]. To understand the influence of different ages at reproduction on the evolution of such populations, Lou and Zhang [10] considered a species (*Ixodes scapularis* ticks) with two classes of individuals, 1 and 2, with different ages at reproduction, τ_1 and τ_2, respectively. Based on the Mckendrick-von Foerster equation, they derived the following reaction-diffusion equation with two nonlocal delays:

$$\begin{aligned} M_t = DM_{xx} - (\mu + g(M)M + pe^{-d_1\tau_1} \int_{\mathbb{R}} \Gamma_1(D_1\tau_1, x-y) f(M(y, t-\tau_1))dy \\ + (1-p)e^{-d_2\tau_2} \int_{\mathbb{R}} \Gamma_2(D_2\tau_2, x-y) f(M(y, t-\tau_2))dy, \quad x \in \mathbb{R},\ t > 0, \end{aligned} \tag{1.1}$$

where $\Gamma_i(D_i\tau_i, x)$, $i = 1, 2$, are the kernel functions which satisfy

(H0) $\Gamma_i(D_i\tau_i, \cdot) \in C(\mathbb{R}, \mathbb{R})$, $\Gamma_i(D_i\tau_i, x) = \Gamma_i(D_i\tau_i, -x) > 0$, $\int_{\mathbb{R}} \Gamma_i(D_i\tau_i, y)dy = 1$, and $\int_{\mathbb{R}} \Gamma_i(D_i\tau_i, y)e^{\lambda|y|}dy < \infty$ for any $\lambda > 0$.

In [10], $\Gamma_i(D_i\tau_i, x) = \sqrt{4\pi D_i\tau_i} e^{-\frac{x^2}{4D_i\tau_i}}$ for $i = 1, 2$. In this model, $M(x, t)$ represents the total adult tick density at time t and location x; $D > 0$ is the diffusion coefficient of mature individuals; $p \in (0, 1)$ and $1 - p$ are the proportions of eggs at birth rate for classes 1 and 2, respectively; $d_i > 0$ and $D_i > 0$ are the death rate and diffusion coefficients of immature individuals for class i, respectively; μ and the function $g(\cdot)$ denote the density-independent and density-dependent death rate of adults, respectively. The functions $f(\cdot)$ and $g(\cdot)$ satisfy the following assumptions

2020 *Mathematics Subject Classification.* 35R10, 35B40, 92D30.
Key words and phrases. Regular traveling fronts; reaction-diffusion equation; nonlocal delay; uniqueness; stability.

Submitted August 26, 2022. Published December 12. 2022.

(H1) $f, g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $f(0) = g(0) = 0$, $g'(u) > 0$ for $u > 0$, and there exists $u_* > 0$ such that $f'(u) > 0$ for $u \in (0, u_*)$ and $f'(u) < 0$ for $u \in (u_*, +\infty)$.

(H2) $\big(pe^{-d_1\tau_1} + (1-p)e^{-d_2\tau_2}\big)f'(0) > \mu$ and $f(u)/u$ is non-increasing in $u \in (0, +\infty)$.

It is clear that f is an unimodal function. In biological literature, two types of such functions which have been widely used are the Richer type function: $f(N) = aNe^{-bN}$, $a, b > 0$, and the Beverton-Holt type function: $f(N) = \frac{aN}{1+bN^m}$, $m > 1, a, b > 0$.

From (H1) and (H2), equation (1.1) has exactly two equilibria 0 and $M_* > 0$. As mentioned in [14], a traveling wave is called *a regular traveling wave* if it decays exponentially at minus infinity (see also Definition 2.1 below). Under the assumptions (H1) and (H2), the authors in [10] established the existence and non-existence of monostable regular traveling waves of (1.1) in the case where $M_* \le u_*$ (i.e. quasi-monotone case) and $M_* > u_*$ (i.e. non-quasi-monotone case). In particular, the regular traveling wave is increasing in the quasi-monotone case, while it may be non-monotone in the non-quasi-monotone case. They also give some results on the existence, uniqueness and stability of bistable traveling fronts of (1.1) with quasi-monotonicity. However, to the best of our knowledge, there has been no results on uniqueness and stability of the monostable regular traveling waves for such reaction-diffusion equations with two nonlocal delays. The aim of this paper is to solve these problems.

We remark that, in recent years, there have been a few of interesting results devoted to the uniqueness of *monotone* traveling waves for various diffusion equations [2, 3, 6, 16, 18, 20, 22]. For example, Wang et al. [16] proved the uniqueness of monotone traveling waves for a class of delayed reaction-diffusion equations. However, such method, if not impossible, can not be applied to study the uniqueness of non-monotone traveling waves. In this paper, we adapt nontrivially the technique developed in Diekmann and Kapper [5] and Aguerrea [1] to study the uniqueness of the regular traveling waves of (1.1) which may not be monotone. More precisely, based on establishing a necessary condition on the regular traveling waves (Lemma 2.2), we prove the uniqueness of all noncritical regular traveling waves regardless of whether they are monotone or not (Theorem 2.4).

In addition to the uniqueness, another interesting problem is the stability of traveling waves. In recent years, the squeezing technique and the weighted energy method have been widely used to study the stability of monostable traveling waves for reaction-diffusions with *single* delay, see [4, 8, 9, 11, 12, 16]. We will use a different approach [7, 13, 17] to study the stability of the traveling waves of the reaction-diffusion equation (1.1) with two nonlocal delays. More precisely, by establishing two comparison theorems and constructing a lower auxiliary equation, we show that all noncritical monotone traveling waves (traveling fronts for short) of (1.1) are exponentially stable. In particular, the exponential convergence rate is also obtained (Theorem 3.3).

The rest of this article is organized as follows. In Section 2, we first establish a necessary condition on the regular traveling waves. Then, we prove the uniqueness of all noncritical regular traveling waves of (1.1) with or without quasi-monotone assumptions. Section 3 is devoted to the exponential stability of noncritical traveling waves.

2. Uniqueness

In this section, we prove the uniqueness of all noncritical regular traveling waves. We first give the definition of (regular) traveling waves [14, 21].

Definition 2.1. A bounded solution $u(x,t)$ of (1.1) is called a traveling wave (solution) with speed c, if $u(x,t) = U(x+ct)$ for some function $U(\cdot)$ which satisfies

$$U(-\infty) = 0 \text{ and } \liminf_{\xi\to+\infty} U(\xi) \geq \varepsilon \text{ for some constant } \varepsilon > 0, \tag{2.1}$$

where c and $U(\cdot)$ are called the wave speed and the wave profile, respectively.

Further, if there exists a constant $\beta > 0$ such that $\lim_{\xi\to-\infty} U(\xi)e^{-\beta\xi} = q$ for some $q > 0$, then $U(x+ct)$ is called a *regular traveling wave solution*.

It is clear that the wave profile $U(\cdot)$ of the traveling wave of (1.1) satisfies

$$\begin{aligned} &DU''(\xi) - cU'(\xi) - \mu U(\xi) - g(U(\xi))U(\xi) \\ &+ \omega_1 \int_{\mathbb{R}} \Gamma_1(D_1\tau_1, y) f(U(\xi - y - c\tau_1))dy \\ &+ \omega_1 \int_{\mathbb{R}} \Gamma_2(D_2\tau_2, y) f(U(\xi - y - c\tau_2))dy = 0, \end{aligned} \tag{2.2}$$

where $\omega_1 := pe^{-d_1\tau_1}$ and $\omega_2 := (1-p)e^{-d_2\tau_2}$.

As in [10] one can easily verify that the characteristic equation

$$\lambda = D\nu^2 - \mu + \omega_1 f'(0)e^{-\tau_1\lambda}\gamma_1(\nu) + \omega_2 f'(0)e^{-\tau_2\lambda}\gamma_2(\nu)$$

admits a principal eigenvalue $\lambda(\nu) > 0$ with $\lim_{\nu\to0+} \frac{\lambda(\nu)}{\nu} = \lim_{\nu\to+\infty} \frac{\lambda(\nu)}{\nu} = +\infty$, where

$$\gamma_i(\nu) := \int_{\mathbb{R}} \Gamma_i(D_i\tau_i, y)e^{-\nu y}dy, \quad i = 1, 2.$$

Moreover, there exist $c_*, \nu_* > 0$ such that

$$c_* = \frac{\lambda(\nu_*)}{\nu_*} = \inf_{\nu>0} \frac{\lambda(\nu)}{\nu},$$

and for any $c > c_*$, there exists a unique $\nu_1 := \nu_1(c) \in (0, \nu_*)$ such that $\lambda(\nu_1) = c\nu_1$, and $\lambda(\nu) < c\nu$ for any $\nu \in (\nu_1, \nu_*)$.

We define

$$\Delta(c,\nu) := D\nu^2 - c\nu - \nu + \omega_1 f'(0)e^{-c\tau_1\nu}\gamma_1(\nu) + \omega_2 f'(0)e^{-c\tau_2\nu}\gamma_2(\nu).$$

It follows from [10, Lemma 4.3] that

$$\Delta(c_*, \nu_*) = 0, \quad \frac{\partial}{\partial\nu}\Delta(c_*, \nu)\big|_{\nu=\nu_*} = 0.$$

Furthermore, if $c > c_*$, then the equation $\Delta(c,\nu) = 0$ has two positive real roots $\nu_1 := \nu_1(c) < \nu_* < \nu_2 := \nu_2(c)$ such that $\Delta(c,\nu) > 0$ for $\nu \in \mathbb{R} \setminus (\nu_1, \nu_2)$ and $\Delta(c,\nu) < 0$ for $\nu \in (\nu_1, \nu_2)$.

Using the method in the proof of Lou and Zhang [10, Theorem 4.5], one has the following result.

Lemma 2.2. *Assume* (H1) *and* (H2). *Then for each $c > c_*$, equation* (1.1) *admits a traveling wave $U(x+ct)$ which satisfies*

$$M^- \leq \liminf_{\xi\to+\infty} U(\xi) \leq \limsup_{\xi\to+\infty} U(\xi) \leq M^+$$

for two constants $M^{\pm} > 0$ *and* $\lim_{\xi\to-\infty} U(\xi)e^{-\nu_1\xi} = 1$. *Moreover, if* $M_* \le u_*$, *then* $U(\xi)$ *is non-decreasing and* $U(+\infty) = u_*$.

From Definition 2.1, we see that the traveling waves obtained in Lemma 2.2 are regular traveling waves. We now give a necessary condition on the regular traveling waves, which will play a critical role in proving uniqueness.

Lemma 2.3. *Assume* (H1), (H2), *and* $c > c_*$. *If* $W(x + ct)$ *is a regular traveling wave of* (1.1) *with* $\lim_{\xi\to-\infty} W(\xi)e^{-\beta\xi} = q$ *for some* $\beta, q > 0$, *then*

$$\lim_{\xi\to-\infty} W(\xi)e^{-\nu_1\xi} = q.$$

Proof. It suffices to show $\beta = \nu_1$, which is done in two steps.

Step 1. We show that $\Delta(c, \beta) = 0$. Denote $\lambda_\pm = \frac{c\pm\sqrt{c^2+4D(\mu+L)}}{2}$ for some constant $L > 0$. It is clear that $\lambda_- < 0 < \lambda_+$ are two roots of the equation: $D\lambda^2 - c\lambda - (\mu + L) = 0$. Take $L > 0$ large enough such that $\lambda_+ > \alpha$. Let $\mu_1 := \mu + L$ and define

$$\begin{aligned} H(W)(\xi) &:= [L - g(W(\xi))]W(\xi) + \omega_1 \int_{\mathbb{R}} \Gamma_1(D_1\tau_1, y)f(W(\xi - y - c\tau_1))dy \\ &\quad + \omega_1 \int_{\mathbb{R}} \Gamma_2(D_2\tau_2, y)f(W(\xi - y - c\tau_2))dy. \end{aligned}$$

Then by (2.2), we obtain

$$DW''(\xi) - cW'(\xi) - \mu_1 W(\xi) + H(W)(\xi) = 0. \tag{2.3}$$

It follows that

$$W(\xi) = \frac{1}{D(\lambda_+ - \lambda_-)}\Big[\int_{-\infty}^{\xi} e^{\lambda_-(\xi-s)}H(W)(s)ds + \int_{\xi}^{\infty} e^{\lambda_+(\xi-s)}H(W)(s)ds\Big],$$

which implies that

$$\begin{aligned} W'(\xi) = \frac{1}{D(\lambda_+ - \lambda_-)}\Big[\lambda_- \int_{-\infty}^{\xi} e^{\lambda_-(\xi-s)}H(W)(s)ds \\ + \lambda_+ \int_{\xi}^{\infty} e^{\lambda_+(\xi-s)}H(W)(s)ds\Big]. \end{aligned} \tag{2.4}$$

Since $W(-\infty) = 0$, by (2.4), it is easy to verify that $W'(-\infty) = 0$. Integrating both sides of (2.3), we obtain

$$DW'(\xi) = cW(\xi) - \int_{-\infty}^{\xi} [-\mu_1 W(s) + H(W)(s)]ds. \tag{2.5}$$

Let $V(\xi) = W(\xi)e^{-\beta\xi}$. Then $V(-\infty) = q > 0$ and $W'(\xi) = V'(\xi)e^{\beta\xi} + \beta V(\xi)e^{\beta\xi}$. From (2.5), we infer that

$$DV'(\xi) = -D\beta V(\xi) + cV(\xi) - e^{-\beta\xi}\int_{-\infty}^{\xi} [-\mu_1 W(s) + H(W)(s)]ds. \tag{2.6}$$

It is easy to verify that

$$\begin{aligned} &\lim_{\xi\to-\infty} e^{-\beta\xi}H(W)(\xi) \\ &= \lim_{\xi\to-\infty} e^{-\beta\xi}\Big\{[L - g(W(\xi))]W(\xi) + \omega_1 \int_{\mathbb{R}} \Gamma_1(D_1\tau_1, y)f(W(\xi - y - c\tau_1))dy \end{aligned}$$

$$
\begin{aligned}
&+\omega_2\int_{\mathbb{R}}\Gamma_2(D_2\tau_2,y)f(W(\xi-y-c\tau_2))dy\Big\}\\
&=Lq+q\varpi(\beta),
\end{aligned}
$$

where $\varpi(\beta):=\omega_1 f'(0)e^{-c\tau_1\beta}\gamma_1(\beta)+\omega_2 f'(0)e^{-c\tau_2\beta}\gamma_2(\beta)$, by which we obtain

$$
\begin{aligned}
&\lim_{\xi\to-\infty}e^{-\beta\xi}\int_{-\infty}^{\xi}[-\mu_1 W(s)+H(W)(s)]ds\\
&=\frac{1}{\beta}\lim_{\xi\to-\infty}e^{-\beta\xi}[-\mu_1 W(\xi)+H(W)(\xi)]\\
&=\frac{q}{\beta}[-\mu+\varpi(\beta)].
\end{aligned}
\tag{2.7}
$$

From (2.6) it follows that

$$
\lim_{\xi\to-\infty}V'(\xi)=\frac{q}{D\beta}[-D\beta^2+c\beta+\mu-\varpi(\beta)]=-\frac{q}{D\beta}\Delta(c,\beta). \tag{2.8}
$$

Since $V(\xi)$ is bounded, $\lim_{\xi\to-\infty}V'(\xi)=0$; therefore $\Delta(c,\beta)=0$.

Step 2. We complete the proof by showing $\Delta(c,\alpha)>0$ for any $\alpha\in(0,\beta)$. Denote $Z(\xi)=W(\xi)e^{-\alpha\xi}$ for $\alpha\in(0,\beta)$. Then $Z(\pm\infty)=0$. Thus, we may assume that $Z(\xi)$ attains its maxmimum at $\xi_1\in\mathbb{R}$. Hence,

$$
0=Z'(\xi_1)=W'(\xi_1)e^{-\alpha\xi_1}-\alpha W(\xi_1)e^{-\alpha\xi_1}
$$

and

$$
W(s-y-c\tau_i)=Z(s-y-c\tau_i)e^{\alpha(s-y-c\tau_i)}\le Z(\xi_1)e^{\alpha s}e^{-\alpha(y+c\tau_i)},
$$

for each $x,y\in\mathbb{R}$, $i=1,2$. By assumption (H2), we see that $f(u)\le f'(0)u$ for $u\ge 0$, and hence, for any $s\in\mathbb{R}$,

$$
\begin{aligned}
&H(W)(s)\\
&\le LW(s)+\omega_1\int_{\mathbb{R}}\Gamma_1(D_1\tau_1,y)f'(0)W(s-y-c\tau_1)dy\\
&\quad+\omega_2\int_{\mathbb{R}}\Gamma_2(D_2\tau_2,y)f'(0)W(s-y-c\tau_2)dy\\
&\le\Big\{L+f'(0)\int_{\mathbb{R}}\Gamma_1(D_1\tau_1,y)[\omega_1 e^{-\alpha c\tau_1}+\omega_2 e^{-\alpha c\tau_2}]e^{-\alpha y}dy\Big\}Z(\xi_1)e^{\alpha s}\\
&=\Big\{L+\big[\omega_1 e^{-\alpha c\tau_1}\gamma_1(\alpha)+\omega_2 e^{-\alpha c\tau_2}\gamma_2(\alpha)\big]f'(0)\Big\}Z(\xi_1)e^{\alpha s}=:L_1 Z(\xi_1)e^{\alpha s}.
\end{aligned}
$$

Combining (2.4) and noting that $\lambda_+>\alpha$, it holds

$$
\begin{aligned}
\alpha W(\xi_1)&=W'(\xi_1)\\
&\le\frac{L_1 Z(\xi_1)}{D(\lambda_+-\lambda_-)}\Big[\lambda_-\int_{-\infty}^{\xi_1}e^{\lambda_-(\xi_1-s)}e^{\alpha s}ds+\lambda_+\int_{\xi_1}^{\infty}e^{\lambda_+(\xi_1-s)}e^{\alpha s}ds\Big]\\
&=\frac{L_1 Z(\xi_1)e^{\alpha\xi_1}}{D(\lambda_+-\lambda_-)}\Big[\frac{\lambda_-}{\alpha-\lambda_-}+\frac{\lambda_+}{\lambda_+-\alpha}\Big]\\
&=\frac{L_1 W(\xi_1)\alpha}{D(\alpha-\lambda_-)(\lambda_+-\alpha)}.
\end{aligned}
\tag{2.9}
$$

Then

$$
L+\big[\omega_1 e^{-\alpha c\tau_1}\gamma_1(\alpha)+\omega_2 e^{-\alpha c\tau_2}\gamma_2(\alpha)\big]f'(0)
$$

$$\geq D(\alpha-\lambda_-)(\lambda_+-\alpha)=-D\alpha^2-c\alpha+\mu+L,$$

and hence, $\Delta(c,\alpha)\geq 0$ for any $\alpha\in(0,\beta)$. Since $\Delta(c,\alpha)$ is strictly convex, we obtain $\Delta(c,\alpha)>0$ for any $\alpha\in(0,\beta)$. Therefore, it must be $\beta=\nu_1$. This completes the proof. □

We are now ready to give the uniqueness of the regular traveling waves of (1.1).

Theorem 2.4. *Assume* (H1), (H2), *and that $f'(u)\leq f'(0)$ for $u\geq 0$. Let $U_1(x+ct)$ and $U_2(x+ct)$ be two regular traveling waves of* (1.1) *with speed $c>c_*$. Then there exists ξ_0 such that $U_1(\xi+\xi_0)=U_2(\xi)$.*

Proof. By a translation in Lemma 2.2, we can assume that $\lim_{\xi\to-\infty}U_i(\xi)/e^{\nu_1\xi}=1$ for $i=1,2$. We define

$$\Pi(\xi):=[U_1(\xi)-U_2(\xi)]/e^{\nu_1\xi},\quad \xi\in\mathbb{R}.$$

Obviously, we have $W(\pm\infty)=0$.

Next, we show that $\Pi(\cdot)\equiv 0$. We first prove $\Pi(\xi)\leq 0$, $\forall\xi\in\mathbb{R}$. Suppose for the contrary that $\max_{\xi\in\mathbb{R}}\Pi(\xi)>0$. Since $\Pi(\pm\infty)=0$, there exists ξ_1 such that $\Pi(\xi_1)=\max_{\xi\in\mathbb{R}}\Pi(\xi)>0$ and $\Pi(\xi)<\Pi(\xi_1)$ for any $\xi<\xi_1$. Then $\Pi'(\xi_1)=0$ and $\Pi''(\xi_1)\leq 0$. Those imply that

$$\begin{gathered}(U_1-U_2)(\xi_1)=\Pi(\xi_1)e^{\nu_1\xi_1}>0,\\ (U_1-U_2)'(\xi_1)=[\Pi'(\xi_1)+\nu_1\Pi(\xi_1)]e^{\nu_1\xi_1}=\nu_1\Pi(\xi_1)e^{\nu_1\xi_1},\\ (U_1-U_2)''(\xi_1)=[\Pi''(\xi_1)+2\nu_1\Pi'(\xi_1)+\nu_1^2\Pi(\xi_1)]e^{\nu_1\xi_1}\leq\nu_1^2\Pi(\xi_1)e^{\nu_1\xi_1}.\end{gathered}$$

Nothing that $g(u)u$ is increasing, it holds $g(U_1(\xi_1))U_1(\xi_1)-g(U_2(\xi_1))U_2(\xi_1)\geq 0$. Using $f'(u)\leq f'(0)$ for $u\geq 0$, direct computations show that

$$\begin{aligned}&(c\nu_1-D\nu_1^2)\Pi(\xi_1)e^{\nu_1\xi_1}\\ &\leq c(U_1-U_2)'(\xi_1)-D(U_1-U_2)''(\xi_1)\\ &=-\mu(U_1-U_2)(\xi_1)-[g(U_1(\xi_1))U_1(\xi_1)-g(U_2(\xi_1))U_2(\xi_1)]\\ &\quad+\omega_1\int_{\mathbb{R}}\Gamma_1(D_1\tau_1,y)[f(U_1(\xi_1-y-c\tau_1))-f(U_2(\xi_1-y-c\tau_1))]dy\\ &\quad+\omega_2\int_{\mathbb{R}}\Gamma_2(D_2\tau_2,y)[f(U_1(\xi_1-y-c\tau_2))-f(U_2(\xi_1-y-c\tau_2))]dy\\ &\leq-\mu\Pi(\xi_1)e^{\nu_1\xi_1}+\omega_1 f'(0)\int_{\mathbb{R}}\Gamma_1(D_1\tau_1,y)\\ &\quad\times\max\{0,U_1(\xi_1-y-c\tau_1)-U_2(\xi_1-y-c\tau_1)\}dy\\ &\quad+\omega_2 f'(0)\int_{\mathbb{R}}\Gamma_2(D_2\tau_2,y)\max\{0,U_1(\xi_1-y-c\tau_2)-U_2(\xi_1-y-c\tau_2)\}dy\\ &=-\mu\Pi(\xi_1)e^{\nu_1\xi_1}+\omega_1 f'(0)\int_{\mathbb{R}}\Gamma_1(D_1\tau_1,y)\\ &\quad\times\max\{0,\Pi(\xi_1-y-c\tau_1)e^{-\nu_1 y}\}dye^{-c\tau_1\nu_1}e^{\nu_1\xi_1}\\ &\quad+\omega_2 f'(0)\int_{\mathbb{R}}\Gamma_2(D_2\tau_2,y)\max\{0,\Pi(\xi_1-y-c\tau_2)e^{-\nu_1 y}\}dye^{-c\tau_2\nu_1}e^{\nu_1\xi_1},\end{aligned}$$

which yields

$$(c\nu_1-D\nu_1^2)\Pi(\xi_1)$$

$$
\begin{aligned}
&\le -\mu\Pi(\xi_1) + \omega_1 f'(0)\int_{\mathbb{R}} \Gamma_1(D_1\tau_1, y)\max\{0, \Pi(\xi_1 - y - c\tau_1))e^{-\nu_1 y}\}dye^{-c\tau_1\nu_1}\\
&\quad + \omega_2 f'(0)\int_{\mathbb{R}} \Gamma_2(D_2\tau_2, y)\max\{0, \Pi(\xi_1 - y - c\tau_2))e^{-\nu_1 y}\}dye^{-c\tau_2\nu_1}.
\end{aligned}
$$

In view of

$$
\begin{aligned}
&(c\nu_1 - D\nu_1^2)\Pi(\xi_1)\\
&= -\mu\Pi(\xi_1) + \omega_1 f'(0)e^{-c\tau_1\nu_1}\gamma_1(\nu_1)\Pi(\xi_1) + \omega_2 f'(0)e^{-c\tau_2\nu_1}\gamma_2(\nu_1)\Pi(\xi_1)\\
&= -\mu\Pi(\xi_1) + \omega_1 f'(0)e^{-c\tau_1\nu_1}\int_{\mathbb{R}} \Gamma_1(D_1\tau_1, y)e^{-\nu_1 y}\Pi(\xi_1)dy\\
&\quad + \omega_2 f'(0)e^{-c\tau_2\nu_1}\int_{\mathbb{R}} \Gamma_2(D_2\tau_2, y)e^{-\nu_1 y}\Pi(\xi_1)dy,
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
&\omega_1 f'(0)e^{-c\tau_1\nu_1}\int_{\mathbb{R}} \Gamma_1(D_1\tau_1, y)[e^{-\nu_1 y}\Pi(\xi_1) - \max\{0, \Pi(\xi_1 - y - c\tau_1))e^{-\nu_1 y}\}]dy\\
&+ \omega_2 f'(0)e^{-c\tau_2\nu_1}\int_{\mathbb{R}} \Gamma_2(D_2\tau_2, y)[e^{-\nu_1 y}\Pi(\xi_1) - \max\{0, \Pi(\xi_1 - y - c\tau_2))e^{-\nu_1 y}\}]dy\\
&\le 0.
\end{aligned}
$$

In view of $\Pi(y) \le \Pi(\xi_1)$ for all $y \in \mathbb{R}$, we obtain

$$
\omega_1 f'(0)e^{-c\tau_1\nu_1}\int_{\mathbb{R}} \Gamma_1(D_1\tau_1, y)[e^{-\nu_1 y}\Pi(\xi_1) - \max\{0, \Pi(\xi_1 - y - c\tau_1))e^{-\nu_1 y}\}]dy = 0.
$$

Consequently, there exists $y_1 > 0$ such that

$$
e^{-\nu_1 y_1}\Pi(\xi_1) = \max\{0, \Pi(\xi_1 - y_1 - c\tau_1)e^{-\nu_1 y_1}\}.
$$

Since $\Pi(\xi_1) > 0$, we obtain $\Pi(\xi_1) = \Pi(\xi_1 - y_1 - c\tau_1)$, which contradicts $\Pi(\xi) < \Pi(\xi_1)$ for $\xi < \xi_1$. Hence, $\Pi(\xi) \le 0$, $\forall \xi \in \mathbb{R}$. Similarly, one can easily show that $\Pi(\xi) \ge 0$, $\forall \xi \in \mathbb{R}$. Thus, $\Pi(\cdot) \equiv 0$, i.e. $U_1(\cdot) \equiv U_2(\cdot)$. Since we have made possible translations of $U_1(\xi)$ and $U_2(\xi)$, it holds $U_1(\cdot + \xi_0) \equiv U_2(\cdot)$ for some $\xi_0 \in \mathbb{R}$. This completes the proof. □

3. Stability

This section is devoted to the stability of monotone traveling waves under the quasi-monotone assumption, i.e. $M_* \le u_*$. In this case, the traveling wave $U(x+ct)$ with speed $c > c_*$ given in Lemma 2.2 is non-decreasing and $U(+\infty) = M_*$.

For studying the stability of the traveling front $U(x + ct)$ with speed $c > c_*$, in addition to (H1)-(H2), we need the following assumption:

- (H3) $M_* \le u_*$ and $\mu > -g'(M_*)M_* - g(M_*) + (\omega_1 + \omega_2)f'(M_*)$;
- (H4) $\Gamma_1(\cdot)$ and $\Gamma_2(\cdot)$ have compact supports.

Recall that $\omega_1 := pe^{-d_1\tau_1}$ and $\omega_2 := (1 - p)e^{-d_2\tau_2}$. For simplicity, we may assume $[-r_i, r_i] = \operatorname{supp}\Gamma_i$ for some $r_i > 0$, $i = 1, 2$. It is easy to see that the second part of (H3) means that the unique positive equilibrium is stable.

Consider the linear equation

$$
M'(t) = -(\mu + g_1'(M_*))M(t) + f'(M_*)[\omega_1 M(t - \tau_1) + \omega_2 M(t - \tau_2)]. \tag{3.1}
$$

Clearly, its characteristic equation has the from

$$
\lambda = -(\mu + g_1'(M_*)) + f'(M_*)[\omega_1 e^{-\lambda\tau_1} + \omega_2 e^{-\lambda\tau_2}]. \tag{3.2}
$$

Lemma 3.1. *The characteristic problem* (3.2) *has a principal eigenvalue* $\check{\lambda} < 0$.

Proof. The existence of the principal eigenvalue $\check{\lambda}$ of (3.2) follows from [15, Theorem 5.1]. It remains to show $\check{\lambda} < 0$. Suppose on the contrary that $\check{\lambda} \geq 0$. Then

$$\begin{aligned}\check{\lambda} &= -(\mu + g_1'(M_*)) + f'(M_*)[\omega_1 e^{-\check{\lambda}\tau_1} + \omega_2 e^{-\check{\lambda}\tau_2}] \\ &\leq -(\mu + g_1'(M_*)) + f'(M_*)[\omega_1 + \omega_2] < 0.\end{aligned}$$

This contradiction implies that $\check{\lambda} < 0$, and the lemma follows. □

Let $\mathcal{X}$ be the set of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$ equipped with compact open topology. Denote $\tau = \max\{\tau_1, \tau_2\}$ and $\mathcal{C} = C([-\tau, 0], \mathcal{X})$.

In the sequel, we assume that $U(x + ct)$ is a traveling front of (1.1) connecting 0 and M_* with speed $c > c_*$. Take $\nu \in (\nu_1(c), \nu_*)$. From the discussions in Section 2, we see that $c\nu - \lambda(\nu) > 0$ for any $\nu \in (\nu_1(c), \nu_*)$. We now define the weight function

$$W(x) = e^{-\nu x}. \tag{3.3}$$

We make the following assumptions on the initial data $u_0(x, s)$:

(H5) $u_0 \in \mathcal{C}$, $0 < u_0(x, s) \leq M_*$ for any $x \in \mathbb{R}, s \in [-\tau, 0]$, and for any given $x_0 \in \mathbb{R}$, $\inf\big\{u_0(x, s)\big| x \geq x_0$ and $s \in [-\tau, 0]\big\} > 0$, and

$$\sup_{x\in\mathbb{R}, s\in[-\tau,0]} \big|u_0(x, s) - U(x + cs)\big| W(x) < \infty. \tag{3.4}$$

We first give the results on the existence, uniqueness and positivity of solutions for the Cauchy problem of (1.1).

Lemma 3.2. *Assume* (H1)-(H3), (H5). *Then the Cauchy problem of* (1.1) *with the initial value* $u_0(x, s)$ *admits a unique solution* $u(x, t)$ *satisfying* $0 < u(x, t) \leq M_*$ *for all* $(x, t) \in \mathbb{R} \times [-\tau, +\infty)$.

Proof. The existence and uniqueness of solutions can be found in [10, Section 4]. Moreover, $\mathcal{C}_{M_*} := \{\psi \in \mathcal{C} : 0 \leq \psi(\cdot) \leq M_*\}$ is positively invariant. It remains to show that $u(x, t) > 0$ for all $(x, t) \in \mathbb{R} \times [-\tau, +\infty)$. Suppose for the contrary that there exists $(x_0, t_0) \in \mathbb{R} \times [0, \tau]$ such that $u(x_0, t_0) = 0$. Clearly, $t_0 \in (0, \tau]$. Then $u_t(x_0, t_0) \leq 0$ and $u_{xx}(x_0, t_0) \geq 0$. Consequently, it follows from (1.1) that

$$\begin{aligned}0 \geq \omega_1 &\int_{\mathbb{R}} \Gamma_1(D_1\tau_1, x_0 - y) f(u(y, t_0 - \tau_1)) dy \\ &+ \omega_2 \int_{\mathbb{R}} \Gamma_2(D_2\tau_2, x_0 - y) f(u(y, t_0 - \tau_2)) dy,\end{aligned}$$

which yields

$$\begin{aligned}0 = \omega_1 &\int_{\mathbb{R}} \Gamma_1(D_1\tau_1, x_0 - y) f(u(y, t_0 - \tau_1)) dy \\ &+ \omega_2 \int_{\mathbb{R}} \Gamma_2(D_2\tau_2, x_0 - y) f(u(y, t_0 - \tau_2)) dy.\end{aligned} \tag{3.5}$$

Without loss of generality, we assume $\tau_2 = \tau = \max\{\tau_1, \tau_2\}$. By (3.5), we have

$$\int_{\mathbb{R}} \Gamma_2(D_2\tau_2, y) f(u_0(x_0 - y, t_0 - \tau_2)) dy = 0.$$

Thus, $f(u_0(x_0, t_0 - \tau_2)) = 0$, which implies that $u_0(x_0, t_0 - \tau_2) = 0$. This contradicts to the assumption of u_0. Then, $u(x, t) > 0$ for all $(x, t) \in \mathbb{R} \times [0, \tau]$. By induction, we conclude that $u(x, t) > 0$ for all $(x, t) \in \mathbb{R} \times [-\tau, +\infty)$. The proof is complete. □

Now, we are ready to state the stability of the traveling front.

Theorem 3.3. *Assume* (H0)–(H5). *Then the unique solution $u(x,t)$ of* (1.1) *with the initial data $u_0(x,s), s \in [-\tau, 0]$ satisfies $0 < u(x,t) \le M_*$ for $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, and there exist positive constants ρ, C and T_1 such that*

$$\sup_{x\in\mathbb{R}} |u(x,t) - U(x+ct)| \le Ce^{-\rho t} \quad \textit{for all } t \ge T_1.$$

To prove this theorem, we need some lemmas. In particular, we need to establish two comparison theorems and construct an auxiliary sub-system. In the sequel, we always make all the assumptions in Theorem 3.3.

Given $c_1 \ge 0$, $\zeta \in \mathbb{R} \cup \{-\infty\}$, and $0 \le t_1 < T_1 \le +\infty$. Take $\chi = \max\{r_1, r_2\} + c\max\{\tau_1, \tau_2\}$. We further define the following regions: For $\zeta = -\infty$,

$$\Omega_\zeta^1 = \emptyset, \quad \Omega_\zeta^2 = \mathbb{R} \times [t_1 - \tau, t_1], \quad \Omega_\zeta^3 = \mathbb{R} \times (t_1, T_1];$$

and for $\zeta \in \mathbb{R}$,

$$\begin{gathered}
\Omega_\zeta^1 = \big\{(x,t) \in \mathbb{R}^2 \big| \zeta - \chi \le x + c_1 t \le \zeta, t \in [t_1 - \tau, T_1]\big\},\\
\Omega_\zeta^2 = \big\{(x,t) \in \mathbb{R}^2 \big| x + c_1 t > \zeta, t \in [t_1 - \tau, t_1]\big\},\\
\Omega_\zeta^3 = \big\{(x,t) \in \mathbb{R}^2 \big| x + c_1 t > \zeta, t \in (t_1, T_1]\big\}.
\end{gathered}$$

We denote $\Omega_\zeta = \Omega_\zeta^1 \cup \Omega_\zeta^2 \cup \Omega_\zeta^3$. Then, we have the following comparison theorems, which can be proved by using similar methods as in [19, Lemma 5.2].

Lemma 3.4. *Assume* (H0)–(H4), *and that $w^\pm(x,t) : \Omega_\zeta \to \mathbb{R}_+$ are two continuous functions satisfying*

(i) *$0 \le w^+(x,t), w^-(x,t) \le M_*$ for all $(x,t) \in \Omega_\zeta$;*
(ii) *$w^+(x,t) \ge w^-(x,t)$ for all $(x,t) \in \Omega_\zeta^1 \cup \Omega_\zeta^2$;*
(iii) *$\mathcal{G}(w^+)(x,t) \ge 0 \ge \mathcal{G}(w^-)(x,t)$ for all $(x,t) \in \Omega_\zeta^3$, where*

$$\begin{aligned}
&\mathcal{G}(w^\pm)(x,t)\\
&:= \frac{\partial w^\pm}{\partial t} - Dw^\pm_{xx} - \omega_1 \int_{\mathbb{R}} \Gamma_1(D_1\tau_1, y) f(w^\pm(x-y, t-\tau_1))dy\\
&\quad + (\mu + g(w^\pm))w^\pm - \omega_2 \int_{\mathbb{R}} \Gamma_2(D_2\tau_2, y) f(w^\pm(x-y, t-\tau_2))dy.
\end{aligned}$$

Then $w^+(x,t) \ge w^-(x,t)$ for all $(x,t) \in \Omega_\zeta^3$.

Lemma 3.5. *Assume* (H0), (H4), *and that for all $Q_1, Q_2 > 0$, the functions $w^\pm(x,t) : \Omega_\zeta \to \mathbb{R}_+$ are continuous and satisfying:*

(i) *$0 \le w^+(x,t)$ and $w^-(x,t) \le M_*$ for all $(x,t) \in \Omega_\zeta$;*
(ii) *$w^+(x,t) \ge w^-(x,t)$ for all $(x,t) \in \Omega_\zeta^1 \cup \Omega_\zeta^2$;*
(iii) *$\mathcal{F}(w^+)(x,t) \ge 0 \ge \mathcal{F}(w^-)(x,t)$ for all $(x,t) \in \Omega_\zeta^3$, where*

$$\begin{aligned}
\mathcal{F}(w^\pm)(x,t) &:= \frac{\partial w^\pm}{\partial t} - Dw^\pm_{xx} - Q_1 \int_{\mathbb{R}} \Gamma_1(D_1\tau_1, y) w^\pm(x-y, t-\tau_1)dy\\
&\quad + \mu w^\pm - Q_2 \int_{\mathbb{R}} \Gamma_2(D_2\tau_2, y) w^\pm(x-y, t-\tau_2)dy.
\end{aligned}$$

Then $w^+(x,t) \ge w^-(x,t)$ for all $(x,t) \in \Omega_\zeta^3$.

Let us define

$$u_0^+(x,s) = \max\{u_0(x,s),\ U(x+cs)\}, \quad u_0^-(x,s) = \min\{u_0(x,s),\ U(x+cs)\}$$

for $(x,s) \in \mathbb{R} \times [-\tau, 0]$. One can easily see that

$$0 \le u_0^-(x,s) \le u_0(x,s), \quad U(x+cs) \le u_0^+(x,s) \le u_*(s) \tag{3.6}$$

for $(x,s) \in \mathbb{R} \times [-\tau, 0]$. Let $u^-(x,t)$ and $u^+(x,t)$ be the solution of (1.1) with the initial data $u_0^-(x,s)$ and $u_0^+(x,s)$, respectively. Applying the comparison principle for quasi-monotone systems, we have

$$0 \le u^-(x,t) \le u(x,t), \quad U(x+ct) \le u^+(x,t) \le M_*. \tag{3.7}$$

Hence,

$$|u(x,t) - U(x+ct)| \le \max\big\{|u^+(x,t) - U(x+ct)|, |u^-(x,t) - U(x+ct)|\big\} \tag{3.8}$$

for any $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. We denote

$$W^+(x,t) = u^+(x,t) - U(x+ct), \quad W^-(x,t) = U(x+ct) - u^-(x,t) \tag{3.9}$$

for any $(x,t) \in \mathbb{R} \times [-\tau, \infty)$. Thus, we only need to show the exponential convergence of $W^\pm(x,t)$ to 0.

Fix any $\epsilon_0 \in (0, -\check{\lambda})$ and choose $\epsilon > 0$ such that

$$\epsilon_0 \ge \epsilon + \omega_1 \epsilon e^{\epsilon_0 \tau_1} + \omega_2 \epsilon e^{\epsilon_0 \tau_2}. \tag{3.10}$$

We define the function $g_1(u) = g(u)u$ for $u \ge 0$. Then we can take $\delta_0 \in (0, M_*)$ such that for any $u_2 \ge u_1 \ge M_* - \delta_0$,

$$\begin{gathered} g_1(u_2) - g_1(u_1) \ge [g_1'(M_*) - \epsilon](u_2 - u_1), \\ f(u_2) - f(u_1) \le [f'(M_*) + \epsilon](u_2 - u_1). \end{gathered} \tag{3.11}$$

Since $M_* \ge u^+(x,t) \ge U(x+ct)$ and $U(+\infty) = M_*$, it is clear that $U(x+ct)$ and $u^+(x,t)$ are close to M_* as $x + ct \gg 1$. To show that $u^-(x,t)$ is also close to M_* when $x+ct$ and t are sufficiently large, we need to construct an auxiliary equation. Take $\kappa_0 \in (0,1]$ such that for any $\kappa \in [\kappa_0, 1]$,

$$\mu > g'(M_*)M_* + g(M_*) + \big(\omega_1 + \omega_2\big)\kappa f'(M_*).$$

Consider the equation

$$M'(t) = -(\mu + g(M))M + \omega_1 \kappa f(M(t-\tau_1)) + \omega_2 \kappa f(M(t-\tau_2)), \tag{3.12}$$

where $\kappa \in [\kappa_0, 1]$. It is easy to see that, for each $\kappa \in [\kappa_0, 1]$, (3.12) has a unique positive equilibrium $M_*^\kappa \in (0, M_*]$ which is globally stable. We can further choose $\kappa_1 \in [\kappa_0, 1)$ such that

$$M_* - \frac{\delta_0}{2} < M_*^{\kappa_1} < M_*. \tag{3.13}$$

Proof of Theorem 3.3. In view of $0 \le V^\pm(x,s) \le |u_0(x,s) - U(x+cs)|$, $\forall x \in \mathbb{R}, s \in [-\tau, 0]$, it follows from (A1) that $V^\pm(x,s)W(x)$ is uniformly bounded on $\mathbb{R}$. To prove the convergence of $U^\pm(x,t)$ to $U(x+ct)$, we use the following 2 claims:

Claim 1. There exists $L_1 > 0$ such that

$$0 \le W^\pm(x,t) \le L_1 e^{\nu(x+ct) - \rho_1 t} \quad \text{for all } x \in \mathbb{R}, t > 0, \tag{3.14}$$

where $\rho_1 = c\nu - \lambda(\nu) > 0$.

We only prove the assertion for $W^+(x,t)$, since the assertion for $W^-(x,t)$ can be obtained similarly. By (3.4), there exists sufficiently large $L_1 > 0$ such that
$$0 \le W^+(x,s)W(x) \le \sup_{x\in\mathbb{R},s\in[-\tau,0]} \big|u_0(x,s) - U(x+cs)\big|W(x) \le L_1 e^{-\lambda(\nu)\tau},$$
which implies that $W^+(x,s) \le L_1 e^{\nu x} e^{\lambda(\nu)s}$ for all $x \in \mathbb{R}$, $s \in [-\tau,0]$. We define
$$\bar W(x,t) = L_1 e^{\nu x+\lambda(\nu)t}, \quad \forall x \in \mathbb{R}, t \ge -\tau.$$
It is obvious that $\bar W(x,s) \ge W^+(x,s)$ for $\forall x \in \mathbb{R}$, $s \in [-\tau,0]$. In view of
$$\lambda(\nu) = D\nu^2 - \mu + \omega_1 f'(0)e^{-\tau_1\lambda(\nu)}\gamma_1(\nu) + \omega_2 f'(0)e^{-\tau_2\lambda(\nu)}\gamma_2(\nu),$$
one can easily verify that
$$\begin{aligned} \bar W_t = D\bar W_{xx} - \mu\bar W + \omega_1 f'(0)\int_{\mathbb{R}} \Gamma_1(D_1\tau_1,y)\bar W(x-y,t-\tau_1)dy \\ + \omega_2 f'(0)\int_{\mathbb{R}} \Gamma_2(D_2\tau_2,y)\bar W(x-y,t-\tau_2)dy. \end{aligned} \tag{3.15}$$
On the other hand, from (1.1) and (3.9) it follows that
$$\begin{aligned} W_t^+ &= DW_{xx}^+ - \mu W^+ - [g(u^+(x,t))u^+(x,t) - g(U(x+ct))U(x+ct)] \\ &\quad + \omega_1 \int_{\mathbb{R}} \Gamma_1(D_1\tau_1,y)[f(u^+(x-y,t-\tau_1)) - f(U(x-y+ct-c\tau_1))]dy \\ &\quad + \omega_2 \int_{\mathbb{R}} \Gamma_2(D_2\tau_2,y)[f(u^+(x-y,t-\tau_2)) - f(U(x-y+ct-c\tau_2))]dy. \end{aligned} \tag{3.16}$$
Since g is increasing on $(0,+\infty)$ and $f'(u) \le f'(0)$ for all $u \ge 0$, we have
$$\begin{aligned} W_t^+ \le DW_{xx}^+ - \mu W^+ + \omega_1 f'(0)\int_{\mathbb{R}} \Gamma_1(D_1\tau_1,y)W^+(x-y,t-\tau_1)dy \\ + \omega_2 f'(0)\int_{\mathbb{R}} \Gamma_2(D_2\tau_2,y)W^+(x-y,t-\tau_2)dy. \end{aligned} \tag{3.17}$$
Consequently, using Lemma 3.5 with $c_1 = c > 0$, $\zeta = -\infty$, $T_1 = \infty$, $Q_1 = \omega_1 f'(0)$, and $Q_2 = \omega_2 f'(0)$, we obtain $W^+(x,t) \le \bar W(x,t)$ for $(x,t) \in \mathbb{R}\times(0,\infty)$, that is,
$$W^+(x,t) \le L_1 e^{\nu x+\lambda(\nu)t} = L_1 e^{\nu(x+ct)-\rho_1 t} \quad \text{for } (x,t) \in \mathbb{R}\times(0,\infty).$$
This proves claim 1.

Claim 2. There exist $\gamma_* \in \mathbb{R}$ and $t_* > 0$ such that $U(x+ct), u^\pm(x,t) \ge M_* - \delta_0$ for any $x + ct \ge \gamma_*$ and $t \ge t_*$.

In view of $u^+(x,t) \ge U(x+ct)$ and $U(+\infty) = M_*$, it is clear that there exists $\gamma_* \in \mathbb{R}$ such that
$$u^+(x,t) \ge U(x+ct) \ge \tfrac{M_*+M_*^{\kappa_1}}{2} \ge M_*^{\kappa_1} = M_* - \delta_0 \text{ for } x+ct \ge \gamma_* \text{ and } t > 0.$$
To prove Claim 2, it remains to show that there exists $t_* > 0$ such that $u^-(x,t) \ge M_* - \delta_0$ for any $x + ct \ge \gamma_*$ and $t \ge t_*$.

We first consider the case $\gamma_* \le x + ct \le \gamma_* + \chi$. By Claim 1,
$$|u^-(x,t) - U(x+ct)| \le L_1 e^{\nu(\gamma_*+\chi)-\rho_1 t} \le L_1 e^{\nu(\gamma_*+\chi)} e^{-\rho_1(\gamma_*-x)/c}, \tag{3.18}$$
from which, we can take $Y_0 < 0$ so that
$$L_1 e^{\nu(\gamma_*+\chi)} e^{-\rho_1(\gamma_*-x)/c} \le (M_* - M_*^{\kappa_1})/2 \quad \text{for } x \le Y_0. \tag{3.19}$$

Thus, we conclude from (3.13), (3.18), and (3.19) that for $x \le Y_0$ and $t \ge 0$ with $x + ct \in [\gamma_*, \gamma_* + \chi]$, we have

$$\begin{aligned} u^-(x,t) &\ge U(x+ct) - L_1 e^{\nu(\gamma_*+\chi)} e^{-\rho_1(\gamma_*-x)/c} \\ &\ge \frac{M_* + M_*^{\kappa_1}}{2} - \frac{M_* - M_*^{\kappa_1}}{2} = M_*^{\kappa_1} \ge M_* - \delta_0, \end{aligned}$$

which implies that $u^-(x,t) \ge M_*^{\kappa_1} \ge M_* - \delta_0$ for $(x,t) \in \Omega_0$, where $t_0 := (\gamma_* + \chi - Y_0)/c$ and

$$\Omega_0 := \{(x,t) \in \mathbb{R}^2 | \gamma_* \le x + ct \le \gamma_* + \chi \text{ and } t \ge t_0\}.$$

We now define the following regions:

$$\begin{gathered} \Omega_1 = \{(x,t) \in \mathbb{R}^2 \big| Y_0 - \chi \le x \le Y_0,\ t \in [-\tau, t_0 + \tau]\}, \\ \Omega_2 = \{(x,t) \in \mathbb{R}^2 \big| x \ge Y_0, t \in [-\tau, 0]\}, \\ \Omega_3 = \{(x,t) \in \mathbb{R}^2 \big| x > Y_0, t \in (0, t_0 + \tau]\}. \end{gathered}$$

By (H5) and Lemma 3.1, there exists $\zeta \in (0, M_*^{\kappa_1})$ such that $u^-(x,t) \ge \zeta$ for $(x,t) \in \Omega_1 \cup \Omega_2$, and

$$(\omega_1 + \omega_2) f(\zeta) - (\mu + g(\zeta)\zeta > 0.$$

We define

$$\underline{u}(x,t) = \zeta \quad \text{for all } (x,t) \in \Omega_1 \cup \Omega_2 \cup \Omega_3.$$

One can easily verify that $u^-(x,t) \ge \underline{u}(x,t)$ for $(x,t) \in \Omega_1 \cup \Omega_2$ and

$$\begin{aligned} &\underline{u}_t - D\underline{u}_{xx} + (\mu + g(\underline{u}(x,t))\underline{u}(x,t) - \omega_1 \int_{\mathbb{R}} \Gamma_1(D_1\tau_1, x-y) f(\underline{u}(y, t-\tau_1))dy \\ &- \omega_2 \int_{\mathbb{R}} \Gamma_2(D_2\tau_2, x-y) f(\underline{u}(y, t-\tau_2))dy \\ &= (\mu + g(\zeta)\zeta - (\omega_1 + \omega_2) f(\zeta) \le 0. \end{aligned}$$

According to Lemma 3.4 with $c_1 = 0$, $t_1 = 0$, $T_1 = t_0 + \tau$ and $\zeta = \gamma_* + \chi$, we have $u^-(x,t) \ge \underline{u}(x,t) = \zeta$ for $(x,t) \in \Omega_3$. Let us now define the following regions:

$$\begin{gathered} \Omega_4 = \{(x,t) \in \mathbb{R}^2 \big| \gamma_* \le x + ct \le \gamma_* + \chi, t \ge t_0\}, \\ \Omega_5 = \{(x,t) \in \mathbb{R}^2 \big| x + ct \ge \gamma_* + \chi, t \in [t_0, t_0 + \tau]\}, \\ \Omega_6 = \{(x,t) \in \mathbb{R}^2 \big| x + ct > \gamma_* + \chi, t > t_0 + \tau\}, \end{gathered}$$

and the function $u_1(x,t) = T(t)$ for $(x,t) \in \Omega_4 \cup \Omega_5 \cup \Omega_6$, where $T(t)$ is the unique solution of the initial value problem

$$\begin{gathered} T'(t) = -(\mu + g(T(t))T(t) + \kappa_1\omega_1 f(T(t-\tau_1)) + \kappa_1\omega_2 f(T(t-\tau_2)), \quad t > t_0 + \tau, \\ T(\theta) = \zeta, \quad \theta \in [t_0, t_0 + \tau]. \end{gathered}$$

It is obvious that $T(t) \in [0, M_*^{\kappa_1}]$ for any $t > 0$ and $\lim_{t\to\infty} T(t) = M_*^{\kappa_1}$. Further, we have

$$\begin{gathered} u^-(x,t) \ge M_*^{\kappa_1} \ge T(t) \quad \text{for } (x,t) \in \Omega_4 = \Omega_0, \\ u^-(x,t) \ge \zeta = T(t) \quad \text{for } (x,t) \in \Omega_5 \subseteq \Omega_1 \cup \Omega_3, \end{gathered}$$

and

$$(u_1)_t - D(u_1)_{xx} + (\mu + g(u_1)u_1 - \omega_1 \int_{\mathbb{R}} \Gamma_1(D_1\tau_1, x-y) f(u_1(y, t-\tau_1))dy$$

$$
\begin{aligned}
&- \omega_2 \int_{\mathbb{R}} \Gamma_2(D_2\tau_2, x-y) f(u_1(y, t-\tau_2))dy \\
&= T'(t) + (\mu + g(T(t))T(t) - \omega_1 f(T(t-\tau_1)) - \omega_2 f(u_1(t-\tau_2)) \\
&= (\kappa_1 - 1)\omega_1 f(T(t-\tau_1)) + (\kappa_1 - 1)\omega_2 f(u_1(t-\tau_2)) \\
&\le 0, \quad \forall t > t_0 + \tau.
\end{aligned}
$$

Using Lemma 3.4 again with $c_1 = c$, $t_1 = t_0 + \tau$, $T_1 = \infty$, and $\zeta = \gamma_* + \chi$, we have

$$
u^-(x,t) \ge u_1(x,t) = T(t) \quad \text{for } (x,t) \in \Omega_6.
$$

In view of $\lim_{t\to\infty} T(t) = M_*^{\kappa_1}$, there exists $t_* > t_0 + \tau$ such that

$$
u^-(x,t) \ge M_*^{\kappa_1} - \frac{\delta_0}{2} \ge M_* - \delta_0 \quad \text{for } x + ct > \gamma_* + \chi, t \ge t_*.
$$

Therefore, $U^-(x,t) \ge M_* - \delta_0$ for any $x + ct \ge \gamma_*$ and $t \ge t_*$.

Completion of the proof. We first consider the case $x + ct \le \gamma_* + \chi$. In this case, from claim 1, there exists $C_0 > 0$ such that

$$
|u^{\pm}(x,t) - U(x+ct)| \le L_1 e^{\nu(x+ct) - \rho_1 t} \le C_0 e^{-\rho_1 t}, \quad t > 0. \tag{3.20}
$$

Next, we consider the case $x + ct > \gamma_* + \chi$. We denote

$$
\begin{aligned}
\Omega^1_{\gamma_*} &= \{(x,t) \in \mathbb{R}^2 \big| \gamma_* \le x + ct \le \gamma_* + \chi, \ t \ge t_* t\}, \\
\Omega^2_{\gamma_*} &= \{(x,t) \in \mathbb{R}^2 \big| x + ct > \gamma_* + \chi, t \in [t_*, t_* + \tau]\}, \\
\Omega^3_{\gamma_*} &= \{(x,t) \in \mathbb{R}^2 \big| x + ct > \gamma_* + \chi, t > t_* + \tau\}.
\end{aligned}
$$

Noting that $\chi = \max\{r_1, r_2\} + c\max\{\tau_1, \tau_2\}$ and $[-r_i, r_i] = \operatorname{supp}\Gamma_i$, $i = 1, 2$, we have

$$
x - y + c(t - \tau_i) = x + ct - y - c\tau_i \ge \gamma_* + \chi - r_i - c\tau_i \ge \gamma_*
$$

for any $y \in [-r_i, r_i]$ and $(x,t) \in \mathbb{R} \times [0, \infty)$ with $x + ct > \gamma_* + \chi$. It then follows from Claim 2 and (3.11) that

$$
\begin{aligned}
\frac{\partial W^-}{\partial t} &= U_\xi(x+ct) - u_t^-(x,t) \\
&= DW^-_{xx} - \mu W^- - (g_1(U(x+ct)) - g_1(u^-(x,t))) \\
&\quad + \omega_1 \int_{\mathbb{R}} \Gamma_1(D_1\tau_1, y)[f(U(x-y+c(t-\tau_1))) - f(u^-(x-y, t-\tau_1))]dy \\
&\quad + \omega_2 \int_{\mathbb{R}} \Gamma_2(D_2\tau_2, y)[f(U(x-y+c(t-\tau_2))) - f(u^-(x-y, t-\tau_2))]dy \\
&\le DW^-_{xx} - (\mu + g_1'(M_*) - \epsilon)W^- \\
&\quad + \omega_1[f'(M_*) + \epsilon] \int_{\mathbb{R}} \Gamma_1(D_1\tau_1, x-y) W^-(y, t-\tau_1)dy \\
&\quad + \omega_2[f'(M_*) + \epsilon] \int_{\mathbb{R}} \Gamma_2(D_2\tau_2, x-y) W^-(y, t-\tau_2)dy.
\end{aligned}
$$

Similarly, we obtain

$$\begin{aligned}\frac{\partial W^+}{\partial t} &\le DW^+_{xx} - (\mu + g_1'(M_*) - \epsilon)W^+ \\ &\quad + \omega_1[f'(M_*) + \epsilon]\int_{\mathbb{R}} \Gamma_1(D_1\tau_1, x-y)W^+(y, t-\tau_1)dy \\ &\quad + \omega_2[f'(M_*) + \epsilon]\int_{\mathbb{R}} \Gamma_2(D_2\tau_2, x-y)W^+(y, t-\tau_2)dy.\end{aligned} \tag{3.21}$$

Let $\epsilon_0 = \min\{\rho_1, -\breve{\lambda} - \epsilon_0\}$, and choose $L_2 > 0$ such that $L_2 \ge \max\{C_0, M_* e^{\epsilon_0(t_*+\tau)}\}$. Now, we define

$$\tilde{V}(x,t) = L_2 e^{-\epsilon_0 t}, \quad \forall (x,t) \in \Omega^1_{\xi_*} \cup \Omega^2_{\xi_*} \cup \Omega^3_{\xi_*}.$$

By (3.20), we obtain

$$\begin{gathered} W^{\pm}(x,t) \le C_0 e^{-\rho_1 t} \quad \text{for } (x,t) \in \Omega^1_{\xi_*}, \\ W^{\pm}(x,t) \le M_* \quad \text{for } (x,t) \in \Omega^2_{\xi_*}. \end{gathered} \tag{3.22}$$

Thus $W^{\pm}(x,t) \le \tilde{V}(x,t)$ for all $(x,t) \in \Omega^1_{\xi_*} \cup \Omega^2_{\xi_*}$. From (3.10), we obtain

$$\begin{aligned}\epsilon_0 \tilde{V}(x,t) &= \epsilon_0 L_2 e^{-\epsilon_0 t} \\ &\ge \epsilon L_2 e^{-\epsilon_0 t} + \omega_1 \epsilon e^{-\epsilon_0(t-\tau_1)} + \omega_2 \epsilon e^{-\epsilon_0(t-\tau_2)} \\ &= \epsilon \tilde{V}(x,t) + \omega_1 \epsilon \int_{\mathbb{R}} \Gamma_1(D_1\tau_1, x-y)\tilde{V}(y, t-\tau_1)dy \\ &\quad + \omega_2 \epsilon \int_{\mathbb{R}} \Gamma_2(D_2\tau_2, x-y)\tilde{V}(y, t-\tau_2)dy.\end{aligned} \tag{3.23}$$

Noting that

$$\breve{\lambda} = -(\mu + g_1'(M_*)) + f'(M_*)[\omega_1 e^{-\breve{\lambda}\tau_1} + \omega_2 e^{-\breve{\lambda}\tau_2}],$$

it follows from (3.23) and $\epsilon_0 < -\breve{\lambda} - \epsilon_0 < -\breve{\lambda}$ that

$$\begin{aligned}&\tilde{V}_t(x,t) \\ &= -\epsilon_0 L_2 e^{-\epsilon_0 t} \\ &\ge (\epsilon_0 + \breve{\lambda}) L_2 e^{-\epsilon_0 t} \\ &= L_2 e^{-\epsilon_0 t}\big\{ -(\mu + g_1'(M_*)) + f'(M_*)[\omega_1 e^{-\breve{\lambda}\tau_1} + \omega_2 e^{-\breve{\lambda}\tau_2}] + \epsilon_0\big\} \\ &= -(\mu + g_1'(M_*))\tilde{V}(x,t) + f'(M_*)\omega_1 \int_{\mathbb{R}} \Gamma_1(D_1\tau_1, x-y)\tilde{V}(y, t-\tau_1)dy \\ &\quad + f'(M_*)\omega_2 \int_{\mathbb{R}} \Gamma_2(D_2\tau_2, x-y)\tilde{V}(y, t-\tau_2)dy + \epsilon_0 \tilde{V}(x,t) \\ &\ge -(\mu + g_1'(M_*) - \epsilon)\tilde{V}(x,t) + [f'(M_*) + \epsilon]\omega_1 \int_{\mathbb{R}} \Gamma_1(D_1\tau_1, x-y)\tilde{V}(y, t-\tau_1)dy \\ &\quad + [f'(M_*) + \epsilon]\omega_2 \int_{\mathbb{R}} \Gamma_2(D_2\tau_2, x-y)\tilde{V}(y, t-\tau_2)dy + \epsilon_0 \tilde{V}(x,t),\end{aligned}$$

by which and (3.23), we obtain

$$\frac{\partial \tilde{V}}{\partial t} \ge D\tilde{V}_{xx} - (\mu + g_1'(M_*) - \epsilon)\tilde{V}$$

$$+\omega_1[f'(M_*)+\epsilon]\int_{\mathbb{R}}\Gamma_1(D_1\tau_1,x-y)\tilde{V}(y,t-\tau_1)dy$$
$$+\omega_2[f'(M_*)+\epsilon]\int_{\mathbb{R}}\Gamma_2(D_2\tau_2,x-y)\tilde{V}(y,t-\tau_2)dy.$$

Consequently, using Lemma 3.5 with $c_1=c$, $t_1=t_0+\tau$, $T_1=\infty$ and $\zeta=\gamma_*+\chi$, $Q_1=\omega_1[f'(M_*)+\epsilon]$, and $Q_2=\omega_2[f'(M_*)+\epsilon]$, we deduce that

$$W^{\pm}(x,t)\le \tilde{V}(x,t)=L_2e^{-\epsilon_0 t}\quad \text{for all } (x,t)\in\Omega^3_{\xi_*}.$$

Take $M:=\max\{C_0,L_2\}$. From the above discussions, it holds

$$|u^{\pm}(x,t)-U(x+ct)|=W^{\pm}(x,t)\le Me^{-\epsilon_0 t} \text{ for all } x\in\mathbb{R}, t\ge T_*:=t_*+\tau.$$

The proof of Theorem 3.3 is complete. □

Acknowledgments. We thank the referees for their valuable comments. This research was partially supported by the NSF of China (No. 12171381), by the Natural Science Basic Research Program of Shaanxi (No. 2020JC-24), and by the Fundamental Research Funds for the Central Universities (No. QTZX22050).

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