

## MONOTONICITY PROPERTIES OF THE EIGENVALUES OF NONLOCAL FRACTIONAL OPERATORS AND THEIR APPLICATIONS

GIOVANNI MOLICA BISCI, RAFFAELLA SERVADEI, BINLIN ZHANG

ABSTRACT. In this article we study an equation driven by the nonlocal integrodifferential operator  $-\mathcal{L}_K$  in presence of an asymmetric nonlinear term  $f$ . Among the main results of the paper we prove the existence of at least a weak solution for this problem, under suitable assumptions on the asymptotic behavior of the nonlinearity  $f$  at  $\pm\infty$ . Moreover, we show the uniqueness of this solution, under additional requirements on  $f$ . We also give a non-existence result for the problem under consideration. All these results were obtained using variational techniques and a monotonicity property of the eigenvalues of  $-\mathcal{L}_K$  with respect to suitable weights, that we prove along the present paper. This monotonicity property is of independent interest and represents the nonlocal counterpart of a famous result obtained by de Figueiredo and Gossez [14] in the setting of uniformly elliptic operators.

### 1. INTRODUCTION

In recent years, a great attention has been focused on the study of fractional and nonlocal operators of elliptic type, both for the pure mathematical research and for concrete real-world applications. Fractional and nonlocal operators appear naturally in applications in many fields such as optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves, see [3, 4, 6, 7, 8, 19, 31, 32, 33, 34] and the references therein.

Also thanks to all these applications nonlocal fractional problems are widely studied in the literature. Many authors have considered nonlocal fractional Laplacian equations (and their generalizations) with different growth assumptions on the nonlinear term, such as superlinear and subcritical, critical, asymptotically linear and many others. We refer to the monograph [20] for an overview on these topics.

There are a lot of interesting problems in the standard framework of the Laplacian and, more generally, of uniformly elliptic operators, widely studied in the literature. A natural question is whether or not the results got in this classical

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2020 *Mathematics Subject Classification*. 35A01, 35S15, 47G20, 45G05.

*Key words and phrases*. Fractional Laplacian; integrodifferential operator; nonlocal problems; eigenvalue and eigenfunction; asymmetric nonlinearities; variational methods; critical point theory; saddle point theorem.

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Submitted December 8, 2022. Published December 21, 2022.

context can be extended to the nonlocal framework of the fractional Laplacian type operators.

In this spirit, in this article we are concerned with the existence, non-existence and uniqueness of solutions for the following nonlocal fractional equation with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} -\mathcal{L}_K u &= f(x, u) + g(x) \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{aligned} \quad (1.1)$$

Here  $\Omega$  is an open bounded subset of  $\mathbb{R}^n$  with Lipschitz boundary,  $n > 2s$ ,  $s \in (0, 1)$ , while  $\mathcal{L}_K$  is the integrodifferential operator defined as

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} \left( u(x+y) + u(x-y) - 2u(x) \right) K(y) dy, \quad x \in \mathbb{R}^n, \quad (1.2)$$

where the kernel  $K : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$  is such that

$$mK \in L^1(\mathbb{R}^n), \quad \text{with } m(x) = \min\{|x|^2, 1\}; \quad (1.3)$$

$$\text{there exists } \theta > 0 \text{ such that } K(x) \geq \theta|x|^{-(n+2s)} \text{ for all } x \in \mathbb{R}^n \setminus \{0\}. \quad (1.4)$$

A typical model for  $K$  is the singular kernel  $K(x) = |x|^{-(n+2s)}$  which gives rise to the fractional Laplace operator  $-(-\Delta)^s$ , widely studied in the recent literature (see, for instance, the seminal papers [6, 32, 33]).

Moreover, we suppose that  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function satisfying the following conditions

$$\begin{aligned} &\text{there exist } a_1 \in L^2(\Omega) \text{ and } a_2 \in L^\infty(\Omega) \text{ with } a_2 \geq 0 \text{ such that} \\ &|f(x, t)| \leq a_1(x) + a_2(x)|t| \quad \text{for a.e. } x \in \Omega \text{ and for every } t \in \mathbb{R}; \end{aligned} \quad (1.5)$$

$$\begin{aligned} -\infty \leq \alpha(x) := \lim_{t \rightarrow -\infty} \frac{f(x, t)}{t} \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{f(x, t)}{t} =: \beta(x) \leq +\infty \\ \text{for a.e. } x \in \Omega. \end{aligned} \quad (1.6)$$

Note that the asymptotic growth condition (1.6) on  $f$  at  $-\infty$  and  $+\infty$  includes also the case when the condition at  $-\infty$  is different from the one at  $+\infty$ , in which case  $f$  is called *asymmetric nonlinearity*.

Finally, the function  $g$  is such that

$$g \in L^2(\Omega). \quad (1.7)$$

In the classical context of the Laplacian and uniformly elliptic operators problems like (1.1) were widely studied in the literature, see, for instance, [1, 9]. In the context of nonlinear integral problems this kind of studies goes back to the classical results of Dolph [11].

Note that  $u \equiv 0$  may not be a solution of (1.1), since  $f(x, 0) + g(x)$  may not vanish. In this paper we prove the existence of weak solutions of (1.1) using variational methods.

We denote by

$$\lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots$$

the eigenvalues of the operator  $-\mathcal{L}_K$  in  $\Omega$  with homogeneous Dirichlet boundary condition (see Subsection 2.2 for more details), and by  $X_0^s(\Omega)$  the fractional functional space where we look for solutions (see Subsection 2.1 for a precise definition). The main results of this article can be stated as follows.

**Theorem 1.1.** *Let  $s \in (0, 1)$ ,  $n > 2s$  and  $\Omega$  be an open bounded subset of  $\mathbb{R}^n$  with Lipschitz boundary. Let  $K : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$  satisfy assumptions (1.3) and (1.4) and let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function satisfying (1.5) and (1.6), and  $g : \Omega \rightarrow \mathbb{R}$  be a function satisfying (1.7). Then, the following assertions hold:*

- (i) *if  $\alpha(x), \beta(x) < \lambda_1$  for a.e.  $x \in \Omega$ , then (1.1) has at least one weak solution in  $X_0^s(\Omega)$ .*

*Moreover, the solution is unique if, in addition, for a.e.  $x \in \Omega$  and for all  $t, t' \in \mathbb{R}, t \neq t'$ ,*

$$0 < \frac{f(x, t) - f(x, t')}{t - t'} < \lambda_1 \tag{1.8}$$

*and the eigenfunctions of  $-\mathcal{L}_K$  corresponding to  $\lambda_1$  enjoy the unique continuation property;*

- (ii) *if  $\lambda_k < \alpha(x), \beta(x) < \lambda_{k+1}$  for a.e.  $x \in \Omega$  for some  $k \in \mathbb{N}$  and the eigenfunctions of  $-\mathcal{L}_K$  corresponding to  $\lambda_k$  and the ones corresponding to  $\lambda_{k+1}$  enjoy the unique continuation property, then problem (1.1) has at least one weak solution in  $X_0^s(\Omega)$ .*

*Moreover, the solution is unique if, in addition, for a.e.  $x \in \Omega$  and for all  $t, t' \in \mathbb{R}, t \neq t'$ ,*

$$\lambda_k < \frac{f(x, t) - f(x, t')}{t - t'} < \lambda_{k+1}; \tag{1.9}$$

- (iii) *if either*

$$f(x, t) + g(x) < \lambda_1 t \tag{1.10}$$

*or*

$$f(x, t) + g(x) > \lambda_1 t \tag{1.11}$$

*for a.e.  $x \in \Omega$  and for all  $t \in \mathbb{R}$ , then (1.1) has no weak solution in  $X_0^s(\Omega)$ .*

Fiscella [16] obtained a similar result for the case  $\alpha \equiv \beta$  and  $g \equiv 0$ . Therefore, [16, Theorem 1] can be viewed as a particular case of Theorem 1.1.

In the setting of the fractional Laplacian problem (1.1) reads as follows

$$\begin{aligned} (-\Delta)^s u &= f(x, u) + g(x) && \text{in } \Omega \\ u &= 0 && \text{in } \mathbb{R}^n \setminus \Omega, \end{aligned} \tag{1.12}$$

where  $(-\Delta)^s$  is the fractional Laplace operator defined, up to normalization factors, as

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy \quad x \in \mathbb{R}^n. \tag{1.13}$$

In this article for  $k \in \mathbb{N}$  we denote by  $\lambda_{k,s}$  the eigenvalues of  $(-\Delta)^s$  in  $\Omega$  with homogeneous Dirichlet boundary datum (see Subsection 2.2). In the framework of problem (1.12) we can state Theorem 1.1 as follows.

**Theorem 1.2.** *Let  $s \in (0, 1)$ ,  $n > 2s$  and  $\Omega$  be an open bounded subset of  $\mathbb{R}^n$  with Lipschitz boundary. Let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function satisfying (1.5) and (1.6), and  $g : \Omega \rightarrow \mathbb{R}$  be a function satisfying (1.7). Then the following assertions hold:*

- (i) *if  $\alpha(x), \beta(x) < \lambda_{1,s}$  for a.e.  $x \in \Omega$ , then (1.12) has at least one weak solution in  $H^s(\mathbb{R}^n)$ .*

Moreover, the solution is unique if, in addition, for a.e.  $x \in \Omega$  and for all  $t, t' \in \mathbb{R}, t \neq t'$ ,

$$0 < \frac{f(x, t) - f(x, t')}{t - t'} < \lambda_{1,s}; \quad (1.14)$$

(ii) if  $\lambda_{k,s} < \alpha(x), \beta(x) < \lambda_{k+1,s}$  for a.e.  $x \in \Omega$  for some  $k \in \mathbb{N}$ , then problem (1.12) has at least one weak solution in  $H^s(\mathbb{R}^n)$ .

Moreover, the solution is unique if, in addition, for a.e.  $x \in \Omega$  and for all  $t, t' \in \mathbb{R}, t \neq t'$ ,

$$\lambda_{k,s} < \frac{f(x, t) - f(x, t')}{t - t'} < \lambda_{k+1,s}; \quad (1.15)$$

(iii) if either

$$f(x, t) + g(x) < \lambda_{1,s}t \quad (1.16)$$

or

$$f(x, t) + g(x) > \lambda_{1,s}t \quad (1.17)$$

for a.e.  $x \in \Omega$  and for all  $t \in \mathbb{R}$ , then (1.12) has no weak solution in  $H^s(\mathbb{R}^n)$ .

In the classical setting it is well known that the interaction of  $\alpha$  and  $\beta$  with the spectrum of the Laplace operator is closely related with the existence of weak solutions. Actually, Theorems 1.1 and 1.2 state that the absence of this interaction implies the existence of weak solutions for the fractional equations: this represents the nonlocal counterpart of the classical results by Dolph [11] (for other details we refer also to [1, 13]).

However, there are a lot of cases where this interaction appears. In the seminal paper [2], Ambrosetti and Prodi firstly studied the case in which the derivative of nonlinearity jumps the first eigenvalue of the Laplacian operator (see also [1, 15, 23]). Hence a natural question arises: is there a weak solution in  $X_0^s(\Omega)$  for problem (1.1) if  $\alpha(x) < \lambda_1 < \beta(x)$ , or if  $\lambda_k < \alpha(x) < \lambda_{k+1} < \beta(x)$  a.e. in  $\Omega$ ? The answer is more delicate and still remains an open problem.

This article is organized as follows. Section 2 is devoted to some preliminaries. In Section 3 we investigate an eigenvalues problem for  $-\mathcal{L}_K$  with weights, focusing on a comparison property of the eigenvalues which will be crucial in the proof of our main result. Here we also give a new result on the monotonicity property of the eigenvalues of the fractional Laplacian  $(-\Delta)^s$ , which is of independent interest. In Section 4 we provide the proof of Theorem 1.1, using variational methods, together with the monotonicity result got in Subsection 3.1. Finally, in Section 5 we consider the case of the fractional Laplacian.

## 2. PRELIMINARIES

In this section we briefly introduce the notation and we recall some results for Sobolev fractional functional spaces used in this article.

**2.1. Functional space  $X_0^s(\Omega)$  and its properties.** This subsection is devoted to the definition of the functional space  $X_0^s(\Omega)$  introduced in [27] (see also [28, 29]) and we give some well known properties of it.

The space  $X_0^s(\Omega)$  is defined as

$$X_0^s(\Omega) := \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\},$$

where  $X$  denotes the linear space of Lebesgue measurable functions from  $\mathbb{R}^n$  to  $\mathbb{R}$  such that the restriction to  $\Omega$  of any function  $g$  in  $X$  belongs to  $L^2(\Omega)$  and

the map  $(x, y) \mapsto (g(x) - g(y))\sqrt{K(x - y)}$  is in  $L^2((\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega))$

(here  $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$ ).

Note that the label  $s$  recalls that the kernel  $K$  satisfies (1.4) and that  $X$  and  $X_0^s(\Omega)$  are non-empty, since  $C_0^2(\Omega) \subseteq X_0^s(\Omega)$ , as proved in [27, Lemma 5.1]. We define a norm in  $X_0^s(\Omega)$  as

$$X_0^s(\Omega) \ni g \mapsto \|g\|_{X_0^s(\Omega)} := \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} |g(x) - g(y)|^2 K(x - y) dx dy \right)^{1/2}. \quad (2.1)$$

In this way,  $(X_0^s(\Omega), \|\cdot\|_{X_0^s(\Omega)})$  is a Hilbert space (for this see [28, Lemma 7]), with scalar product

$$\langle u, v \rangle_{X_0^s(\Omega)} := \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(v(x) - v(y))K(x - y) dx dy. \quad (2.2)$$

In the following we denote by  $H^s(\Omega)$  the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo* norm)

$$\|g\|_{H^s(\Omega)} := \|g\|_{L^2(\Omega)} + \left( \int_{\Omega \times \Omega} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2}. \quad (2.3)$$

We remark that, even in the model case in which  $K(x) = |x|^{-(n+2s)}$ , the norms in (2.1) and (2.3) are not the same, because  $\Omega \times \Omega$  is strictly contained in  $\mathbb{R}^n \times \mathbb{R}^n$ . This is the reason why the classical fractional Sobolev space approach is not sufficient for studying problem (1.1).

In [30, Lemma 7] (see also [28, Lemma 5 and Lemma 6]) the authors proved the following result, which states a relation between the space  $X_0^s(\Omega)$  and the usual fractional Sobolev spaces  $H^s(\mathbb{R}^n)$ .

**Lemma 2.1.** *The following assertions hold:*

- (i) *let  $K : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$  satisfy assumptions (1.3) and (1.4). Then  $X_0^s(\Omega) \subseteq H^s(\mathbb{R}^n)$  and, moreover, for all  $v \in X_0^s(\Omega)$*

$$\|v\|_{H^s(\Omega)} \leq \|v\|_{H^s(\mathbb{R}^n)} \leq C \|v\|_{X_0^s(\Omega)},$$

*where  $C$  is a positive constant depending only on  $n, s, \theta$  and  $\Omega$ ;*

- (ii) *let  $K(x) = |x|^{-(n+2s)}$ . Then*

$$X_0^s(\Omega) = \{v \in H^s(\mathbb{R}^n) : v = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

The following embedding result, proved in [28, Lemma 8] and in [30, Lemma 9], holds.

**Lemma 2.2.** *Let  $s \in (0, 1)$ ,  $n > 2s$  and  $\Omega$  be an open bounded subset of  $\mathbb{R}^n$ . Let  $K : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$  satisfy (1.3) and (1.4). Then, the following assertions hold:*

- (i) *if  $\Omega$  has a Lipschitz boundary, then the embedding  $X_0^s(\Omega) \hookrightarrow L^\nu(\mathbb{R}^n)$  is compact for all  $\nu \in [1, 2_s^*)$ , where  $2_s^* = 2n/(n - 2s)$  is the fractional critical Sobolev exponent;*
- (ii) *the embedding  $X_0^s(\Omega) \hookrightarrow L^{2_s^*}(\mathbb{R}^n)$  is continuous.*

The counterpart of Lemma 2.2 in the usual fractional Sobolev spaces is given by the following one proved in [10, Theorem 6.5].

**Lemma 2.3.** *The embedding  $H^s(\mathbb{R}^n) \hookrightarrow L^\nu(\mathbb{R}^n)$  is continuous for all  $\nu \in [2, 2_s^*]$ .*

**2.2. Eigenvalues and eigenfunctions of the operator  $-\mathcal{L}_K$ .** This subsection deals with the following eigenvalue problem associated with the integrodifferential operator  $-\mathcal{L}_K$ :

$$\begin{aligned} -\mathcal{L}_K u &= \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{aligned} \quad (2.4)$$

We denote by  $\{\lambda_k\}_k$  the sequence of the eigenvalues of (2.4), with

$$0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots, \quad \lambda_k \rightarrow +\infty \text{ as } k \rightarrow +\infty \quad (2.5)$$

and by  $e_k$  the eigenfunction corresponding to  $\lambda_k$ . When  $\mathcal{L}_K = -(-\Delta)^s$ , the eigenvalues and the eigenfunctions are denoted by  $\lambda_{k,s}$  and  $e_{k,s}$  for all  $k \in \mathbb{N}$ , respectively.

Moreover, we normalize  $e_k$  in such a way that the sequence  $\{e_k\}_k$  provides an orthonormal basis of  $L^2(\Omega)$  and an orthogonal basis of  $X_0^s(\Omega)$ . We know that  $\lambda_1$  is simple and  $e_1$  is non-negative ( $e_{1,s}$  is positive as proved in [26, Corollary 8]). Finally,  $\lambda_1$  can be characterized as follows

$$\lambda_1 = \min_{u \in X_0 \setminus \{0\}} \frac{\int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dx dy}{\int_{\Omega} |u(x)|^2 dx}. \quad (2.6)$$

For a complete study of the spectrum of the integrodifferential operator  $-\mathcal{L}_K$  we refer to [24, Proposition 2.3], [29, Proposition 9 and Appendix A], and [25, Proposition 4].

Furthermore, we say that the eigenvalue  $\lambda_k$ ,  $k \geq 2$ , has multiplicity  $m \in \mathbb{N}$  if

$$\lambda_{k-1} < \lambda_k = \dots = \lambda_{k+m-1} < \lambda_{k+m}.$$

Then, the set of all the eigenvalues corresponding to  $\lambda_k$  agrees with

$$\text{span}\{e_k, \dots, e_{k+m-1}\}.$$

Finally, for all  $k \in \mathbb{N}$  in the sequel we denote the spaces

$$\mathbb{H}_k := \text{span}\{e_1, \dots, e_k\}, \quad (2.7)$$

$$\mathbb{P}_{k+1} := \{u \in X_0^s(\Omega) : \langle u, e_j \rangle_{X_0^s(\Omega)} = 0 \text{ for } j = 1, \dots, k\}, \quad (2.8)$$

where  $\langle \cdot, \cdot \rangle_{X_0^s(\Omega)}$  is defined by (2.2). When  $\mathcal{L}_K = -(-\Delta)^s$ , the spaces  $\mathbb{H}_k$  and  $\mathbb{P}_{k+1}$  are denoted by  $\mathbb{H}_{k,s}$  and  $\mathbb{P}_{k+1,s}$  for all  $k \in \mathbb{N}$ , respectively.

Using definitions (2.7) and (2.8), the variational characterization of the eigenvalues of  $-\mathcal{L}_K$  (see [29, Proposition 9] and [24, Proposition 2.3]) implies that

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dx dy \geq \lambda_{k+1} \int_{\Omega} |u(x)|^2 dx \quad \text{for all } u \in \mathbb{P}_{k+1} \quad (2.9)$$

and

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dx dy \leq \lambda_k \int_{\Omega} |u(x)|^2 dx \quad \text{for all } u \in \mathbb{H}_k \quad (2.10)$$

for all  $k \in \mathbb{N}$ .

### 3. AN EIGENVALUE PROBLEM FOR $-\mathcal{L}_K$ WITH WEIGHTS

In this section we are concerned with an eigenvalue problem for  $-\mathcal{L}_K$  with weights, whose properties will be used all along the paper. Precisely, we are interested in the eigenvalue problem

$$\begin{aligned} -\mathcal{L}_K u &= \lambda r(x)u \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{aligned} \quad (3.1)$$

where  $r : \overline{\Omega} \rightarrow \mathbb{R}$  is such that

$$r \in \text{Lip}(\overline{\Omega}), \quad (3.2)$$

$$r > 0 \quad \text{in } \Omega. \quad (3.3)$$

The weak formulation of problem (3.1) is

$$\begin{aligned} & \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x - y) dx dy \\ &= \lambda \int_{\Omega} r(x)u(x)\varphi(x) dx \quad \forall \varphi \in X_0^s(\Omega) \\ & u \in X_0^s(\Omega). \end{aligned} \quad (3.4)$$

We say that  $\lambda[r] \in \mathbb{R}$  is an eigenvalue of  $-\mathcal{L}_K$  with weight  $r$  if there exists a non-trivial solution  $u \in X_0^s(\Omega)$  of (3.1) with  $\lambda = \lambda[r]$ . In this case,  $u$  is called eigenfunction corresponding to the eigenvalue  $\lambda[r]$ .

The existence of a sequence of eigenvalues

$$\lambda_1[r] < \lambda_2[r] \leq \dots \leq \lambda_k[r] \leq \dots$$

and of the corresponding eigenfunctions  $e_k[r]$  of (3.1) was proved in [17, Proposition 2.1] (see also [29, Proposition 9 and Appendix A]), under the assumption that the weight  $r$  satisfies (3.2) and (3.3).

We refer to [17, Proposition 2.1] also for the following variational characterization of the eigenvalues and properties of the related eigenfunctions. The first eigenvalue  $\lambda_1[r]$  of problem (3.1) is simple and it is given by

$$\lambda_1[r] = \min_{u \in X_0^s(\Omega) \setminus \{0\}} \frac{\int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dx dy}{\int_{\Omega} r(x)|u(x)|^2 dx} \quad (3.5)$$

and there exists a non-negative function  $e_1[r] \in X_0^s(\Omega)$ , which is an eigenfunction corresponding to  $\lambda_1[r]$ , attaining the minimum in (3.5), that is

$$\begin{aligned} & \int_{\Omega} r(x)|e_1[r](x)|^2 dx = 1 \\ \lambda_1[r] &= \int_{\mathbb{R}^n \times \mathbb{R}^n} |e_1[r](x) - e_1[r](y)|^2 K(x - y) dx dy. \end{aligned} \quad (3.6)$$

Furthermore, for all  $k \in \mathbb{N}$  the eigenvalues of (3.1) can be characterized as follows:

$$\lambda_{k+1}[r] = \min_{u \in \mathbb{P}_{k+1}[r] \setminus \{0\}} \frac{\int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dx dy}{\int_{\Omega} r(x)|u(x)|^2 dx}, \quad (3.7)$$

where

$$\mathbb{P}_{k+1}[r] := \{u \in X_0^s(\Omega) : \langle u, e_j[r] \rangle_{X_0^s(\Omega)} = 0, \forall j = 1, \dots, k\},$$

and by

$$\lambda_k[r] = \max_{u \in \mathbb{H}_k[r] \setminus \{0\}} \frac{\int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dx dy}{\int_{\Omega} r(x) |u(x)|^2 dx}, \quad (3.8)$$

where

$$\mathbb{H}_k[r] := \text{span} \{e_1[r], \dots, e_k[r]\}.$$

Finally, we say that the eigenvalue  $\lambda_k[r]$ ,  $k \geq 2$ , has multiplicity  $m \in \mathbb{N}$  if

$$\lambda_{k-1}[r] < \lambda_k[r] = \dots = \lambda_{k+m-1}[r] < \lambda_{k+m}[r],$$

and, in this case, the set of all the eigenfunctions corresponding to  $\lambda_k[r]$  agrees with

$$\text{span}\{e_k[r], \dots, e_{k+m-1}[r]\}.$$

In [18, Proposition 2.1] the authors obtained another interesting characterization of the eigenvalues  $\lambda_k[r]$ , which is the extension to the nonlocal fractional setting of [14, formula (3)] valid for uniformly elliptic operators.

### 3.1. Monotonicity properties of the eigenvalues of nonlocal operators with respect to the weights.

This subsection is devoted to another important property of the eigenvalues of nonlocal fractional operators. In particular we deal with a monotonicity property of eigenvalues of problem (3.1) with respect to the weights and we obtain a result of independent interest. Firstly we recall that, as a consequence of [18, Proposition 2.1], Frassu and Iannizzotto proved a property of monotone dependence of the eigenvalues  $\lambda_k[r]$  with respect to the weights, provided the eigenvalues  $e_k[r]$  satisfy a unique continuation property.

We say that a family of functions has the *unique continuation property* if no function, besides possibly the zero function, vanishes on a set of positive Lebesgue measure. With this definition, we can state the following monotonicity property (with respect to the weights) for the eigenvalues, proved in [18, Theorem 3.2].

**Proposition 3.1.** *Let  $s \in (0, 1)$ ,  $n > 2s$  and  $\Omega$  be an open bounded subset of  $\mathbb{R}^n$  with Lipschitz boundary and let  $K : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$  satisfy assumptions (1.3) and (1.4). Let  $r_1, r_2 \in L^\infty(\Omega)$  be such that  $0 \not\equiv r_1 \leq r_2$  a.e. in  $\Omega$  and  $r_1 \not\equiv r_2$ . Assume that either the eigenfunctions of (3.1) corresponding to  $\lambda_k[r_1]$  or the ones corresponding to  $\lambda_k[r_2]$  enjoy the unique continuation property for some  $k \in \mathbb{N}$ . Then,  $\lambda_k[r_1] > \lambda_k[r_2]$ .*

Now, let us consider the eigenvalue problem (3.1) in the case  $K(x) = |x|^{-(n+2s)}$ , that is the following one

$$\begin{aligned} (-\Delta)^s u &= \lambda r(x)u && \text{in } \Omega \\ u &= 0 && \text{in } \mathbb{R}^n \setminus \Omega. \end{aligned} \quad (3.9)$$

The main result of this subsection can be stated as follows.

**Proposition 3.2.** *Let  $s \in (0, 1)$ ,  $n > 2s$  and let  $\Omega$  be a bounded domain of  $\mathbb{R}^n$  with Lipschitz boundary. Let  $r_1, r_2 \in C^1(\Omega) \cap L^\infty(\Omega)$  be such that  $0 \not\equiv r_1 \leq r_2$  in  $\Omega$  and  $r_1 \not\equiv r_2$ , and let  $\lambda_{k,s}[r_i]$  be the eigenvalue of (3.9) with  $r = r_i$ ,  $i = 1, 2$ , for some  $k \in \mathbb{N}$ . Then  $\lambda_{k,s}[r_1] > \lambda_{k,s}[r_2]$ .*

*Proof.* By [12, Theorem 1.4] and the regularity assumptions on  $r_1$  and  $r_2$  we know that the eigenfunctions  $e_{k,s}[r_1]$  and  $e_{k,s}[r_2]$  satisfy the unique continuation property. Hence, by Proposition 3.1 we obtain the assertion of Proposition 3.2.  $\square$



Proposition 3.2 can be seen as the nonlocal counterpart of the well known result due to de Figueiredo and Gossez [14, Proposition 1]. Note that Proposition 3.2 improves [18, Corollary 4.2], where the authors consider just the case when  $s \in [1/4, 1)$ .

It is an open question if Proposition 3.2 holds for a general nonlocal fractional operator. As far as we know there are no result in the literature on the unique continuation property for the eigenfunctions of problem (3.1)

#### 4. VARIATIONAL FORMULATION OF THE PROBLEM AND BEYOND

This section is devoted to problem (1.1). In Theorem 1.1 we state existence, uniqueness and non-existence results for it. These results are obtained by using variational methods, together with the monotonicity property given for the eigenvalues of  $-\mathcal{L}_K$  in Proposition 3.1.

First of all, we recall that, thanks to [27, Lemma 5.6] (see also [30, footnote 3]), by a weak solutions of (1.1) we mean a solution of the problem

$$\begin{aligned} & \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x - y)dx dy \\ &= \int_{\Omega} f(x, u(x))\varphi(x)dx + \int_{\Omega} g(x)\varphi(x)dx \quad \forall \varphi \in X_0^s(\Omega) \\ & u \in X_0^s(\Omega). \end{aligned} \quad (4.1)$$

We observe that problem (4.1) has a variational structure, indeed it is the Euler-Lagrange equation of the functional  $\mathcal{J} : X_0^s(\Omega) \rightarrow \mathbb{R}$  defined as follows

$$\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dx dy - \int_{\Omega} F(x, u(x))dx - \int_{\Omega} g(x)u(x)dx,$$

where

$$F(x, t) = \int_0^t f(x, \tau) d\tau. \quad (4.2)$$

Note that the functional  $\mathcal{J}$  is well-defined in  $X_0^s(\Omega)$ , because of assumptions (1.5) on the nonlinear term  $f$ , (1.7) on  $g$ , and Lemma 2.2. Furthermore, it is well known that the functional  $\mathcal{J}$  is Frechét differentiable in  $X_0^s(\Omega)$  and for all  $\varphi \in X_0^s(\Omega)$ ,

$$\begin{aligned} \langle \mathcal{J}'(u), \varphi \rangle &= \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x - y) dx dy \\ &\quad - \int_{\Omega} f(x, u(x))\varphi(x)dx - \int_{\Omega} g(x)\varphi(x)dx. \end{aligned}$$

Thus, critical points of the functional  $\mathcal{J}$  are solutions of (4.1). Hence, our goal consists in looking for critical points of  $\mathcal{J}$ . For this purpose we use Weierstrass Theorem in the case in which the functions  $\alpha$  and  $\beta$  given in (1.6) are less than the first eigenvalue of problem (2.4), while we perform the Saddle Point Theorem by Rabinowitz (see [21, 22]) when  $\alpha$  and  $\beta$  lie between two consecutive eigenvalues of  $-\mathcal{L}_K$ .

Before going on, we observe that, as a consequence of (1.5), it is easy to prove that

$$|F(x, t)| \leq a_1(x)|t| + \frac{a_2(x)}{2}|t|^2 \quad \text{for a.e. } x \in \Omega \text{ and for all } t \in \mathbb{R}. \quad (4.3)$$

Moreover, by (1.6) we obtain that

$$-\infty \leq \frac{\alpha(x)}{2} = \lim_{t \rightarrow -\infty} \frac{F(x,t)}{t^2} \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{F(x,t)}{t^2} = \frac{\beta(x)}{2} \leq +\infty \quad (4.4)$$

for a.e.  $x \in \Omega$ . Now we are ready to prove Theorem 1.1.

**4.1. Proof of Theorem 1.1-(i).** In this subsection we consider the case when

$$\alpha(x), \beta(x) < \lambda_1 \quad \text{for a.e. } x \in \Omega, \quad (4.5)$$

where  $\lambda_1$  is defined as in (2.6). Note that in this setting  $\beta < +\infty$ , while  $\alpha$  may possibly take the value  $-\infty$ .

*Proof of Theorem 1.1-(i).* Our strategy consists in applying the Weierstrass Theorem to the functional  $\mathcal{J}$ . We proceed by steps.

**Step 4.1.** *The functional  $\mathcal{J}$  is weakly lower semicontinuous in  $X_0^s(\Omega)$ .*

For this purpose we claim that the maps

$$u \mapsto \int_{\Omega} F(x, u(x)) dx \quad \text{and} \quad u \mapsto \int_{\Omega} g(x)u(x) dx \quad (4.6)$$

are continuous in the weak topology of  $X_0^s(\Omega)$ .

For this, let  $\{u_j\}_j$  be a sequence in  $X_0^s(\Omega)$  such that  $u_j \rightarrow u_{\infty}$  weakly in  $X_0^s(\Omega)$  as  $j \rightarrow +\infty$ . Then, by Lemma 2.2, up to a subsequence, still denoted by  $u_j$ , we have

$$u_j \rightarrow u_{\infty} \quad \text{in } L^{\nu}(\mathbb{R}^n) \quad \text{for all } \nu \in [1, 2_s^*), \quad (4.7)$$

$$u_j \rightarrow u_{\infty} \quad \text{a.e. in } \mathbb{R}^n \quad (4.8)$$

as  $j \rightarrow +\infty$  and, for all  $\nu \in [1, 2_s^*)$ , there exists  $q_{\nu} \in L^{\nu}(\mathbb{R}^n)$  such that

$$|u_j(x)| \leq q_{\nu}(x) \quad \text{for a.e. in } \mathbb{R}^n \quad (4.9)$$

see, for instance [5, Theorem IV.9]. Hence, by (1.7), (4.3), (4.7), (4.8), and the Dominated Convergence Theorem we deduce (4.6).

Taking into account that the function  $u \mapsto \|u\|_{X_0^s(\Omega)}$  is weakly lower semicontinuous in  $X_0^s(\Omega)$ , by (4.6) we obtain that  $\mathcal{J}$  is weakly lower semicontinuous in  $X_0^s(\Omega)$  and this proves Step 4.1.

**Step 4.2.** *The functional  $\mathcal{J}$  is coercive in  $X_0^s(\Omega)$ .*

Let  $\{u_j\}_j$  be a sequence in  $X_0^s(\Omega)$  such that

$$\|u_j\|_{X_0^s(\Omega)} \rightarrow +\infty \quad (4.10)$$

as  $j \rightarrow +\infty$ . Thus, up to a subsequence, there exists  $v_{\infty} \in X_0^s(\Omega)$  such that

$$v_j := \frac{u_j}{\|u_j\|_{X_0^s(\Omega)}} \rightarrow v_{\infty} \quad \text{weakly in } X_0^s(\Omega) \quad (4.11)$$

as  $j \rightarrow +\infty$  and, by the weak lower semicontinuity of the norm,

$$\|v_{\infty}\|_{X_0^s(\Omega)} \leq 1. \quad (4.12)$$

By (4.11) and Lemma 2.2 it easily follows that

$$\begin{aligned} v_j &\rightarrow v_{\infty} \quad \text{weakly in } X_0^s(\Omega) \\ v_j &\rightarrow v_{\infty} \quad \text{in } L^{\nu}(\mathbb{R}^n) \quad \text{for all } \nu \in [1, 2_s^*) \\ v_j &\rightarrow v_{\infty} \quad \text{a.e. in } \mathbb{R}^n \end{aligned} \quad (4.13)$$

as  $j \rightarrow +\infty$  and, for all  $\nu \in [1, 2_s^*)$ , there exists  $q_\nu \in L^\nu(\Omega)$  such that

$$\frac{|u_j(x)|}{\|u_j\|_{X_0^s(\Omega)}} \leq q_\nu(x) \quad \text{for a.e. } x \in \mathbb{R}^n \tag{4.14}$$

for all  $j \in \mathbb{N}$  (see, for example [5, Theorem IV.9]).

By (4.3) we obtain that

$$0 \leq \frac{|F(x, u_j(x))|}{\|u_j\|_{X_0^s(\Omega)}^2} \leq a_1(x) \frac{|u_j(x)|}{\|u_j\|_{X_0^s(\Omega)}^2} + \frac{a_2(x)}{2} \frac{|u_j(x)|^2}{\|u_j\|_{X_0^s(\Omega)}^2},$$

which implies that

$$\lim_{j \rightarrow +\infty} \frac{F(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}^2} = 0 \quad \text{for a.e. } x \in \Omega \text{ provided that } v_\infty(x) = 0, \tag{4.15}$$

thanks to (4.10) and (4.13).

Now, suppose that  $v_\infty(x) \neq 0$ . Then, again by (4.10) and (4.13) we obtain that

$$|u_j(x)| = \frac{|u_j(x)|}{\|u_j\|_{X_0^s(\Omega)}} \|u_j\|_{X_0^s(\Omega)} \rightarrow +\infty \tag{4.16}$$

as  $j \rightarrow +\infty$ . Hence, by (4.4), (4.13), and (4.16), we have

$$\frac{F(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}^2} = \frac{F(x, u_j(x))}{|u_j(x)|^2} \frac{|u_j(x)|^2}{\|u_j\|_{X_0^s(\Omega)}^2} \rightarrow \frac{r(x)}{2} |v_\infty(x)|^2 \tag{4.17}$$

as  $j \rightarrow +\infty$ , where

$$r(x) := \begin{cases} \beta(x) & \text{if } v_\infty(x) > 0 \\ \alpha(x) & \text{if } v_\infty(x) < 0. \end{cases} \tag{4.18}$$

By (4.17) and the fact that  $r(x) < \lambda_1$  for a.e.  $x \in \Omega$  (see (4.5)), we deduce that

$$\lim_{j \rightarrow +\infty} \frac{F(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}^2} < \frac{\lambda_1}{2} |v_\infty(x)|^2 \quad \text{for a.e. } x \in \Omega \text{ provided that } v_\infty(x) \neq 0. \tag{4.19}$$

All in all (4.15) and (4.19) give

$$\lim_{j \rightarrow +\infty} \frac{F(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}^2} \leq \frac{\lambda_1}{2} |v_\infty(x)|^2 \tag{4.20}$$

for a.e.  $x \in \Omega$ , with strict inequality if  $v_\infty(x) \neq 0$ .

Finally, by (2.6), (4.3), (4.10), (4.13), (4.14), (4.20), and the Dominated Convergence Theorem, we obtain

$$\begin{aligned} L &:= \lim_{j \rightarrow +\infty} \frac{\mathcal{J}(u_j)}{\|u_j\|_{X_0^s(\Omega)}^2} \\ &= \lim_{j \rightarrow +\infty} \left( \frac{1}{2} - \int_\Omega \frac{F(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}^2} dx - \int_\Omega \frac{g(x)u_j(x)}{\|u_j\|_{X_0^s(\Omega)}^2} dx \right) \\ &\geq \frac{1}{2} - \frac{\lambda_1}{2} \int_\Omega |v_\infty(x)|^2 dx \quad (\text{with strict inequality if } v_\infty \not\equiv 0) \\ &\geq \frac{1}{2} - \frac{1}{2} \|v_\infty\|_{X_0^s(\Omega)}^2 \geq 0 \quad (\text{with strict inequality if } v_\infty \equiv 0), \end{aligned} \tag{4.21}$$

since (4.12) holds. Thus  $L > 0$  and by (4.21) we obtain that

$$\mathcal{J}(u_j) > \frac{L}{2} \|u_j\|_{X_0^s(\Omega)}^2$$

for  $j$  sufficiently large. This, together with (4.10), yields that the functional  $\mathcal{J}$  is coercive. This completes the proof of Step 4.2.

By Steps 4.1 and 4.2, it is easy to see that  $\mathcal{J}$  has a minimum  $u \in X_0^s(\Omega)$ , thanks to Weierstrass Theorem. Of course  $u$  is a weak solution of problem (1.1).

Now, it remains to prove that  $u$  is unique, provided (1.8) is satisfied. For this, let  $u_1, u_2 \in X_0^s(\Omega)$  be two distinct weak solutions of problem (1.1) and let  $v = u_1 - u_2$ . Then,  $v \not\equiv 0$  in  $\Omega$  and

$$\begin{aligned} & \int_{\mathbb{R}^n \times \mathbb{R}^n} (v(x) - v(y))(\varphi(x) - \varphi(y))K(x - y)dx dy \\ &= \int_{\Omega} \left( f(x, u_1(x)) - f(x, u_2(x)) \right) \varphi(x) dx \\ &= \int_{\Omega} r(x) \left( u_1(x) - u_2(x) \right) \varphi(x) dx \\ &= \int_{\Omega} r(x)v(x)\varphi(x) dx \end{aligned} \tag{4.22}$$

for all  $\varphi \in X_0^s(\Omega)$ , where

$$r(x) := \begin{cases} \frac{f(x, u_1(x)) - f(x, u_2(x))}{u_1(x) - u_2(x)} & \text{if } u_1(x) \neq u_2(x) \\ 0 & \text{if } u_1(x) = u_2(x). \end{cases}$$

Note that  $r$  is a measurable function non identically zero and, by (1.8),

$$0 \leq r(x) < \lambda_1 \quad \text{for a.e. } x \in \Omega. \tag{4.23}$$

Hence,  $r \in L^\infty(\Omega)$ , being  $\Omega$  bounded.

Since  $v \not\equiv 0$  in  $\Omega$ , by (4.22) we deduce that  $v$  is an eigenfunction of problem (3.1) whose corresponding eigenvalue is 1. Thus, there exists  $\tilde{k} \in \mathbb{N}$  such that  $v = e_{\tilde{k}}[r]$  and the corresponding eigenvalue  $\lambda_{\tilde{k}}[r]$  is such that  $\lambda_{\tilde{k}}[r] = 1$ .

Proposition 3.1, the fact that the eigenfunction of  $-\mathcal{L}_K$  corresponding to  $\lambda_1$  enjoys the unique continuation property (by assumption), and (4.23) yield that

$$1 = \lambda_{\tilde{k}}[r] > \lambda_{\tilde{k}}[\lambda_1] = \frac{\lambda_{\tilde{k}}}{\lambda_1}, \tag{4.24}$$

thanks to the definition of  $\lambda_{\tilde{k}}[\lambda_1]$ .

By (4.24) we obtain the contradiction  $\lambda_1 > \lambda_{\tilde{k}}$  and so  $v \equiv 0$  in  $\Omega$  and the proof of Theorem 1.1-(i) is complete.  $\square$

**4.2. Proof of Theorem 1.1-(ii).** In this subsection we focus on the situation when  $\alpha$  and  $\beta$  in (1.6) satisfy the condition

$$\lambda_k < \alpha(x), \beta(x) < \lambda_{k+1} \quad \text{for a.e. } x \in \Omega \text{ and some } k \in \mathbb{N}, \tag{4.25}$$

where  $\lambda_j$  is the  $j$ -th eigenvalue of problem (2.4),  $j \in \mathbb{N}$ .

*Proof of Theorem 1.1-(ii).* First of all, let us prove the existence of at least one weak solution for problem (1.1): our aim in this setting is to perform the Saddle Point Theorem (see [22]). For this purpose, we proceed by steps.

**Step 4.3.** *All the Palais-Smale sequences for  $\mathcal{J}$  are bounded in  $X_0^s(\Omega)$ .*

Let  $\{u_j\}_j$  be a Palais-Smale sequence for  $\mathcal{J}$ . Assume, by contradiction, that

$$\|u_j\|_{X_0^s(\Omega)} \rightarrow +\infty \tag{4.26}$$

as  $j \rightarrow +\infty$ . We have

$$|\langle \mathcal{J}'(u_j), \varphi \rangle| \leq \varepsilon_j \|\varphi\|_{X_0^s(\Omega)} \quad \text{for all } \varphi \in X_0^s(\Omega), \tag{4.27}$$

where  $\varepsilon_j \rightarrow 0$  as  $j \rightarrow +\infty$ . In particular, setting  $v_j := u_j/\|u_j\|_{X_0^s(\Omega)}$ , by (4.26) and (4.27) we have

$$\begin{aligned} & \lim_{j \rightarrow +\infty} \int_{\mathbb{R}^n \times \mathbb{R}^n} \left( (v_j(x) - v_j(y)) (\varphi(x) - \varphi(y)) K(x - y) dx dy \right. \\ & \left. - \int_{\Omega} \frac{f(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}} \varphi(x) dx - \int_{\Omega} \frac{g(x)}{\|u_j\|_{X_0^s(\Omega)}} \varphi(x) dx \right) = 0. \end{aligned} \tag{4.28}$$

From the fact that  $\{v_j\}_j$  is bounded in  $X_0^s(\Omega)$  and Lemma 2.2, up to a subsequence, still denoted by  $v_j$ , we can assume that there exists  $v_\infty \in X_0^s(\Omega)$  such that

$$\begin{aligned} v_j &\rightarrow v_\infty \quad \text{weakly in } X_0^s(\Omega) \\ v_j &\rightarrow v_\infty \quad \text{in } L^2(\mathbb{R}^n) \\ v_j &\rightarrow v_\infty \quad \text{a.e. in } \mathbb{R}^n \end{aligned} \tag{4.29}$$

as  $j \rightarrow +\infty$ , and there exists  $q \in L^2(\Omega)$  such that

$$|v_j(x)| \leq q(x) \quad \text{for a.e. } x \in \mathbb{R}^n \tag{4.30}$$

for all  $j \in \mathbb{N}$ , see, for instance [5, Theorem IV.9].

Taking into account (1.5) we have

$$0 \leq \frac{|f(x, u_j(x))|}{\|u_j\|_{X_0^s(\Omega)}} \leq \frac{a_1(x)}{\|u_j\|_{X_0^s(\Omega)}} + \frac{a_2(x)|u_j(x)|}{\|u_j\|_{X_0^s(\Omega)}},$$

which implies that

$$\lim_{j \rightarrow +\infty} \frac{f(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}} = 0 \quad \text{for a.e. } x \in \Omega \text{ provided that } v_\infty(x) = 0, \tag{4.31}$$

thanks to (4.26) and (4.29).

Now, let us consider the case when  $v_\infty(x) \neq 0$ . By (1.6) there exists a function  $h$  such that

$$f(x, t) = \beta(x)t^+ + \alpha(x)t^- + h(x, t), \tag{4.32}$$

with

$$\lim_{|t| \rightarrow +\infty} \frac{h(x, t)}{t} = 0, \tag{4.33}$$

where  $t^+ := \max\{t, 0\}$  and  $t^- := \min\{t, 0\}$ . Moreover, by (4.26) we obtain that

$$|u_j(x)| \rightarrow +\infty \quad \text{a.e. } x \in \Omega \quad \text{as } j \rightarrow +\infty. \tag{4.34}$$

Therefore, by (4.29), (4.32), (4.33) and (4.34), we deduce that

$$\begin{aligned} \frac{f(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}} &= \beta(x)v_j^+(x) + \alpha(x)v_j^-(x) + \frac{h(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}} \\ &= \beta(x)v_j^+(x) + \alpha(x)v_j^-(x) + \frac{h(x, u_j(x))}{|u_j(x)|} \frac{|u_j(x)|}{\|u_j\|_{X_0^s(\Omega)}} \\ &\rightarrow \beta(x)v_\infty^+(x) + \alpha(x)v_\infty^-(x) \end{aligned} \tag{4.35}$$

a.e.  $x \in \Omega$ , provided that  $v_\infty(x) \neq 0$ .

All in all, by (4.31) and (4.35), we have

$$\frac{f(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}} \rightarrow \beta(x)v_\infty^+(x) + \alpha(x)v_\infty^-(x) \tag{4.36}$$

a.e.  $x \in \Omega$ . Then, by (4.36), the Dominated Convergence Theorem and again by (4.29) and (4.30), we obtain that

$$\lim_{j \rightarrow +\infty} \int_\Omega \frac{f(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}} \varphi(x) dx = \int_\Omega (\beta(x)v_\infty^+(x) + \alpha(x)v_\infty^-(x)) \varphi(x) dx \tag{4.37}$$

for all  $\varphi \in X_0^s(\Omega)$ .

Taking into account (4.26), (4.28), and (4.37), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n \times \mathbb{R}^n} (v_\infty(x) - v_\infty(y)) (\varphi(x) - \varphi(y)) K(x - y) dx dy \\ &= \int_\Omega (\beta(x)v_\infty^+(x) + \alpha(x)v_\infty^-(x)) \varphi(x) dx \end{aligned}$$

for all  $\varphi \in X_0^s(\Omega)$ . This means that the function  $v_\infty$  satisfies weakly (3.1) with  $\lambda = 1$  and the weight  $r$  given by

$$r(x) := \begin{cases} \beta(x) & \text{if } v_\infty(x) \geq 0 \\ \alpha(x) & \text{if } v_\infty(x) < 0. \end{cases} \tag{4.38}$$

Note that, since (4.25) holds, it follows that

$$\lambda_k < r(x) < \lambda_{k+1} \quad \text{for a.e. } x \in \Omega \tag{4.39}$$

and so  $r \in L^\infty(\Omega)$ , since  $\Omega$  is bounded.

Now we claim that

$$v_\infty \equiv 0 \quad \text{in } \Omega. \tag{4.40}$$

To prove this we argue by contradiction and we suppose that  $v_\infty \not\equiv 0$ . Then  $v_\infty$  is an eigenfunction of (3.1) whose corresponding eigenvalue is 1, that is there exists  $\tilde{k} \in \mathbb{N}$  such that  $v_\infty = e_{\tilde{k}}[r]$  and the corresponding eigenvalue  $\lambda_{\tilde{k}}[r] = 1$ .

Since, by assumption, the eigenfunctions of  $-\mathcal{L}_K$  corresponding to  $\lambda_k$  and the ones corresponding to  $\lambda_{k+1}$  enjoy the unique continuation property, by Proposition 3.1 and (4.39), we deduce that

$$\frac{\lambda_{\tilde{k}}}{\lambda_{k+1}} = \lambda_{\tilde{k}}[\lambda_{k+1}] < \lambda_{\tilde{k}}[r] = 1 < \lambda_{\tilde{k}}[\lambda_k] = \frac{\lambda_{\tilde{k}}}{\lambda_k}, \tag{4.41}$$

taking into account the definition of  $\lambda_{\tilde{k}}[\lambda_j]$ ,  $j \in \mathbb{N}$ .

Hence, (4.41) yields that  $\lambda_{\tilde{k}} \in (\lambda_k, \lambda_{k+1})$ . This is a contradiction, since  $(\lambda_k, \lambda_{k+1})$  does not contain any eigenvalue of  $-\mathcal{L}_K$ . Thus, (4.40) holds and our claim is proved.

By (4.28) with  $\varphi = v_j$ , (4.29), (4.37), and (4.40), we deduce that

$$\begin{aligned} 0 &= \lim_{j \rightarrow +\infty} \frac{\langle \mathcal{J}'(u_j), u_j \rangle}{\|u_j\|_{X_0^s(\Omega)}^2} \\ &= 1 - \lim_{j \rightarrow +\infty} \int_\Omega \frac{f(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}^2} u_j(x) dx - \lim_{j \rightarrow +\infty} \int_\Omega \frac{g(x)}{\|u_j\|_{X_0^s(\Omega)}^2} u_j(x) dx = 1 \end{aligned}$$

which is absurd. Therefore, the sequence  $\{u_j\}_j$  is bounded in  $X_0^s(\Omega)$  and this proves Step 4.3.

**Step 4.4.** *All Palais-Smale sequences for  $\mathcal{J}$  have a convergent subsequence in  $X_0^s(\Omega)$ .*

The proof is quite standard. We repeat it just to make this article selfcontained. Let  $\{u_j\}_j$  be a Palais-Smale sequence for  $\mathcal{J}$  in  $X_0^s(\Omega)$ . By Step 4.3, the sequence  $\{u_j\}_j$  is bounded in  $X_0^s(\Omega)$ . Hence, since  $X_0^s(\Omega)$  is a Hilbert space, up to a subsequence, still denoted by  $u_j$ , there exists  $u_\infty \in X_0^s(\Omega)$  such that

$$\begin{aligned} u_j &\rightarrow u_\infty \quad \text{weakly in } X_0^s(\Omega) \\ u_j &\rightarrow u_\infty \quad \text{in } L^\nu(\mathbb{R}^n) \text{ for all } \nu \in [1, 2_s^*) \\ u_j &\rightarrow u_\infty \quad \text{a.e. in } \mathbb{R}^n \end{aligned} \tag{4.42}$$

as  $j \rightarrow +\infty$  and, for all  $\nu \in [1, 2_s^*)$ , there exists  $q_\nu \in L^\nu(\Omega)$  such that

$$|u_j(x)| \leq q_\nu(x) \text{ a.e. } x \in \mathbb{R}^n \tag{4.43}$$

for all  $j \in \mathbb{N}$ .

By (1.5), (4.42), (4.43), and the Lebesgue Dominated Convergence Theorem, we obtain that

$$\begin{aligned} \int_\Omega f(x, u_j(x))u_j(x)dx &\rightarrow \int_\Omega f(x, u_\infty(x))u_\infty(x) dx \\ \int_\Omega f(x, u_j(x))u_\infty(x) dx &\rightarrow \int_\Omega f(x, u_\infty(x))u_\infty(x) dx \end{aligned} \tag{4.44}$$

as  $j \rightarrow +\infty$ , while, by (1.7) and (4.42) we obtain

$$\int_\Omega g(x)u_j(x) dx \rightarrow \int_\Omega g(x)u_\infty(x) dx \tag{4.45}$$

as  $j \rightarrow +\infty$ .

Moreover, since  $\{u_j\}_j$  is a Palais-Smale sequence, we know that

$$\begin{aligned} \langle \mathcal{J}'(u_j), u_j \rangle_{X_0^s(\Omega)} &\rightarrow 0 \\ \langle \mathcal{J}'(u_j), u_\infty \rangle_{X_0^s(\Omega)} &\rightarrow 0 \end{aligned} \tag{4.46}$$

as  $j \rightarrow +\infty$ . Thus, by (4.44), (4.45), and (4.46), it is easy to see that

$$\|u_j\|_{X_0^s(\Omega)} \rightarrow \|u_\infty\|_{X_0^s(\Omega)}$$

as  $j \rightarrow +\infty$ . This, together with the fact that the sequence  $\{u_j\}_j$  weakly converges to  $u_\infty$  in  $X_0^s(\Omega)$ , gives the desired assertion. This concludes the proof of Step 4.4.

**Step 4.5.** *The functional  $\mathcal{J}$  has the Saddle Point Theorem geometry.*

Let  $k$  be as in (4.25) and let us split the space  $X_0^s(\Omega)$  as  $X_0^s(\Omega) = \mathbb{H}_k \oplus \mathbb{P}_{k+1}$ . First of all, we show that  $\mathcal{J}$  is bounded from below in  $\mathbb{P}_{k+1}$ . For this purpose, let  $\{u_j\}_j$  be a sequence in  $\mathbb{P}_{k+1}$  such that

$$\|u_j\|_{X_0^s(\Omega)} \rightarrow +\infty \tag{4.47}$$

as  $j \rightarrow +\infty$ .

Arguing as in Step 4.2 and taking into account that (4.25) holds (instead of (4.5)), it is easily seen that

$$\lim_{j \rightarrow +\infty} \frac{F(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}^2} \leq \frac{\lambda_{k+1}}{2} |v_\infty(x)|^2 \tag{4.48}$$

for a.e.  $x \in \Omega$ , where  $v_\infty \in \mathbb{P}_{k+1}$  is such that  $u_j/\|u_j\|_{X_0^s(\Omega)} \rightarrow v_\infty$  weakly in  $X_0^s(\Omega)$  as  $j \rightarrow +\infty$ .

Arguing as in Step 4.2, (4.48) yields that

$$L := \lim_{j \rightarrow +\infty} \frac{\mathcal{J}(u_j)}{\|u_j\|_{X_0^s(\Omega)}^2} > 0. \quad (4.49)$$

Hence, by (4.49), we have

$$\mathcal{J}(u_j) \geq \frac{L}{2} \|u_j\|_{X_0^s(\Omega)}^2 \quad (4.50)$$

for  $j$  sufficiently large. By (4.47) and (4.50) we obtain that  $\mathcal{J}(u_j) \rightarrow +\infty$  as  $j \rightarrow +\infty$ , which yields

$$\lim_{u \in \mathbb{P}_{k+1}, \|u\|_{X_0^s(\Omega)} \rightarrow +\infty} \mathcal{J}(u) = +\infty. \quad (4.51)$$

By (4.51) we deduce that there exists  $M$  such that

$$\mathcal{J}(u) \geq 1 \quad \text{for all } u \in \mathbb{P}_{k+1} \text{ with } \|u\|_{X_0^s(\Omega)} \geq M. \quad (4.52)$$

Now, let  $u \in \mathbb{P}_{k+1}$  be such that  $\|u\|_{X_0^s(\Omega)} < M$ . By (2.6) and (4.3) we have

$$\begin{aligned} \mathcal{J}(u) &= \frac{1}{2} \|u\|_{X_0^s(\Omega)}^2 - \int_{\Omega} F(x, u(x)) \, dx - \int_{\Omega} g(x)u(x) \, dx \\ &\geq -\|a_1\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} - \frac{1}{2} \|a_2\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}^2 - \|g\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \quad (4.53) \\ &\geq -\frac{\|a_1\|_{L^2(\Omega)}}{\sqrt{\lambda_1}} M - \frac{1}{2} \frac{\|a_2\|_{L^\infty(\Omega)}}{\lambda_1} M^2 - \frac{\|g\|_{L^2(\Omega)}}{\sqrt{\lambda_1}} M =: -K. \end{aligned}$$

By (4.52) and (4.53) we obtain that  $\mathcal{J}(u) \geq -K$  for all  $u \in \mathbb{P}_{k+1}$ , that is  $\mathcal{J}$  is bounded from below in  $\mathbb{P}_{k+1}$ .

Now, we have to show that there exists  $R > 0$  such that

$$\sup_{u \in \mathbb{H}_k, \|u\|_{X_0^s(\Omega)} = R} \mathcal{J}(u) < -K. \quad (4.54)$$

where  $K$  is given in (4.53).

For this purpose, let  $\{u_j\}_j$  be a sequence in  $\mathbb{H}_k$  such that

$$\|u_j\|_{X_0^s(\Omega)} \rightarrow +\infty \quad (4.55)$$

as  $j \rightarrow +\infty$ . With the same arguments used for proving (4.48) and taking into account that (2.10) and (4.25) hold, we have

$$\lim_{j \rightarrow +\infty} \frac{F(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}^2} \geq \frac{\lambda_k}{2} |v_\infty(x)|^2 \quad (4.56)$$

for a.e.  $x \in \Omega$ , with strict inequality when  $v_\infty(x) \neq 0$ . Here  $v_\infty \in \mathbb{H}_k$  is such that  $u_j / \|u_j\|_{X_0^s(\Omega)} \rightarrow v_\infty$  weakly in  $X_0^s(\Omega)$  as  $j \rightarrow +\infty$ . Since  $\mathbb{H}_k$  is a finite dimensional space,  $\|v_\infty\|_{X_0^s(\Omega)} = 1$  and so

$$v_\infty \neq 0. \quad (4.57)$$



Now, by (2.10), (4.55), (4.56), and (4.57), we have

$$\begin{aligned}
 L &:= \lim_{j \rightarrow +\infty} \frac{\mathcal{J}(u_j)}{\|u_j\|_{X_0^s(\Omega)}^2} \\
 &= \lim_{j \rightarrow +\infty} \left( \frac{1}{2} - \int_{\Omega} \frac{F(x, u_j(x))}{\|u_j\|_{X_0^s(\Omega)}^2} dx - \int_{\Omega} \frac{g(x)u_j(x)}{\|u_j\|_{X_0^s(\Omega)}^2} dx \right) \\
 &< \frac{1}{2} - \frac{\lambda_k}{2} \int_{\Omega} |v_{\infty}(x)|^2 dx \\
 &\leq \frac{1}{2} - \frac{1}{2} \|v_{\infty}\|_{X_0^s(\Omega)}^2 = 0,
 \end{aligned} \tag{4.58}$$

since  $\|v_{\infty}\|_{X_0^s(\Omega)} = 1$ .

By (4.58) we have

$$\mathcal{J}(u_j) < \frac{L}{2} \|u_j\|_{X_0^s(\Omega)}^2 \tag{4.59}$$

for  $j$  sufficiently large. By (4.55), (4.58), and (4.59), we obtain that  $\mathcal{J}(u_j) \rightarrow -\infty$  as  $j \rightarrow +\infty$ . Hence,

$$\lim_{u \in \mathbb{H}_k, \|u\|_{X_0^s(\Omega)} \rightarrow +\infty} \mathcal{J}(u) = -\infty.$$

Thus, there exists  $R > 0$  such that (4.54) is verified. This concludes the proof of Step 4.5.

Thanks to Steps 4.3, 4.4, and 4.5, the functional  $\mathcal{J}$  satisfies the assumptions of the Saddle Point Theorem. Hence,  $\mathcal{J}$  admits a critical point and this proves the existence of a weak solution for problem (1.1).

Now, it remains to prove that this solution is unique, provided (1.9) is satisfied. For this, let  $u_1, u_2 \in X_0^s(\Omega)$  be two distinct weak solutions of (4.1). A simple calculation shows that

$$\begin{aligned}
 \langle u_1 - u_2, \varphi \rangle_{X_0^s(\Omega)} &= \int_{\Omega} \left( f(x, u_1(x)) - f(x, u_2(x)) \right) \varphi(x) dx \\
 &= \int_{\Omega} r(x) \left( u_1(x) - u_2(x) \right) \varphi(x) dx
 \end{aligned}$$

for all  $\varphi \in X_0^s(\Omega)$ , where

$$r(x) := \begin{cases} \frac{f(x, u_1(x)) - f(x, u_2(x))}{u_1(x) - u_2(x)} & \text{if } u_1(x) \neq u_2(x) \\ \frac{1}{2}(\lambda_k + \lambda_{k+1}) & \text{if } u_1(x) = u_2(x). \end{cases}$$

Note that  $r$  is a measurable function and  $\lambda_k < r(x) < \lambda_{k+1}$  for a.e.  $x \in \Omega$  by (4.25).

Arguing as in Step 4.3 (see the proof of (4.40)), we can show that  $u_1 - u_2 \equiv 0$  in  $X_0^s(\Omega)$ . This is a contradiction and this completes the proof of Theorem 1.1-(ii).  $\square$

### 4.3. Non-existence of solutions.

*Proof of Theorem 1.1-(iii).* We argue by contradiction and we suppose that problem (1.1) admits one weak solution  $u \in X_0^s(\Omega)$ .

Taking  $\varphi = e_1$  as a test function in (4.1), where  $e_1$  is the eigenfunction associated to the first eigenvalue  $\lambda_1$  of  $-\mathcal{L}_K$ , we obtain that

$$\int_{\Omega} \left( f(x, u(x)) + g(x) - \lambda_1 u(x) \right) e_1(x) dx = 0. \tag{4.60}$$

Taking into account that  $e_1 \neq 0$  and  $e_1 \geq 0$  in  $\Omega$ , (4.60) is in contradiction to both (1.10) and (1.11). This completes the proof of Theorem 1.1.  $\square$

We end this section with the following comment.

**Remark 4.6.** Note that in the setting (i) of Theorem 1.1, we need the unique continuation property of the eigenfunctions of  $-\mathcal{L}_K$  corresponding to  $\lambda_1$  just for proving the uniqueness of solution of (1.1). While, in the framework (ii) of Theorem 1.1 the same assumption on the eigenfunctions of  $\lambda_k$  and the ones of  $\lambda_{k+1}$  is necessary both for the proof of the existence and of the uniqueness of weak solution of problem (1.1).

## 5. FRACTIONAL LAPLACIAN PROBLEM WITH ASYMMETRIC NONLINEARITY

In this section we consider problem (1.12) and we prove Theorem 1.2. First of all we observe that Theorem 1.2 can not be derived completely by Theorem 1.1. This is so because in the proof of Theorem 1.2 we can not use Proposition 3.2 due to the regularity assumption on the weight  $r$ : note that, while in Proposition 3.1 the weight  $r$  is a  $L^\infty$ -function, in Proposition 3.2 we require that  $r \in C^1(\Omega) \cap L^\infty(\Omega)$ .

*Proof of Theorem 1.2.* For assertion (i) we can use the same arguments of the proof of Theorem 1.1-(i), to prove the existence of at least a weak solution of problem (1.12). We have to make some changes when proving the uniqueness of the solution.

For this purpose, assume by contradiction that problem (1.12) admits two distinct weak solutions  $u_1, u_2 \in X_0^s(\Omega)$  and let  $v = u_1 - u_2$ . Of course  $v \neq 0$  and

$$\begin{aligned} & \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(v(x) - v(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dx dy \\ &= \int_{\Omega} \left( f(x, u_1(x)) - f(x, u_2(x)) \right) \varphi(x) dx \\ &= \int_{\Omega} r(x)(u_1(x) - u_2(x)) \varphi(x) dx \\ &= \int_{\Omega} r(x)v(x)\varphi(x) dx \end{aligned} \tag{5.1}$$

for all  $\varphi \in X_0^s(\Omega)$ , where

$$r(x) := \begin{cases} \frac{f(x, u_1(x)) - f(x, u_2(x))}{u_1(x) - u_2(x)} & \text{if } u_1(x) \neq u_2(x) \\ 0 & \text{if } u_1(x) = u_2(x). \end{cases}$$

Note that  $r$  is a measurable function not identically zero and, by (1.14),

$$0 \leq r(x) < \lambda_{1,s} \quad \text{for a.e. } x \in \Omega. \tag{5.2}$$

Testing (5.1) with  $\varphi = v$  and taking into account (5.2), we obtain that

$$\|v\|_{X_0^s(\Omega)}^2 = \int_{\Omega} r(x)|v(x)|^2 dx < \lambda_{1,s} \|v\|_{L^2(\Omega)}^2 \leq \|v\|_{X_0^s(\Omega)}^2,$$

thanks to the variational characterization of  $\lambda_{1,s}$ . This is a contradiction and this completes the proof of Theorem 1.2-(i).

Now, let us show Theorem 1.2-(ii). Also in this setting we can argue as in the proof of Theorem 1.1-(ii). We have to change something in the proof of Step 4.3 when proving that (4.40) holds. We have  $v_\infty$  satisfies

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(v_\infty(x) - v_\infty(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dx dy = \int_{\Omega} r(x)v_\infty(x)\varphi(x)dx \quad (5.3)$$

for all  $\varphi \in X_0^s(\Omega)$ , where the weight  $r$  is the measurable function given in (4.38), which, by assumption, satisfies the condition

$$\lambda_{k,s} < r(x) < \lambda_{k+1,s} \quad \text{for a.e. } x \in \Omega. \quad (5.4)$$

Since the space  $X_0^s(\Omega)$  can be split as  $X_0^s(\Omega) = \mathbb{H}_{k,s} \oplus \mathbb{P}_{k+1,s}$ , it follows that  $v_\infty$  can be written as  $v_\infty = \bar{v}_\infty + \tilde{v}_\infty$ , where  $\bar{v}_\infty \in \mathbb{H}_{k,s}$  and  $\tilde{v}_\infty \in \mathbb{P}_{k+1,s}$ . Taking  $\varphi = \bar{v}_\infty$  and then  $\varphi = \tilde{v}_\infty$  as test functions in (5.3) and taking into account the definitions of  $\mathbb{H}_{k,s}$  and  $\mathbb{P}_{k+1,s}$ , the orthogonality between  $\bar{v}_\infty$  and  $\tilde{v}_\infty$  in  $X_0^s(\Omega)$  and (5.4), we have

$$\begin{aligned} \|\bar{v}_\infty\|_{X_0^s(\Omega)}^2 &= \int_{\Omega} r(x)|\bar{v}_\infty(x)|^2 dx + \int_{\Omega} r(x)\bar{v}_\infty(x)\tilde{v}_\infty(x) dx \\ &\geq \lambda_{k,s}\|\bar{v}_\infty\|_{L^2(\Omega)}^2 + \int_{\Omega} r(x)\bar{v}_\infty(x)\tilde{v}_\infty(x) dx \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} \|\tilde{v}_\infty\|_{X_0^s(\Omega)}^2 &= \int_{\Omega} r(x)|\tilde{v}_\infty(x)|^2 dx + \int_{\Omega} r(x)\tilde{v}_\infty(x)\bar{v}_\infty(x) dx \\ &\leq \lambda_{k+1,s}\|\tilde{v}_\infty\|_{L^2(\Omega)}^2 + \int_{\Omega} r(x)\tilde{v}_\infty(x)\bar{v}_\infty(x) dx. \end{aligned} \quad (5.6)$$

If  $v_\infty \not\equiv 0$ , then at least one of the functions  $\bar{v}_\infty$  and  $\tilde{v}_\infty$  is not identically zero. Thus, at least one of the inequalities (5.5) and (5.6) has to be strict, also thanks to (5.4). Hence, using the variational characterization of the eigenvalues given in (2.9) and (2.10) (with  $K(x) = |x|^{-(n+2s)}$ ), by (5.5) and (5.6) we have

$$\begin{aligned} \|\bar{v}_\infty\|_{X_0^s(\Omega)}^2 - \|\tilde{v}_\infty\|_{X_0^s(\Omega)}^2 &> \lambda_{k,s}\|\bar{v}_\infty\|_{L^2(\Omega)}^2 - \lambda_{k+1,s}\|\tilde{v}_\infty\|_{L^2(\Omega)}^2 \\ &\geq \|\bar{v}_\infty\|_{X_0^s(\Omega)}^2 - \|\tilde{v}_\infty\|_{X_0^s(\Omega)}^2, \end{aligned}$$

which is a contradiction. This means that  $v_\infty \equiv 0$ . From here on we can follow the proof of Step 4.3.

The remaining part of the proof needs no changes with respect to the proof of Theorem 1.1 and this shows Theorem 1.2.  $\square$

**Acknowledgements.** G. Molica Bisci and R. Servadei are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). B. Zhang was supported by National Natural Science Foundation of China (No. 11871199 and No. 12171152).

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GIOVANNI MOLICA BISCI

DIPARTIMENTO DI SCIENZE PURE E APPLICATE (DISPEA), UNIVERSITÀ DEGLI STUDI DI URBINO  
CARLO BO, PIAZZA DELLA REPUBBLICA 13, 61029 URBINO (PESARO E URBINO), ITALY  
*Email address:* [giovanni.molicabisci@uniurb.it](mailto:giovanni.molicabisci@uniurb.it)

RAFFAELLA SERVADEI

DIPARTIMENTO DI SCIENZE PURE E APPLICATE (DISPEA), UNIVERSITÀ DEGLI STUDI DI URBINO  
CARLO BO, PIAZZA DELLA REPUBBLICA 13, 61029 URBINO (PESARO E URBINO), ITALY  
*Email address:* [raffaella.servadei@uniurb.it](mailto:raffaella.servadei@uniurb.it)

BINLIN ZHANG

COLLEGE OF MATHEMATICS AND SYSTEMS SCIENCE, SHANDONG UNIVERSITY OF SCIENCE AND  
TECHNOLOGY, QINGDAO 266590, CHINA  
*Email address:* [zhangbinlin2012@163.com](mailto:zhangbinlin2012@163.com)