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WELL-POSEDNESS OF STOCHASTIC TIME FRACTIONAL 2D-STOKES MODELS WITH FINITE AND INFINITE DELAY

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Abstract. We analyze the well-posedness of two versions of a stochastic time delay fractional 2D-Stokes model with nonlinear multiplicative noise. The main tool to prove the existence and uniqueness of mild solutions is a fixed point argument. The results for the first model can only be proved for $\alpha \in (1/2,1)$, and the global existence in time is shown only when the noise is additive. As for the second model, all results are true for $\alpha \in (0,1)$, and the global solutions in time is shown for general nonlinear multiplicative noise. The analyzes for the finite and infinite delay cases, follow the same lines, but they require different phase spaces and estimates. This article can be considered as a first approximation to the challenging model of stochastic time fractional Navier-Stokes (with or without delay) which so far remains as an open problem.

1. Introduction

Xu et al. [16] studied the well-posedness of the stochastic time delay fractional 2D-Stokes equations of order $\alpha \in (0,1)$ with nonlinear multiplicative noise,

$$\begin{gathered} \partial_t^\alpha u - \kappa\Delta u + \nabla p = f(t,u_t) + g(t,u_t)\frac{dW(t)}{dt} \quad \text{in } \mathbb{R}^2,\ t>0, \\ \nabla\cdot u = 0 \quad \text{in } \mathbb{R}^2,\ t>0, \\ u(t,x) = \varphi(t,x) \quad \text{in } \mathbb{R}^2,\ t\in[-h,0]\,. \end{gathered} \tag{1.1}$$

Here f and g are external forcing terms containing some hereditary or delay characteristics, φ is the initial datum on the time interval $[-h,0]$ and h is a fixed positive number (in the case of bounded delay) or $h=\infty$ (for unbounded delay), $W(t)$ is a standard Brownian motion/Wiener process on an underlying complete filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$.

This model can be considered as a first linear approximation of the most challenging problem of the stochastic time fractional 2D-Navier-Stokes equations,

$$\begin{gathered} \partial_t^\alpha u - \kappa\Delta u + u\cdot\nabla u + \nabla p = f(t,u_t) + g(t,u_t)\frac{dW(t)}{dt} \quad \text{in } \mathbb{R}^2,\ t>0, \\ \nabla\cdot u = 0 \quad \text{in } \mathbb{R}^2,\ t>0, \\ u(t,x) = \varphi(t,x) \quad \text{in } \mathbb{R}^2,\ t\in[-h,0], \end{gathered} \tag{1.2}$$

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which is our future final goal because of the importance of Navier-Stokes problems in the research of turbulent fluids. The deterministic model (i.e. $f = g = 0$) has been analyzed by Carvalho-Neto and Planas [4], and the existence (and eventual uniqueness) of mild solutions were proved by them. However, because of the reasons mentioned in [16], new techniques are necessary to handle the stochastic problem (1.2) in an appropriate way, mainly because of the difficulties generated by the nonlinear term $u \cdot \nabla u$. In [16], a detailed introduction and background with many references related to problem (1.2) were described, thus we recommend the reader to read it first for a better understanding of the analysis in this article.

Let us now recall that in [16], the authors proved the existence and uniqueness of mild solutions to problem (1.1), for $\alpha \in (0,1)$, based on [16, Definition 2]. However, this definition given in [16] does not match the standard one of mild solution for an evolution equation with time fractional derivative like (1.1). Despite of the fact that all results in [16] were correctly proved according to such definition, it is desirable and necessary to state a more accurate definition that matches the usual ones for time fractional derivative which requires $\alpha \in (1/2, 1)$. Hence, our first objective in this paper is to re-establish the results in [16] according to the new definition that we will state in this paper (see Definition 3.2 in Section 3). In addition, we will analyze another problem which contains more time regularity in the coefficients, ensuring better results for the asymptotic behavior of the system. In particular, the well-posedness can be proved for $\alpha \in (0,1)$. More precisely, we will consider the problem

$$
\begin{gathered}
\partial_t^\alpha u - \kappa\Delta u + \nabla p = J_t^{1-\alpha}[f(t,u_t)] + J_t^{1-\alpha}[g(t,u_t)\frac{dW(t)}{dt}] \quad \text{in } \mathbb{R}^2,\ t>0,\\
\nabla \cdot u = 0 \quad \text{in } \mathbb{R}^2,\ t>0,\\
u(t,x) = \varphi(t,x) \quad \text{in } \mathbb{R}^2,\ t \in [-h,0],
\end{gathered} \tag{1.3}
$$

where $J_t^{1-\alpha}$ is the Riemann-Liouville fractional integral of order $1-\alpha$ defined in Section 2. The main reasons supporting this kind of models with more regular coefficient can be found in the previous literature (see, e.g. Li and Wang [7] and the references therein), but to mention some of them we can say that, in addition that the well-posedness is proved for $\alpha \in (0,1)$, the expression for the mild solution (see Definition 4.1 in Section 4) only involves one of the Mittag-Leffler operators and does not contain singular kernel, which allows us to obtain better estimates in the computations, also to prove global in time existence of solutions for general multiplicative noise (not only for additive one as in the first case). Furthermore, one can have a much better asymptotic behavior of solutions because of the compactness properties of the Mittag-Leffler operator involved in the formulation.

Although there are several possibilities to handle the problems with time fractional derivative, we have chosen the Caputo fractional time derivative, whose advantage, amongst others, is that the derivative of a constant function is zero. Thus, time-independent solutions are also solutions of the time-dependent problem [1]. Also, compared with Riemann-Liouville derivative [12], Caputo derivative removes singularities at the origin and shares many similarities with the classical derivative, so that they are suitable for initial value problems.

We have structured our paper as follows. Section 2 is devoted to briefly recall some relevant preliminaries for our analysis. In Section 3, we first investigate the well-posedness of (1.1) by considering the case of bounded/finite delay and proving the existence, uniqueness and continuous dependence of mild solutions with respect to initial data. We can only prove that the mild solution is globally defined in time when the noise is additive. The results are shown under two sets of assumptions. Next we consider the case of unbounded/infinite delay and prove similar results but with substantial differences concerning the phase space and estimates. In contrast, these results are only valid for $\alpha \in (1/2, 1)$. In Section 4, a parallel analysis is carried out but for (1.3). In this case, since the external forcing terms possess more regularity, we can perform the complete analysis of well-posedness and global existence in time for general nonlinear multiplicative noise term, as well as for any $\alpha \in (0, 1)$. Eventually, some conclusions and comments about future directions are included in Section 5.

2. Preliminaries

Let us recall some basic background, notation and properties of Mittag-Leffler functions; see [2, 3, 4, 11, 12, 13, 14] and the references therein for more information.

2.1. **Stochastic theory and notation.**

First we fix a stochastic basis,

$$\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W),$$

where $\mathbb{P}$ is a probability measure on Ω, $\mathcal{F}$ is a σ-algebra, $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration on $(\Omega, \mathcal{F})$ such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible subsets and $W(t) = W(\omega, t)$, $\omega \in \Omega$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

To set problems (1.1) and (1.3) in an abstract framework, we consider the standard notation L^2_σ to describe the subspace of the divergence-free vector fields in L^2:

$$L^2_\sigma = \{u \in L^2 : \nabla \cdot u = 0 \text{ in } \mathbb{R}^2\},$$

with norm $\|\cdot\|$, where L^2 denotes the vector-valued Lebesgue space and for $u \in L^2$,

$$\|u\|^2 = \sum_{j=1}^{2} \int_{\mathbb{R}^2} |u_j(x)|^2 dx.$$

In a similar way, we define L^r_σ for $r > 1$. Besides, let $S \subset \mathbb{R}$ and X be a Banach space. We denote the space of continuous functions from S to X by $C(S; X)$ (equipped with its usual supremum norm). $L^2(S; X)$ denotes the Banach space of L^2 integrable functions $u : S \to X$. $H^1(S; X) = W^{1,2}(S; X)$ is the subspace of $L^2(S; X)$ consisting of functions such that the weak derivative $\frac{\partial u}{\partial t}$ belongs to $L^2(S; X)$. Both spaces $L^2(S; X)$ and $W^{1,2}(S; X)$ are endowed with their standard norms. Moreover, we denote $a \wedge b = \min\{a, b\}$.

Consider a fixed $T > 0$, given $u : [-h, T] \to L^2_\sigma$, for each $t \in [0, T]$, we denote by u_t the function defined on $[-h, 0]$ as

$$u_t(s) = u(t + s), \quad s \in [-h, 0],$$

where $h > 0$ denotes the delay. When $h = \infty$, it denotes an infinite or unbounded delay. Furthermore, let $L^2(\Omega; X)$ be the Banach space of X-valued random variables with norm $\|u(\cdot)\|^2_{L^2(\Omega;X)} = \mathbb{E}\|u(\cdot)\|^2_X$, where the expectation $\mathbb{E}$ is defined by $\mathbb{E}u = \int_\Omega u(\cdot) d\mathbb{P}$.

2.2. **Fractional setting and Mittag-Leffler operators.** Let us now introduce some basic and useful results concerning the fractional calculus theory. For $\alpha > 0$, define the function $g_\alpha : \mathbb{R} \to \mathbb{R}$ by

$$g_\alpha(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $\Gamma(\alpha)$ denotes the Euler Gamma function. Assume that $T > 0$, for a function $u \in L^1([0,T]; X)$, the Riemann-Liouville fractional integral of order α of u is given by

$$J_t^\alpha u(t) := g_\alpha * u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad t \in [0,T].$$

Thus, based on the definition of Riemann-Liouville fractional integral operator, we present the Caputo fractional differential operator (see [4, 6, 12] and the references therein for more details).

Definition 2.1 ([4, Def. 1])**.** Let $\alpha \in (0,1)$ and $T > 0$. Consider $u \in C([0,T]; X)$ such that the convolution $g_{1-\alpha} * u \in W^{1,1}([0,T]; X)$. The expression

$$D_t^\alpha u(t) := \frac{d}{dt}\big\{ J_t^{1-\alpha}[u(t) - u(0)] \big\} = \frac{d}{dt}\big\{ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}[u(s) - u(0)] ds \big\}$$

is called the Caputo fractional derivative of order α of the function u.

We recall now some properties of the Mainardi function [3] denoted by M_α. This function is a particular case of the Wright type function introduced by Mainardi in [11]. More precisely, for $\alpha \in (0,1)$, the entire function $M_\alpha : \mathbb{C} \to \mathbb{C}$ is

$$M_\alpha(z) := \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(1 - \alpha(1+n))}.$$

Some basic properties of the Mainardi function are necessary in this paper to deal with most estimates in the proofs.

Proposition 2.2 ([4, Pro. 2])**.** *For $\alpha \in (0,1)$ and $-1 < r < \infty$, when we restrict M_α to the positive real line, it holds that*

$$M_\alpha(t) \geq 0 \quad \text{for all } t \geq 0, \quad \text{and} \quad \int_0^\infty t^r M_\alpha(t) dt = \frac{\Gamma(r+1)}{\Gamma(\alpha r + 1)}.$$

The next results are classical computations in the literature dealing with Mittag-Leffler operators, as can be seen, for instance, in [4]. Indeed, let X be a Banach space and $-\mathcal{A} : D(\mathcal{A}) \subset X \to X$ be the infinitesimal generator of an analytic semigroup $\{T(t) : t \geq 0\}$. Then, for each $\alpha \in (0,1)$, we define the Mittag-Leffler families $\{\mathbf{E}_\alpha(-t^\alpha \mathcal{A}) : t \geq 0\}$ and $\{\mathbf{E}_{\alpha,\alpha}(-t^\alpha \mathcal{A}) : t \geq 0\}$ by

$$\mathbf{E}_\alpha(-t^\alpha \mathcal{A}) = \int_0^\infty M_\alpha(s) T(st^\alpha) ds,$$

$$\mathbf{E}_{\alpha,\alpha}(-t^\alpha \mathcal{A}) = \int_0^\infty \alpha s M_\alpha(s) T(st^\alpha) ds.$$

Note that the Mainardi function acts as a bridge between the fractional and the classical abstract theories. This relation is based on the inversion of certain Laplace transforms in order to obtain the fundamental solutions of the fractional diffusion-wave equations (see, e.g., [3, 4, 8] and the references therein). The following lemma contains the main assertions on Mittag-Leffler operators.

Lemma 2.3 ([4, Theorem 3]). *Operators* $\mathbf{E}_\alpha(-t^\alpha\mathcal{A})$ *and* $\mathbf{E}_{\alpha,\alpha}(-t^\alpha\mathcal{A})$ *are well defined from* X *to* X*. Moreover, for* $x\in X$*, it holds:*

(i) $\mathbf{E}_\alpha(-t^\alpha\mathcal{A})x|_{t=0}=x$;

(ii) *The functions* $t\in[0,+\infty)\mapsto\mathbf{E}_\alpha(-t^\alpha\mathcal{A})x$ *and* $t\in[0,+\infty)\mapsto\mathbf{E}_{\alpha,\alpha}(-t^\alpha\mathcal{A})x$ *are analytic from* $[0,+\infty)$ *to* X*.*

3. Analysis of problem (1.1)

We can rewrite the time fractional stochastic 2D-Stokes delay differential equations (1.1) in the abstract form

$$\begin{gathered} D_t^\alpha u=-Au+F(t,u_t)+G(t,u_t)\frac{dW(t)}{dt},\quad t>0,\\ u(t)=\varphi(t),\quad t\in(-h,0], \end{gathered}\tag{3.1}$$

where $A=-P\Delta=-\Delta P$, $F(t,u_t)=Pf(t,u_t)$, and $G(t,u_t)=Pg(t,u_t)$. Here, $P:L^2\to L^2_\sigma$ is the Helmholtz-Leray projector and $A:D(A)\subset L^2_\sigma\to L^2_\sigma$ is the Stokes operator.

Before stating the correct definition of mild solutions to problem (3.1), let us recall the properties of both families of Mittag-Leffler operators, which furnish the essential tools used throughout the whole article, see [4, 5] for more details.

Lemma 3.1. *Consider* $\alpha\in(0,1)$*, and* r_1*,* r_2 *real numbers satisfying*

$$1<r_1\le r_2<\infty\quad\text{and}\quad r_2N/(2r_2+N)<r_1.$$

Then, for any $v\in L^{r_1}_\sigma$*, there exists a constant* $C_1=C_1(r_1,r_2,N,\alpha)>0$ *such that*

$$\text{(i)}\quad \|\mathbf{E}_\alpha(-t^\alpha A_{r_1})v\|_{L^{r_2}}\le C_1t^{-\alpha(N/r_1-N/r_2)/2}\|v\|_{L^{r_1}},\quad t>0,$$

$$\text{(ii)}\quad \|\mathbf{E}_{\alpha,\alpha}(-t^\alpha A_{r_1})v\|_{L^{r_2}}\le C_1t^{-\alpha(N/r_1-N/r_2)/2}\|v\|_{L^{r_1}},\quad t>0,$$

where A_r *denotes the Stokes operator from* $D(A_r)\subset L^r_\sigma\to L^r_\sigma$*.*

Definition 3.2. Let $\mathcal{S}=(\Omega,\mathcal{F},\{\mathcal{F}\}_{t\ge0},\mathbb{P})$ be a fixed stochastic basis generated by a standard Brownian motion W and $T>0$. Consider $\alpha\in(1/2,1)$. Let φ be an initial function such that $\varphi(t)$ is $\mathcal{F}_0$-measurable for $t\in(-h,0]$. A mild solution to problem (3.1) on $(-h,T]$ is a stochastic process such that $u(t)=\varphi(t)$ for $t\in(-h,0]$, and fulfills, for $t\in[0,T]$,

$$\begin{aligned} u(t)&=\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)+\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds\\ &\quad+\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s),\quad \mathbb{P}\text{-a.s.} \end{aligned}\tag{3.2}$$

Remark 3.3. The Stokes operator $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}:t\ge0\}$. Hence, the Mittag-Leffler families $\mathbf{E}_\alpha(-t^\alpha A)$ and $\mathbf{E}_{\alpha,\alpha}(-t^\alpha A)$ are well defined.

It is worth mentioning that the analysis in this paper can be easily extended to the case in which system (3.1) is driven by a Hilbert valued Brownian motion/Wiener process in infinite dimensions. However we prefer to consider this simpler formulation for the sake of clarity to the reader.

The definition of mild solutions established in [16] omitted the kernel $(t-s)^{\alpha-1}$ in both integrals in (3.2). Consequently, the concept of mild solution used in [16] does not provide the usual one of mild solution in the deterministic case, i.e., $G \equiv 0$.

3.1. **Well-posedness of problem** (3.1) **with bounded delay.** In this subsection, we justify the well-posedness of the fractional stochastic 2D-Stokes equations with bounded delay

$$\begin{gathered} D_t^\alpha u = -Au + F(t,u_t) + G(t,u_t)\frac{dW(t)}{dt}, \quad t>0, \\ u(t) = \varphi(t), \quad t \in [-h,0], \end{gathered} \tag{3.3}$$

where h is a positive fixed constant (finite delay).

First, we need to introduce suitable Banach spaces, which aim to capture the essence of the problem. For any fixed $T>0$, consider the Banach space $\mathcal{X}_2^T$ which is the set of continuous functions $u:[-h,T] \to L^2(\Omega; L^2_\sigma)$ equipped with its natural norm

$$\|u\|_{\mathcal{X}_2^T} = \Big(\sup_{t\in[-h,T]} \mathbb{E}\|u(t)\|^2\Big)^{1/2}.$$

When no confusion is possible we will omit T in $\mathcal{X}_2^T$.

Next, we state the hypotheses imposed on both external forcing terms in our problem. Let F, $G : [0,+\infty) \times C([-h,0]; L^2(\Omega; L^2_\sigma)) \to L^2(\Omega; L^2_\sigma)$.

- (A1) For each $\xi \in C([-h,0]; L^2(\Omega; L^2_\sigma))$, the mappings $t \in [0,+\infty) \to F(t,\xi)$ and $t \in [0,+\infty) \to G(t,\xi)$ are measurable.
- (A2) $F(\cdot,0) = G(\cdot,0) = 0$ (for simplicity).
- (A3) There exist positive constants L_F and L_G, such that, for all $t \in [0,\infty)$ and $\xi, \eta \in C([-h,0];\ L^2(\Omega; L^2_\sigma))$,
$$\begin{aligned} \|F(t,\xi) - F(t,\eta)\|^2_{L^2(\Omega;L^2_\sigma)} &\le L_F \|\xi - \eta\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))}, \\ \|G(t,\xi) - G(t,\eta)\|^2_{L^2(\Omega;L^2_\sigma)} &\le L_G \|\xi - \eta\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))}. \end{aligned}$$
- (A4) There exists a constant $L_f > 0$, such that the function $F : [0,\infty) \times C([-h,0]; L^2(\Omega; L^2_\sigma)) \to L^2(\Omega; L^2_\sigma)$ satisfies
$$\int_0^t \mathbb{E}\|F(s,u_s) - F(s,v_s)\|^2 ds \le L_f \int_{-h}^t \mathbb{E}\|u(s) - v(s)\|^2 ds,$$
for all u, $v \in C([-h,T]; L^2(\Omega; L^2_\sigma))$.
- (A5) There exists a constant $L_g > 0$, such that the function $G : [0,\infty) \times C([-h,0]; L^2(\Omega; L^2_\sigma)) \to L^2(\Omega; L^2_\sigma)$ satisfies
$$\int_0^t \mathbb{E}\|G(s,u_s) - G(s,v_s)\|^2 ds \le L_g \int_{-h}^t \mathbb{E}\|u(s) - v(s)\|^2 ds,$$
for all $u, v \in C([-h,T]; L^2(\Omega; L^2_\sigma))$.

We can now establish a first result on local existence and uniqueness of mild solutions to problem (3.3) by a fixed point argument.

Theorem 3.4. *Let $\alpha \in (1/2,1)$, (A1)–(A3) hold, and $\varphi \in C([-h,0]; L^2(\Omega; L^2_\sigma))$ such that $\varphi(t)$ is $\mathcal{F}_0$-measurable for all $t \in [-h,0]$. Then, there exists $T>0$ (small enough) such that problem* (3.3) *admits a unique mild solution in the sense of Definition 3.2 on $[-h,T]$.*

Proof. Let $\varphi \in C([-h,0];L^2(\Omega;L^2_\sigma))$ be the initial function, and choose $R>0$ such that

$$(3C_1^2+1)\|\varphi\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))} \le \frac{R^2}{2},$$

where C_1 is the constant in Lemma 3.1 for $r_1=r_2=2$.

For some $T>0$ which will be fixed later on, we define the space $\mathcal{C}_R^\varphi$ for the previous R:

$$\mathcal{C}_R^\varphi = \big\{u\in C([-h,T];L^2(\Omega;L^2_\sigma)) : u(t)=\varphi(t) \text{ for all } t\in[-h,0],\ \|u\|_{\mathcal{X}_2}\le R\big\}.$$

Let us moreover define the operator $\mathcal{N}$ on $\mathcal{C}_R^\varphi$ with $\alpha\in(1/2,1)$ as follows,

$$(\mathcal{N}u)(t)=\begin{cases}\varphi(t), \quad t\in[-h,0],\\ \mathbf{E}_\alpha(-t^\alpha A)\varphi(0)+\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds\\ +\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s), \quad t\in(0,T],\ \mathbb{P}\text{-a.s.}\end{cases}$$

Assertion 1: $\mathcal{N}u\in C([-h,T];L^2(\Omega;L^2_\sigma))$, for every $u\in C([-h,T];L^2(\Omega;L^2_\sigma))$. Observe that, if $t\in[-h,0]$, then $(\mathcal{N}u)(t)=\varphi(t)$ and $\varphi\in C([-h,0];L^2(\Omega;L^2_\sigma))$. It only remains to prove the continuity of $\mathcal{N}u$ on $[0,T]$. But this follows immediately from the proof of [4, Lemma 11], with the help of the analytical property of the Mittag-Leffler operators in time (see Lemma 2.3(ii)). We omit the details here.

Assertion 2: There exists $T>0$ (sufficiently small) such that $\|\mathcal{N}u\|_{\mathcal{X}_2}\le R$, for all $u\in\mathcal{C}_R^\varphi$. Indeed, for any $u\in\mathcal{C}_R^\varphi$, it holds

$$\|\mathcal{N}u\|_{\mathcal{X}_2}=\Big(\sup_{t\in[-h,T]}\mathbb{E}\|(\mathcal{N}u)(t)\|^2\Big)^{1/2}. \tag{3.4}$$

On the one hand, for $t\in[-h,0]$, we have

$$\mathbb{E}\|(\mathcal{N}u)(t)\|^2=\mathbb{E}\|\varphi(t)\|^2\le\sup_{t\in[-h,0]}\mathbb{E}\|\varphi(t)\|^2=\|\varphi\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))}. \tag{3.5}$$

On the other hand, for $t\in(0,T]$, we have

$$\begin{aligned}\mathbb{E}\|(\mathcal{N}u)(t)\|^2 &\le 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2\\ &\quad+3\mathbb{E}\Big\|\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds\Big\|^2\\ &\quad+3\mathbb{E}\Big\|\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s)\Big\|^2\\ &:=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3.\end{aligned} \tag{3.6}$$

Now we estimate each term on the right-hand side of (3.6). For $\mathcal{I}_1$, by Lemma 3.1(i), it is obvious that

$$\mathcal{I}_1=3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2\le 3C_1^2\mathbb{E}\|\varphi(0)\|^2\le 3C_1^2\sup_{t\in[-h,0]}\mathbb{E}\|\varphi(t)\|^2. \tag{3.7}$$

For $\mathcal{I}_2$, by Lemma 3.1(i), (A1)–(A3), the Cauchy-Schwarz inequality and Fubini's theorem, we obtain

$$
\begin{aligned}
\mathcal{I}_2 &= 3\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds\Big\|^2 \\
&\le 3\mathbb{E}\Big(\int_0^t (t-s)^{\alpha-1}\|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)\|ds\Big)^2 \\
&\le \frac{3C_1^2 t^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\|F(s,u_s)\|^2 ds \\
&\le \frac{3C_1^2 L_F t^{2\alpha-1}}{2\alpha-1}\int_0^t \|u_s\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))}ds \\
&\le \frac{3C_1^2 L_F t^{2\alpha-1}}{2\alpha-1}\int_0^t \sup_{\theta\in[-h,0]}\mathbb{E}\|u(s+\theta)\|^2 ds \\
&\le \frac{3C_1^2 L_F t^{2\alpha-1}}{2\alpha-1}\int_0^t \sup_{\theta\in[-h,T]}\mathbb{E}\|u(\theta)\|^2 ds \\
&\le \frac{3C_1^2 L_F t^{2\alpha}}{2\alpha-1}R^2.
\end{aligned} \tag{3.8}
$$

For $\mathcal{I}_3$, by Lemma 3.1(i), Itô's isometry and (A1)–(A3), we have

$$
\begin{aligned}
\mathcal{I}_3 &= 3\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s)\Big\|^2 \\
&\le 3\mathbb{E}\int_0^t (t-s)^{2\alpha-2}\|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)\|^2 ds \\
&\le 3C_1^2 L_G\int_0^t (t-s)^{2\alpha-2}\|u_s\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))}ds \\
&\le \frac{3C_1^2 L_G t^{2\alpha-1}}{2\alpha-1}R^2.
\end{aligned} \tag{3.9}
$$

Replacing (3.7)–(3.9) into (3.6), combining with (3.5), we obtain

$$
\mathbb{E}\|(\mathcal{N}u)(t)\|^2 \le 3C_1^2\|\varphi\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))} + \frac{3C_1^2 L_F t^{2\alpha}R^2}{2\alpha-1} + \frac{3C_1^2 L_G t^{2\alpha-1}R^2}{2\alpha-1}.
$$

Consequently, thanks to the choice of R, we can choose T small enough such that

$$
\begin{aligned}
\|\mathcal{N}u\|_{\mathcal{X}_2} &= \Big(\sup_{t\in[-h,T]}\mathbb{E}\|(\mathcal{N}u)(t)\|^2\Big)^{1/2} \\
&\le \Big((3C_1^2+1)\|\varphi\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))} + \frac{3C_1^2 L_F T^{2\alpha}R^2}{2\alpha-1} \\
&\quad + \frac{3C_1^2 L_G T^{2\alpha-1}R^2}{2\alpha-1}\Big)^{1/2} \le R.
\end{aligned} \tag{3.10}
$$

Assertion 3: Operator $\mathcal{N}:\mathcal{C}_R^\varphi\to\mathcal{C}_R^\varphi$ is a contraction. To this end, for any u, $v\in\mathcal{C}_R^\varphi$, it follows that

$$
\|\mathcal{N}u-\mathcal{N}v\|_{\mathcal{X}_2} := \Big(\sup_{t\in[-h,T]}\mathbb{E}\|(\mathcal{N}u)(t)-(\mathcal{N}v)(t)\|^2\Big)^{1/2}. \tag{3.11}
$$

Noticing that, for $t \in [-h, 0]$, one has $(\mathcal{N}u)(t) = (\mathcal{N}v)(t) = \varphi(t)$, it is sufficient to consider $t \in (0, T]$. Observe that

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{N}u)(t) - (\mathcal{N}v)(t)\|^2 \\
&\le 2\mathbb{E}\Big\| \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s) - F(s,v_s))ds\Big\|^2 \\
&\quad + 2\mathbb{E}\Big\| \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s) - G(s,v_s))dW(s)\Big\|^2 \\
&:= \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned} \tag{3.12}$$

For $\mathcal{J}_1$, by Lemma 3.1(i), (A3), the Cauchy-Schwarz inequality and Fubini's theorem, we obtain

$$\begin{aligned}
\mathcal{J}_1 &= 2\mathbb{E}\Big\| \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s) - F(s,v_s))ds\Big\|^2 \\
&\le 2\mathbb{E}\Big(\int_0^t (t-s)^{\alpha-1}\|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s) - F(s,v_s))\|ds\Big)^2 \\
&\le \frac{2C_1^2 L_F t^{2\alpha-1}}{2\alpha-1}\int_0^t \|u_s - v_s\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))}ds \\
&\le \frac{2C_1^2 L_F t^{2\alpha}}{2\alpha-1}\sup_{s\in[0,t]}\mathbb{E}\|u(s) - v(s)\|^2.
\end{aligned} \tag{3.13}$$

As for $\mathcal{J}_2$, by Lemma 3.1(i), (A3) and Itô's isometry, we deduce

$$\begin{aligned}
\mathcal{J}_2 &= 2\mathbb{E}\Big\| \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s) - G(s,v_s))dW(s)\Big\|^2 \\
&\le 2C_1^2 L_G \int_0^t (t-s)^{2\alpha-2}\|u_s - v_s\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))}ds \\
&\le \frac{2C_1^2 L_G t^{2\alpha-1}}{2\alpha-1}\sup_{s\in[0,t]}\mathbb{E}\|u(s) - v(s)\|^2.
\end{aligned} \tag{3.14}$$

Hence, substituting (3.13)-(3.14) into (3.12), it follows that

$$\begin{aligned}
\|\mathcal{N}u - \mathcal{N}v\|_{\mathcal{X}_2} &\le \Big(\Big(\frac{2C_1^2 L_F T^{2\alpha}}{2\alpha-1} + \frac{2C_1^2 L_G T^{2\alpha-1}}{2\alpha-1}\Big)\sup_{t\in[0,T]}\mathbb{E}\|u(t) - v(t)\|^2\Big)^{1/2}, \\
&:= \mathbf{M}\|u - v\|_{\mathcal{X}_2},
\end{aligned}$$

where

$$\mathbf{M}^2 = \frac{2C_1^2 L_F T^{2\alpha}}{2\alpha-1} + \frac{2C_1^2 L_G T^{2\alpha-1}}{2\alpha-1}.$$

Therefore, we can choose T small enough such that $0 < \mathbf{M} < 1$. In other words, operator $\mathcal{N}$ maps $\mathcal{C}_R^\varphi$ into itself and it is a contraction. The Banach fixed-point Theorem ensures that operator $\mathcal{N}$ possesses a fixed point in $\mathcal{C}_R^\varphi$. Namely, problem (3.3) has a unique local mild solution on $[-h, T]$. □

With similar arguments as in the proof of the previous theorem, we can prove the local existence and uniqueness of mild solutions to problem (3.3) under assumptions (A4) and (A5) instead of (A3). We establish the result in the next theorem but we omit the proof.

Theorem 3.5. *Let $\alpha \in (1/2, 1)$. Assume that* (A1), (A2), (A4), (A5) *hold, and $\varphi \in C([-h,0]; L^2(\Omega; L^2_\sigma))$ such that $\varphi(t)$ is $\mathcal{F}_0$-measurable for all $t \in [-h,0]$. Then, there exists $T > 0$ (small enough) such that problem* (3.3) *admits a unique mild solution in the sense of Definition 3.2 on $[-h, T]$.*

As far as we are aware, none of the known techniques can be used to ensure existence (and uniqueness) of a global mild solutions to problem (3.3) when it is driven by a general nonlinear term $G(t, u_t)$. However, in the case of additive noise (i.e. $G \equiv \phi \in L^2_\sigma$), we can prove the existence and uniqueness of global solutions.

Theorem 3.6. *Under the assumptions of Theorem 3.4 (or Theorem 3.5), suppose that $G(t,\xi) = \phi \in L^2_\sigma$ for all $\xi \in C([-h,0]; L^2(\Omega; L^2_\sigma))$ and $t \in [0,T]$. Then, for every initial value $\varphi \in C([-h,0]; L^2(\Omega; L^2_\sigma))$ such that $\varphi(t)$ is $\mathcal{F}_0$-measurable for all $t \in [-h,0]$, the initial value problem* (3.3) *has a unique mild solution defined globally in the sense of Definition 3.2.*

Proof. We prove the result under assumptions of Theorem 3.4 since the proof under conditions of Theorem 3.5 is similar.

Initially, assume that there are two solutions to problem (3.3), u and v on $[0, T_1]$ and $[0, T_2]$ respectively. Next, let us prove that $u = v$ on $[-h, T_1 \wedge T_2]$. It is clear that $u(t) = v(t) = \varphi(t)$ on $[-h, 0]$, hence, we only need to prove that $u(t) = v(t)$ for any $t \in (0, T_1 \wedge T_2]$. Notice that

$$\|u - v\|^2_{\mathcal{X}_2} := \sup_{t\in[-h,T_1\wedge T_2]} \mathbb{E}\|u(t) - v(t)\|^2 = \sup_{t\in[0,T_1\wedge T_2]} \mathbb{E}\|u(t) - v(t)\|^2. \tag{3.15}$$

By Lemma 3.1(i), (A3) and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
&\mathbb{E}\|u(t) - v(t)\|^2 \\
&\le \mathbb{E}\Big\| \int_0^t (t-s)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s) - F(s,v_s))ds\Big\|^2 \\
&\le \mathbb{E}\Big(\int_0^t (t-s)^{\alpha-1} \|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s) - F(s,v_s))\|ds\Big)^2 \\
&\le \frac{C_1^2 t^{2\alpha-1} L_F}{2\alpha - 1} \int_0^t \|u_s - v_s\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))} ds \\
&\le \frac{C_1^2 L_F t^{2\alpha-1}}{2\alpha - 1} \int_0^t \sup_{\sigma\in[0,s]} \mathbb{E}\|u(\sigma) - v(\sigma)\|^2 ds.
\end{aligned} \tag{3.16}$$

Denoting $\mathbf{M}_1 = \frac{C_1^2 L_F T^{2\alpha-1}}{2\alpha-1}$, we have

$$\sup_{s\in[-h,t]} \mathbb{E}\|u(s) - v(s)\|^2 \le \mathbf{M}_1 \int_0^t \Big(\sup_{\sigma\in[-h,s]} \mathbb{E}\|u(\sigma) - v(\sigma)\|^2\Big) ds,$$

for all $t \in [0, T_1 \wedge T_2]$. Then Gronwall's Lemma implies that

$$\sup_{s\in[-h,t]} \mathbb{E}\|u(s) - v(s)\|^2 = 0, \quad \text{for all } t \in [0, T_1 \wedge T_2].$$

Therefore, $u \equiv v$ on $[-h, T_1 \wedge T_2]$.

Now we prove that for each given $T > 0$, the mild solution to problem (3.3) with additive noise is bounded with $\mathcal{X}_2$ norm. Taking into account Lemma 3.1(i), (A1)–(A3), Itô's isometry, the Cauchy-Schwarz inequality and Fubini's theorem, for $t \in [0,T]$, we have

$$
\begin{aligned}
&\mathbb{E}\|u(t)\|^2 \\
&\le 3\mathbb{E}\big\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2 + 3\mathbb{E}\big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds\big\|^2 \\
&\quad + 3\mathbb{E}\big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)\phi dW(s)\big\|^2 \\
&\le 3C_1^2 \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|^2 + \frac{3C_1^2 t^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\|F(s,u_s)\|^2 ds + \frac{3C_1^2\|\phi\|^2 t^{2\alpha-1}}{2\alpha-1} \\
&\le 3C_1^2 \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|^2 + \frac{3C_1^2\|\phi\|^2 t^{2\alpha-1}}{2\alpha-1} \\
&\quad + \frac{3C_1^2 t^{2\alpha-1}L_F}{2\alpha-1}\int_0^t \|u_s\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))}ds.
\end{aligned}
$$

Therefore, for all $t \in [0,T]$,

$$
\begin{aligned}
\sup_{s\in[-h,t]} \mathbb{E}\|u(s)\|^2 &\le (3C_1^2+1)\sup_{s\in[-h,0]} \mathbb{E}\|\varphi(s)\|^2 \\
&\quad + \frac{3C_1^2 L_F T^{2\alpha-1}}{2\alpha-1}\int_0^t \sup_{\sigma\in[-h,s]}\mathbb{E}\|u(\sigma)\|^2 ds + \frac{3C_1^2\|\phi\|^2 T^{2\alpha-1}}{2\alpha-1} \\
&:= A_1(\varphi,T,\phi) + \mathbf{M}_2\int_0^t \sup_{\sigma\in[-h,s]}\mathbb{E}\|u(\sigma)\|^2 ds,
\end{aligned}
$$

where

$$
A_1(\varphi,T,\phi) := (3C_1^2+1)\sup_{t\in[-h,0]}\mathbb{E}\|\varphi(t)\|^2 + \frac{3C_1^2\|\phi\|^2 T^{2\alpha-1}}{2\alpha-1},
$$

$$
\mathbf{M}_2 := \frac{3C_1^2 L_F T^{2\alpha-1}}{2\alpha-1}.
$$

Applying the Gronwall lemma, for any fixed $T > 0$, we obtain

$$
\|u\|^2_{\mathcal{X}_2} \le A_1(\varphi,T,\phi)\exp(\mathbf{M}_2 T).
$$

Because of the arbitrariness of T, together with the conclusion of uniqueness of u on $[-h,T]$, it is straightforward that the mild solution to (3.3) driven by additive noise is defined globally. □

Now, we complete our analysis of well-posedness to (3.3) by proving the continuous dependence of global mild solutions on the initial data.

Proposition 3.7. *Under the assumptions of Theorem 3.4 (or Theorem 3.5), assume that the mild solution to* (3.3) *is globally defined. Then, it is continuous with respect to the initial data in* $C([-h,0];L^2(\Omega;L^2_\sigma))$. *In particular, if* $u(t)$, $w(t)$ *are the corresponding mild solutions to the initial data* ζ *and* ψ *on* $[-h,T]$, *then the following estimate holds*

$$
\|u-w\|_{\mathcal{X}_2} \le \|\zeta-\psi\|_{C([-h,0];L^2(\Omega;L^2_\sigma))}\exp\Big(\frac{C_1^2 L_F T^{2\alpha}}{2\alpha-1}\Big).
$$

The above proposition is proved similarly to the proof of uniqueness in the previous theorem, so we omit the details here.

3.2. **Well-posedness of mild solutions to (3.3) with unbounded delay.** In this subsection, we consider the following stochastic time fractional 2D-Stokes equation with unbounded delay,

$$\begin{gathered} D_t^\alpha u = -Au + F(t,u_t) + G(t,u_t)\frac{dW(t)}{dt}, \quad t > 0, \\ u(t) = \varphi(t), \quad t \in (-\infty, 0]. \end{gathered} \tag{3.17}$$

Before going a step further to prove the main results, we first introduce a suitable space motivated by unbounded delay. Let $\mathbb{H}$ be a separable Hilbert space, then the space $\mathcal{C}_X$ on $\mathbb{H}$ is defined as

$$\mathcal{C}_X(\mathbb{H}) = \big\{\varphi \in C((-\infty,0];\mathbb{H}) : \lim_{\theta\to-\infty} \varphi(\theta) \text{ exists in } \mathbb{H}\big\},$$

which is a Banach space equipped with the norm

$$\|\varphi\|_{\mathcal{C}_X} = \sup_{\theta\in(-\infty,0]} \|\varphi(\theta)\|_{\mathbb{H}}.$$

Let F, $G : [0,\infty) \times \mathcal{C}_X(L^2(\Omega;L^2_\sigma)) \to L^2(\Omega;L^2_\sigma)$ and assume that:

- (A1') For any $\xi \in \mathcal{C}_X(L^2(\Omega;L^2_\sigma))$, the mappings $[0,\infty) \ni t \mapsto F(t,\xi) \in L^2(\Omega;L^2_\sigma)$ and $[0,\infty) \ni t \mapsto G(t,\xi) \in L^2(\Omega;L^2_\sigma)$ are measurable.
- (A2') $F(\cdot,0) = G(\cdot,0) = 0$ (for simplicity).
- (A3') There exist two positive constants L'_F and L'_G, such that, for all $t \in [0,\infty)$ and ξ, $\eta \in \mathcal{C}_X(L^2(\Omega;L^2_\sigma))$,

$$\begin{gathered} \|F(t,\xi) - F(t,\eta)\|^2_{L^2(\Omega;L^2_\sigma)} \le L'_F \|\xi-\eta\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}, \\ \|G(t,\xi) - G(t,\eta)\|^2_{L^2(\Omega;L^2_\sigma)} \le L'_G \|\xi-\eta\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \end{gathered}$$

It is quite usual, when dealing with unbounded delay differential equations, to adopt a different space for the initial data [17], namely,

$$C^\gamma(\mathbb{H}) = \big\{\varphi \in C((-\infty,0];\mathbb{H}) : \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}\|\varphi(\theta)\|_{\mathbb{H}} < +\infty\big\}.$$

However, if we consider this space, then hypothesis (A3') is not fulfilled when the delay in F or G is a variable delay one. For instance, $F(t,u_t) = F_0(u(t-\rho(t)))$, where ρ is a measurable function taking nonnegative values and $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is a Lipschitz function (see [9]). Therefore, this new space $\mathcal{C}_X(\mathbb{H})$, although it is a bit more restrictive than the usual one, allows us to consider more general delay terms in the functional formulation.

We can now state our main results on well-posedness of problem (3.17). The proof is similar to the case of bounded delay by replacing the norm in $C([-h,0]; L^2(\Omega;L^2_\sigma))$ by the one in $\mathcal{C}_X(L^2(\Omega;L^2_\sigma))$. For the sake of completeness, we will include the proof in the Appendix.

Theorem 3.8. *Let $\alpha \in (1/2,1)$ and (A1')–(A3') hold. Then, for each initial function $\varphi \in \mathcal{C}_X(L^2(\Omega;L^2_\sigma))$ such that $\varphi(t)$ is $\mathcal{F}_0$-measurable for all $t \in [-h,0]$, problem (3.17) admits a unique mild solution in the sense of Definition 3.2 on $(-\infty,T]$, for sufficiently small $T > 0$.*

As in the case of bounded delay, we can prove the existence of global mild solution when the noise is additive.

Theorem 3.9. *Assume the hypotheses of Theorem 3.8 hold. Then for every initial value* $\varphi \in C((-\infty,0];L^2(\Omega;L^2_\sigma))$ *such that* $\varphi(t)$ *is* $\mathcal{F}_0$*-measurable for all* $t\in[-h,0]$*, the initial value problem* (3.17) *driven by an additive noise* $G\equiv\phi\in L^2_\sigma$ *has a unique mild solution defined globally in the sense of Definition 3.2.*

Proof. The proof follows the same lines as the case of bounded delay, but with differences in the estimates which make interesting to include it. We will shift it to the Appendix section. □

Proposition 3.10. *Under the assumptions of Theorem 3.8, suppose that the mild solutions to* (3.17) *are globally defined in time. Then it is continuous with respect to the initial data* $\varphi\in C((-\infty,0];L^2(\Omega;L^2_\sigma))$*. In particular, if* $u(t)$*,* $w(t)$ *are the mild solutions on* $(-\infty,T]$ *corresponding to the initial data* ζ *and* ψ*, then*

$$\|u_T-w_T\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}\le 3C_1^2\|\zeta-\psi\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}\exp\Big(\frac{C_1^2L_F'T^{2\alpha}}{2\alpha-1}\Big).$$

This proposition is proved by similar arguments to those concerning the uniqueness of the previous theorem. We omit it here.

4. Analysis of problem (1.3)

In this section, we will study the time fractional stochastic delay 2D-Stokes equations (4.1) below which contain more regular coefficients,

$$\begin{gathered}D_t^\alpha u=-Au+J_t^{1-\alpha}[F(t,u_t)]+J_t^{1-\alpha}[G(t,u_t)\frac{dW(t)}{dt}],\quad t>0,\\ u(t)=\varphi(t),\quad t\in(-h,0].\end{gathered}\tag{4.1}$$

As we mentioned in the Introduction, in this case the expression for the mild solutions only involves one of the Mittag-Leffler operators, yielding to a more usual variation of constants formula without singular kernel in the integrals. Moreover, the results can be proved now for $\alpha\in(0,1)$ instead of $\alpha\in(1/2,1)$, and the existence of global solutions can be shown for general multiplicative noise. Also the compactness properties of the Mittag-Leffler operator allow us to analyze the asymptotic behavior of the system in a forthcoming paper. All these reasons justify the interest of this model (see [15, 18] and the references therein for more details).

Definition 4.1. Let $\mathcal{S}=(\Omega,\mathcal{F},\{\mathcal{F}\}_{t\ge0},\mathbb{P})$ be a fixed stochastic basis generated by a standard Brownian motion W and $T>0$. Consider $\alpha\in(0,1)$ and an initial function φ such that $\varphi(t)$ is $\mathcal{F}_0$-measurable (relative to $\mathcal{S}$) for all $t\in[-h,0]$. A mild solution to problem (4.1) on $(-h,T]$ is a stochastic process such that $u(t)=\varphi(t)$ for $t\in(-h,0]$, and fulfills

$$\begin{aligned}u(t)&=\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)+\int_0^t\mathbf{E}_\alpha(-(t-s)^\alpha A)F(s,u_s)ds\\&\quad+\int_0^t\mathbf{E}_\alpha(-(t-s)^\alpha A)G(s,u_s)dW(s),\ \mathbb{P}\text{-a.s.},\quad\text{for every } t\in[0,T].\end{aligned}\tag{4.2}$$

4.1. **Well-posedness of mild solution to** (4.1) **with bounded delay.** We now analyze the well-posedness of the model below with bounded delay (i.e. $h>0$ is fixed),

$$\begin{aligned} D_t^\alpha u &= -Au + J_t^{1-\alpha}[F(t,u_t)] + J_t^{1-\alpha}[G(t,u_t)\frac{dW(t)}{dt}], \quad t>0, \\ u(t) &= \varphi(t), \quad t\in[-h,0]. \end{aligned} \tag{4.3}$$

The first result established below is about the existence and uniqueness of local mild solutions to problem (4.3), by a fixed point argument. We can prove the local existence and uniqueness by the same arguments as before: either under conditions (A1)–(A3) or by replacing (A3) by (A4) and (A5). In fact, the computations are easier now since the kernel $(t-s)^{\alpha-1}$ does not appear in the expression of mild solutions, which makes possible to remove the restriction $\alpha \in (1/2, 1)$.

Theorem 4.2. *Let* $\alpha \in (0,1)$. *Assume that* (A1), (A2), (A4), (A5) *hold (or* (A1)–(A3)*), and* $\varphi \in C([-h,0]; L^2(\Omega; L^2_\sigma))$ *such that* $\varphi(t)$ *is* $\mathcal{F}_0$*-measurable for all* $t \in [-h,0]$. *Then there exists* $T>0$ *(small enough) such that* (4.3) *admits a unique mild solution in the sense of Definition 4.1 on* $[-h,T]$.

Proof. We prove this result under assumptions (A1), (A2), (A4), (A5) (the other case is similar). Let $\varphi \in C([-h,0]; L^2(\Omega; L^2_\sigma))$ be such that $\varphi(t)$ is $\mathcal{F}_0$-measurable for all $t\in[-h,0]$, choose $R>0$ (large enough) such that

$$3(C_1^2 + 1 + C_1^2 h L_g)\|\varphi\|^2_{C([-h,0];L^2(\Omega;L^2_\sigma))} \le \frac{R^2}{2}.$$

Consider the space

$$\mathcal{B}_R^\varphi = \big\{u \in C([-h,T]; L^2(\Omega; L^2_\sigma)) : u(t) = \varphi(t) \text{ for all } t\in[-h,0],\ \|u\|_{\mathcal{X}_2} \le R\big\},$$

and define the operator $\mathcal{L}$ on $\mathcal{B}_R^\varphi$ as

$$(\mathcal{L}u)(t) = \begin{cases} \varphi(t), \quad t\in[-h,0], \\ \mathbf{E}_\alpha(-t^\alpha A)\varphi(0) + \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)F(s,u_s)ds \\ + \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)G(s,u_s)dW(s), \quad t\in(0,T],\ \mathbb{P}\text{-a.s.} \end{cases} \tag{4.4}$$

Assertion 1: $\mathcal{L}u \in C([-h,T]; L^2(\Omega; L^2_\sigma))$, for every $u \in C([-h,0]; L^2(\Omega; L^2_\sigma))$. If $t\in[-h,0]$, then $(\mathcal{L}u)(t) = \varphi(t)$ and $\varphi \in C([-h,0]; L^2(\Omega; L^2_\sigma))$. Therefore, we only need to check the continuity of $\mathcal{L}u$ on $[0,T]$. By slightly modifying the proof of the [4, Lemma 11], with the help of the analytical property of the Mittag-Leffler operator in time (see Lemma 2.3(ii)), the result holds immediately.

Assertion 2: There exists $T>0$ (sufficiently small) such that $\|\mathcal{L}u\|_{\mathcal{X}_2} \le R$, for all $u \in \mathcal{B}_R^\varphi$. To this end, we have to prove that, for any $u\in\mathcal{B}_R^\varphi$ and $t\in[0,T]$,

$$\|\mathcal{L}u\|_{\mathcal{X}_2} = \Big(\sup_{t\in[-h,T]} \mathbb{E}\|(\mathcal{L}u)(t)\|^2\Big)^{1/2} \le R. \tag{4.5}$$

For $t\in[-h,0]$, we have

$$\mathbb{E}\|(\mathcal{L}u)(t)\|^2 = \mathbb{E}\|\varphi(t)\|^2 \le \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|^2. \tag{4.6}$$

If $t \in (0, T]$, it follows that

$$
\begin{aligned}
\mathbb{E}\|(\mathcal{L}u)(t)\|^2 &\le 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2 + 3\mathbb{E}\Big\| \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)F(s,u_s)ds\Big\|^2 \\
&\quad + 3\mathbb{E}\Big\| \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)G(s,u_s)dW(s)\Big\|^2 \\
&:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned} \tag{4.7}
$$

We now estimate each term on the right-hand side of (4.7). For $\mathcal{I}_1$, by Lemma 3.1(i), it is obvious that

$$
\mathcal{I}_1 = 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2 \le 3C_1^2\mathbb{E}\|\varphi(0)\|^2 \le 3C_1^2 \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|^2. \tag{4.8}
$$

For $\mathcal{I}_2$, by Lemma 3.1(i), $(A_2), (A_4)$, the Cauchy-Schwarz inequality and Fubini's theorem, we obtain

$$
\begin{aligned}
\mathcal{I}_2 &= 3\mathbb{E}\Big\| \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)F(s,u_s)ds\Big\|^2 \\
&\le 3\mathbb{E}\Big(\int_0^t \|\mathbf{E}_\alpha(-(t-s)^\alpha A)F(s,u_s)\|ds\Big)^2 \\
&\le 3C_1^2 t \int_0^t \mathbb{E}\|F(s,u_s)\|^2 ds \\
&\le 3C_1^2 L_f t \int_{-h}^t \mathbb{E}\|u(s)\|^2 ds \\
&\le 3C_1^2 L_f t \Big(\int_{-h}^0 \mathbb{E}\|\varphi(s)\|^2 ds + \int_0^t \mathbb{E}\|u(s)\|^2 ds\Big) \\
&\le 3C_1^2 h L_f t \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|^2 + 3C_1^2 L_f t \int_0^t \mathbb{E}\|u(s)\|^2 ds \\
&\le 3C_1^2 h L_f t \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|^2 + 3C_1^2 t^2 L_f R^2.
\end{aligned} \tag{4.9}
$$

For $\mathcal{I}_3$, by Lemma 3.1(i), Itô's isometry and (A2), (A5),

$$
\begin{aligned}
\mathcal{I}_3 &= 3\mathbb{E}\Big\| \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)G(s,u_s)dW(s)\Big\|^2 \\
&\le 3C_1^2 \int_0^t \mathbb{E}\|G(s,u_s)\|^2 ds \\
&\le 3C_1^2 L_g \Big(\int_{-h}^0 \mathbb{E}\|\varphi(s)\|^2 ds + \int_0^t \mathbb{E}\|u(s)\|^2 ds\Big) \\
&\le 3C_1^2 h L_g \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|^2 + 3C_1^2 t L_g \sup_{s\in[0,t]} \mathbb{E}\|u(s)\|^2 \\
&\le 3C_1^2 h L_g \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|^2 + 3C_1^2 t L_g R^2.
\end{aligned} \tag{4.10}
$$

Substituting (4.8)-(4.10) into (4.7), combining with (4.6), it is obvious that

$$
\begin{aligned}
\mathbb{E}\|(\mathcal{L}u)(t)\|^2 &\le 3(C_1^2 + C_1^2 h L_f t + C_1^2 h L_g) \sup_{s\in[-h,0]} \mathbb{E}\|\varphi(s)\|^2 \\
&\quad + 3C_1^2 t^2 L_f R^2 + 3C_1^2 t L_g R^2.
\end{aligned}
$$

Consequently, thanks to the choice of R, we can choose T small enough such that

$$\begin{aligned} \|\mathcal{L}u\|_{\mathcal{X}_2} &= \Big(\sup_{t\in[-h,T]} \mathbb{E}\|(\mathcal{L}u)(t)\|^2\Big)^{1/2} \\ &\le \Big(3\big(C_1^2+1+C_1^2hL_fT+C_1^2hL_g\big)\sup_{t\in[-h,0]}\mathbb{E}\|\varphi(t)\|^2 \\ &\quad + 3C_1^2T^2L_fR^2+3C_1^2TL_gR^2\Big)^{1/2} \le R. \end{aligned} \tag{4.11}$$

Assertion 3: Operator $\mathcal{L}:\mathcal{B}_R^{\varphi}\to\mathcal{B}_R^{\varphi}$ is a contraction. To this end, for any u, $v\in\mathcal{B}_R^{\varphi}$, it follows that

$$\|\mathcal{L}u-\mathcal{L}v\|_{\mathcal{X}_2} := \Big(\sup_{t\in[-h,T]}\mathbb{E}\|(\mathcal{L}u)(t)-(\mathcal{L}v)(t)\|^2\Big)^{1/2}. \tag{4.12}$$

For $t\in[-h,0]$, one has $(\mathcal{L}u)(t)=(\mathcal{L}v)(t)=\varphi(t)$. Thus, it is sufficient to consider the case $t\in(0,T]$. Observe that

$$\begin{aligned} &\mathbb{E}\|(\mathcal{L}u)(t)-(\mathcal{L}v)(t)\|^2 \\ &\le 2\mathbb{E}\Big\|\int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds\Big\|^2 \\ &\quad + 2\mathbb{E}\Big\|\int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s)\Big\|^2 \\ &:= \mathcal{J}_1+\mathcal{J}_2. \end{aligned} \tag{4.13}$$

For $\mathcal{J}_1$, by Lemma 3.1(i), (A4), the Cauchy-Schwarz inequality and Fubini's theorem, we obtain

$$\begin{aligned} \mathcal{J}_1 &= 2\mathbb{E}\Big\|\int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds\Big\|^2 \\ &\le 2\mathbb{E}\Big(\int_0^t \|\mathbf{E}_\alpha(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))\|ds\Big)^2 \\ &\le 2C_1^2L_ft\int_{-h}^t \mathbb{E}\|u(s)-v(s)\|^2ds \\ &= 2C_1^2L_ft\int_0^t \mathbb{E}\|u(s)-v(s)\|^2ds \\ &\le 2C_1^2L_ft^2\sup_{s\in[0,t]}\mathbb{E}\|u(s)-v(s)\|^2. \end{aligned} \tag{4.14}$$

For $\mathcal{J}_2$, by Lemma 3.1(i), (A5) and Itô's isometry,

$$\begin{aligned} \mathcal{J}_2 &= 2\mathbb{E}\Big\|\int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s)\Big\|^2 \\ &\le 2C_1^2L_g\int_{-h}^t \mathbb{E}\|u(s)-v(s)\|^2ds \\ &= 2C_1^2L_g\int_0^t \mathbb{E}\|u(s)-v(s)\|^2ds \\ &\le 2C_1^2L_gt\sup_{s\in[0,t]}\mathbb{E}\|u(s)-v(s)\|^2. \end{aligned} \tag{4.15}$$

Hence, substituting (4.14)-(4.15) into (4.13), it follows that

$$\begin{aligned}\|\mathcal{L}u-\mathcal{L}v\|_{\mathcal{X}_2} &\le \Big(2C_1^2T(L_fT+L_g)\sup_{t\in[0,T]}\mathbb{E}\|u(t)-v(t)\|^2\Big)^{1/2}\\ &:= \mathcal{M}\|u-v\|_{\mathcal{X}_2},\end{aligned}\tag{4.16}$$

where $\mathcal{M}^2=2C_1^2T(L_fT+L_g)$. Therefore, we can choose T small enough such that $0<\mathcal{M}<1$, and the Banach fixed-point theorem implies that operator $\mathcal{L}$ possesses a fixed point in $\mathcal{B}_R^{\varphi}$. □

Unlike the result proved in Section 3, where we could only prove global existence for additive noise, in this situation we can prove this global in time result for a general multiplicative noise term.

Theorem 4.3. *Assume hypotheses of Theorem 4.2 hold. Then, for every initial value $\varphi\in C([-h,0];L^2(\Omega;L^2_\sigma))$ such that $\varphi(t)$ is $\mathcal{F}_0$-measurable for all $t\in[-h,0]$, the initial value problem (4.3) has a unique mild solution defined globally in the sense of Definition 4.1.*

Proof. We proceed with the proof under assumptions (A1), (A2), (A4),(A5). The proof under assumptions (A1)–(A3) is similar and we omit it.

Assume that there exist two solutions to problem (4.3), u and v on $[0,T_1]$ and $[0,T_2]$, respectively. Let us prove that $u=v$ on $[-h,T_1\wedge T_2]$. It is clear that $u(t)=v(t)=\varphi(t)$ on $[-h,0]$, so we only need to show that $u(t)=v(t)$ for any $t\in[0,T_1\wedge T_2]$. Notice that

$$\|u-v\|^2_{\mathcal{X}_2}:=\sup_{t\in[-h,T_1\wedge T_2]}\mathbb{E}\|u(t)-v(t)\|^2,\tag{4.17}$$

and

$$\begin{aligned}\mathbb{E}\|u(t)-v(t)\|^2 &\le 2\mathbb{E}\Big\|\int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds\Big\|^2\\ &\quad+2\mathbb{E}\Big\|\int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s)\Big\|^2\\ &:= I_1+I_2.\end{aligned}\tag{4.18}$$

For I_1, by Lemma 3.1(i), (A4) and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}I_1 &\le 2\mathbb{E}\Big(\int_0^t\|\mathbf{E}_\alpha(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))\|ds\Big)^2\\ &\le 2C_1^2\mathbb{E}\Big(\int_0^t\|F(s,u_s)-F(s,v_s)\|ds\Big)^2\\ &\le 2C_1^2L_ft\int_0^t\mathbb{E}\|u(s)-v(s)\|^2ds\\ &\le 2C_1^2L_ft\int_0^t\sup_{\sigma\in[0,s]}\mathbb{E}\|u(\sigma)-v(\sigma)\|^2ds.\end{aligned}\tag{4.19}$$

For I_2, by Lemma 3.1(i), (A5) and Itô's isometry, we have

$$\begin{aligned} I_2 &\le 2\int_0^t \mathbb{E}\|\mathbf{E}_\alpha(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))\|^2 ds \\ &\le 2C_1^2 L_g \int_0^t \mathbb{E}\|u(s)-v(s)\|^2 ds \\ &\le 2C_1^2 L_g \int_0^t \sup_{\sigma\in[0,s]} \mathbb{E}\|u(\sigma)-v(\sigma)\|^2 ds. \end{aligned} \tag{4.20}$$

Substituting (4.19)-(4.20) to (4.18), we deduce that

$$\mathbb{E}\|u(t)-v(t)\|^2 \le 2C_1^2(L_f t + L_g)\int_0^t \sup_{\sigma\in[0,s]} \mathbb{E}\|u(\sigma)-v(\sigma)\|^2 ds.$$

Denoting $\mathcal{M}_1 = 2C_1^2(L_f(T_1\wedge T_2)+L_g)$, for all $t\in[0,T_1\wedge T_2]$, we have

$$\sup_{s\in[-h,t]} \mathbb{E}\|u(s)-v(s)\|^2 \le \mathcal{M}_1 \int_0^t \Big(\sup_{\sigma\in[-h,s]} \mathbb{E}\|u(\sigma)-v(\sigma)\|^2\Big) ds.$$

Then Gronwall's Lemma implies that

$$\sup_{s\in[-h,t]} \mathbb{E}\|u(s)-v(s)\|^2 = 0, \quad \text{for all } t\in[0,T_1\wedge T_2].$$

Therefore, $u=v$ on $[-h,T_1\wedge T_2]$.

Now we prove that for each given $T>0$, the mild solution to problem (4.3) is bounded with $\mathcal{X}_2$ norm. Taking into account Lemma 3.1(i), (A2), (A4), (A5), Itô's isometry, the Cauchy-Schwarz inequality, and Fubini's theorem, we have

$$\begin{aligned} \mathbb{E}\|u(t)\|^2 &\le 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2 + 3\mathbb{E}\Big\|\int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)F(s,u_s)ds\Big\|^2 \\ &\quad + 3\mathbb{E}\Big\|\int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)G(s,u_s)dW(s)\Big\|^2 \\ &\le 3C_1^2 \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|^2 + 3C_1^2 t\mathbb{E}\Big(L_f\int_{-h}^0 \|\varphi(s)\|^2 ds + L_f\int_0^t \|u(s)\|^2 ds\Big) \\ &\quad + 3C_1^2\mathbb{E}\Big(L_g\int_{-h}^0 \|\varphi(s)\|^2 ds + L_g\int_0^t \|u(s)\|^2 ds\Big) \\ &\le 3C_1^2(1+L_f th + L_g h) \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|^2 \\ &\quad + 3C_1^2(L_f t + L_g)\int_0^t \sup_{\sigma\in[0,s]} \mathbb{E}\|u(\sigma)\|^2 ds. \end{aligned}$$

Therefore, for all $t\in[0,T]$,

$$\begin{aligned} \sup_{s\in[-h,t]} \mathbb{E}\|u(s)\|^2 &\le 3(C_1^2 + 1 + C_1^2 L_f Th + C_1^2 L_g h) \sup_{s\in[-h,0]} \mathbb{E}\|\varphi(s)\|^2 \\ &\quad + 3C_1^2(L_f T + L_g)\int_0^t \sup_{\sigma\in[0,s]} \mathbb{E}\|u(\sigma)\|^2 ds \\ &:= A(\varphi,T,F,G) + \mathcal{M}_2 \int_0^t \sup_{\sigma\in[-h,s]} \mathbb{E}\|u(\sigma)\|^2 ds, \end{aligned}$$

where

$$A(\varphi, T, F, G) := 3(C_1^2 + 1 + C_1^2 L_f T h + C_1^2 L_g h) \sup_{t\in[-h,0]} \mathbb{E}\|\varphi(t)\|^2,$$

$$\mathcal{M}_2 := 3C_1^2(L_f T + L_g).$$

Applying the Gronwall lemma, for any fixed $T > 0$, we obtain

$$\|u\|_{\mathcal{X}_2}^2 \leq A(\varphi, T, F, G)\exp(\mathcal{M}_2 T).$$

Because of the arbitrariness of T, together with the conclusion of uniqueness of u on $[-h, T]$, it is straightforward that the mild solution to problem (4.3) is defined globally. □

As a consequence, we can now state the continuous dependence on initial values.

Proposition 4.4. *Under the assumptions of Theorem 4.2, the mild solution to* (4.3) *is continuous with respect to the initial data* $\varphi \in C([-h, 0]; L^2(\Omega; L^2_\sigma))$*. In particular, if* $u(\cdot)$, $w(\cdot)$ *are the corresponding mild solutions on* $[-h, T]$ *to the initial data* ζ *and* ψ*, then the following estimate holds,*

$$\|u - w\|_{\mathcal{X}_2} \leq 3\|\zeta - \psi\|_{C([-h,0];L^2(\Omega;L^2_\sigma))} \exp(2C_1^2(L_f T + L_g)T).$$

This proposition is proved by using similar arguments as those concerning the uniqueness of previous theorem.

4.2. **Well-posedness of problem** (4.1) **with unbounded delay.** In this section, we analyze the well-posedness of the following stochastic time fractional 2D-Stokes equations with unbounded delay:

$$\begin{gathered} D_t^\alpha u = -Au + J_t^{1-\alpha}[F(t, u_t)] + J_t^{1-\alpha}[G(t, u_t)\frac{dW(t)}{dt}], \quad t > 0, \\ u(t) = \varphi(t), \quad t \in (-\infty, 0]. \end{gathered} \tag{4.21}$$

Theorem 4.5. *Let* $\alpha \in (0, 1)$ *and* (A1')–(A3') *hold. Then for each initial function* $\varphi \in \mathcal{C}_X(L^2(\Omega; L^2_\sigma))$ *such that* $\varphi(t)$ *is* $\mathcal{F}_0$*-measurable for all* $t \in [-h, 0]$*, problem* (4.21) *admits a unique mild solution on* $(-\infty, T]$ *in the sense of Definition 4.1, for* $T > 0$ *small enough.*

Proof. Let $\varphi \in \mathcal{C}_X(L^2(\Omega; L^2_\sigma))$ be such an initial value and choose $R > 0$ such that

$$3(C_1^2 + 1)\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \leq \frac{R^2}{3}.$$

Now consider the space

$$\mathcal{V}_R^\varphi = \{u \in C((-\infty, T]; L^2(\Omega; L^2_\sigma)) : u(t) = \varphi(t) \text{ for all } t \in (-\infty, 0], \text{ and } |u_T\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \leq R\},$$

and define the operator $\mathcal{K}$ on $\mathcal{V}_R^\varphi$ as

$$(\mathcal{K}u)(t) = \begin{cases} \varphi(t), \quad t \in (-\infty, 0], \\ \mathbf{E}_\alpha(-t^\alpha A)\varphi(0) + \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)F(s, u_s)ds \\ + \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)G(s, u_s)dW(s), \quad t \in (0, T], \ \mathbb{P}\text{-a.s.} \end{cases} \tag{4.22}$$

Note that if $u \in \mathcal{V}_R^\varphi$, then $\|u_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \leq R$ for all $t \in [0, T]$.

Assertion 1: $\mathcal{K}u \in C((-\infty,T];L^2(\Omega;L^2_\sigma))$ for all $u \in C((-\infty,T];L^2(\Omega;L^2_\sigma))$. Observe that if $t\in(-\infty,0]$, then $(\mathcal{K}u)(t)=\varphi(t)$. Therefore, we only need to check the continuity of $\mathcal{K}u$ on $[0,T]$. By slightly modifying the proof in [4, Lemma 11], with the help of the analyticity in time of Mittag-Leffler operators (see Lemma 2.3(ii)), the result holds.

Assertion 2: There exists $T>0$ such that $\|(\mathcal{K}u)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}\le R$ for all $t\in[0,T]$ and $u\in\mathcal{V}_R^\varphi$. For every $u\in\mathcal{V}_R^\varphi$, we have to show that

$$\|(\mathcal{K}u)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}:=\Big(\sup_{\theta\in(-\infty,0]}\mathbb{E}\|(\mathcal{K}u)(t+\theta)\|^2\Big)^{1/2}\le R.$$

For $t+\theta\in(-\infty,0]$, namely, $t\in(-\infty,-\theta)$, we have

$$\mathbb{E}\|(\mathcal{K}u)(t+\theta)\|^2=\mathbb{E}\|\varphi(t+\theta)\|^2\le\sup_{t\in(-\infty,0]}\mathbb{E}\|\varphi(t)\|^2. \tag{4.23}$$

If $t+\theta\in(0,T]$, namely $t\in(-\theta,T]$ (for convenience, here we denote by $t:=t+\theta\in[0,T]$), then it follows that

$$\begin{aligned}\mathbb{E}\|(\mathcal{K}u)(t)\|^2&\le 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2\\&\quad+3\mathbb{E}\Big\|\int_0^t\mathbf{E}_\alpha(-(t-s)^\alpha A)F(s,u_s)ds\Big\|^2\\&\quad+3\mathbb{E}\Big\|\int_0^t\mathbf{E}_\alpha(-(t-s)^\alpha A)G(s,u_s)dW(s)\Big\|^2\\&:=\mathcal{I}^1+\mathcal{I}^2+\mathcal{I}^3.\end{aligned}\tag{4.24}$$

We estimate now each term. For $\mathcal{I}^1$, by Lemma 3.1(i), it is obvious that

$$\mathcal{I}^1=3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2\le 3C_1^2\mathbb{E}\|\varphi(0)\|^2\le 3C_1^2\sup_{t\in(-\infty,0]}\mathbb{E}\|\varphi(t)\|^2. \tag{4.25}$$

For $\mathcal{I}^2$, by Lemma 3.1(i), (A2')-(A3'), the Cauchy-Schwarz inequality and Fubini's theorem, we obtain

$$\begin{aligned}\mathcal{I}^2&=3\mathbb{E}\Big\|\int_0^t\mathbf{E}_\alpha(-(t-s)^\alpha A)F(s,u_s)ds\Big\|^2\\&\le 3\mathbb{E}\Big(\int_0^t\|\mathbf{E}_\alpha(-(t-s)^\alpha A)F(s,u_s)\|ds\Big)^2\\&\le 3C_1^2t\int_0^t\mathbb{E}\|F(s,u_s)\|^2ds\\&\le 3C_1^2L_F't\int_0^t\|u_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds\\&\le 3C_1^2L_F't^2\|u_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}\\&\le 3C_1^2L_F't^2R^2.\end{aligned}\tag{4.26}$$

For $\mathcal{I}^3$, by Lemma 3.1(i), Itô's isometry, (A2'), and (A3'), we have

$$
\begin{aligned}
\mathcal{I}^3 &= 3\mathbb{E}\Big\| \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)G(s,u_s)dW(s)\Big\|^2 \\
&\le 3C_1^2 \int_0^t \mathbb{E}\|G(s,u_s)\|^2 ds \\
&\le 3C_1^2 L_G' \int_0^t \|u_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} ds \\
&\le 3C_1^2 L_G' t \|u_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \\
&\le 3C_1^2 L_G' t R^2.
\end{aligned} \tag{4.27}
$$

Replacing (4.25)-(4.27) into (4.24), combining with (4.23), it is obvious that

$$
\mathbb{E}\|(\mathcal{K}u)_t\|^2 \le 3\Big((C_1^2+1)\sup_{t\in(-\infty,0]} \mathbb{E}\|\varphi(t)\|^2 + C_1^2 t^2 L_F' R^2 + C_1^2 t L_G' R^2\Big).
$$

Consequently, by the choice of R, we can choose T such that

$$
\begin{aligned}
&\|(\mathcal{K}u)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \\
&= \Big(\sup_{\theta\in(-\infty,0]} \mathbb{E}\|(\mathcal{K}u)(t+\theta)\|^2\Big)^{1/2} \\
&\le \Big(3(C_1^2+1)\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} + 3C_1^2 T^2 L_F' R^2 + 3C_1^2 T L_G' R^2\Big)^{1/2} \le R,
\end{aligned}
$$

for all $t \in (0,T]$.

Assertion 3: Operator $\mathcal{K}: \mathcal{V}_R^\varphi \to \mathcal{V}_R^\varphi$ is a contraction. To this end, for each u, $v \in \mathcal{V}_R^\varphi$ and $t\in[0,T]$, it follows that

$$
\begin{aligned}
\|(\mathcal{K}u)_t - (\mathcal{K}v)_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} &:= \sup_{\theta\in(-\infty,0]} \mathbb{E}\|(\mathcal{K}u)(t+\theta) - (\mathcal{K}v)(t+\theta)\|^2 \\
&= \sup_{t\in(-\infty,T]} \mathbb{E}\|(\mathcal{K}u)(t) - (\mathcal{K}v)(t)\|^2.
\end{aligned} \tag{4.28}
$$

For $t\in(-\infty,0]$, one has $(\mathcal{K}u)(t) = (\mathcal{K}v)(t) = \varphi(t)$. Thus, we only need to consider the case $t\in(0,T]$. Observe that

$$
\begin{aligned}
&\mathbb{E}\|(\mathcal{K}u)(t) - (\mathcal{K}v)(t)\|^2 \\
&\le 2\mathbb{E}\Big\| \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)(F(s,u_s) - F(s,v_s))ds\Big\|^2 \\
&\quad + 2\mathbb{E}\Big\| \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)(G(s,u_s) - G(s,v_s))dW(s)\Big\|^2 \\
&:= \mathcal{J}^1 + \mathcal{J}^2.
\end{aligned} \tag{4.29}
$$

For $\mathcal{J}^1$, by Lemma 3.1(i), (A3'), the Cauchy-Schwarz inequality, and Fubini's theorem, we obtain

$$\begin{aligned} \mathcal{J}^1 &= 2\mathbb{E}\Big\| \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds\Big\|^2 \\ &\le 2\mathbb{E}\Big(\int_0^t \|\mathbf{E}_\alpha(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))\|ds\Big)^2 \\ &\le 2C_1^2 L_F' t \int_0^t \|u_s - v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} ds \\ &\le 2C_1^2 L_F' t^2 \|u_t - v_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}. \end{aligned} \tag{4.30}$$

For $\mathcal{J}^2$, by Lemma 3.1(i), (A3'), and Itô's isometry, one has

$$\begin{aligned} \mathcal{J}^2 &= 2\mathbb{E}\Big\| \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s)\Big\|^2 \\ &\le 2C_1^2 L_G' \int_0^t \|u_s - v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} ds \\ &\le 2C_1^2 L_G' t \|u_t - v_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}. \end{aligned} \tag{4.31}$$

Hence, substituting (4.29)-(4.31) into (4.28), it follows that

$$\begin{aligned} \|(\mathcal{K}u)_t - (\mathcal{K}v)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} &\le \Big(2C_1^2(L_F'T^2 + L_G'T)\|u_t - v_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}\Big)^{1/2} \\ &:= \mathcal{W}\|u_t - v_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}, \end{aligned}$$

where $\mathcal{W}^2 = 2C_1^2(L_F'T^2 + L_G'T)$. Therefore, we can choose T small enough such that $0 < \mathcal{W} < 1$, which means that, the operator $\mathcal{K}$ maps $\mathcal{V}_R^\varphi$ into itself. This is a contraction. The Banach fixed-point theorem yields that operator $\mathcal{K}$ has a fixed point in $\mathcal{V}_R^\varphi$. Namely, problem (4.21) has a unique local mild solution on $(-\infty, T]$. □

Theorem 4.6. *Assume the hypotheses of Theorem 4.5 hold. Then, for every initial value $\varphi \in \mathcal{C}_X(L^2(\Omega;L^2_\sigma))$ such that $\varphi(t)$ is $\mathcal{F}_0$-measurable for all $t \in [-h,0]$, the initial value problem* (4.21) *has a unique mild solution defined globally in the sense of Definition 4.1.*

Proof. The proof follows the same lines as in Section 3, but with differences in the estimates. Assume that there exist two solutions to problem (4.21), u and v on $[0,T_1]$ and $[0,T_2]$, respectively. Let us prove that $u = v$ on $(-\infty, T_1 \wedge T_2]$. It is remarkable that $u(t) = v(t) = \varphi(t)$ on $(-\infty, 0]$, so we only need to prove that $u(t) = v(t)$ for any $t \in (0, T_1 \wedge T_2]$. Observe that

$$\|u_t - v_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} := \sup_{s\in(-\infty,t]} \mathbb{E}\|u(s) - v(s)\|^2, \tag{4.32}$$

and

$$\begin{aligned} \mathbb{E}\|u(t)-v(t)\|^2 &\le 2\mathbb{E}\Big\| \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds\Big\|^2 \\ &\quad + 2\mathbb{E}\Big\| \int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s)\Big\|^2 \\ &:= I^1 + I^2. \end{aligned} \tag{4.33}$$

For I^1, by Lemma 3.1(i), (A3') and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} I^1 &\le 2\mathbb{E}\Big(\int_0^t \|\mathbf{E}_\alpha(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))\|\Big)^2 \\ &\le 2C_1^2\mathbb{E}\Big(\int_0^t \|F(s,u_s)-F(s,v_s)\|ds\Big)^2 \\ &\le 2C_1^2L'_F t\int_0^t \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds. \end{aligned} \tag{4.34}$$

For I^2, by Lemma 3.1(i), (A3') and Itô's isometry, we have

$$\begin{aligned} I^2 &\le 2\int_0^t \mathbb{E}\|\mathbf{E}_\alpha(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))\|^2 ds \\ &\le 2C_1^2L'_G\int_0^t \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds. \end{aligned} \tag{4.35}$$

Substituting (4.34)-(4.35) to (4.33), we have

$$\mathbb{E}\|u(t)-v(t)\|^2 \le 2C_1^2(L'_F t+L'_G)\int_0^t \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds.$$

Denoting by $\mathcal{W}_1 = 2C_1^2(L'_F(T_1\wedge T_2)+L'_G)$, we have for all $t\in[0,T_1\wedge T_2]$,

$$\|u_t-v_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \le \mathcal{W}_1\int_0^t \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds.$$

The Gronwall Lemma implies that

$$\|u_t-v_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} = 0, \quad \text{for all } t\in[0,T_1\wedge T_2],$$

and therefore $u=v$ on $(-\infty,T_1\wedge T_2]$.

Now we prove that for each given $T>0$, the mild solution to (4.21) is bounded with $\mathcal{C}_X(L^2(\Omega;L^2_\sigma))$ norm. Taking into account Lemma 3.1(i), (A1')–(A3'), Itô's isometry, the Cauchy-Schwarz inequality, and Fubini's theorem, we have

$$\begin{aligned} \mathbb{E}\|u(t)\|^2 &\le 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2 \\ &\quad + 3\mathbb{E}\Big\|\int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)F(s,u_s)ds\Big\|^2 \\ &\quad + 3\mathbb{E}\Big\|\int_0^t \mathbf{E}_\alpha(-(t-s)^\alpha A)G(s,u_s)dW(s)\Big\|^2 \\ &\le 3C_1^2\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} + 3C_1^2L'_F t\int_0^t \|u_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds \\ &\quad + 3C_1^2L'_G\int_0^t \|u_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds \\ &\le 3C_1^2\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} + 3C_1^2(L'_F t+L'_G)\int_0^t \|u_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds. \end{aligned}$$

Applying the Gronwall lemma to the above inequality, for any fixed $T>0$ and $t\in[0,T]$, we obtain

$$\|u_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \le (3C_1^2+1)\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}\exp(3C_1^2(L'_F T+L'_G)T).$$

Thank to the arbitrariness of T, together with the conclusion of uniqueness of u on $(-\infty, T]$, it is straightforward that the mild solution to (4.21) is defined globally. $\square$

As a consequence, we can state the continuous dependence of mild solutions with respect to the initial data.

Proposition 4.7. *Under the assumptions of Theorem 4.5, the mild solution to* (4.21) *is continuous with respect to the initial data* $\varphi \in \mathcal{C}_X(L^2(\Omega; L^2_\sigma))$*. In particular, if* $u(\cdot)$*,* $w(\cdot)$ *are the mild solutions corresponding to the initial data* ζ *and* ψ *on* $(-\infty, T]$*, then*

$$\|u_T - w_T\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \leq 3C_1^2 \|\zeta - \psi\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \exp(3C_1^2(L'_F T + L'_G)T).$$

This proposition is proved by similar arguments to those for the uniqueness of previous theorem.

5. Conclusion and final remarks

In this article, we considered quite general time-fractional stochastic Stokes models driven by multiplicative Brownian motion with finite and infinite delay. As we said, this is only a first approach to our goal concerning the case of stochastic time fractional delay Navier-Stokes equations with multiplicative noise. However, to that end, a new technique has to be designed because the fixed point theorem used in our proofs is not appropriate to handle the nonlinear term: the appearance of expectation in the norm does not allow us to bound that term in an appropriate way, as it is done in the deterministic case, especially for the contraction property. Therefore, this is a challenging problem to be analyzed shortly. But, it is not surprising that the problem cannot be analyzed with this technique since, to the best of our knowledge, even the non-fractional stochastic time derivative system has not been solved for the multiplicative noise case. We plan to work on this first case and combine the ideas of both techniques to achieve our goal for the time- fractional stochastic Navier-Stokes equations with delays.

On a different note, concerning the models analyzed in the current paper, we are interested in analyzing the long time behavior of both problems (1.1) and (1.3) with general multiplicative noise. To this end, it is necessary to ensure first the existence of solutions globally defined in time. Motivated by this fact, we will investigate, in a forthcoming paper, the idea of approximating our problems with nonlinear multiplicative noise by the so-called colored noise. This means to consider Wong-Zakai approximations to provide information about the stochastic problem (see Lu et al. [10] for more details).

6. Appendix

Proof of Theorem 3.8. Consider an initial function $\varphi \in \mathcal{C}_X(L^2(\Omega; L^2_\sigma))$ and choose $R > 0$ such that

$$(3C_1^2 + 1)\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \leq \frac{R^2}{3}.$$

Let $\alpha \in (1/2, 1)$ and define the following space $\mathcal{U}_R^\varphi$ for $R > 0$,

$$\mathcal{U}_R^\varphi = \big\{u \in C((-\infty, T] : L^2(\Omega; L^2_\sigma)) : u(t) = \varphi(t),\ \forall t \in (-\infty, 0] \text{ and } \|u_T\|_{\mathcal{C}_X} \leq R\big\}.$$

Notice that if $u \in \mathcal{U}_R^\varphi$, then $u_t \in \mathcal{C}_X(L^2(\Omega; L^2_\sigma))$ for all $t \in [0,T]$. Define now the operator $\mathcal{Q}$ on $\mathcal{U}_R^\varphi$ as

$$(\mathcal{Q}u)(t) = \begin{cases} \varphi(t), \quad t \in (-\infty, 0], \\ \mathbf{E}_\alpha(-t^\alpha A)\varphi(0) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds \\ + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s), \quad t \in [0,T], \ \mathbb{P}\text{-a.s.} \end{cases}$$

Assertion 1: $\mathcal{Q}u \in C((-\infty,T]; L^2(\Omega; L^2_\sigma))$ for all $u \in C((-\infty,0]; L^2(\Omega; L^2_\sigma))$. Observe that if $t \in (-\infty,0]$, then $(\mathcal{Q}u)(t) = \varphi(t)$. Therefore, we only need to check the continuity of $\mathcal{Q}u$ on $[0,T]$. By slightly modifying the proof in [4, Lemma 11], with the help of the analyticity of Mittag-Leffler operator in time (see Lemma 2.3 (ii)), the result follows.

Assertion 2: There exists $T > 0$ such that $\|(\mathcal{Q}u)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \le R$, for all $t \in [0,T]$ and $u \in \mathcal{U}_R^\varphi$. For every $u \in \mathcal{U}_R^\varphi$ and $t \in [0,T]$, we have to show that

$$\|(\mathcal{Q}u)_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} := \sup_{\theta \in (-\infty,0]} \mathbb{E}\|(\mathcal{Q}u)(t+\theta)\|^2 = \sup_{t \in (-\infty,T]} \mathbb{E}\|(\mathcal{Q}u)(t)\|^2 \le R^2.$$

For $t \in (-\infty,0]$, we have

$$\mathbb{E}\|(\mathcal{Q}u)(t)\|^2 = \mathbb{E}\|\varphi(t)\|^2 \le \sup_{t \in (-\infty,0]} \mathbb{E}\|\varphi(t)\|^2. \tag{6.1}$$

If $t \in (0,T]$, then

$$\begin{aligned} \mathbb{E}\|(\mathcal{Q}u)(t)\|^2 &\le 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2 \\ &\quad + 3\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds\Big\|^2 \\ &\quad + 3\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s)\Big\|^2 \\ &:= \mathcal{I}^1 + \mathcal{I}^2 + \mathcal{I}^3. \end{aligned} \tag{6.2}$$

We estimate now each term. For $\mathcal{I}^1$, by Lemma 3.1(i), it is obvious that

$$\mathcal{I}^1 = 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2 \le 3C_1^2\mathbb{E}\|\varphi(0)\|^2 \le 3C_1^2 \sup_{t \in (-\infty,0]} \mathbb{E}\|\varphi(t)\|^2. \tag{6.3}$$

For $\mathcal{I}^2$, by Lemma 3.1(i), (A1')–(A3'), the Cauchy-Schwarz inequality, and Fubini's theorem, we obtain

$$\begin{aligned} \mathcal{I}^2 &= 3\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds\Big\|^2 \\ &\le 3\mathbb{E}\Big(\int_0^t (t-s)^{\alpha-1}\|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)\|ds\Big)^2 \\ &\le \frac{3C_1^2 t^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\|F(s,u_s)\|^2 ds \\ &\le \frac{3C_1^2 L'_F t^{2\alpha-1}}{2\alpha-1}\int_0^t \|u_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds \\ &\le \frac{3C_1^2 L'_F t^{2\alpha}}{2\alpha-1}\|u_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \\ &\le \frac{3C_1^2 L'_F t^{2\alpha}}{2\alpha-1}R^2. \end{aligned} \tag{6.4}$$

For $\mathcal{I}^3$, by Lemma 3.1(i), Itô's isometry, and (A_1')-(A_3'), one has

$$\begin{aligned}
\mathcal{I}^3 &= 3\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)G(s,u_s)dW(s)\Big\|^2 \\
&\le 3C_1^2\mathbb{E}\int_0^t (t-s)^{2\alpha-2}\|G(s,u_s)\|^2 ds \\
&\le \frac{3C_1^2L_G't^{2\alpha-1}}{2\alpha-1}\|u_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \\
&\le \frac{3C_1^2L_G't^{2\alpha-1}}{2\alpha-1}R^2.
\end{aligned} \tag{6.5}$$

Substituting (6.3)-(6.5) into (6.2), combining with (6.1), we obtain

$$\mathbb{E}\|(\mathcal{Q}u)(t)\|^2 \le 3C_1^2\sup_{t\in(-\infty,0]}\mathbb{E}\|\varphi(t)\|^2 + \frac{3C_1^2L_F't^{2\alpha}}{2\alpha-1}R^2 + \frac{3C_1^2L_G't^{2\alpha-1}}{2\alpha-1}R^2.$$

Consequently, because of the choice of R, we can choose T small enough such that

$$\begin{aligned}
&\|(\mathcal{Q}u)_t\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \\
&= \Big(\sup_{\theta\in(-\infty,0]}\mathbb{E}\|(\mathcal{Q}u)(t+\theta)\|^2\Big)^{1/2} \\
&\le \Big((3C_1^2+1)\|\varphi\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} + \frac{3C_1^2L_F'T^{2\alpha}}{2\alpha-1}R^2 + \frac{3C_1^2L_G'T^{2\alpha-1}}{2\alpha-1}R^2\Big)^{1/2} \le R.
\end{aligned}$$

Assertion 3: Operator $\mathcal{Q}:\mathcal{U}_R^\varphi\to\mathcal{U}_R^\varphi$ is a contraction. To this end, for any u, $v\in\mathcal{U}_R^\varphi$ and $t\in[0,T]$, notice first that

$$\begin{aligned}
\|(\mathcal{Q}u)_T-(\mathcal{Q}v)_T\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} &= \sup_{\theta\in(-\infty,0]}\mathbb{E}\|(\mathcal{Q}u)(T+\theta)-(\mathcal{Q}v)(T+\theta)\|^2 \\
&= \sup_{t\in(-\infty,T]}\mathbb{E}\|(\mathcal{Q}u)(t)-(\mathcal{Q}v)(t)\|^2.
\end{aligned} \tag{6.6}$$

For $t\in(-\infty,0]$, one has $(\mathcal{Q}u)(t)=(\mathcal{Q}v)(t)=\varphi(t)$. Thus, we only need to consider the case $t\in[0,T]$. Observe that

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{Q}u)(t)-(\mathcal{Q}v)(t)\|^2 \\
&\le 2\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds\Big\|^2 \\
&\quad + 2\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s)-G(s,v_s))dW(s)\Big\|^2 \\
&:= \mathcal{J}^1+\mathcal{J}^2.
\end{aligned} \tag{6.7}$$

For $\mathcal{J}^1$, by Lemma 3.1(i), (A1')–(A3'), the Cauchy-Schwarz inequality, and Fubini's theorem, we obtain

$$\begin{aligned}\mathcal{J}^1 &= 2\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds\Big\|^2\\ &\le 2\mathbb{E}Big(\int_0^t (t-s)^{\alpha-1}\|\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))\|ds\Big)^2\\ &\le \frac{2C_1^2L'_Ft^{2\alpha-1}}{2\alpha-1}\int_0^t \|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds\\ &\le \frac{2C_1^2L'_Ft^{2\alpha}}{2\alpha-1}\|u_t-v_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}.\end{aligned}\tag{6.8}$$

For $\mathcal{J}^2$, by Lemma 3.1(i), Itô's isometry, and (A1')–(A3'), we have

$$\begin{aligned}\mathcal{J}^2 &= 2\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(G(s,u_s))-G(s,v_s))dW(s)\Big\|^2\\ &\le 2C_1^2L'_G\int_0^t (t-s)^{2\alpha-2}\|u_s-v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}ds\\ &\le \frac{2C_1^2L'_Gt^{2\alpha-1}}{2\alpha-1}\|u_t-v_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}.\end{aligned}\tag{6.9}$$

Hence, substituting (6.7)-(6.9) into (6.6), it follows that

$$\begin{aligned}&\|(\mathcal{Q}u)_T-(\mathcal{Q}v)_T\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}\\ &\le \Big(\frac{2C_1^2L'_FT^{2\alpha}}{2\alpha-1}+\frac{2C_1^2L'_GT^{2\alpha-1}}{2\alpha-1}\Big)\|u_T-v_T\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}\\ &:= \mathbf{W}^2\|u_T-v_T\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))},\end{aligned}$$

where

$$\mathbf{W}^2=\frac{2C_1^2L'_FT^{2\alpha}}{2\alpha-1}+\frac{2C_1^2L'_GT^{2\alpha-1}}{2\alpha-1}.$$

Therefore, we can choose T small enough such that $0<\mathbf{W}<1$, which means that the operator $\mathcal{Q}$ maps $\mathcal{U}_R^\varphi$ into itself and also it is a contraction. The Banach fixed-point theorem implies the operator $\mathcal{Q}$ has a fixed point in $\mathcal{U}_R^\varphi$. Namely, problem (3.17) has a unique local mild solution on $(-\infty,T]$. □

Proof of Theorem 3.9. Assume that there exist two solutions to problem (4.21), u and v on $[0,T_1]$ and $[0,T_2]$, respectively. Let us prove that $u=v$ on $(-\infty,T_1\wedge T_2]$. Since $u(t)=v(t)=\varphi(t)$ on $(-\infty,0]$, we only need to prove that $u(t)=v(t)$ for any $t\in[0,T_1\wedge T_2]$. Observe that

$$\|u_{T_1\wedge T_2}-v_{T_1\wedge T_2}\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))}:=\sup_{t\in(-\infty,T_1\wedge T_2]}\mathbb{E}\|u(t)-v(t)\|^2.\tag{6.10}$$

By Lemma 3.1(i), (A1')–(A3'), and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\mathbb{E}\|u(t)-v(t)\|^2\\ &\le \mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)(F(s,u_s)-F(s,v_s))ds\Big\|^2\\ &\le C_1^2\mathbb{E}\Big(\int_0^t (t-s)^{\alpha-1}\|F(s,u_s)-F(s,v_s)\|ds\Big)^2\end{aligned}$$

$$\leq \frac{C_1^2 L_F' t^{2\alpha-1}}{2\alpha-1} \int_0^t \|u_s - v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} ds.$$

Denoting $\mathbf{W}_1 = \frac{C_1^2 L_F'(T_1\wedge T_2)^{2\alpha-1}}{2\alpha-1}$ we have

$$\|u_t - v_t\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \leq \mathbf{W}_1 \int_0^t \|u_s - v_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} ds, \quad \forall t \in [0, T_1 \wedge T_2].$$

The Gronwall Lemma implies

$$\|u_{T_1\wedge T_2} - v_{T_1\wedge T_2}\|_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} = 0.$$

Therefore, $u = v$ on $(-\infty, T_1 \wedge T_2]$.

Now we prove that for each given $T > 0$, the mild solution to (3.17) is bounded with $C((-\infty,T];L^2(\Omega;L^2_\sigma))$ norm. Taking into account Lemma 3.1(i), (A1')-(A3'), Itô's isometry, the Cauchy-Schwarz inequality, and Fubini's theorem, we have

$$\begin{aligned}
\mathbb{E}\|u(t)\|^2 &\leq 3\mathbb{E}\|\mathbf{E}_\alpha(-t^\alpha A)\varphi(0)\|^2 \\
&\quad + 3\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)F(s,u_s)ds\Big\|^2 \\
&\quad + 3\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(-(t-s)^\alpha A)\phi(s)dW(s)\Big\|^2 \\
&\leq 3C_1^2\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} + \frac{3C_1^2\|\phi\|^2 t^{2\alpha-1}}{2\alpha-1} \\
&\quad + \frac{3C_1^2 L_F' t^{2\alpha-1}}{2\alpha-1}\int_0^t \|u_s\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} ds.
\end{aligned}$$

Applying the Gronwall lemma to the above inequality, for any fixed $T > 0$ and all $t \in [0, T]$, we deduce that

$$\begin{aligned}
&\|u_T\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} \\
&\leq \Big((3C_1^2+1)\|\varphi\|^2_{\mathcal{C}_X(L^2(\Omega;L^2_\sigma))} + \frac{3C_1^2\|\phi\|^2 T^{2\alpha-1}}{2\alpha-1}\Big)\exp\Big(\frac{3C_1^2 L_F' T^{2\alpha}}{2\alpha-1}\Big).
\end{aligned}$$

Because of the arbitrariness of T, together with the conclusion of uniqueness of u on $(-\infty, T]$, it is straightforward to show that the mild solution to (3.17) is defined globally. □

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References

[1] M. Allen, L. Caffarelli, A. Vasseur; A parabolic problem with a fractional time derivative, *Arch. Ration. Mech. Anal.,* **221** (2016), 603–630.

[2] N. T. Bao, T. Caraballo, N. H. Tuan, Y. Zhou; Existence and regularity results for terminal value problem for nonlinear fractional wave equations, *Nonlinearity*, **34** (2021), 1448–1503.

[3] P. M. Carvalho-Neto; *Fractional differential equations: a novel study of local and global solutions in Banach spaces*, PhD thesis, Universidade de São Paulo, São Carlos, 2013.

[4] P. M. Carvalho-Neto, G. Planas; Mild solutions to the time fractional Navier-Stokes equations in $\mathbb{R}^N$, *J. Differential Equations*, **259** (2015), 2948–2980.

[5] T. Kato; Strong L^p-solutions of the Navier-Stokes equation in R^m, with applications to weak solution, *Math. Z.*, **187** (1984), 471–480.

[6] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies 204, Elsevier, Amsterdam, 2006.

[7] Y. J. Li, Y. J. Wang; The existence and asymptotic behavior of solutions to fractional stochastic evolution equations with infinite delay, *J. Differential Equations,* **266** (2019), 3514–3558.

[8] Y. J. Li, Y. J. Wang, W. H. Deng; Galerkin finite element approximations for stochastic space-time fractional wave equations, *SIAM J. Numer. Anal.*, **55** (2017), 3173–3202.

[9] L. F. Liu, T. Caraballo, P. Marín-Rubio; Stability results for 2D Navier–Stokes equations with unbounded delay, *J. Differential Equations*, **265** (2018), 5685–5708.

[10] K. N. Lu, B. X. Wang; Wong-Zakai approximations and long term behavior of stochastic partial differential equations, *J. Dynam. Differential Equations*, **31** (2019), 1341–1371.

[11] F. Mainardi; On the initial value problem for the fractional diffusion-wave equation, *Ser. Adv. Math. Appl. Sci.*, **23** (1994), 246–251.

[12] I. Podlubny; *Fractional Differential Equations*, Mathematics in Science and Engineering 198, Academic Press, San Diego, 1999.

[13] N. H. Tuan, T. Caraballo; On initial and terminal value problems for fractional nonclassical diffusion equations, *Proc. Amer. Math. Soc.*, **149** (2021), 143-161.

[14] A. Tuan Nguyen, T. Caraballo, N. H. Tuan, Nguyen Huy; On the initial value problem for a class of nonlinear biharmonic equation with time-fractional derivative, *Proc. Roy. Soc. Edinburgh Sect. A*, **152** (2022), 989–1031.

[15] R. N. Wang, D. H. Chen, T. J. Xiao; Abstract fractional Cauchy problems with almost sectorial operators, *J. Differential Equations*, **252** (2012), 202–235.

[16] J. H. Xu, Z. Zhang, T. Caraballo; Mild solutions to time fractional stochastic 2D-Stokes equations with bounded and unbounded delay, *J. Dynam. Differential Equations*, **34** (2022), 583–603.

[17] J. H. Xu, T. Caraballo; Long time behavior of fractional impulsive stochastic differential equations with infinite delay, *Discrete Contin. Dyn. Syst. Ser. B*, **24** (2019), 2719–2743.

[18] J. H. Xu, Z. Zhang, T. Caraballo; Non-autonomous nonlocal partial differential equations with delay and memory, *J. Differential Equations*, **270** (2021), 505–546.

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