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FOURTH-ORDER DIFFERENTIAL OPERATORS WITH INTERIOR DEGENERACY AND GENERALIZED WENTZELL BOUNDARY CONDITIONS

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Abstract. In this article we consider the fourth-order operators $A_1 u := (au'')''$ and $A_2 u := au''''$ in divergence and non divergence form, where $a : [0,1] \to \mathbb{R}_+$ degenerates in an interior point of the interval. Using the semigroup technique, under suitable assumptions on a, we study the generation property of these operators associated to generalized Wentzell boundary conditions. We prove the well posedness of the corresponding parabolic problems.

1. Introduction

In mathematical analysis the boundary conditions associated with a differential operator usually involve the function and its derivatives (including Dirichlet, Neumann and Robin conditions). In some cases, as in Markov process theory, it is natural to include boundary conditions involving the operator itself (see [26, 34] for a detailed exposition). for a detailed exposition. In particular, if A denotes an elliptic operator, the parabolic problem

$$\frac{\partial u}{\partial t} - Au = 0 \quad \text{in } \Omega \subset \mathbb{R}^n,\ t \geq 0,$$

is said to be equipped with the *Wentzell boundary condition* if $Au(t,x) = 0$ for $x \in \partial\Omega$ and all $t \geq 0$. In the literature a more general boundary condition which arises naturally in the context of the heat equation is the *generalized Wentzell boundary condition* (GWBC)

$$\alpha Au(x) + \beta \frac{\partial u}{\partial n}(x) + \gamma u(x) = 0, \quad x \in \partial\Omega, \tag{1.1}$$

where $(\alpha,\beta,\gamma) \neq (0,0,0)$. Note that (α,β,γ) can depend on x as well. Surprisingly, these boundary conditions arise naturally as part of the formulation of the problem and are incorporated in the derivation of the heat equation itself. Additional motivation for the study of evolution equations with (GWBC) comes from their possible interpretation as evolution equations with dynamical boundary conditions (for a general view on the role of Wentzell boundary conditions we refer to

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[11]). It is worth to mention that, in the case of heat equations, (GWBC) allow to take into account the action and the effect of heat sources on the boundary (see [26]). For a systematic study of the derivations and physical interpretations of Wentzell boundary conditions we refer, e.g., to [26], which covers heat and wave equations. On the other hand, for beam equations, Cahn-Hilliard equations and related models one can see, e.g., [23, 24, 25] and the references therein. Because of their importance and physical interpretation, we mention briefly other mathematical problems and contexts in which these boundary conditions appear. For example, if Ω is a smooth bounded domain in $\mathbb{R}^n$ and B is a formally symmetric differential operator on $L^2(\Omega)$ with domain $D(B) \subset \mathcal{C}_0^\infty(\Omega)$, a classical problem is to find all self-adjoint extensions of B (or all self-adjoint restrictions of B^*). This problem is solved abstractly by von Neumann and has been worked out in detail in some concrete cases. For example, if $\Omega = (0,1)$ and $B = \frac{1}{i}\frac{d}{dx}$, the self-adjoint extensions of B are determined by boundary conditions and are parametrized by the unit circle in $\mathbb{C}$. An analogous problem is to consider $B = \Delta$ with $D(B) = \mathcal{C}_0^\infty(\Omega)$. On the space $\mathcal{C}_0(\Omega)$, in which $\mathcal{C}_0^\infty(\Omega)$ is dense, $\overline{B}$ generates a positive contraction semigroup and an interesting question is to understand which extensions of B on $\mathcal{C}(\overline{\Omega})$ have this property. In his pioneering work [34] Wentzell showed that such extensions are characterized by (Wentzell) boundary conditions of the form (1.1), where $(\alpha(x), \beta(x), \gamma(x)) \neq (0,0,0)$ for all $x \in \partial\Omega$ with $\alpha > 0$, $\beta \geq 0$ and $\gamma \geq 0$ (Wentzell's work generalizes previous results by Feller in one space dimension, see, e.g., [18]). Furthermore, other multiple applications are possible. For example, in [32] the authors consider a Dirichlet problem that describes the basic diffusion of particles in a locally compact space, endowed with a Radon measure. Moreover, in a very general setting, they present an abstract version of the Wentzell boundary conditions (see also [31]). In the context of classical and quantum field theory on asymptotically anti-de Sitter spacetimes (AdS) and other spacetimes with boundaries, [12] studies a massive scalar field in AdS in $d+1$ spacetime dimensions subject to (GWBC) and it is highlighted that they are dynamical boundary conditions invariant under the action of the isometry group of the AdS boundary. The treatment of Wentzell boundary conditions in the classical and quantum field theoretic literature appears also in [3], where the classical mechanical system of a finite string with point masses subject to harmonic potentials in the extrema is solved. (GWBC) are also considered in [33] in $(d+1)$-dimensional Minkowski spacetime with one or two timelike boundaries. In particular, in [33] the author shows that the Wentzell boundary conditions ensure that the short-distance singularities of the two-point function for the boundary field has the form expected of a field living in a d-dimensional spacetime, contrary to other boundary conditions, for which the two-point function inherits the short-singularity of the $(d+1)$-dimensional bulk. This seems to be a very desired feature for holographic purposes. In Biology a significant amount of interest has been devoted to the analysis of mathematical models arising in structured population dynamics and a very important problem is the choice of suitable boundary conditions for a biologically plausible and mathematically sound model. In this field [6] introduces and analyzes a structured population model, with so called distributed recruitment term and (GWBC), describing the dynamics of a population infected with a certain type of bacteria. We refer to [15, 30] for a model of structured populations with generalized Wentzell-Robin boundary conditions. On the other hand, for more general operators one can see [6, 15, 29, 30].

Finally, it is well known that degenerate parabolic equations are widely used as mathematical models in the applied sciences to describe the evolution in time of a given system. For this reason, in recent years an increasing interest has been devoted to the study of differential degenerate operators in divergence or in non divergence form. In particular, after the new directions opened in [16, 20], great attention is given to the operators

$$\mathcal{A}_1 u := (au')',$$
$$\mathcal{A}_2 u := au''$$

with general Wentzell boundary conditions.

Actually, in this article we are interested in *fourth order operators* since many problems that are relevant for applications are described by these operators. Among these applications we can find dealloying (corrosion processes), population dynamics, bacterial films, thin film, tumor growth, clustering of mussels and so on (see [7, Introduction] for some detailed references). Moreover, operators of this type with suitable domains involving different boundary conditions arise in a natural way in several contexts as beam analysis and Euler-Bernoulli beam theory (see [8, Introduction] for some related references in this field).

The novelty of this article is that we prove the generation property for the *degenerate* fourth order operators

$$A_1 u := (au'')'',$$
$$A_2 u := au''''$$

equipped with (GWBC) (see the main theorems in Sections 3-4), obtaining the existence of solutions for the associated parabolic Cauchy problems. Here $a \in \mathcal{C}[0,1]$ is such that there exists $x_0 \in (0,1)$ such that $a(x_0) = 0$ and $a(x) > 0$ if $x \neq x_0$. As far as we know, this is the first paper that deals with this problem; thus we intend to fill this gap following the ideas of [17] and [20].

The paper is organized as follows: in Section 2 we prove some preliminary results that hold if a degenerates in a general point $x_0 \in [0,1]$. In Section 3 and in Section 4 we assume that the degeneracy point x_0 belongs to $(0,1)$ and we prove that the operators A_i, $i = 1, 2$, equipped with (GWBC) are non negative and self-adjoint with dense domain, obtaining the well posedness for the associated Cauchy problems. It is worth noting that in this paper we deal with real function spaces, but the assertions can be easily extended to the complex case.

Notation: C denotes universal positive constants which are allowed to vary from line to line; $'$ denotes the derivative of a function depending on the real space variable x.

2. Preliminary results

In this section we recall some suitable weighted spaces and preliminary results given in [8], that will be crucial for the rest of the paper. For simplicity, we distinguish between the case of a weakly degenerate function and of a strongly degenerate one.

2.1. **Weakly degenerate case.** First of all, we give the following definition on the function a.

Definition 2.1. A function $a \in \mathcal{C}[0,1]$ is said to be weakly degenerate if there exists $x_0 \in [0,1]$ such that $a(x_0)=0$, $a(x)>0$ for all $x \in [0,1]\setminus\{x_0\}$ and $\frac{1}{a} \in L^1(0,1)$.

As an example of a weakly degenerate function we can take $a(x)=|x-x_0|^K$, with $0<K<1$. To deal with the divergence case, for any weakly degenerate function $a \in \mathcal{C}[0,1]$, we introduce the weighted space

$$\begin{aligned} H^i_a(0,1) := \big\{&u \in H^{i-1}(0,1) : u^{(i-1)} \text{ is absolutely continuous in [0,1]}, \\ &\sqrt{a}u^{(i)} \in L^2(0,1)\big\}, \end{aligned} \tag{2.1}$$

endowed with the norm

$$\|u\|^2_{H^i_a(0,1)} := \sum_{j=0}^{i-1} \|u^{(j)}\|^2_{L^2(0,1)} + \|\sqrt{a}u^{(i)}\|^2_{L^2(0,1)} \quad \forall u \in H^i_a(0,1), \tag{2.2}$$

$i=1,2$; here $H^0(0,1) := L^2(0,1)$ and $u^{(0)}=u$. The following proposition holds.

Proposition 2.2. *For $u \in H^i_a(0,1)$, $i=1,2$, let*

$$\|u\|^2_{i,a} := \|u\|^2_{L^2(0,1)} + \|\sqrt{a}u^{(i)}\|^2_{L^2(0,1)}.$$

Then $\|u\|_{H^i_a(0,1)}$ and $\|u\|_{i,a}$ are equivalent.

To prove this proposition, the next result is essential.

Lemma 2.3 ([27, Theorem 7.37])**.** *Let $I=(a,b)$, with $a,b \in \mathbb{R}$, $a<b$, and let $1 \le p,q,r \le +\infty$ be such that*

$$\frac{1}{2q}+\frac{1}{2p} \ge \frac{1}{r}.$$

Let $u \in W^{2,1}_{\rm loc}(I)$, then there exists $c=c(p,q,r)>0$ such that

$$\|u'\|_{L^r(I)} \le cl^{\frac{1}{r}-1-\frac{1}{q}}\|u\|_{L^q(I)} + cl^{1-\frac{1}{p}+\frac{1}{r}}\|u''\|_{L^p(I)} \tag{2.3}$$

for every $0<l<\mathcal{L}^1(I)$, where $\mathcal{L}^1(I)$ is the one-dimensional measure of I.

Proof of Proposition 2.2. For $i=1$ the statement is obvious. Now, take $i=2$ and $u \in H^2_a(0,1)$. Clearly,

$$\|u\|^2_{2,a} \le \|u\|^2_{H^2_a(0,1)}.$$

For the other estimate, it is sufficient to prove

$$\|u'\|^2_{L^2(0,1)} \le C\big(\|u\|^2_{L^2(0,1)} + \|\sqrt{a}u''\|^2_{L^2(0,1)}\big), \tag{2.4}$$

for a positive constant C. To this aim, observe that $u \in W^{2,1}_{\rm loc}(0,1)$. Thus, by Lemma 2.3 with $I=(0,1)$, $p=1$, $q=2$ and $r=2$, one has

$$\|u'\|_{L^2(0,1)} \le cl^{-1}\|u\|_{L^2(0,1)} + cl^{1/2}\|u''\|_{L^1(0,1)}. \tag{2.5}$$

Now, by Hölder's inequality,

$$\begin{aligned} \|u''\|_{L^1(0,1)} &= \int_0^1 \frac{\sqrt{a}|u''|}{\sqrt{a}}dx \\ &\le \Big(\int_0^1 (a(u'')^2)(x)dx\Big)^{1/2}\Big(\int_0^1 \frac{1}{a(x)}dx\Big)^{1/2} \\ &= \|\sqrt{a}u''\|_{L^2(0,1)}\Big\|\frac{1}{a}\Big\|^{1/2}_{L^1(0,1)}. \end{aligned}$$

Using this inequality in (2.5), one has

$$\|u'\|^2_{L^2(0,1)} \le C\big(\|u\|^2_{L^2(0,1)} + \|\sqrt{a}u''\|^2_{L^2(0,1)}\big)$$

for some suitable constant $C > 0$ and the thesis follows. $\square$

Moreover, observe that $u \in H^2_a(0,1)$ implies $u' \in H^1_a(0,1)$. Using the space $H^2_a(0,1)$, we define

$$\mathcal{Z}_w(0,1) := \{u \in H^2_a(0,1) : au'' \in H^2(0,1)\}. \tag{2.6}$$

To deal with the non divergence case, in place of $L^2(0,1)$ and $H^i_a(0,1)$, we consider the spaces

$$L^2_{1/a}(0,1) := \Big\{u \in L^2(0,1) : \int_0^1 \frac{u^2}{a}\,dx < +\infty\Big\},$$
$$H^i_{1/a}(0,1) := L^2_{1/a}(0,1) \cap H^i(0,1),$$

with the respective norms

$$\|u\|^2_{L^2_{1/a}(0,1)} := \int_0^1 \frac{u^2}{a}\,dx \quad \forall u \in L^2_{1/a}(0,1),$$

and

$$\|u\|^2_{H^i_{1/a}(0,1)} := \|u\|^2_{L^2_{1/a}(0,1)} + \sum_{j=1}^i \|u^{(j)}\|^2_{L^2(0,1)} \quad \forall u \in H^i_{1/a}(0,1), \tag{2.7}$$

for $i = 1,2$. Observe that for all $u \in H^i_{1/a}(0,1)$, one can prove that $\|u\|^2_{H^i_{1/a}(0,1)}$ is equivalent to

$$\|u\|^2_{i,1/a} := \|u\|^2_{L^2_{1/a}(0,1)} + \|u^{(i)}\|^2_{L^2(0,1)}.$$

For $i = 1$ the equivalence is obvious; for $i = 2$ it follows by [5, Chapter VIII]. Indeed, for all $u \in H^i(0,1)$, one has that $\|u\|_{H^i(0,1)}$ is equivalent to $\|u\|_{L^2(0,1)} + \|u^{(i)}\|_{L^2(0,1)}$, $i = 1,2$. However,

$$\|u\|^2_{L^2(0,1)} = \int_0^1 \frac{u^2(x)}{a(x)}a(x)dx \le C\int_0^1 \frac{u^2(x)}{a(x)}dx = C\|u\|^2_{L^2_{1/a}(0,1)}$$

for all $u \in L^2_{1/a}(0,1)$ and, in particular, for all $u \in H^i_{1/a}(0,1)$. Hence $\|u\|_{i,1/a}$ is equivalent to $\|u\|_{H^i_{1/a}(0,1)}$, for all $u \in H^i_{1/a}(0,1)$ and for $i = 1,2$.

Finally, we consider the space

$$\mathcal{W}_w(0,1) := \big\{u \in H^2_{1/a}(0,1) : au'''' \in L^2_{1/a}(0,1)\big\}. \tag{2.8}$$

Clearly, it is a trivial fact that, if $au'''' \in L^2_{1/a}(0,1)$ then $u'''' \in L^1(0,1)$ (since $\frac{1}{a} \in L^1(0,1)$) and, by [7, Lemma 2.1], $u \in W^{4,1}(0,1)$.

Using the same considerations as in [7, Proposition 3.1], the spaces $H^i_{1/a}(0,1)$ and $H^i(0,1)$, $i = 1,2$, coincide algebraically and the two norms are equivalent. Indeed, for all $u \in H^i(0,1)$ it is sufficient to prove that there exists a positive constant C such that

$$\|u\|^2_{L^2_{1/a}(0,1)} \le C\Big\|\frac{1}{a}\Big\|_{L^1(0,1)};$$

but this follows immediately taking $C := \|u\|^2_{\mathcal{C}[0,1]}$.

Hence, if $u \in \mathcal{W}_w(0,1)$, then $u \in \mathcal{C}^3[0,1]$, and, in particular, $(au^{(k)})(x_0) = 0$, for all $k = 0,1,2,3$, being $a(x_0) = 0$ and $u^{(k)} \in \mathcal{C}[0,1]$, for all $k = 0,1,2,3$.

In the above spaces the following Green formulas hold.

Lemma 2.4 ([8, Lemmas 2.1 and 3.1])**.** *If a is weakly degenerate, then*

(i) *for all $(u,v) \in \mathcal{Z}_w(0,1) \times H^2_a(0,1)$*
$$\int_0^1 (au'')''v\,dx = [(au'')'v]_{x=0}^{x=1} - [au''v']_{x=0}^{x=1} + \int_0^1 au''v''dx; \tag{2.9}$$

(ii) *for all $(u,v) \in \mathcal{W}_w(0,1) \times H^2_{1/a}(0,1)$*
$$\int_0^1 u''''v\,dx = [u'''v]_{x=0}^{x=1} - [u''v']_{x=0}^{x=1} + \int_0^1 u''v''dx. \tag{2.10}$$

We underline that in [8], to prove the previous lemma, the requirement $\frac{1}{a} \in L^1(0,1)$ is not used; actually it is sufficient to require $a \in \mathcal{C}[0,1]$.

2.2. **Strongly degenerate case.**

In this subsection we consider another notion of degeneracy: the strongly one.

Definition 2.5. A function $a \in \mathcal{C}[0,1]$ is called strongly degenerate if there exists $x_0 \in [0,1]$ such that $a(x_0) = 0$, $a(x) > 0$ for all $x \in [0,1] \setminus \{x_0\}$, and $\frac{1}{a} \notin L^1(0,1)$.

As an example of a strongly degenerate function a we can take $a(x) = |x - x_0|^K$, with $K \geq 1$. For any strongly degenerate function a let us introduce the corresponding weighted spaces
$$\begin{aligned} H^i_a(0,1) := \big\{u \in H^{i-1}(0,1) :\ & u^{(i-1)} \text{ is locally absolutely continuous in} \\ & [0,1] \setminus \{x_0\} \text{ and } \sqrt{a}u^{(i)} \in L^2(0,1)\big\}, \end{aligned} \tag{2.11}$$
equipped with the norms (2.2), $i = 1,2$. Also in this case, with an additional assumption on a, the analogous of Proposition 2.2 holds.

Hypothesis 2.6. Assume that there exists $K \in [1,2)$ such that the function $x \mapsto |x - x_0|^K/a$ is

(1) non increasing on the left of x_0 and non decreasing on the right of x_0, if $x_0 \in (0,1)$;
(2) non decreasing on the right of 0, if $x_0 = 0$;
(3) non increasing on the left of 1, if $x_0 = 1$.

Observe that the previous assumption on a is not surprising because it was already used in other papers; moreover this assumption and the requirement $K \geq 1$ are satisfied by the prototype that we have in mind $a(x) = |x - x_0|^K$, with $K \geq 1$. More precisely, since in the previous hypothesis we require that $K < 2$, as prototype we consider $a(x) = |x - x_0|^K$, with $K \in [1,2)$.

Proposition 2.7. *Assume Hypothesis 2.6, and for $u \in H^2_a(0,1)$ set*
$$\|u\|^2_{2,a} := \|u\|^2_{L^2(0,1)} + \|\sqrt{a}u''\|^2_{L^2(0,1)}.$$
Then $\|u\|_{H^2_a(0,1)}$ and $\|u\|_{2,a}$ are equivalent.

Proof. Obviously, there exists a positive constant C such that for all $u \in H^2_a(0,1)$,

$$\|u\|_{2,a} \le C\|u\|_{H^2_a(0,1)}.$$

Now, we will prove the other inequality. Assume, for simplicity, $x_0 = 0$. As a first step we prove that there exists a positive constant C such that

$$\int_0^1 (v'(x))^2 dx \le C\|\sqrt{a}v''\|^2_{L^2(0,1)}, \tag{2.12}$$

for all

$$v \in \mathcal{X} := \{v \in H^2_a(0,1) : \exists\, y_0 \in (0,1) \text{ such that } v'(y_0) = 0\}.$$

Take $x \in (0,1]$ and $v \in \mathcal{X}$; then there exists $y_0 \in (0,1)$ such that $v'(y_0) = 0$ and

$$v'(x) = v'(x) - v'(y_0) = \int_{y_0}^x \frac{v''(y)\sqrt{a}}{\sqrt{a}} dy \le \|\sqrt{a}v''\|_{L^2(0,1)} \Big(\int_{y_0}^x \frac{1}{a(t)}dt\Big)^{1/2}.$$

Hence

$$\int_0^1 (v'(x))^2 dx \le \|\sqrt{a}v''\|^2_{L^2(0,1)} \int_0^1 \Big(\int_{y_0}^x \frac{1}{a(t)}dt\Big)dx.$$

Thus, it is sufficient to estimate the integral in the right-hand side of the above inequality. To this aim, we split the integral as follows:

$$\int_0^1 \Big(\int_{y_0}^x \frac{1}{a(t)}dt\Big)dx = \int_0^{y_0} \Big(\int_{y_0}^x \frac{1}{a(t)}dt\Big)dx + \int_{y_0}^1 \Big(\int_{y_0}^x \frac{1}{a(t)}dt\Big)dx.$$

Using the assumption on a, one has

$$\int_{y_0}^1 \Big(\int_{y_0}^x \frac{1}{a(t)}dt\Big)dx \le C\int_{y_0}^1 \Big(\int_{y_0}^1 \frac{1}{t^K}dt\Big)dx \le \begin{cases} C\frac{y_0^{1-K}}{a(1)(K-1)}, & K \ne 1,\\ C\frac{-\log(y_0)}{a(1)}, & K = 1,\end{cases}$$

for a positive constant C. Now, we consider the term $\int_0^{y_0}\big(\int_{y_0}^x \frac{1}{a(t)}dt\big)dx$. Using again the assumptions on a and the fact that the constant in Hypothesis 2.6 is strictly less than 2, one has

$$\begin{aligned}
\Big|\int_0^{y_0} \Big(\int_{y_0}^x \frac{1}{a(t)}dt\Big)dx\Big| &= \Big|-\int_0^{y_0} \Big(\int_x^{y_0} \frac{1}{a(t)}dt\Big)dx\Big| \\
&= \int_0^{y_0} \Big(\int_x^{y_0} \frac{1}{a(t)}dt\Big)dx \\
&= \int_0^{y_0} \int_0^t \frac{1}{a(t)}dx\, dt \\
&= \int_0^{y_0} \frac{t}{a(t)}dt \\
&= \int_0^{y_0} \frac{t^K}{a(t)t^{K-1}}dt \\
&\le C\int_0^{y_0} \frac{1}{t^{K-1}}dt \\
&\le C\frac{1}{(2-K)}y_0^{2-K}.
\end{aligned}$$

Thus,

$$\int_0^1 \left(\int_{y_0}^x \frac{1}{a(t)}dt\right) dx \le C \quad \text{and}$$
$$\int_0^1 (v'(x))^2 dx \le C\|\sqrt{a}v''\|^2_{L^2(0,1)},$$

for a positive constant C. Now, we will prove the thesis for all $u \in H^2_a(0,1)$. To this aim, consider $u \in H^2_a(0,1)$ and let P be the subspace of polynomials of degree one. Then, we can find a polynomial p_1 of degree one such that

$$\|u-p_1\|_{L^2(0,1)} = \min_{p\in P} \|u-p\|_{L^2(0,1)}.$$

Set $v := u - p_1$; then $v \in H^2_a(0,1)$, v has at least two zeros and its derivative vanishes at least once (see Lemma 2.8 below). Hence $v \in \mathcal{X}$ and, by (2.12), there exists a positive constant C such that

$$\|v'\|^2_{L^2(0,1)} = \int_0^1 (v'(x))^2 dx \le C\|\sqrt{a}v''\|^2_{L^2(0,1)}.$$

Hence

$$\|u'\|_{L^2(0,1)} \le \|v'\|_{L^2(0,1)} + \|p_1'\|_{L^2(0,1)} \le C\|\sqrt{a}v''\|_{L^2(0,1)} + \|p_1'\|_{L^2(0,1)}. \tag{2.13}$$

It remains to estimate $\|p_1'\|_{L^2(0,1)}$. To this aim, observe that obviously there exists a positive constant C such that

$$\|p_1'\|_{L^2(0,1)} \le C\|p_1\|_{L^2(0,1)}. \tag{2.14}$$

Moreover,

$$\|v\|_{L^2(0,1)} = \|u-p_1\|_{L^2(0,1)} \le \|u\|_{L^2(0,1)}$$

and

$$\|p_1\|_{L^2(0,1)} \le \|u-p_1\|_{L^2(0,1)} + \|u\|_{L^2(0,1)} \le 2\|u\|_{L^2(0,1)}. \tag{2.15}$$

Hence, by (2.14) and (2.15), it follows that

$$\|p_1'\|_{L^2(0,1)} \le C\|u\|_{L^2(0,1)}, \tag{2.16}$$

for a positive constant C. By (2.13) and (2.16), we obtain

$$\|u'\|_{L^2(0,1)} \le C\|u\|_{2,a}, \tag{2.17}$$

thus the statement follows.

The proof in the case $x_0 \neq 0$ is similar, so we omit it. □

Clearly, the analogous of Proposition 2.2 holds in the case $i = 1$ with a strongly degenerate without additional assumption on the function a itself.

Lemma 2.8. *Let $X := L^2(0,1) \cap \mathcal{C}[0,1]$ and let P be the subspace of polynomials of degree one. For all $u \in H^2_a(0,1)$, if $p_1 \in P$ is such that*

$$\|u-p_1\|_{L^2(0,1)} = \min_{p\in P} \|u-p\|_{L^2(0,1)},$$

then the function $v := u - p_1$ has at least two zeros.

Proof. Assume that $v(x) := u(x) - p_1(x) \neq 0$ for all $x \in [0,1]$. Without loss of generality, we can assume $v(x) > 0$. Clearly, $v(x) \geq \min_{[0,1]}(u - p_1) =: \alpha > 0$. Assume that $p_1(x) = mx + q$, $m, q \in \mathbb{R}$. Then $p_1 + \alpha \in P$ and

$$\|u - p_1\|^2_{L^2(0,1)} - \|u - p_1 - \alpha\|^2_{L^2(0,1)} = \alpha \int_0^1 (-\alpha + 2u - 2mx - 2q)dx.$$

Recalling that $u(x) - mx - q \geq \alpha$ and $\alpha > 0$, we have

$$\|u - p_1\|^2_{L^2(0,1)} - \|u - p_1 - \alpha\|^2_{L^2(0,1)} \geq 0.$$

But this is not possible since, by assumption, p_1 is such that

$$\|u - p_1\|_{L^2(0,1)} = \min_{p \in P} \|u - p\|_{L^2(0,1)}.$$

Now, assume that there exists only a point $y_0 \in [0,1]$ such that $v(y_0) = 0$ and $v(x) \neq 0$ for all $x \neq y_0$. For $\epsilon \in \big(0, 2\int_0^1 v(x)dx\big)$, consider $g_\epsilon(x) = v(x) - \epsilon$. Then one can prove that

$$\|g_\epsilon\|^2_{L^2(0,1)} < \|v\|^2_{L^2(0,1)}.$$

Indeed

$$\|g_\epsilon\|^2_{L^2(0,1)} = \int_0^1 (v^2 + \epsilon^2 - 2\epsilon v)dx < \int_0^1 v^2 dx$$

if and only if

$$\epsilon\Big(\epsilon - 2\int_0^1 v\,dx\Big) < 0 \iff \epsilon < 2\int_0^1 v\,dx.$$

Again, we find $p(x) := p_1(x) + \epsilon \in P$, such that

$$\|u - p\|_{L^2(0,1)} \leq \|u - p_1\|_{L^2(0,1)}$$

and this is not possible. Thus, the proof is complete. □

Also in the strongly degenerate case we consider the space $\mathcal{Z}_w(0,1)$ given in (2.6), where $H^2_a(0,1)$ is the one defined in (2.11). To distinguish the two spaces, we use the notation $\mathcal{Z}_s(0,1)$ if a is strongly degenerate. Thus, if $u \in \mathcal{Z}_s(0,1)$, u' is locally absolutely continuous in $[0,1] \setminus \{x_0\}$ and not absolutely continuous in $[0,1]$ as for the weakly degenerate case; so equality (2.9) is not true a priori. For this reason in [8] we characterize the space $\mathcal{Z}_s(0,1)$. In particular, we introduce the space

$$\begin{aligned} X := \big\{&u \in H^1(0,1) : \ u' \text{ is locally absolutely continuous in } [0,1] \setminus \{x_0\},\\ &au, au' \in H^1(0,1),\ au'' \in H^2(0,1),\ \sqrt{a}u'' \in L^2(0,1),\\ &(au^{(k)})(x_0) = 0,\ \text{for all } k = 0,1,2\big\}. \end{aligned}$$

Using the definition of X one can easily obtain the following property, see [8, Lemma 2.2].

Lemma 2.9. *For all $u \in X$ we have that*

1. $|a(x)u(x)| \leq \|(au)'\|_{L^2(0,1)}\sqrt{|x - x_0|}$,
2. $|a(x)u'(x)| \leq \|(au')'\|_{L^2(0,1)}\sqrt{|x - x_0|}$,
3. $|a(x)u''(x)| \leq \|(au'')'\|_{L^2(0,1)}\sqrt{|x - x_0|}$

for all $x \in [0,1]$.

Thanks to the previous estimates and the fact that $\frac{1}{a} \notin L^1(0,1)$, one can prove the following characterization.

Proposition 2.10 ([8, Proposition 2.1])**.** *The spaces X and $\mathcal{Z}_s(0,1)$ coincide.*

For the non divergence case we consider the same spaces as for the weakly degenerate case but, to prove a formula similar to (2.10), we have to characterize the space $H^2_{1/a}(0,1)$. Thus, we introduce

$$Y := \{u \in H^2_{1/a}(0,1) : u(x_0) = (au')(x_0) = 0\}$$

and, proceeding as in [9] and [21] (if $x_0 \in \{0,1\}$) or as in [7] (if $x_0 \in (0,1)$), one can prove the following result.

Proposition 2.11. *If Hypothesis 2.6 is satisfied, then*

$$H^2_{1/a}(0,1) = Y.$$

Hence, if Hypothesis 2.6 is satisfied, we can rewrite the space $\mathcal{W}_s(0,1)$ defined as in (2.8) in the following way

$$\mathcal{W}_s(0,1) = \{u \in H^2_{1/a}(0,1) : u(x_0) = (au')(x_0) = 0 \text{ and } au'''' \in L^2_{1/a}(0,1)\}.$$

As for the weakly degenerate case, one can prove the following Green formulas:

Lemma 2.12 ([8, Lemmas 2.3, 3.2])**.** *If a is strongly degenerate, then*

(1) *equality* (2.9) *holds for all $(u,v) \in \mathcal{Z}_s(0,1) \times H^2_a(0,1)$;*
(2) *assume Hypothesis 2.6:*
- *if $x_0 \in (0,1)$, then for all $(u,v) \in \mathcal{W}_s(0,1) \times H^2_{1/a}(0,1)$, the equality*
$$\int_0^1 u''''v dx = [u'''v]_{x=0}^{x=1} - [u''v']_{x=0}^{x=1} + [u''v']_{x_0^-}^{x_0^+} + \int_0^1 u''v''dx,$$
holds. Here $u''(x_0^+) = \lim_{\delta\to 0^+} u''(x_0+\delta)$, $u''(x_0^-) = \lim_{\delta\to 0^+} u''(x_0-\delta)$ and $v'(x_0^+) = v'(x_0^-) = v'(x_0)$;
- *if $x_0 = 0$, then for all $(u,v) \in \mathcal{W}_s(0,1) \times H^2_{1/a}(0,1)$*
$$\int_0^1 u''''v\,dx = u'''(1)v(1) - [u''v']_{x=0}^{x=1} + \int_0^1 u''v''dx;$$
- *if $x_0 = 1$, then for all $(u,v) \in \mathcal{W}_s(0,1) \times H^2_{1/a}(0,1)$*
$$\int_0^1 u''''v\,dx = -u'''(0)v(0) - [u''v']_{x=0}^{x=1} + \int_0^1 u''v''dx.$$

Actually Proposition 2.11 and Lemma 2.12.2 are proved in [8] under a weaker assumption (see [8, Hypothesis 3.1]; naturally, Hypothesis 2.6 implies this assumption). However, here we consider a stronger assumption since we have to apply Lemma 2.12.2 under this hypothesis.

3. Operators in divergence form with generalized Wentzell boundary conditions and interior degeneracy

Let us fix $\beta_j, \gamma_j \in \mathbb{R}$ such that $\beta_j > 0$ and $\gamma_j \leq 0$, $j = 0,1$. Consider a weakly or a strongly degenerate function a and assume that the degeneracy point x_0 is in the *interior* of the domain. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 |f(x)|^2 dx + \frac{a(0)|f(0)|^2}{\beta_0} + \frac{a(1)|f(1)|^2}{\beta_1} \in \mathbb{R}$$

and define X_μ to be the completion of $\mathcal{C}[0,1]$ with respect to the norm $\|\cdot\|_{X_\mu}$, where

$$\|f\|^2_{X_\mu} = \int_0^1 |f(x)|^2 dx + \frac{a(0)|f(0)|^2}{\beta_0} + \frac{a(1)|f(1)|^2}{\beta_1}.$$

From [16] and [20], it follows that $X_\mu := L^2([0,1], d\mu)$, where

$$d\mu := dx|_{(0,1)} \oplus \frac{adS}{\beta}|_{\{0,1\}},$$

dx denotes the Lebesgue measure on $(0,1)$, $\beta = (\beta_0, \beta_1)$, and $\frac{adS}{\beta}$ denotes the natural Dirac measure dS on $\{0,1\}$ with weight $\frac{a}{\beta}$. More precisely, X_μ is a Hilbert space with respect to the inner product given by

$$\langle f, g\rangle_{X_\mu} = \int_0^1 f(x)g(x)\,dx + \frac{a(0)f(0)g(0)}{\beta_0} + \frac{a(1)f(1)g(1)}{\beta_1},$$

where $f, g \in X_\mu$ are written as $(f\chi_{(0,1)}, (f(0), f(1)))$, $(g\chi_{(0,1)}, (g(0), g(1)))$. We recall that, as usual, $\chi_{(0,1)}$ denotes the characteristic function of the interval $(0,1)$. Clearly, $H^i(0,1) \subseteq X_\mu$, $i = 0,1,2$, where we recall $H^0(0,1) = L^2(0,1)$. In particular, $\mathcal{C}[0,1] \subseteq X_\mu \subseteq L^2(0,1)$. Now, define the operator in divergence form $A_1 u := (au'')''$ equipped with the following generalized Wentzell boundary conditions

$$A_1 u(j) + (-1)^{j+1}\frac{\beta_j}{a(j)}(au'')'(j) + \gamma_j u(j) = 0, \quad j = 0,1, \tag{3.1}$$

and the additional boundary conditions

$$u''(0) = u''(1) = 0. \tag{3.2}$$

We distinguish between the weakly degenerate case and the strongly degenerate one.

If a is weakly degenerate we define the weighted space:

$$\mathcal{Z}_\mu(0,1) := \{u \in H^2_a(0,1) : au'' \in H^2(0,1) \text{ and } (au'')'' \in X_\mu\}.$$

Clearly $\mathcal{Z}_\mu(0,1) \subseteq \mathcal{Z}_w(0,1)$, and using the definition of the space $H^2_a(0,1)$, $\mathcal{Z}_\mu(0,1)$ can be rewritten as

$$\begin{aligned}\mathcal{Z}_\mu(0,1) = \big\{&u \in H^1(0,1) : u' \text{ is absolutely continuous in [0,1]},\\ &\sqrt{a}u'' \in L^2(0,1),\ au'' \in H^2(0,1), (au'')'' \in X_\mu\big\}.\end{aligned}$$

Let us observe that for any $(u,v) \in \mathcal{Z}_\mu(0,1) \times H^2_a(0,1)$ the Green formula (2.9) holds.

Now, let us define the domain of the operator A_1 through the following subspace of $Z_\mu \subseteq X_\mu$:

$$D_w(A_1) := \big\{u \in \mathcal{Z}_\mu(0,1) :\ (3.1) \text{ and } (3.2) \text{ hold}\big\}.$$

Then we can prove the following theorem.

Theorem 3.1. *The operator $A_1 : D_w(A_1) \to X_\mu$ is non negative, self-adjoint with dense domain. Thus $-A_1$ generates a contraction semigroup.*

Proof. First of all we prove that A_1 is symmetric and non negative on X_μ.

A_1 is symmetric: take $u, v \in D_w(A_1)$. Then $(u, v) \in \mathcal{Z}_\mu(0,1) \times H^2_a(0,1)$ and (2.9) holds. Consequently

$$\begin{aligned}
\langle A_1 u, v\rangle_{X_\mu} &= \int_0^1 (au'')''v\,dx + \frac{a(0)(au'')''(0)v(0)}{\beta_0} + \frac{a(1)(au'')''(1)v(1)}{\beta_1} \\
&= [(au'')'v]_{x=0}^{x=1} - [au''v']_{x=0}^{x=1} + \int_0^1 au''v''dx \\
&\quad + \frac{a(0)v(0)}{\beta_0}\Big(\frac{\beta_0}{a(0)}(au'')'(0) - \gamma_0 u(0)\Big) \\
&\quad - \frac{a(1)v(1)}{\beta_1}\Big(\frac{\beta_1}{a(1)}(au'')'(1) + \gamma_1 u(1)\Big) \\
&= v(1)(au'')'(1) - v(0)(au'')'(0) + \int_0^1 au''v''dx - \frac{\gamma_0}{\beta_0}a(0)v(0)u(0) \\
&\quad - \frac{\gamma_1}{\beta_1}a(1)v(1)u(1) + v(0)(au'')'(0) - v(1)(au'')'(1) \\
&= \int_0^1 au''v''dx - \frac{\gamma_0}{\beta_0}a(0)v(0)u(0) - \frac{\gamma_1}{\beta_1}a(1)v(1)u(1) \\
&= \langle u, A_1 v\rangle_{X_\mu}.
\end{aligned}$$

A_1 is non negative: for any $u \in D_w(A_1)$, according to the above calculations, one has

$$\langle A_1 u, u\rangle_{X_\mu} = \int_0^1 a|u''|^2dx - \frac{\gamma_0}{\beta_0}a(0)|u(0)|^2 - \frac{\gamma_1}{\beta_1}a(1)|u(1)|^2 \ge 0.$$

Now we prove that $\lambda I + A_1$ is surjective for sufficiently large $\lambda \in \mathbb{R}$.

$\lambda I + A_1$ is surjective: consider the space $H^2_a(0,1)$ with the inner product

$$(u, v)_1 := \langle u, v\rangle_{X_\mu} + \langle \sqrt{a}u'', \sqrt{a}v''\rangle_{L^2(0,1)} \quad \forall u, v \in H^2_a(0,1),$$

which induces the norm

$$\|u\|^2_\circ := \|u\|^2_{X_\mu} + \|\sqrt{a}u''\|^2_{L^2(0,1)} \quad \forall u \in H^2_a(0,1).$$

It is well known that for all $u \in H^1(0,1)$

$$|u(x)| \le C\|u\|_{H^1(0,1)}, \quad \forall x \in [0,1] \tag{3.3}$$

for a suitable positive constant C. Thus, thanks to (2.4) and (3.3), one can prove that the norm $\|\cdot\|_\circ$ is equivalent to $\|\cdot\|_{H^2_a(0,1)}$. Moreover,

$$H^2_a(0,1) \hookrightarrow X_\mu \hookrightarrow (H^2_a(0,1))^*,$$

where $(H^2_a(0,1))^*$ is the dual space of $H^2_a(0,1)$ with respect to X_μ. Let $f \in X_\mu$ and define $F : H^2_a(0,1) \to \mathbb{R}$ such that

$$F(v) = \int_0^1 fv\,dx + \frac{a(0)f(0)v(0)}{\beta_0} + \frac{a(1)f(1)v(1)}{\beta_1} \quad \forall v \in H^2_a(0,1).$$

From $H^2_a(0,1) \hookrightarrow X_\mu$, it follows that $F \in (H^2_a(0,1))^*$. Now, we define

$$L(u, v) := \lambda\int_0^1 uv\,dx + \int_0^1 au''v''dx + \frac{(\lambda-\gamma_1)a(1)}{\beta_1}u(1)v(1) + \frac{(\lambda-\gamma_0)a(0)}{\beta_0}u(0)v(0),$$

for all $u, v \in H^2_a(0,1)$. Clearly $L(u,v)$ is a continuous bilinear form. Moreover, it is coercive. Indeed, if we take $\lambda > \gamma_i$, for $i = 0, 1$, then

$$\begin{aligned} L(u,u) &= \lambda \int_0^1 u^2 dx + \int_0^1 a(u'')^2 dx + \frac{(\lambda - \gamma_1)a(1)}{\beta_1} u^2(1) + \frac{(\lambda - \gamma_0)a(0)}{\beta_0} u^2(0) \\ &\geq \alpha \|u\|_\circ^2, \end{aligned}$$

for all $u \in H^2_a(0,1)$, where $\alpha := \min\{\lambda, 1, \lambda - \gamma_0, \lambda - \gamma_1\}$. As a consequence, by the Lax-Milgram Theorem, there exists a unique $u \in H^2_a(0,1)$ such that

$$L(u,v) = F(v)$$

for any $v \in H^2_a(0,1)$. Hence, setting for simplicity $C_i := \frac{a(i)}{\beta_i}$, $i = 0, 1$, the previous equality can be rewritten as

$$\begin{aligned} &\int_0^1 a u'' v'' dx + (\lambda - \gamma_0) C_0 u(0) v(0) + (\lambda - \gamma_1) C_1 u(1) v(1) \\ &= \int_0^1 (f - \lambda u) v \, dx + C_0 f(0) v(0) + C_1 f(1) v(1). \end{aligned} \tag{3.4}$$

In particular, (3.4) holds for all $v \in C_c^\infty(0,1) \subseteq H^2_a(0,1)$. Hence

$$\int_0^1 a u'' v'' dx = \int_0^1 (f - \lambda u) v \, dx.$$

This implies that $(au'')'' = f - \lambda u$ a.e. in $(0,1)$. Hence, using the fact that $f - \lambda u \in X_\mu \subseteq L^2(0,1)$, one has that $(au'')'' \in X_\mu$ and $au'' \in H^2(0,1)$. Thus $u \in \mathcal{Z}_\mu(0,1)$.

Now, we come back to (3.4). Using (2.9), (3.4) becomes

$$\begin{aligned} &\int_0^1 (au'')'' v \, dx - [(au'')' v]_{x=0}^{x=1} + [a u'' v']_{x=0}^{x=1} - \gamma_0 C_0 u(0) v(0) - \gamma_1 C_1 u(1) v(1) \\ &= \int_0^1 (f - \lambda u) v \, dx + C_0 (f - \lambda u)(0) v(0) + C_1 (f - \lambda u)(1) v(1), \end{aligned} \tag{3.5}$$

for all $v \in H^2_a(0,1)$. Since $(au'')'' = f - \lambda u$ a.e. in $(0,1)$, (3.5) becomes

$$\begin{aligned} &- [(au'')' v]_{x=0}^{x=1} + [a u'' v']_{x=0}^{x=1} - \gamma_0 C_0 u(0) v(0) - \gamma_1 C_1 u(1) v(1) \\ &= C_0 (f - \lambda u)(0) v(0) + C_1 (f - \lambda u)(1) v(1), \end{aligned} \tag{3.6}$$

for all $v \in H^2_a(0,1)$. Recalling that $a(0) \neq 0$ and $a(1) \neq 0$, we have immediately that $u''(0) = u''(1) = 0$. Thus (3.6) becomes

$$\begin{aligned} &- [(au'')' v](1) + [(au'')' v](0) - \gamma_0 C_0 u(0) v(0) - \gamma_1 C_1 u(1) v(1) \\ &= C_0 (au'')''(0) v(0) + C_1 (au'')''(1) v(1) \end{aligned}$$

for all $v \in H^2_a(0,1)$. Hence

$$A_1 u(j) + (-1)^{j+1} \frac{\beta_j}{a(j)} (au'')'(j) + \gamma_j u(j) = 0, \quad j = 0, 1.$$

This implies that $u \in D_w(A_1)$. Thus $\lambda I + A_1$ is surjective for λ sufficiently large.

By Theorems 5.3 and 5.5 (see the Appendix), we know that $-A_1$ is self-adjoint with upper bound 0, has dense domain and $(-A_1, D_w(A_1))$ generates a contraction semigroup. $\square$

By Theorem 3.1 (see the Appendix), the problem

$$\begin{gathered} u_t(t,x)+A_1u(t,x)=h(t,x), \quad (t,x)\in(0,T)\times(0,1), \\ A_1u(t,j)+(-1)^{j+1}\frac{\beta_j}{a(j)}(au_{xx})_x(t,j)+\gamma_j u(t,j)=0, \quad t\in(0,T), j=0,1, \\ u_{xx}(t,0)=u_{xx}(t,1)=0, \quad t\in(0,T), \\ u(0,x)=u_0(x), \quad x\in(0,1), \end{gathered} \tag{3.7}$$

is well posed in the sense of Theorem 3.3 below. As a first step, we recall the following concept.

Definition 3.2. If $u_0 \in X_\mu$ and $h\in L^2(0,T;X_\mu)$, a function u is said to be a weak solution of (3.7) if

$$u\in \mathcal{C}\big([0,T];X_\mu\big)\cap L^2\big(0,T;H^2_a(0,1)\big)$$

and

$$\begin{aligned} &\int_0^1 u(T,x)\varphi(T,x)\,dx-\int_0^1 u_0(x)\varphi(0,x)\,dx-\int_{(0,T)\times(0,1)}u(t,x)\varphi_t(t,x)\,dx\,dt \\ &+\frac{a(1)u(T,1)\varphi(T,1)}{\beta_1}-\frac{a(1)u_0(1)\varphi(0,1)}{\beta_1}-\frac{a(1)}{\beta_1}\int_0^T u(t,1)\varphi_t(t,1)dt \\ &+\frac{a(0)u(T,0)\varphi(T,0)}{\beta_0}-\frac{a(0)u_0(0)\varphi(0,0)}{\beta_0}-\frac{a(0)}{\beta_0}\int_0^T u(t,0)\varphi_t(t,0)dt \\ &=-\int_{(0,T)\times(0,1)}a(x)u_{xx}(t,x)\varphi_{xx}(t,x)\,dx\,dt-\frac{\gamma_1}{\beta_1}\int_0^T a(1)u(t,1)\varphi(t,1)dt \\ &-\frac{\gamma_0}{\beta_0}\int_0^T a(0)u(t,0)\varphi(t,0)dt+\int_{(0,T)\times(0,1)}h(t,x)\varphi(t,x)\,dx\,dt \\ &+\int_0^T\frac{a(1)h(t,1)\varphi(t,1)}{\beta_1}dt+\int_0^T\frac{a(0)h(t,0)\varphi(t,0)}{\beta_0}dt \end{aligned}$$

for all $\varphi\in H^1(0,T;X_\mu)\cap L^2(0,T;H^2_a(0,1))$.

Theorem 3.3. *For all $h\in L^2(0,T;X_\mu)$ and $u_0\in X_\mu$, there exists a unique solution*

$$u\in \mathcal{C}\big([0,T];X_\mu\big)\cap L^2\big(0,T;H^2_a(0,1)\big)$$

of (3.7) *such that*

$$\sup_{t\in[0,T]}\|u(t)\|^2_{X_\mu}+\int_0^T\|u(t)\|^2_{H^2_a(0,1)}dt\le C_T\Big(\|u_0\|^2_{X_\mu}+\|h\|^2_{L^2(0,T;X_\mu)}\Big) \tag{3.8}$$

for some positive constant C_T. Moreover, if $h\in W^{1,1}(0,T;X_\mu)$ and $u_0\in D_w(A_1)$, then

$$u\in\mathcal{C}^1\big([0,T];X_\mu\big)\cap\mathcal{C}\big([0,T];D_w(A_1)\big). \tag{3.9}$$

Proof. The assertion concerning the assumption $u_0\in X_\mu$ and the regularity of the solution u when $u_0\in D_w(A_1)$ is a consequence of the results in [2], [28, Chapter 3, Section 4, Theorem 4.1 and Remark 4.3] and of [10, Lemma 4.1.5, Proposition 4.1.6 and Proposition 4.3.9], [4, Propositions 3.2 and 3.3]. We only need to prove (3.8). Let us fix $u_0\in D_w(A_1)$ and consider the corresponding weak solution $u\in$

$\mathcal{C}^1\big([0,T];X_\mu\big)\cap\mathcal{C}\big([0,T];D_w(A_1)\big)$. Now, we multiply the equation of (3.7) by u considering the inner product in X_μ; thus

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{X_\mu}+\|\sqrt{a}u_{xx}(t)\|^2_{L^2(0,1)}-\frac{\gamma_0}{\beta_0}a(0)u^2(t,0)-\frac{\gamma_1}{\beta_1}a(1)u^2(t,1)\\&\le\frac{1}{2}\|u(t)\|^2_{X_\mu}+\frac{1}{2}\|h(t)\|^2_{X_\mu}.\end{aligned}$$

Hence we deduce that

$$\frac{d}{dt}\|u(t)\|^2_{X_\mu}\le\frac{d}{dt}\|u(t)\|^2_{X_\mu}+2\|\sqrt{a}u_{xx}(t)\|^2_{X_\mu}\le\|u(t)\|^2_{X_\mu}+\|h(t)\|^2_{X_\mu}. \tag{3.10}$$

By Gronwall's Lemma for every $t\in[0,T]$, we obtain

$$\|u(t)\|^2_{X_\mu}\le e^T\big(\|u_0\|^2_{X_\mu}+\|h\|^2_{L^2(0,T;X_\mu)}\big).$$

Thus, there exists a positive constant C such that

$$\sup_{t\in[0,T]}\|u(t)\|^2_{X_\mu}\le C\big(\|u_0\|^2_{X_\mu}+\|h\|^2_{L^2(0,T;X_\mu)}\big). \tag{3.11}$$

Integrating the second inequality of (3.10) over $(0,T)$ and using (3.11), we have

$$\int_0^T\|\sqrt{a}u_{xx}(t)\|^2_{L^2(0,1)}dt\le C\big(\|u_0\|^2_{X_\mu}+\|h\|^2_{L^2(0,T;X_\mu)}\big),$$

for a suitable positive constant C, and the conclusion follows.

Clearly, (3.8) and (3.9) hold also if $u_0\in H^2_a(0,1)$, since $D_w(A_1)$ is dense in $H^2_a(0,1)$. $\square$

Now we assume that a is strongly degenerate and we consider the operator $(A_1,D_s(A_1))$. Here A_1 is defined as in the weakly degenerate case and $D_s(A_1)$ is defined as $D_w(A_1)$ where we have to consider $H^2_a(0,1)$ defined in (2.11) in place of $H^2_a(0,1)$ defined in (2.1). In particular,

$$D_s(A_1):=\big\{u\in\mathcal{Z}_{s,\mu}(0,1):\text{(3.1) and (3.2) hold}\big\},$$

where

$$\begin{aligned}\mathcal{Z}_{s,\mu}(0,1):=\big\{&u\in H^1(0,1):u'\text{ is locally absolutely continuous in}[0,1]\setminus\{x_0\}\\&\text{and }\sqrt{a}u''\in L^2(0,1),au''\in H^2(0,1),(au'')''\in X_\mu\big\}.\end{aligned}$$

Clearly, the Green formula given in Lemma 2.12 for the divergence case still holds.

Now, we define

$$\begin{aligned}\widetilde{X}:=\big\{&u\in H^1(0,1):u'\text{ is locally absolutely continuous in }[0,1]\setminus\{x_0\},\\&au,au'\in H^1(0,1),\ au''\in H^2(0,1),\ \sqrt{a}u''\in L^2(0,1),\\&(au'')''\in X_\mu,\ (au^{(k)})(x_0)=0,\ \text{for all }k=0,1,2\big\}.\end{aligned}$$

Analogously to Proposition 2.10 one has that

$$\widetilde{X}=\mathcal{Z}_{s,\mu}(0,1).$$

As for the weakly degenerate case, one has the next result which contains the generation property in the strongly degenerate context.

Theorem 3.4. *Assume Hypothesis 2.6. The operator $A_1:D_s(A_1)\to X_\mu$ is non negative, self-adjoint with dense domain. Thus $-A_1$ generates a contraction semigroup.*

The proof of this theorem is analogous to the one of Theorem 3.1, so we omit it. In any case we underline that, thanks to (2.17) and (3.3), the two norms $\|\cdot\|_\circ$ and $\|\cdot\|_{H^2_a(0,1)}$ are equivalent. Thus we can prove that $\lambda I + A_1$ is surjective, for λ sufficiently large, and the analogous of Theorem 3.3 holds for (3.7) if a is strongly degenerate.

4. Operators in non divergence form with generalized Wentzell boundary conditions and interior degeneracy

As in the previous section, let us fix β_j, $\gamma_j \in \mathbb{R}$ such that $\beta_j > 0$, $\gamma_j \le 0$, $j = 0, 1$. Consider a weakly degenerate function a, and assume that the degeneracy point x_0 belongs to $(0,1)$. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 \frac{|f(x)|^2}{a} dx + \frac{|f(0)|^2}{\beta_0} + \frac{|f(1)|^2}{\beta_1} \in \mathbb{R}$$

and define Y_μ to be the completion of $\mathcal{C}[0,1]$ with respect to the norm $\|\cdot\|_{Y_\mu}$, where

$$\|f\|^2_{Y_\mu} = \int_0^1 \frac{|f(x)|^2}{a} dx + \frac{|f(0)|^2}{\beta_0} + \frac{|f(1)|^2}{\beta_1}.$$

By [16] and [20], it follows that $Y_\mu := L^2_{1/a}([0,1], d\mu)$, where

$$d\mu := \frac{dx}{a}|_{(0,1)} \oplus \frac{dS}{\beta}|_{\{0,1\}}.$$

As before, dx denotes the Lebesgue measure on $(0,1)$, $\beta = (\beta_0, \beta_1)$, and $\frac{dS}{\beta}$ denotes the natural Dirac measure dS on $\{0,1\}$ with weight $1/\beta$. In this way Y_μ becomes a Hilbert space with the inner product given by

$$\langle f, g\rangle_{Y_\mu} = \int_0^1 \frac{f(x)g(x)}{a} dx + \frac{f(0)g(0)}{\beta_0} + \frac{f(1)g(1)}{\beta_1},$$

in which $f, g \in Y_\mu$ are written as in Section 3.

Clearly, $H^i_{1/a}(0,1) \subseteq Y_\mu$, $i = 1, 2$; in particular, $\mathcal{C}[0,1] \subseteq Y_\mu \subseteq L^2_{1/a}(0,1)$.

Now we introduce the operator in non divergence form $A_2 u := a u''''$ equipped with the general Wentzell boundary conditions

$$A_2 u(j) + (-1)^{j+1}\beta_j u'''(j) + \gamma_j u(j) = 0, \quad j = 0, 1, \tag{4.1}$$

and the additional boundary conditions (3.2). Moreover, we consider the weighted space:

$$\mathcal{W}_\mu(0,1) := \big\{u \in H^2_{1/a}(0,1) : a u'''' \in Y_\mu\big\}.$$

Now, since $au'''' \in Y_\mu$ implies $au'''' \in L^2_{1/a}(0,1)$, the same considerations made before Lemma 2.4 hold; in particular, if $u \in \mathcal{W}_\mu(0,1)$, then $(au^{(k)})(x_0) = 0$ for $k = 0, 1, 2, 3$. Moreover, observe that, for any $(u, v) \in \mathcal{W}_\mu(0,1) \times H^2_{1/a}(0,1)$, the Green formula (2.10) holds and $\mathcal{W}_\mu(0,1) \subseteq \mathcal{W}_w(0,1)$.

Now, let us define the domain of the operator A_2 through the following subspace of $W_\mu \subseteq Y_\mu$:

$$D_w(A_2) := \big\{u \in \mathcal{W}_\mu(0,1) : (4.1) \text{ and } (3.2) \text{ hold}\big\}.$$

Hence, we can prove the following result that establishes the main properties of this operator.

Theorem 4.1. *If a is weakly degenerate, then the operator $(A_2, D_w(A_2))$ is self-adjoint and non-negative on Y_μ with dense domain. Thus $-A_2$ generates a contraction semigroup.*

Proof. First, we prove the symmetry and non-negativity of A_2.
A_2 is symmetric: let $u, v \in D_w(A_2)$; then $(u, v) \in \mathcal{W}_\mu(0,1) \times H^2_{1/a}(0,1)$ and, by (2.10), we have

$$
\begin{aligned}
\langle A_2u, v\rangle_{Y_\mu} &= \int_0^1 \frac{au''''v}{a}\,dx + \frac{a(0)u''''(0)v(0)}{\beta_0} + \frac{a(1)u''''(1)v(1)}{\beta_1} \\
&= [u'''v]_{x=0}^{x=1} - [u''v']_{x=0}^{x=1} + \int_0^1 u''v''dx + \frac{v(0)}{\beta_0}(\beta_0 u'''(0) - \gamma_0 u(0)) \\
&\quad - \frac{v(1)}{\beta_1}\big(\beta_1 u'''(1) + \gamma_1 u(1)\big) \\
&= \int_0^1 u''v''dx - \frac{\gamma_0}{\beta_0}v(0)u(0) - \frac{\gamma_1}{\beta_1}v(1)u(1) \\
&= \langle u, A_2v\rangle_{Y_\mu}.
\end{aligned}
$$

A_2 is non negative: for any $u \in D_w(A_2)$, according to the above calculations, one has

$$
\langle A_2u, u\rangle_{Y_\mu} = \int_0^1 |u''|^2dx - \frac{\gamma_0}{\beta_0}|u(0)|^2 - \frac{\gamma_1}{\beta_1}|u(1)|^2 \geq 0.
$$

Finally, we prove that $\lambda I + A_2$ is surjective for sufficiently large $\lambda \in \mathbb{R}$.
$\lambda I + A_2$ is surjective: consider the space $H^2_{1/a}(0,1)$ with the inner product

$$
(u, v)_2 := \langle u, v\rangle_{Y_\mu} + \langle u'', v''\rangle_{L^2(0,1)} \quad \forall u, v \in H^2_{1/a}(0,1),
$$

which induces the norm

$$
\|u\|^2_\triangle := \|u\|^2_{Y_\mu} + \|u''\|^2_{L^2(0,1)} \quad \forall u \in H^2_{1/a}(0,1).
$$

Thus, thanks to (3.3), the norm $\|\cdot\|_\triangle$ is equivalent to $\|\cdot\|_{H^2_{1/a}(0,1)}$. Moreover,

$$
H^2_{1/a}(0,1) \hookrightarrow Y_\mu \hookrightarrow (H^2_{1/a}(0,1))^*,
$$

where $(H^2_{1/a}(0,1))^*$ is the dual space of $H^2_{1/a}(0,1)$ with respect to Y_μ. Let $f \in Y_\mu$ and define $F : H^2_{1/a}(0,1) \to \mathbb{R}$ such that

$$
F(v) = \int_0^1 \frac{fv}{a}\,dx + \frac{f(0)v(0)}{\beta_0} + \frac{f(1)v(1)}{\beta_1} \quad \forall v \in H^2_{1/a}(0,1).
$$

From $H^2_{1/a}(0,1) \hookrightarrow Y_\mu$, it follows that $F \in (H^2_{1/a}(0,1))^*$. Now, we define

$$
M(u, v) := \lambda \int_0^1 \frac{uv}{a}\,dx + \int_0^1 u''v''dx + \frac{(\lambda - \gamma_1)}{\beta_1}u(1)v(1) + \frac{(\lambda - \gamma_0)}{\beta_0}u(0)v(0),
$$

for all $u, v \in H^2_{1/a}(0,1)$. Clearly $M(u, v)$ is a continuous bilinear form. Moreover, it is coercive. Indeed, taking $\lambda > \gamma_i$, for $i = 0, 1$, and $\delta := \min\{\lambda, 1, \lambda - \gamma_0, \lambda - \gamma_1\}$,

$$
M(u, u) = \lambda \int_0^1 \frac{u^2}{a}\,dx + \int_0^1 (u'')^2dx + \frac{(\lambda - \gamma_1)}{\beta_1}u^2(1) + \frac{(\lambda - \gamma_0)}{\beta_0}u^2(0) \geq \delta\|u\|^2_\triangle,
$$

for all $u \in H^2_{1/a}(0,1)$. As a consequence, by the Lax-Milgram Theorem, there exists a unique $u \in H^2_{1/a}(0,1)$ such that

$$M(u,v) = F(v)$$

for any $v \in H^2_{1/a}(0,1)$. Hence, the previous equality can be rewritten as

$$\begin{aligned} &\int_0^1 u''v''dx + \frac{(\lambda-\gamma_0)}{\beta_0}u(0)v(0) + \frac{(\lambda-\gamma_1)}{\beta_1}u(1)v(1) \\ &= \int_0^1 \frac{f-\lambda u}{a}\, v\, dx + \frac{f(0)v(0)}{\beta_0} + \frac{f(1)v(1)}{\beta_1}. \end{aligned} \tag{4.2}$$

In particular, (4.2) holds for all $v \in C_c^\infty(0,1) \subseteq H^2_{1/a}(0,1)$. Hence

$$\int_0^1 u''v''dx = \int_0^1 \frac{f-\lambda u}{a}\, v\, dx.$$

In other words, the second distributional derivative of u'' is equal to $\frac{f-\lambda u}{a}$ a.e. in $(0,1)$. Hence, using the fact that $f-\lambda u \in Y_\mu \subseteq L^2_{1/a}(0,1)$, one has that $au'''' \in Y_\mu$. Thus $u \in \mathcal{W}_\mu(0,1)$.

Now, we come back to (4.2). Using (2.10), (4.2) becomes

$$\begin{aligned} &\int_0^1 u''''v\, dx - [u'''v]_{x=0}^{x=1} + [u''v']_{x=0}^{x=1} - \frac{\gamma_0}{\beta_0}u(0)v(0) - \frac{\gamma_1}{\beta_1}u(1)v(1) \\ &= \int_0^1 \frac{f-\lambda u}{a}\, v\, dx + \frac{(f-\lambda u)(0)v(0)}{\beta_0} + \frac{(f-\lambda u)(1)v(1)}{\beta_1} \end{aligned} \tag{4.3}$$

for all $v \in H^2_{1/a}(0,1)$. Thanks to the fact that $au'''' = f-\lambda u$ a.e. in $(0,1)$, (4.3) becomes

$$\begin{aligned} &-[u'''v]_{x=0}^{x=1} + [u''v']_{x=0}^{x=1} - \frac{\gamma_0}{\beta_0}u(0)v(0) - \frac{\gamma_1}{\beta_1}u(1)v(1) \\ &= \frac{(f-\lambda u)(0)}{\beta_0}v(0) + \frac{(f-\lambda u)(1)}{\beta_1}v(1), \end{aligned} \tag{4.4}$$

for all $v \in H^2_{1/a}(0,1)$. We have immediately that $u''(0) = u''(1) = 0$. Thus (4.4) becomes

$$-[u'''v](1) + [u'''v](0) - \frac{\gamma_0}{\beta_0}u(0)v(0) - \frac{\gamma_1}{\beta_1}u(1)v(1) = \frac{(au'''')(0)}{\beta_0}v(0) + \frac{(au'''')(1)}{\beta_1}v(1)$$

for all $v \in H^2_{1/a}(0,1)$. Hence $u \in H^2_{1/a}(0,1)$, $au'''' \in Y_\mu$, u satisfies (3.2) and

$$A_2u(j) + (-1)^{j+1}\beta_j u'''(j) + \gamma_j u(j) = 0, \quad j = 0,1.$$

This implies that $u \in D_w(A_2)$. In other words $\lambda I + A_2$ is surjective for λ sufficiently large.

Thanks to Theorems 5.3 and 5.5 (see the Appendix), we know that $-A_2$ is self-adjoint with upper bound 0, has dense domain and $(-A_2, D_w(A_2))$ generates a contraction semigroup. $\square$

As a consequence of Theorem 4.1, one has that the problem

$$
\begin{gathered}
u_t(t,x) + A_2 u(t,x) = h(t,x), \quad (t,x) \in (0,T) \times (0,1), \\
A_2 u(t,j) + (-1)^{j+1} \beta_j u_{xxx}(t,j) + \gamma_j u(t,j) = 0, \quad t \in (0,T), j = 0,1, \\
u_{xx}(t,0) = u_{xx}(t,1) = 0, \quad t \in (0,T), \\
u(0,x) = u_0(x), \quad x \in (0,1),
\end{gathered} \tag{4.5}
$$

is well posed in the following sense.

Definition 4.2. If $u_0 \in Y_\mu$ and $h \in L^2(0,T;Y_\mu)$, a function u is said to be a weak solution of (4.5) if

$$
u \in \mathcal{C}\big([0,T];Y_\mu\big) \cap L^2\big(0,T;H^2_{1/a}(0,1)\big)
$$

and

$$
\begin{aligned}
&\int_0^1 \frac{u(T,x)\varphi(T,x)}{a(x)}\,dx - \int_0^1 \frac{u_0(x)\varphi(0,x)}{a(x)}\,dx - \int_{(0,T)\times(0,1)} \frac{u(t,x)\varphi_t(t,x)}{a(x)}\,dx\,dt \\
&+ \frac{u(T,1)\varphi(T,1)}{\beta_1} - \frac{u_0(1)\varphi(0,1)}{\beta_1} - \frac{1}{\beta_1}\int_0^T u(t,1)\varphi_t(t,1)dt \\
&+ \frac{u(T,0)\varphi(T,0)}{\beta_0} - \frac{u_0(0)\varphi(0,0)}{\beta_0} - \frac{1}{\beta_0}\int_0^T u(t,0)\varphi_t(t,0)dt \\
&= -\int_{(0,T)\times(0,1)} u_{xx}(t,x)\varphi_{xx}(t,x)\,dx\,dt - \frac{\gamma_1}{\beta_1}\int_0^T u(t,1)\varphi(t,1)dt \\
&\quad - \frac{\gamma_0}{\beta_0}\int_0^T u(t,0)\varphi(t,0)dt + \int_{(0,T)\times(0,1)} h(t,x)\frac{\varphi(t,x)}{a(x)}\,dx\,dt \\
&\quad + \int_0^T \frac{h(t,1)\varphi(t,1)}{\beta_1}dt + \int_0^T \frac{h(t,0)\varphi(t,0)}{\beta_0}dt
\end{aligned}
$$

for all $\varphi \in H^1(0,T;Y_\mu) \cap L^2(0,T;H^2_{1/a}(0,1))$.

In particular, the following well posedness theorem holds.

Theorem 4.3. *For all $h \in L^2(0,T;Y_\mu)$ and $u_0 \in Y_\mu$, there exists a unique solution*

$$
u \in \mathcal{C}\big([0,T];Y_\mu\big) \cap L^2\big(0,T;H^2_{1/a}(0,1)\big)
$$

of (4.5) such that

$$
\sup_{t\in[0,T]} \|u(t)\|^2_{Y_\mu} + \int_0^T \|u(t)\|^2_{H^2_{1/a}(0,1)}dt \le C_T\big(\|u_0\|^2_{Y_\mu} + \|h\|^2_{L^2(0,T;L^2_{1/a}(0,1))}\big)
$$

for some positive constant C_T. Moreover, if $h \in W^{1,1}(0,T;L^2_{1/a}(0,1))$ and $u_0 \in D_w(A_2)$, then

$$
u \in \mathcal{C}^1\big([0,T];Y_\mu\big) \cap \mathcal{C}\big([0,T];D_w(A_2)\big).
$$

5. Appendix

In this last section we just give some important results needed for the above proofs. These results are well known, and we write them to make the paper self-contained.

Proposition 5.1 ([14, Chapter 2.3, page 90]). *A linear operator A on the real Hilbert space H is dissipative if and only if*

$$\langle Au, u\rangle_H \leq 0 \quad \forall u \in D(A).$$

Definition 5.2. A linear operator A is bounded above in the Hilbert space H if there exists $\omega \in \mathbb{R}$ such that

$$\langle Au, u\rangle_H \leq \omega\langle u, u\rangle_H$$

for all $u \in D(A)$. In this case ω is called an upper bound of A.

Thus, by the previous proposition, we have that if A is dissipative on the real Hilbert space H, then it is bounded above with upper bound 0.

Theorem 5.3 ([1, Theorem B.14]). *Let A be a linear operator on the Hilbert space H and let $\omega \in \mathbb{R}$. The following assertions are equivalent:*

(1) *A is self-adjoint with upper bound ω;*
(2) (a) *$\langle Au, v\rangle_H = \langle v, Au\rangle_H$ for all $u, v \in D(A)$,*
 (b) *$\langle Au, u\rangle_H \leq \omega\langle u, u\rangle_H$ for all $u \in D(A)$,*
 (c) *there exists $\lambda > \omega$ such that $(\lambda I - A)$ is surjective in H.*

Finally, we recall the following generation results.

Theorem 5.4 ([14, Chapter 2.3, page 91]). *A self-adjoint operator $(A, D(A))$ on a Hilbert space H generates a strongly continuous semigroup (of self-adjoint operators) if and only if it is bounded above.*

Theorem 5.5 ([14, Corollary 3.20]). *Let $(A, D(A))$ be a dissipative operator on a reflexive Banach space such that $\lambda I - A$ is surjective for some $\lambda > 0$. Then A is densely defined and generates a contraction semigroup.*

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