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LINEAR HIGHER-ORDER FRACTIONAL DIFFERENTIAL AND INTEGRAL EQUATIONS

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ABSTRACT. We study the equivalences and the implications between linear (or homogeneous) nth order fractional differential equations (FDEs) and integral equations in the spaces $L^1(a,b)$ and $C[a,b]$ when $n \geq 2$. We establish the equivalence in $C[a,b]$ of the IVP of the nth order FDE subject to the initial condition $u^{(i)}(a) = u_i$ for $i \in \{0,1,\dots,n-1\}$ when $n \geq 2$. The difficulty is that the known conditions for such equivalence for the first order FDEs are not sufficient for equivalence in the nth order FDEs with $n \geq 2$. In this article we provide additional conditions to ensure the equivalence for the nth order FDEs with $n \geq 2$. In particular, we obtain conditions under which solutions of the integral equations are solutions of the linear nth order FDEs. These results are keys for further studying the existence of solutions and nonnegative solutions to initial and boundary value problems of nonlinear nth order FDEs. This is done via the corresponding integral equations by topological methods such as the Banach contraction principle, fixed point index theory, and degree theory.

1. Introduction

One of the important topics in fractional calculus is to establish the equivalence or the implication between linear (or homogeneous) fractional differential equations (FDEs) and the corresponding integral equations. Understanding this topic is a key toward further studying the existence of solutions and nonnegative solutions for initial value problems (IVPs) and boundary value problems of nonlinear FDEs via the corresponding nonlinear integral equations by topological methods such as the Banach contraction principle, fixed point index theory or degree theory. Very recently, the equivalence between linear first order FDEs and integral equations obtained in [18] is employed to study the existence of solutions and nonnegative solutions for the IVPs of nonlinear first order FDEs in [21], where the nonlinearity is an L^p-Carathéodory function and first order FDEs with nonlinearities from combustion theory are considered. Some initial or boundary value problems for linear or nonlinear FDEs and integral equations have been studied, for example in [1, 2, 5, 8, 9, 16, 17, 18, 19, 20, 27, 28, 29, 30].

We consider the linear (or homogeneous) nth order FDE

$$D^{n-\alpha}_{p,a^+}u(x) := (I^{\alpha}_{a^+}(u - P_m))^{(n)}(x) = v(x) \quad \text{for a.e. } x \in [a,b], \tag{1.1}$$

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where $\alpha \in (0,1)$, $n \in \mathbb{N}$ with $n \geq 2$, $m \in \{0,1,\dots n-1\}$ and

$$P_m(x) = \sum_{i=0}^{m} \frac{u_i}{i!}(x-a)^i \quad \text{for each } x \in [a,b]$$

is a polynomial of degree m, and the coefficients $\{u_i : i \in \{0,\dots m\}\}$ are given. The symbols and detailed concepts in the Introduction will be given later.

One of our aims when $m = n-1$ is to establish implications of existence of solutions in $L^1(a,b)$ between the nth order FDE (1.1) without any initial conditions and the integral equation

$$u(x) = P_{n-1}(x) + \sum_{i=0}^{n-1} \frac{(I_{a^+}^{\alpha}(u-P_{n-1}))^{(i)}(a)}{\Gamma(1+i-\alpha)}(x-a)^{i-\alpha} + I_{a^+}^{n-\alpha}v(x)$$

for a.e. $x \in [a,b]$. Also to establish implications of solutions in $L^1(a,b)$ between the IVP of (1.1) subject to the initial condition

$$(I_{a^+}^{\alpha}(u-P_{n-1}))^{(i)}(a) = c_i \quad \text{for each } i \in \{0,1,\dots n-1\} \tag{1.2}$$

and the integral equation

$$u(x) = P_{n-1}(x) + \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(1+i-\alpha)}(x-a)^{i-\alpha} + I_{a^+}^{n-\alpha}v(x) \quad \text{for a.e. } x \in [a,b].$$

These results generalize the results in [18] from $n=1$ to $n>1$, and are related to the IVPs for nonlinear (1.1)-(1.2) with $v(x) = f(x,u(x))$ studied in [12, 15, 26, 29]. In particular, we obtain results on the identity on $I_{a^+}^{n-\alpha}D_{p,a^+}^{n-\alpha}u$, which generalize the corresponding results in [5, 15, 25], where $u_i = 0$ for $i \in \{0,1,\dots n-1\}$.

A more challenging topic is to establish the equivalence in $C[a,b]$ between the IVP of (1.1) subject to the initial conditions

$$u(a) = u_0, \quad u'(a) = u_1, \dots, u^{(m)}(a) = u_m, \tag{1.3}$$

where $m \in \{0,\dots n-1\}$ is given, and suitable integral equations.

When $n \geq 2$, $m \leq n-2$ and $[a,b] = [0,1]$, Lan [17] obtained the equivalence in $C[a,b]$ between the IVP of (1.1)-(1.3), where $u^{(i)}(a)$ is not required to exist for $i \in \{m+1,\dots,n-1\}$, and the integral equation

$$u(x) = P_m(x) + \sum_{i=m+1}^{n-1} c_i x^{i-\alpha} + I_{a^+}^{n-\alpha}v(x) \quad \text{for each } x \in [a,b]. \tag{1.4}$$

The results in [17] allow $v \in L^1(a,b)$. Note that all the results in [17] hold on the general interval $[a,b]$ by simple modifications. It is mentioned in [17, p.5226] that for $n=1$ or $n \geq 2$ and $m = n-1$, if one only assumes $v \in L^1(a,b)$, (1.1)-(1.3) may not be equivalent to the integral equation

$$u(x) = P_{n-1}(x) + I_{a^+}^{n-\alpha}v(x) \quad \text{for each } x \in [a,b]. \tag{1.5}$$

Therefore, additional conditions on v are required for studying the equivalence of the IVP (1.1)-(1.3) and the integral equation (1.5) in $C[a,b]$ when $n=1$ or $n \geq 2$ and $m = n-1$.

When $n=1$, Lan [18] used the following additional condition on v to obtain the equivalence between (1.1)-(1.3) and (1.5) in $C[a,b]$:

(H1) $|I_{a^+}^{1-\alpha}v(x)| < \infty$ and $(I_{a^+}^{2-\alpha}v)'(x) = I_{a^+}^{1-\alpha}v(x)$ for each $x \in [a,b]$.

However, when $n \geq 2$ and $m = n-1$, the condition (H1) is not sufficient for studying the equivalence between (1.1)-(1.3) and (1.5) in $C[a,b]$.

The challenge is that under what additional conditions on v, (1.1)-(1.3) implies (1.5), or (1.5) implies (1.1)-(1.3) or (1.1)-(1.3) is equivalent to (1.5) in $C[a,b]$?

Another aim of this paper is to study the equivalence or the implication in $C[a,b]$ between (1.1)-(1.3) and (1.5) mentioned above when $n \geq 2$ and $m = n-1$. For example, when $n \geq 2$ and $m = n-1$, we shall prove that if the condition $v \in H_0^\alpha(a,b)$, in particular, $v \in L^p(a,b)$ for some $p \in (\frac{1}{1-\alpha}, \infty]$, then solutions of (1.5) in $C[a,b]$ are solutions of (1.1)-(1.3) in $C[a,b]$; and if $v \in C_0^\alpha(a,b)$, then one can obtain equivalence between (1.1)-(1.3) and (1.5) in $C[a,b]$. We shall show that

$$C_\gamma[a,b] \subset L^p(a,b) \cap I_{a^+}^\alpha(L^1(a,b)) \subset C_0^\alpha(a,b) \subset H_0^\alpha(a,b) \quad \text{for } p \in (\frac{1}{1-\alpha}, \infty],$$

where $\gamma > -1$, $C_\gamma[a,b] = \{u \in C[a,b] : \lim_{x\to a^+} x^{-\gamma} u(x) \,\text{exists}\}$,

$$C_0^\alpha(a,b) := \{v \in L^1(a,b) : I_{a^+}^{1-\alpha} v \in C[a,b] \quad \text{and } I_{a^+}^{1-\alpha} v(a) = 0\}$$

and

$$H_0^\alpha(a,b) = \{v \in L^1(a,b) : v \text{ satisfies (H1) and } I_{a^+}^{1-\alpha} v(a) = 0\}.$$

Our results generalize [15, Theorem 3.24, p.199] and the parts 2 and 3 of [29, Theorem 5.1], where $v \in C_\gamma[a,b]$ with $v(x) = (x-a)^{-\gamma} w(x)$ for each $x \in (a,b]$, $0 \leq \gamma < \alpha$ and $w \in C[a,b]$. We also discuss the equivalence between (1.1)-(1.3) which holds for each $x \in [a,b]$ and (1.5) in $C[a,b]$.

Closely related to the IVP (1.1)-(1.3) is the IVP for the nth order Caputo fractional differential equation

$$D_{C,a^+}^{n-\alpha} u(x) = I_{a^+}^\alpha u^{(n)}(x) = v(x) \quad \text{for a.e. } x \in [a,b] \tag{1.6}$$

subject to (1.3) with $n \geq 2$ and $m = n-1$. We show that (1.6) has no solutions in the nonempty set $C[a,b] \setminus I_{a^+}^\alpha(L^1(a,b))$ and provide conditions on v for (1.6)-(1.3) and (1.5) to be equivalent. In particular, if $v \in I_{a^+}^\alpha(L^1(a,b))$, then one can obtain equivalence between (1.6)-(1.3) and (1.5) in $C[a,b]$ when $n \geq 2$ and $m = n-1$.

The paper is structured in the following way. In section 2, we recall the important properties of the Riemann-Liouville integral operators and higher order fractional derivatives. In sections 3 and 4, we study implications and equivalences of higher order fractional differential and integral equations in $L^1(a,b)$ and in $C[a,b]$, respectively. In section 5, we study implications and equivalences of higher order Caputo fractional differential equations and integral equations.

2. Higher order fractional integral and derivatives

Throughout this paper, we always assume $\alpha \in (0,1)$, $a,b \in \mathbb{R}$ with $a < b$ and $n \in \mathbb{N}$ with $n \geq 2$. We use the usual derivative symbols u' and $u^{(i)}$ to denote the first order and the ith order derivatives of a real-valued function u defined on $[a,b]$ for each $i \in \mathbb{N} \setminus \{1\}$. For each $n \in \mathbb{N}$, let $\mathbb{N}_n = \{1, \dots, n\}$ and $\mathbb{N}_{0,n} = \{0, 1, \dots, n\}$.

The Riemann-Liouville (R-L) fractional integral of order $\beta \in (0,\infty)$ of a function $u \in L^1(a,b)$ is defined by a Volterra integral operator

$$(I_{a^+}^\beta u)(x) = \frac{1}{\Gamma(\beta)} \begin{cases} \int_a^x \frac{u(y)}{(x-y)^{1-\beta}}\, dy & \text{for a.e. } x \in [a,b] \quad \text{if } \beta < 1, \\ \int_a^x (x-y)^{\beta-1} u(y)\, dy & \text{for each } x \in [a,b], \quad \text{if } \beta \geq 1, \end{cases} \tag{2.1}$$

where $\Gamma(\beta)=\int_0^\infty x^{\beta-1}e^{-x}\,dx$ is the well-known Euler's Gamma function, see [5, p.13], [15, p.69] and [25, p.33]. Following [17, 18], for each $n\in\mathbb{N}$, the integral operator $I^n_{a^+}$ and the fractional integral operator $I^{n-\alpha}_{a^+}$ are said to be the nth order R-L integral operator and the nth order R-L fractional integral operator with fraction α, respectively.

We list some properties of $I^\beta_{a^+}$, $I^n_{a^+}$ and $I^{n-\alpha}_{a^+}$, which will be used later. Most of these properties can be found in [5, 17, 18, 25, 29], where some properties are considered when $[a,b]=[0,1]$, but hold for the general interval $[a,b]$.

Lemma 2.1. *Let $\beta,\gamma\in(0,\infty)$ satisfy $\beta+\gamma\geq 1$ and $v\in L^1(a,b)$. Then*

$$I^\beta_{a^+}I^\gamma_{a^+}v(x)=I^\gamma_{a^+}I^\beta_{a^+}v(x)=I^{\beta+\gamma}_{a^+}v(x)\quad\text{for each } x\in[a,b].$$

Lemma 2.2. *Let $\beta>0$ and $\gamma>-1$. Then*

$$I^\beta_{a^+}(x-a)^\gamma=\frac{\Gamma(1+\gamma)}{\Gamma(1+\beta+\gamma)}(x-a)^{\beta+\gamma}\quad\text{for each } x\in(a,b]$$

If assume further that $\beta+\gamma\geq 0$, then the above result holds for each $x\in[a,b]$.

Lemma 2.3. (1) *$I^1_{a^+}$ maps $L^1(a,b)$ to $AC[a,b]$. Moreover, for each $v\in L^1(a,b)$,*

$$(I^1_{a^+}v)'(x)=v(x)\quad\text{for a.e. } x\in[a,b]$$

and for $v\in C[a,b]$,

$$(I^1_{a^+}v)'(x)=v(x)\quad\text{for each } x\in[a,b]. \tag{2.2}$$

(2) *For $n\geq 2$, $I^n_{a^+}$ maps $L^1(a,b)$ to $C^{n-1}[a,b]$. Moreover, for $v\in L^1(a,b)$,*

$$(I^n_{a^+}v)^{(i)}(x)=I^{n-i}_{a^+}v(x)\quad\text{for each } x\in[a,b] \text{ and } i\in\mathbb{N}_{n-1}$$

and

$$I^{n-i}_{a^+}v(a)=0\quad\text{for each } i\in\mathbb{N}_{n-1}. \tag{2.3}$$

Lemma 2.4. (1) *$I^{1-\alpha}_{a^+}$ maps $L^1(a,b)$ to $L^1(a,b)$, $I^{1-\alpha}_{a^+}$ maps $I^\alpha_{a^+}(L^1(a,b))$ to $AC[a,b]$. For each $p\in(\frac{1}{1-\alpha},\infty]$, $I^{1-\alpha}_{a^+}$ maps $L^p(a,b)$ to $C[a,b]$ and*

$$(I^{1-\alpha}_{a^+}v)(a)=0\quad\text{for } v\in L^p(a,b).$$

(2) *$I^{2-\alpha}_{a^+}$ maps $L^1(a,b)$ to $AC[a,b]$. Moreover, for each $v\in L^1(a,b)$,*

$$(I^{2-\alpha}_{a^+}v)'(x)=I^{1-\alpha}_{a^+}v(x)\quad\text{for a.e } x\in[a,b],$$

and if $I^{1-\alpha}_{a^+}v\in C[a,b]$, then

$$(I^{2-\alpha}_{a^+}v)'(x)=I^{1-\alpha}_{a^+}v(x)\quad\text{for each } x\in[a,b].$$

(3) *For $n\geq 3$, $I^{n-\alpha}_{a^+}$ maps $L^1(a,b)$ to $C^{n-2}[a,b]$ and for each $v\in L^1(a,b)$,*

$$(I^{n-\alpha}_{a^+}v)^{(i)}(x)=(I^{n-\alpha-i}_{a^+}v)(x)\quad\text{for each } x\in[a,b] \text{ and } i\in\mathbb{N}_{n-2}. \tag{2.4}$$

(4) *For $n\geq 2$, $(I^{n-\alpha}_{a^+}v)(a)=0$ for each $v\in L^1(a,b)$.*

(5) *For $n\geq 1$, $I^{n-\alpha}_{a^+}:L^1(a,b)\to L^1(a,b)$ is one to one.*

Proof. We only prove (5). Assume that $v_1,v_2\in L^1(a,b)$ satisfy

$$I^{n-\alpha}_{a^+}v_1(x)=I^{n-\alpha}_{a^+}v_2(x)\quad\text{for a.e. } x\in[a,b].$$

Applying $I^\alpha_{a^+}$ to both sides of the above equation and using Lemma 2.1, we have

$$I^n_{a^+}v_1(x)=I^\alpha_{a^+}I^{n-\alpha}_{a^+}v_1(x)=I^\alpha_{a^+}I^{n-\alpha}_{a^+}v_2(x)=I^n_{a^+}v_2(x)\quad\text{for each } x\in[a,b].$$

Differentiating both sides of the above equation n times and using Lemma 2.3, (2) implies that

$$v_1(x) = v_2(x) \quad \text{for a.e. } x \in [a,b]$$

and $v_1 = v_2$ in $L^1(a,b)$. □

Remark 2.5. The result: $I_{a^+}^{1-\alpha}$ maps $I_{a^+}^{\alpha}(L^1(a,b))$ to $AC[a,b]$ is proved in [18, Proposition 2.2] and generalizes the result: $I_{a^+}^{1-\alpha}$ maps $AC[a,b]$ to $AC[a,b]$ in [20, Lemma 2.3], the first part of [25, Lemma 2.1] and [29, Proposition 3.2 (6)]. When $n = 1$, Lemma 2.4 (5) is given in [25, Theorem 2.1] (also, see [20, p.65]).

Let $n \in \mathbb{N}$ and $m \in \mathbb{N}_{0,n}$. For each $i \in \mathbb{N}_{0,n-1}$, let $u_i \in \mathbb{R}$ be given. We define a function $P_m : [a,b] \to \mathbb{R}$ by

$$P_m(x) = \sum_{i=0}^{m} \frac{u_i}{i!}(x-a)^i \quad \text{for each } x \in [a,b]. \tag{2.5}$$

Definition 2.6. The nth order fractional derivative with fraction α (relative to P_m) of a function $u \in L^1(a,b)$ is defined by

$$D_{p,a^+}^{n-\alpha}u(x) = (I_{a^+}^{\alpha}(u - P_m))^{(n)}(x) \quad \text{for a.e. } x \in [a,b].$$

The letter p in the symbol $D_{p,a^+}^{n-\alpha}u$ is the first letter of polynomial and is used to indicate that the fractional differential operator $D_{p,a^+}^{n-\alpha}$ contains a polynomial. A sufficient condition for the derivative $D_{p,a^+}^{n-\alpha}u$ to exist a.e. on $[a,b]$ is $(I_{a^+}^{\alpha}(u - P_m))^{(n-1)} \in AC[a,b]$. In Definition 2.6, we require neither the existence of $u^{(i)}(a)$ nor $u_i = u^{(i)}(a)$ for $i \in \mathbb{N}_{0,n-1}$ since the polynomial P_m is independent of u.

When $n = 1$, the first order fractional derivative $D_{p,a^+}^{1-\alpha}u$ is used in [18]. For $n \geq 2$ and $m \in \mathbb{N}_{0,n-2}$, the nth order fractional derivative $D_{p,a^+}^{n-\alpha}u$ is considered in [17], where $[a,b] = [0,1]$. If $u_i = 0$ for each $i \in \mathbb{N}_{0,m}$, then $D_{p,a^+}^{n-\alpha} = D_{a^+}^{n-\alpha}$ is the usual R-L fractional differential operator of order $n - \alpha$, see [5, Definition 2.2], [15, p. 70], [22, p.88], [25, p.37], and [29, Definition 4.8]. If $m = n-1$, $u^{(i)}(a)$ exists and $u_i = u^{(i)}(a)$ for each $i \in \mathbb{N}_{0,n-1}$, then $D_{p,a^+}^{n-\alpha} = D_{*a^+}^{n-\alpha}$ is the Caputo differential operator of order $n - \alpha$ in [5, Definition 3.2] and [15, Section 2.4, p.90].

3. Equivalences of the nth order fractional differential and integral equations in $L^1(a,b)$

We study solutions in $L^1(a,b)$ of the nth order FDE

$$D_{p,a^+}^{n-\alpha}u(x) := (I_{a^+}^{\alpha}(u - P_{n-1}))^{(n)}(x) = v(x) \tag{3.1}$$

for a.e. $x \in [a,b]$, where $n \in \mathbb{N}$ with $n \geq 2$, $P_{n-1}(x) = \sum_{i=0}^{n-1} \frac{u_i}{i!}(x-a)^i$ for each $x \in [a,b]$, $u_i \in \mathbb{R}$ for each $i \in \mathbb{N}_{0,n-1}$ is given and $v \in L^1(a,b)$.

Definition 3.1. A function $u \in L^1(a,b)$ is said to be a solution of (3.1) (relative to P_{n-1}) if u satisfies (3.1).

We shall see that the condition $(I_{a^+}^{\alpha}(u - P_{n-1}))^{(n-1)} \in AC[a,b]$ is needed to ensure that u is a solution of (3.1) when we study the equivalence between the FDE (3.1) and the integral equation

$$u(x) = P_{n-1}(x) + \sum_{i=0}^{n-1} \frac{(I_{a^+}^{\alpha}(u - P_{n-1}))^{(i)}(a)}{\Gamma(1+i-\alpha)}(x-a)^{i-\alpha} + I_{a^+}^{n-\alpha}v(x) \tag{3.2}$$

for a.e. $x \in [a,b]$.

A function $u \in L^1(a,b)$ is said to be a solution of (3.2) if u satisfies (3.2).

The following new result provides the implications between (3.1) and (3.2) in $L^1(a,b)$.

Theorem 3.2. *Let $u, v \in L^1(a,b)$. Then the following assertions hold.*

(1) *If $(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)} \in AC[a,b]$ and u is a solution of (3.1), then u is a solution of (3.2).*
(2) *If $I^{\alpha}_{a^+}(u - P_{n-1}) \in C[a,b]$ and u is a solution of (3.2), then u is a solution of (3.1)*

Proof. (1) Assume that $(I^{\alpha}_{a^+}(u-P_{n-1}))^{(n-1)} \in AC[a,b]$ and u is a solution of (3.1). Integrating both sides of (3.1) from a to x implies

$$(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)}(x) = (I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)}(a) + I^1_{a^+}v(x) \quad \text{for each } x \in [a,b].$$

Since $(I^{\alpha}_{a^+}(u-P_{n-1}))^{(n-2)} \in AC[a,b]$, integrating both sides of the above equation from a to x implies for each $x \in [a,b]$,

$$\begin{aligned}
&(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-2)}(x) \\
&= (I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-2)}(a) + (I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)}(a)(x-a) + I^2_{a^+}v(x).
\end{aligned}$$

Repeating the process, for each $x \in [a,b]$ we have

$$I^{\alpha}_{a^+}(u - P_{n-1})(x) = \sum_{i=0}^{n-1} \frac{(I^{\alpha}_{a^+}(u - P_{n-1}))^{(i)}(a)}{i!}(x-a)^i + I^n_{a^+}v(x).$$

Taking $I^{1-\alpha}_{a^+}$ on both sides of the above equation and using Lemmas 2.1 and 2.2, we obtain for each $x \in [a,b]$,

$$\begin{aligned}
&I^1_{a^+}(u - P_{n-1})(x) \\
&= \sum_{i=0}^{n-1} \frac{(I^{\alpha}_{a^+}(u - P_{n-1}))^{(i)}(a)}{i!} \frac{\Gamma(1+i)}{\Gamma(2+i-\alpha)}(x-a)^{1+i-\alpha} + I^1_{a^+}I^{n-\alpha}_{a^+}v(x) \\
&= \sum_{i=0}^{n-1} \frac{(I^{\alpha}_{a^+}(u - P_{n-1}))^{(i)}(a)}{\Gamma(2+i-\alpha)}(x-a)^{1+i-\alpha} + I^1_{a^+}I^{n-\alpha}_{a^+}v(x).
\end{aligned}$$

Differentiating both sides of the above equation, we have for a.e. $x \in [a,b]$,

$$\begin{aligned}
&u(x) - P_{n-1}(x) \\
&= \sum_{i=0}^{n-1} \frac{(I^{\alpha}_{a^+}(u - P_{n-1}))^{(i)}(a)(1+i-\alpha)}{\Gamma(2+i-\alpha)}(x-a)^{i-\alpha} + I^{n-\alpha}_{a^+}v(x) \\
&= \sum_{i=0}^{n-1} \frac{(I^{\alpha}_{a^+}(u - P_{n-1}))^{(i)}(a)}{\Gamma(1+i-\alpha)}(x-a)^{i-\alpha} + I^{n-\alpha}_{a^+}v(x).
\end{aligned}$$

(2) Assume that $I^{\alpha}_{a^+}(u-P_{n-1}) \in C[a,b]$ and u is a solution of (3.2). Taking $I^{\alpha}_{a^+}$ on both sides of (3.2) and using Lemmas 2.1 and 2.2, we obtain for each $x \in [a,b]$,

$$I^{\alpha}_{a^+}(u - P_{n-1})(x) = \sum_{i=0}^{n-1} \frac{(I^{\alpha}_{a^+}(u - P_{n-1}))^{(i)}(a)}{\Gamma(1+i-\alpha)} I^{\alpha}_{a^+}(x-a)^{i-\alpha} + I^{\alpha}_{a^+}I^{n-\alpha}_{a^+}v(x)$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1} \frac{(I_{a^+}^{\alpha}(u-P_{n-1}))^{(i)}(a)}{\Gamma(1+i-\alpha)} \frac{\Gamma(1+i-\alpha)}{\Gamma(1+i)}(x-a)^i + I_{a^+}^n v(x) \\
&= \sum_{i=0}^{n-1} \frac{(I_{a^+}^{\alpha}(u-P_{n-1}))^{(i)}(a)}{\Gamma(1+i)}(x-a)^i + I_{a^+}^n v(x).
\end{aligned}
$$

Differentiating both sides of the above equation $n-1$ times and using Lemma 2.3(2), we have for each $x \in [a,b]$,

$$
(I_{a^+}^{\alpha}(u-P_{n-1}))^{(n-1)}(x) = (I_{a^+}^{\alpha}(u-P_{n-1}))^{(n-1)}(a) + I_{a^+}^1 v(x).
$$

This with Lemma 2.3(1) implies $(I_{a^+}^{\alpha}(u-P_{n-1}))^{(n-1)} \in AC[a,b]$. Differentiating both sides of the last equation and using Lemma 2.3(1), we obtain (3.1). □

By using Theorem 3.2, we obtain the following identity.

Theorem 3.3. *If $u \in L^1(a,b)$ satisfies $(I_{a^+}^{\alpha}(u-P_{n-1}))^{(n-1)} \in AC[a,b]$, then for each $x \in [a,b]$,*

$$
I_{a^+}^{n-\alpha} D_{p,a^+}^{n-\alpha} u(x) = u(x) - P_{n-1}(x) - \sum_{i=0}^{n-1} \frac{(I_{a^+}^{\alpha}(u-P_{n-1}))^{(i)}(a)}{\Gamma(1+i-\alpha)}(x-a)^{i-\alpha}.
$$

Proof. Let $v(x) = (I_{a^+}^{\alpha}(u-P_{n-1}))^{(n)}(x)$ for a.e. $x \in [a,b]$. By Theorem 3.2 (1), u is a solution of (3.2). Substituting v into (3.2) the result holds. □

When $u_i = 0$ for each $i \in \mathbb{N}_{0,n-1}$, Theorem 3.3 was proved by different methods, for example, in [5, Theorem 2.23], where $(I_{a^+}^{\alpha}u)^{(i)}(a)$ is replaced by the limit: $\lim_{x\to a^+}(I_{a^+}^{\alpha}u)^{(i)}(x)$, [15, Lemma 2.5 (b), p.75] and the second part of [25, Theorem 2.4, p.44].

Next, we consider (3.1) subject to the initial condition

$$
(I_{a^+}^{\alpha}(u-P_{n-1}))^{(i)}(a) = c_i \quad \text{for } i \in \mathbb{N}_{0,n-1}, \tag{3.3}
$$

where $c_i \in \mathbb{R}$ is given for each $i \in \mathbb{N}_{0,n-1}$.

We show that (3.1)-(3.3) can be studied via the the equation

$$
u(x) = P_{n-1}(x) + \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(1+i-\alpha)}(x-a)^{i-\alpha} + I_{a^+}^{n-\alpha} v(x) \quad \text{for a.e. } x \in [a,b]. \tag{3.4}
$$

When $n=1$, the equivalence in $L^1(a,b)$ between (3.1) and (3.4) was studied in [18, Theorem 3.1]. The IVPs for (3.1)-(3.3) with $n=1$ were studied in [18, Theorem 3.2]. The IVPs for nonlinear (3.1)-(3.3) with $v(x) = f(x,u(x))$ were studied in [26, Theorem 4], [12, Theorem 1], [15, Theorem 3.1, p.145], and [29, Theorem 6.10].

Theorem 3.4.
(1) *If $I_{a^+}^{\alpha}(u-P_{n-1}) \in C[a,b]$ and u is a solution of (3.4), then $(I_{a^+}^{\alpha}(u-P_{n-1}))^{(n-1)} \in AC[a,b]$ and u is a solution of (3.1)-(3.3).*
(2) *If $u \in L^1(a,b)$ satisfies $(I_{a^+}^{\alpha}(u-P_{n-1}))^{(n-1)} \in AC[a,b]$ and u is a solution of (3.1)-(3.3), then u is a solution of (3.4)-(3.3).*
(3) *If $I_{a^+}^{\alpha}(u-P_{n-1}) \in C[a,b]$ and u is a solution of (3.4), then u is a solution of (3.2).*

Proof. (1) Assume that $u \in L^1(a,b)$ is a solution of (3.4) and $I^{\alpha}_{a^+}(u - P_{n-1}) \in C[a,b]$. Applying $I^{\alpha}_{a^+}$ to both sides of (3.4) and using Lemmas 2.2 and 2.1, we have for each $x \in [a,b]$

$$\begin{aligned} I^{\alpha}_{a^+}(u - P_{n-1})(x) &= \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(1+i-\alpha)} I^{\alpha}_{a^+}(x-a)^{i-\alpha} + I^{\alpha}_{a^+} I^{n-\alpha}_{a^+} v(x) \\ &= \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(1+i)}(x-a)^i + I^n_{a^+} v(x). \end{aligned}$$

By Lemma 2.3 (2), $I^n_{a^+}$ maps $L^1(a,b)$ to $C^{n-1}[a,b]$. It follows from the above equation that $I^{\alpha}_{a^+}(u - P_{n-1}) \in C^{n-1}[a,b]$. Since $I^n_{a^+} v \in AC[a,b]$ and $I^n_{a^+} v(a) = 0$, it follows from the above equation that $I^{\alpha}_{a^+}(u - P_{n-1}) \in AC[a,b]$ and $I^{\alpha}_{a^+}(u - P_{n-1})(a) = c_0$. Differentiating both sides of the last equation and repeating the process k times imply for each $x \in [a,b]$,

$$\begin{aligned} (I^{\alpha}_{a^+}(u - P_{n-1}))^{(k)}(x) &= \sum_{i=k}^{n-1} \Pi^i_{j=i-k+1} j \frac{c_i}{\Gamma(1+i)}(x-a)^{i-k} + I^{n-k}_{a^+} v(x) \\ &= \sum_{i=k}^{n-1} \frac{c_i}{(i-k)!}(x-a)^{i-k} + I^{n-k}_{a^+} v(x). \end{aligned} \tag{3.5}$$

Since $I^{n-k}_{a^+} v \in AC[a,b]$ and $I^{n-k}_{a^+} v(a) = 0$, it follows from (3.5) that

$$(I^{\alpha}_{a^+}(u - P_{n-1}))^{(k)} \in AC[a,b] \quad \text{and} \quad (I^{\alpha}_{a^+}(u - P_{n-1}))^{(k)}(a) = c_k.$$

By (3.5) with $k = n-1$, we obtain

$$(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)}(x) = c_{n-1} + I^1_{a^+} v(x) \quad \text{for each } x \in [a,b].$$

Differentiating both sides of the above equation,

$$(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n)}(x) = v(x) \quad \text{for a.e. } x \in [a,b].$$

(2) Since u is a solution of (3.1), by Theorem 3.2, u is a solution of (3.2). Since u satisfy (3.3), by (3.2), (3.4) holds and u is a solution of (3.4)-(3.3).

(3) If $u \in L^1(a,b)$ is a solution of (3.4), then by result (1), (3.3) holds. Hence, (3.3) and (3.4) together imply (3.2). □

Theorem 3.4 generalizes [18, Theorem 3.2] from $n = 1$ to $n \geq 2$. When $u_i = 0$ for $i \in \mathbb{N}_{0,n-1}$ and $[a,b] = [0,1]$, Theorem 3.4 (1) and (2) were obtained in [29, Theorem 6.10] when $v(x) = f(x, u(x))$ for each $x \in [0,1]$ and $v \in L^1(0,1)$. Note that the solutions of (3.4) depend on c_i. Hence, for given $c_i \in \mathbb{R}$ for $i \in \mathbb{N}_{0,n-1}$, the converse of Theorem 3.4 (3) may not be true since $(I^{\alpha}_{a^+}(u - P_{n-1}))^{(i)}(a)$ may not be equal to c_i. Hence, (3.1)-(3.3) in general is not equivalent to (3.2), where $(I^{\alpha}_{a^+}(u - P_{n-1}))^{(i)}(a)$ is required to exist.

Theorem 3.5. *Assume that $u \in L^1(a,b)$ satisfies $(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)} \in AC[a,b]$ and* (3.3). *Then*

$$I^{n-\alpha}_{a^+} D^{n-\alpha}_{p,a^+} u(x) = u(x) - P_{n-1}(x) - \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(1+i-\alpha)}(x-a)^{i-\alpha} \quad \textit{for a.e. } x \in [a,b].$$

Proof. Since $(I_{a^+}^{\alpha}(u-P_{n-1}))^{(n-1)} \in AC[a,b]$, it follows from Theorem 3.3 that the function v defined by $v(x) = (I_{a^+}^{\alpha}(u-P_{n-1}))^{(n)}(x)$ for a.e. $x \in [a,b]$ satisfies (3.2). Since u satisfies (3.3), (3.2) implies that (3.4) holds and the function v satisfies (3.4). Substituting v into (3.4) the result holds. □

4. Equivalences of the nth order fractional differential and integral equations in $C[a,b]$

We study solutions in $C[a,b]$ of the IVPs for the nth order FDE

$$D_{p,a^+}^{n-\alpha}u(x) := (I_{a^+}^{\alpha}(u-P_{n-1}))^{(n)}(x) = v(x) \quad \text{for a.e. } x \in [a,b] \tag{4.1}$$

subject to the initial condition

$$u(a) = u_0, \quad u'(a) = u_1, \dots, u^{(n-1)}(a) = u_{n-1}, \tag{4.2}$$

where $u_i \in \mathbb{R}$ for each $i \in \mathbb{N}_{0,n-1}$ and $P_{n-1}(x) = \sum_{i=0}^{n-1} \frac{u_i}{i!}(x-a)^i$ for each $x \in [a,b]$. We show that under suitable conditions on v, the solutions in $C[a,b]$ of (4.1)-(4.2) can be studied via the integral equation

$$u(x) = P_{n-1}(x) + I_{a^+}^{n-\alpha}v(x) \quad \text{for each } x \in [a,b]. \tag{4.3}$$

Definition 4.1. Let $v \in L^1(a,b)$ be given. A function $u : [a,b] \to \mathbb{R}$ is said to be a solution of (4.3) if u satisfies (4.3).

When $v \in L^1(a,b)$, the equivalence in $C[a,b]$ between the IVP for

$$D_{p,a^+}^{n-\alpha}u(x) := (I_{a^+}^{\alpha}(u-P_m))^{(n)}(x) = v(x) \quad \text{for a.e. } x \in [a,b] \tag{4.4}$$

subject to the initial condition

$$u(a) = u_0, \quad u'(a) = u_1, \dots, u^{(m)}(a) = u_m,$$

and the integral equation

$$u(x) = P_m(x) + \sum_{i=m+1}^{n-1} c_i x^{i-\alpha} + I_{a^+}^{n-\alpha}v(x) \quad \text{for each } x \in [a,b],$$

where $n \geq 2$, $m \leq n-2$ and $[a,b] = [0,1]$, was studied by Lan [17], where $u^{(i)}(a)$ is not required to exist for $i \in \{m+1.\dots,n-1\}$.

It is pointed out in [17, p.5226] that the solutions in $C[a,b]$ of (4.1)-(4.2) with $n \geq 1$ can not be obtained via (4.3) if one only requires $v \in L^1(a,b)$. Therefore, additional conditions on v are required for studying solutions in $C[a,b]$ of (4.1)-(4.2) with $n \geq 1$. When $n = 1$, the additional condition on v used in [18] is (H1). However, when $n \geq 2$, condition (H1) is not sufficient for studying solutions of (4.1)-(4.2) in $C[a,b]$. Next we introduce new conditions on v for studying solutions of (4.1)-(4.2) in $C[a,b]$ when $n \geq 2$. Let

$$H_0^{\alpha}(a,b) = \{v \in L^1(a,b) : v \text{ satisfies (H1) and } (I_{a^+}^{1-\alpha}v)(a) = 0\},$$
$$H_B^{\alpha}(a,b) = \{v \in L^1(a,b) : v \text{ satisfies (H1) and } \limsup_{x\to a^+} |(I_{a^+}^{1-\alpha}v)(x)| < \infty\}.$$

Then both $H_0^{\alpha}(a,b)$ and $H_B^{\alpha}(a,b)$ are linear spaces. Moreover, if $v \in H_0^{\alpha}(a,b)$ and $\lim_{x\to a^+} I_{a^+}^{1-\alpha}v(x) = 0$, then $v \in H_B^{\alpha}(a,b)$.

Remark 4.2. *It is not clear whether one of the following inclusions holds.*

$$H_0^{\alpha}(a,b) \subset H_B^{\alpha}(a,b) \quad \text{or} \quad H_B^{\alpha}(a,b) \subset H_0^{\alpha}(a,b). \tag{4.5}$$

Let

$$\mathscr{R}_\alpha(a,b) = I^{\alpha}_{a^+}(L^1(a,b)), \tag{4.6}$$

$$C^{\alpha}_0(a,b) := \{v \in L^1(a,b) : I^{1-\alpha}_{a^+}v \in C[a,b] \text{ and } I^{1-\alpha}_{a^+}v(a) = 0\}. \tag{4.7}$$

By [18, Propositions 2.2] (see also [29, Proposition 3.6], where $[a,b] = [0,T]$ and the proof of [25, Theorem 2.1]),

$$\mathscr{R}_\alpha(a,b) = \{v \in L^1(a,b) : I^{1-\alpha}_{a^+}v \in AC[a,b] \text{ and } I^{1-\alpha}_{a^+}v(a) = 0\}. \tag{4.8}$$

By [18, Proposition 2.3] and Lemma 2.4 (1), (2), we obtain the following result.

Proposition 4.3. *Let $\alpha \in (0,1)$. Then the following assertions hold.*

(1) $AC[a,b] \subsetneqq \mathscr{R}_\alpha(a,b) \subset C^{\alpha}_0(a,b) \subset H^{\alpha}_0(a,b) \cap H^{\alpha}_B(a,b)$.
(2) $L^p(a,b) \subset C^{\alpha}_0(a,b)$ *for* $p \in (\frac{1}{1-\alpha}, \infty]$.

Proposition 4.4. *The following assertions hold.*

(1) $C[a,b] \setminus \mathscr{R}_\alpha(a,b) \neq \emptyset$ *for each* $\alpha \in (0,1)$.
(2) $C[a,b] \setminus I^{1-\alpha}_{a^+}(L^p(a,b)) \neq \emptyset$ *for each* $\alpha \in (0,1)$ *and* $p \in (\frac{1}{1-\alpha}, \infty]$.

Proof. (1) Let $\delta > 1$. We consider the following Weierstrass function

$$w(x) = \sum_{k=0}^{\infty} \frac{1}{\delta^{\alpha k}} \cos(\delta^k x) \quad \text{for } x \in [a,b]. \tag{4.9}$$

It follows from [29, Addendum (3), p.33] or [11, section 5.5, p.589] that $w \in C[a,b]$ and $I^{1-\alpha}_{a^+}w \notin AC[a,b]$ since $I^{1-\alpha}_{a^+}w$ are not differentiable anywhere on $[a,b]$. By (4.8), we see that $w \notin \mathscr{R}_\alpha(a,b)$. Hence,

$$w \in C[a,b] \setminus \mathscr{R}_\alpha(a,b) \quad \text{for each } \alpha \in (0,1). \tag{4.10}$$

(2) By (4.6), we have for each $\alpha \in (0,1)$ and $p \in (\frac{1}{1-\alpha}, \infty]$,

$$I^{1-\alpha}_{a^+}(L^p(a,b)) \subset I^{1-\alpha}_{a^+}(L^1(a,b)) = \mathscr{R}_{1-\alpha}(a,b).$$

This, together with (4.10), implies that

$$w \in C[a,b] \setminus \mathscr{R}_{1-\alpha}(a,b) \subset C[a,b] \setminus I^{1-\alpha}_{a^+}(L^p(a,b))$$

and the result follows. □

We refer to [4] for another function in $C[a,b] \setminus \mathscr{R}_\alpha(a,b)$.

Lemma 4.5. *If u is a solution of* (4.3)*, then the following assertions hold.*

(1) *If v satisfies (H1), then*

$$u^{(n-1)}(x) = u_{n-1} + I^{1-\alpha}_{a^+}v(x) \quad \textit{for each } x \in [a,b]. \tag{4.11}$$

(2) $I^{1-\alpha}_{a^+}v \in C[a,b]$ *if and only if* $u^{(n-1)} \in C[a,b]$.
(3) $v \in \mathscr{R}_\alpha(a,b)$ *if and only if* $u^{(n-1)} \in AC[a,b]$ *and* $v \in H^{\alpha}_B(a,b) \cap H^{\alpha}_0(a,b)$.

Proof. (1) Using Lemma 2.4 (3) and (H1) and differentiating both sides of (4.3) $n-1$ times imply (4.11).

(2) By (4.11), we see that the result (2) holds.

(3) If $v \in \mathscr{R}_\alpha(a,b)$, then by Proposition 4.3(1), $v \in H^{\alpha}_B(a,b) \cap H^{\alpha}_0(a,b)$. Since $I^{1-\alpha}_{a^+}v \in AC[a,b]$, by (4.11) we have $u^{(n-1)} \in AC[a,b]$. Conversely, if $u^{(n-1)} \in AC[a,b]$ and $v \in H^{\alpha}_B(a,b) \cap H^{\alpha}_0(a,b)$, then v satisfies (H1) and $I^{1-\alpha}_{a^+}v(a) = 0$. By (4.11) and $u^{(n-1)} \in AC[a,b]$, we obtain $I^{1-\alpha}_{a^+}v \in AC[a,b]$. Hence, $v \in \mathscr{R}_\alpha(a,b)$. □

Lemma 4.6. *Let $n \geq 2$, $y \in C[a,b]$ and $v \in L^1(a,b)$. Then the following assertions are equivalent.*

(1) $(I^{\alpha}_{a^+}y)^{(n-1)} \in AC[a,b]$ *and y, v satisfy the equation*

$$D^{n-\alpha}_{a^+}y(x) = v(x) \quad \textit{for a.e. } x \in [a,b].$$

(2) *There exists $\{c_i \in \mathbb{R} : i \in \mathbb{N}_{n-1}\}$ such that*

$$y(x) = \sum_{i=1}^{n-1} c_i(x-a)^{i-\alpha} + I^{n-\alpha}_{a^+}v(x) \quad \textit{for each } x \in [a,b].$$

Lemma 4.6 was proved in [17, Theorem 2.5] when $[a,b] = [0,1]$, but it is easy to see that the result holds on $[a,b]$. Now, we prove the following result on implications between (4.1)-(4.2) and (4.3).

Theorem 4.7. (1) *If $v \in H^{\alpha}_0(a,b)$ and u is a solution of* (4.3)*, then $u \in C[a,b]$, $(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)} \in AC[a,b]$ and u is a solution of* (4.1)-(4.2).

(2) If $v \in H^{\alpha}_B(a,b)$, $u \in C[a,b]$, $(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)} \in AC[a,b]$ and u is a solution of (4.1)-(4.2)*, then u is a solution of* (4.3) *and $v \in H^{\alpha}_0(a,b)$.*

Proof. (1) Assume that $I^{1-\alpha}_{a^+}v(a) = 0$ and u is a solution of (4.3). By Lemma 2.4 (2) and (3), $I^{n-\alpha}_{a^+}v \in C[a,b]$. This, together with (4.3), implies $u \in C[a,b]$. Let $y = u - P_{n-1}$ and $c_i = 0$ for $i \in \mathbb{N}_{n-1}$. Then $y \in C[a,b]$ and by (4.3), we have

$$y(x) = I^{n-\alpha}_{a^+}v(x) = \sum_{i=1}^{n-1} c_i(x-a)^{i-\alpha} + I^{n-\alpha}_{a^+}v(x) \quad \text{for each } x \in [a,b].$$

Since (2) implying (1) of Lemma 4.6,

$$\begin{gathered}(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)} = (I^{\alpha}_{a^+}y)^{(n-1)} \in AC[a,b], \\ D^{n-\alpha}_{p,a^+}u(x) = D^{n-\alpha}_{a^+}y(x) = v(x) \quad \text{for a.e. } x \in [a,b].\end{gathered}$$

Now, we prove that u satisfies (4.2). If $n = 2$, then by (4.3), we have

$$u(x) = u_0 + u_1(x-a) + I^{2-\alpha}_{a^+}v(x) \quad \text{for each } x \in [a,b].$$

By Lemma 2.4 (4), $I^{2-\alpha}_{a^+}v(a) = 0$ and $u(a) = u_0$. Using (H1) and differentiating both sides of the above equation we obtain

$$u'(x) = u_1 + I^{1-\alpha}_{a^+}v(x) \quad \text{for each } x \in [a,b].$$

This, together with $I^{1-\alpha}_{a^+}v(a) = 0$, implies $u'(a) = u_1$ and (4.2) with $n = 2$ holds.

If $n = 3$, then using Lemma 2.4 (3) and differentiating both sides of (4.3) with $n = 3$, we have

$$u'(x) = u_1 + u_2(x-a) + I^{2-\alpha}_{a^+}v(x) \quad \text{for each } x \in [a,b].$$

By Lemma 2.4 (4), $I^{2-\alpha}_{a^+}v(a) = 0$ and $u'(a) = u_1$. Using (H1) and differentiating both sides of the above equation we obtain

$$u''(x) = u_2 + I^{1-\alpha}_{a^+}v(x) \quad \text{for each } x \in [a,b].$$

This, together with $I^{1-\alpha}_{a^+}v(a) = 0$, implies $u''(a) = u_2$. Hence, (4.2) with $n = 3$ holds and (2) holds.

If $n \geq 4$, then by (4.3) and (2.4), we have for each $k \in \mathbb{N}_{n-3}$ and $x \in [a,b]$,

$$\begin{aligned} u^{(k)}(x) &= \Big(\sum_{i=0}^{n-1} \frac{u_{(i)}}{i!}(x-a)^i\Big)^{(k)} + (I_{a^+}^{n-\alpha} v)^{(k)}(x) \\ &= \sum_{j=0}^{n-1-k} \frac{u_{k+j}}{j!}(x-a)^j + I_{a^+}^{n-k-\alpha} v(x) \end{aligned} \tag{4.12}$$

and

$$u^{(n-2)}(x) = u_{n-2} + u_{n-1}(x-a) + I_{a^+}^{2-\alpha} v(x) \quad \text{for each } x \in [a,b]. \tag{4.13}$$

By Lemma 2.4(4), $(I_{a^+}^{n-\alpha} v)^{(k)}(a) = 0$ for $k \in \mathbb{N}_{n-2}$. This, together with (4.12) and (4.13), implies

$$u^{(k)}(a) = u_k \quad \text{for } k \in \mathbb{N}_{n-2}.$$

Differentiating both sides of (4.13) and using (H1) imply that for $n \geq 4$,

$$u^{(n-1)}(x) = u_{n-1} + I_{a^+}^{1-\alpha} v(x) \quad \text{for each } x \in [a,b].$$

This, together with $I_{a^+}^{1-\alpha} v(a) = 0$, implies $u^{(n-1)}(a) = u_{n-1}$ for $n \geq 4$. Hence, u satisfies (4.2).

(2) By Lemma 4.6 with $y = u - P_{n-1}$, there exists $\{c_i \in \mathbb{R} : i \in \mathbb{N}_{n-1}\}$ such that

$$u(x) = \sum_{i=0}^{n-1} \frac{u_i}{i!}(x-a)^i + \sum_{i=1}^{n-1} c_i (x-a)^{i-\alpha} + I_{a^+}^{n-\alpha} v(x) \quad \text{for each } x \in [a,b]. \tag{4.14}$$

We prove that $c_i = 0$ for each $i \in \mathbb{N}_{n-1}$. If $n = 2$, then by (4.14), we have

$$u(x) = u_0 + u_1(x-a) + c_1(x-a)^{1-\alpha} + I_{a^+}^{2-\alpha} v(x) \quad \text{for each } x \in [a,b]. \tag{4.15}$$

Differentiating both sides of (4.15) and using (H1) yields

$$u'(x) = u_1 + \frac{c_1(1-\alpha)}{(x-a)^\alpha} + I_{a^+}^{1-\alpha} v(x) \quad \text{for each } x \in (a,b]. \tag{4.16}$$

Since $u'(a) = u_1$ exists, it follows from the Mean Value Theorem that there exists a sequence $\{x_m\} \subset (a,b)$ such that $\lim_{m\to\infty} x_m = 0$ and

$$\lim_{m\to\infty} u'(x_m) = u'(a). \tag{4.17}$$

Since $\limsup_{x\to a^+} |I_{a^+}^{1-\alpha} v(x)| < \infty$, it follows from (4.16) and (4.17) that $c_1 = 0$. Noting that $u(a) = u_0$ and $I_{a^+}^{2-\alpha} v(a) = 0$, by (4.15) with $c_1 = 0$, we have

$$u(x) = u_0 + u_1(x-a) + I_{a^+}^{2-\alpha} v(x) \quad \text{for each } x \in [a,b]$$

and (4.3) with $n = 2$ holds.

If $n \geq 3$, then by (4.14) and Lemma 2.4 (3), we have for each $k \in \mathbb{N}_{n-2}$,

$$\begin{aligned} &u^{(k)}(x) \\ &= \Big(\sum_{i=0}^{n-1} \frac{u_i}{i!}(x-a)^i\Big)^{(k)} + \Big(\sum_{i=1}^{n-1} c_i (x-a)^{i-\alpha}\Big)^{(k)} + (I_{a^+}^{n-\alpha} v)^{(k)}(x) \\ &= \sum_{j=0}^{n-1-k} \frac{u_{k+j}}{j!}(x-a)^j + \sum_{i=1}^{n-1} c_i \eta(i,k)(x-a)^{i-\alpha-k} + I_{a^+}^{n-k-\alpha} v(x) \end{aligned} \tag{4.18}$$

$$= \sum_{j=0}^{n-1-k} \frac{u_{k+j}}{j!}(x-a)^j + \sum_{i=1}^{k} \frac{c_i\eta(i,k)}{(x-a)^{k-i+\alpha}} + \sum_{i=k+1}^{n-1} c_i\eta(i,k)(x-a)^{i-\alpha-k} + I_{a^+}^{n-k-\alpha}v(x) \quad \text{for each } x \in (a,b],$$

where

$$\eta(i,k) = c_i\Pi_{j=0}^{k-1}(i-\alpha-j) \quad \text{for } k \in \mathbb{N}_{n-2} \text{ and } i \in \mathbb{N}_{n-1}.$$

By (4.18) with $k=1$, we have for each $x \in (a,b]$,

$$u'(x) = \sum_{j=0}^{n-2} \frac{u_{1+j}}{j!}(x-a)^j + \frac{c_1\eta(1,1)}{(x-a)^\alpha} + \sum_{i=2}^{n-1} c_i\eta(i,1)(x-a)^{i-\alpha-1} + I_{a^+}^{n-1-\alpha}v(x). \quad (4.19)$$

By (4.2), $u'(a) = u_1$ exists. It follows from the Mean Value Theorem that there exists a sequence $\{x_m\} \subset (a,b)$ such that $\lim_{m\to\infty} x_m = 0$ and (4.17) holds. Since $I_{a^+}^{n-1-\alpha}v \in C[a,b]$, by (4.17) and (4.19), we obtain $c_1 = 0$. Hence, (4.14) becomes

$$u(x) = \sum_{i=0}^{n-1} \frac{u_i}{i!}(x-a)^i + \sum_{i=2}^{n-1} c_i(x-a)^{i-\alpha} + I_{a^+}^{n-\alpha}v(x) \quad \text{for each } x \in [a,b].$$

Differentiating both sides of the above equation we obtain that for each $x \in [a,b]$,

$$u'(x) = \sum_{j=0}^{n-2} \frac{u_{1+j}}{j!}(x-a)^j + \sum_{i=2}^{n-1} c_i\eta(i,1)(x-a)^{i-\alpha-1} + I_{a^+}^{n-1-\alpha}v(x). \quad (4.20)$$

By Lemma 2.4 (2) and (3), $I_{a^+}^{n-1-\alpha}v \in C[a,b]$. This with (4.20) implies $u' \in C[a,b]$ for $n \geq 3$.

If $n = 3$, then (4.20) with $n = 3$ becomes

$$u'(x) = u_1 + u_2(x-a) + c_2\eta(2,1)(x-a)^{1-\alpha} + I_{a^+}^{2-\alpha}v(x) \quad \text{for each } x \in [a,b].$$

Differentiating both sides of the above equation and using (H1), we obtain

$$u''(x) = u_2 + \frac{c_2\eta(2,1)(1-\alpha)}{(x-a)^\alpha} + I_{a^+}^{1-\alpha}v(x) \quad \text{for each } x \in (a,b]. \quad (4.21)$$

By (4.2), $u''(a) = u_2$ exists. It follows from the Mean Value Theorem that there exists a sequence $\{x_m^*\} \subset (a,b)$ such that $\lim_{m\to\infty} x_m^* = 0$ and

$$\lim_{m\to\infty} u''(x_m^*) = u''(a). \quad (4.22)$$

Since $\limsup_{x\to a^+} |I_{a^+}^{1-\alpha}v(x)| < \infty$, it follows from (4.21) and (4.22) that $c_2 = 0$. Repeating the process we obtain for $n \geq 4$, $c_i = 0$ for each $i \in \mathbb{N}_{n-2}$. Hence, equation (4.14) becomes

$$u(x) = \sum_{i=0}^{n-1} \frac{u_i}{i!}(x-a)^i + c_{n-1}(x-a)^{n-1-\alpha} + I_{a^+}^{n-\alpha}v(x) \quad \text{for each } x \in [a,b]. \quad (4.23)$$

Differentiating both sides of (4.23) $(n-2)$ times implies for each $x \in [a,b]$,

$$u^{(n-2)}(x) = u_{n-2} + u_{n-1}(x-a) + c_{n-1}\eta(n-1,n-2)(x-a)^{1-\alpha} + I_{a^+}^{2-\alpha}v(x). \quad (4.24)$$

By Lemma 2.4 (2), $I_{a^+}^{2-\alpha}v \in C[a,b]$. This with (4.24) implies $u^{(n-2)} \in C[a,b]$. Using (H1) and differentiating both sides of the above equation imply for $x \in (a,b]$,

$$u^{(n-1)}(x) = u_{n-1} + \frac{c_{n-1}\eta(n-1,n-2)(1-\alpha)}{(x-a)^\alpha} + I_{a^+}^{1-\alpha}v(x). \quad (4.25)$$

Since $u^{(n-1)}(a) = u_{n-1}$ exists and $\limsup_{x\to a^+} |I_{a^+}^{1-\alpha} v(x)| < \infty$, by the Mean Value Theorem and (4.25), we have $c_{n-1} = 0$. By (4.23) with $c_{n-1} = 0$, we see that (4.3) holds for $n \ge 3$. Since $v \in H_B^\alpha(a,b)$, (H1) holds. Since (4.3) holds and $u^{(n-1)} = u_{n-1}$, it follows from (4.11) that $I_{a^+}^{1-\alpha} v(a) = 0$ and $v \in H_0^\alpha(a,b)$. □

Remark 4.8. By Proposition 4.3 (1) and Theorem 4.7 (1), we see that the condition $v \in H_0^\alpha(a,b)$, in particular, $v \in L^p(a,b)$ for some $p \in (\frac{1}{1-\alpha}, \infty]$, is a useful condition for studying solutions of (3.1)-(4.2) in $C[a,b]$ via (4.3).

By using Theorem 4.7, we obtain the following equivalence result.

Theorem 4.9. *Let $v \in L^1(a,b)$. Then the following assertions are equivalent.*

(1) $I_{a^+}^{1-\alpha} v(a) = 0$, $u^{(n-1)} \in C[a,b]$, *and u is a solution of* (4.3).
(2) $I_{a^+}^{1-\alpha} v \in C[a,b]$, $u \in C[a,b]$, $(I_{a^+}^\alpha (u - P_{n-1}))^{(n-1)} \in AC[a,b]$ *and and u is a solution of* (4.1)-(4.2).
(3) $v \in H_B^\alpha(a,b)$, $u^{(n-1)} \in C[a,b]$, $(I_{a^+}^\alpha (u - P_{n-1}))^{(n-1)} \in AC[a,b]$ *and and u is a solution of* (4.1)-(4.2).
(4) $v \in C_0^\alpha(a,b)$ *and u is a solution of* (4.3).

Proof. (1) implies (2). Since $I_{a^+}^{1-\alpha} v(a) = 0$ and $u^{(n-1)} \in C[a,b]$, by Lemma 4.5(2), $I_{a^+}^{1-\alpha} v \in C[a,b]$ and $v \in C_0^\alpha(a,b)$. By Proposition 4.3(1), $C_0^\alpha(a,b) \subset H_0^\alpha(a,b)$ and $v \in H_0^\alpha(a,b)$. By Theorem 4.7(1), $u \in C[a,b]$, $(I_{a^+}^\alpha (u - P_{n-1}))^{(n-1)} \in AC[a,b]$ and u is a solution of (4.1)-(4.2) and (2) holds.

(2) implies (1). By Lemma 2.4 (2), $I_{a^+}^{1-\alpha} v \in C[a,b]$ implies $v \in H_B^\alpha(a,b)$. By Theorem 4.7 (2), u is a solution of (4.3) and $v \in H_0^\alpha(a,b)$. Hence, $I_{a^+}^{1-\alpha} v(a) = 0$. Since $I_{a^+}^{1-\alpha} v \in C[a,b]$ and u is a solution of (4.3), by Lemma 4.5 (2), we obtain $u^{(n-1)} \in C[a,b]$ and (1) holds.

(2) implies (3). Assume that (2) holds. By Lemma 2.4, $I_{a^+}^{1-\alpha} v \in C[a,b]$ implies $v \in H_B^\alpha(a,b)$ and (3) holds.

(3) implies (1). Assume that (3) holds. By Theorem 4.7 (2), u is a solution of (4.3) and $v \in H_0^\alpha(a,b)$. Hence, $I_{a^+}^{1-\alpha} v(a) = 0$. It follows that (1) holds.

Equivalence between (1) and (4). By Theorem 4.7 and Lemma 4.5(2), we see that (1) and (4) are equivalent. □

Remark 4.10. The equivalence between (3) and (4) of Theorem 4.9 generalizes the equivalence [15, Theorem 3.24, p.199] and the parts 2 and 3 of [29, Theorem 5.1], where $v(x) = (x-a)^{-\gamma} w(x)$ for each $x \in (a,b]$, $0 \le \gamma < 1 - \alpha$ and $w \in C[a,b]$.

As an application of Theorem 4.7, we obtain the following identity.

Theorem 4.11. *Assume that $u \in C[a,b]$ satisfies the following conditions:*

(i) $(I_{a^+}^\alpha (u - P_{n-1}))^{(n-1)} \in AC[a,b]$.
(ii) *The initial condition* (4.2) *holds.*
(iii) $D_{p,a^+}^{n-\alpha} u \in H_B^\alpha(a,b)$.

Then

$$I_{a^+}^{n-\alpha} D_{p,a^+}^{n-\alpha} u(x) = u(x) - P_{n-1}(x) \quad \text{for each } x \in [a,b]. \tag{4.26}$$

Proof. Let $v(x) = D_{p,a^+}^{n-\alpha} u(x)$ for a.e. $x \in [a,b]$. By (*iii*), $v \in H_B^\alpha(a,b)$ and $u \in C[a,b]$ satisfies (4.1)-(4.2). This with (i)-(iii) and Theorem 4.7(2) implies that u is a solution of (4.3) and $v \in H_0^\alpha(a,b)$. Substituting v into (4.3) implies (4.26). □

Remark 4.12. When $n = 1$, [18, Theorem 5.6] only requires that $D^{1-\alpha}_{p,a^+}u$ satisfies (H1). When $n \ge 2$, Theorem 4.11 suggests that a stronger condition (iii) is needed.

Now, we study solutions in $C[a, b]$ of the IVP of the nth order FDE

$$D^{n-\alpha}_{p,a^+}u(x) := (I^{\alpha}_{a^+}(u - P_{n-1}))^{(n)}(x) = v(x) \quad \text{for each } x \in [a, b] \tag{4.27}$$

subject to (4.2) when $v \in C[a, b]$.

The difference between (4.1) and (4.27) is that (4.1) holds for a.e. $x \in [a, b]$ while (4.27) holds for each $x \in [a, b]$. We need the following result, see [24, Theorem 7.21, p.149].

Lemma 4.13. *Assume that $f : [a, b] \to \mathbb{R}$ satisfies the following conditions.*

(i) *$f'(x)$ exists for each $x \in [a, b]$.*
(ii) *$f' \in L^1(a, b)$.*

Then $f \in AC[a, b]$.

Proposition 4.14. (1) *Assume that the following conditions hold.*

(i) *$v : [a, b] \to \mathbb{R}$ is a function and $v \in L^1(a, b)$.*
(ii) *u is a solution of* (4.27)*.*

Then $(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)} \in AC[a, b]$, and u is a solution of (4.1)*.*

(2) *If $v \in C[a, b]$ and $(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)} \in AC[a, b]$ and u is a solution of* (4.1)*, then u is a solution of* (4.27)

Proof. (1) By conditions (i) and (ii), $(I^{\alpha}_{0^+}(u - P_{n-1}))^{(n-1)}(x)$ exists for each $x \in [a, b]$. We define a function $f : [a, b] \to \mathbb{R}$ by

$$f(x) = (I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)}(x) \quad \text{for each } x \in [a, b].$$

Then by the conditions (i) and (ii), $f'(x) = v(x)$ exists for each $x \in [a, b]$ and $f' = v \in L^1(a, b)$. By Lemma 4.13, $f \in AC[a, b]$ and the result follows.

(2) Since $(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)} \in AC[a, b]$, integrating both sides of (4.1) from a to x, we have

$$(I^{\alpha}_{a^+}(u-P_{n-1}))^{(n-1)}(x)-(I^{\alpha}_{a^+}(u-P_{n-1}))^{(n-1)}(a) = \int_a^x v(y)\,dy \quad \text{for each } x \in [a, b].$$

Differentiating both sides of the above equation yields

$$(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n)}(x) = v(x) \quad \text{for each } x \in [a, b].$$

Hence, u is a solution of (4.27). □

By Proposition 4.14 with $v \in C[a, b]$, we obtain the following equivalence result that generalizes [18, Proposition 4.2] from $n = 1$ to $n \ge 2$.

Corollary 4.15. *Let $u \in L^1(a, b)$ and $v \in C[a, b]$. Then u is a solution of* (4.27) *if and only if $(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)} \in AC[a, b]$ and u is a solution of* (4.1)*.*

Proof. Assume that u is a solution of (4.27). Since $v \in C[a, b]$, the result follows from Proposition 4.14(1). The converse follows from Proposition 4.14(2). □

The following result shows that (4.27)-(4.2) can be studied via (4.3) when $v \in C[a, b]$.

Theorem 4.16. *Let $v \in C[a, b]$. Then u is a solution of* (4.3) *if and only if $u \in C[a, b]$ and u is a solution of* (4.27)-(4.2)*.*

Proof. Assume that u is a solution of (4.3). Since $v \in C[a,b]$, we have $v \in C_0^\alpha(a,b)$. By (4) implying (2) of Theorem 4.9, $u \in C[a,b]$, $(I_{a^+}^\alpha(u-P_{n-1}))^{(n-1)} \in AC[a,b]$ and u is a solution of (4.1)-(4.2). Since $v \in C[a,b]$ and u is a solution of (4.1), by Corollary 4.15, u is a solution of (4.27). Hence, u is a solution of (4.27)-(4.2). Conversely, assume that $u \in C[a,b]$ and u is a solution of (4.27)-(4.2). Since $v \in C[a,b]$, By Lemma 2.4(2), $I_{a^+}^{1-\alpha}v \in C[a,b]$ and by Corollary 4.15, $(I_{a^+}^\alpha(u - P_{n-1}))^{(n-1)} \in AC[a,b]$ and u is a solution of (4.1)-(4.2). By (1) implying (1) of Theorem 4.9, u is a solution of (4.3). □

Remark 4.17. The solutions of (4.27)-(4.2) were studied analytically and numerically in [3, 10] when $v(x) = -\lambda^\alpha u(x) + f(x)$. When $a = 0$, the equivalence of (4.27)-(4.2) and (4.3) is implicitly proved by a different method in [5, Lemma 6.2] (also see [7, Lemma 2.1], [13, Theorem 1], [14, Theorem 1]), where $u \in C[a,b]$, $v(x) = f(x,u(x))$ and f is a suitable continuous function.

As a special case of Theorem 4.11, we obtain the following result.

Corollary 4.18. *Assume that $u \in C[a,b]$ satisfies* (4.2) *and $D_{p,a^+}^{n-\alpha}u \in C[a,b]$. Then* (4.26) *holds.*

Proof. Let $v(x) = D_{p,a^+}^{n-\alpha}u(x)$ for $x \in [a,b]$. Since $D_{p,a^+}^{n-\alpha}u \in C[a,b]$, by Proposition 4.14, $(I_{a^+}^\alpha(u-P_{n-1}))^{(n-1)} \in AC[a,b]$ and Theorem 4.11 (i) holds. By Proposition 4.3(2), $v \in C[a,b]$ implies Theorem 4.11(iii). The result follows from Theorem 4.11. □

Corollary 4.18 generalizes [6, Theorem 2.5], where $u^{(n-1)} \in C[a,b]$, $u_i = u^{(i)}(a)$ and a very different proof is given.

5. Higher-order Caputo fractional differential equations

We consider the IVPs of the nth order Caputo FDE

$$D_{C,a^+}^{n-\alpha}u(x) = I_{a^+}^\alpha u^{(n)}(x) = v(x) \quad \text{for a.e. } x \in [a,b] \tag{5.1}$$

subject to the initial conditions

$$u(a) = u_0, \quad u'(a) = u_1, \dots, u^{(n-1)}(a) = u_{n-1}, \tag{5.2}$$

where $u_i \in \mathbb{R}$ for each $i \in \mathbb{N}_{0,n-1}$ and $v \in L^1(a,b)$ are given.

Definition 5.1. A function $u : [a,b] \to \mathbb{R}$ is said to be a solution of (5.1) if $u^{(n)} \in L^1(a,b)$ and u satisfies (5.1); to be a solution of (5.1)-(5.2) if u is a solution of (5.1) and satisfies (5.2).

The following result provides a continuous function u whose nth order Caputo fractional derivatives are zero a.e. on $[a,b]$ and $u^{(n-1)} \notin AC[a,b]$, which is a solution of (5.1)-(5.2).

Example 5.2. Let f be the Cantor function defined on $[0,1]$ (see [23, Example 7.1, p. 141]). We define a function $u : [0,1] \to \mathbb{R}$ by

$$u(x) = I_{0^+}^{n-1}f(x) \quad \text{for each } x \in [0,1]. \tag{5.3}$$

Then the following assertions hold.

(i) $u^{(n-1)} \notin AC[0,1]$ and u satisfies the initial condition

$$u(0) = 0, \quad u'(0) = 0, \dots, u^{(n-1)}(0) = 0. \tag{5.4}$$

(ii) $u(x) \not\equiv 0$ on $[0,1]$ and u is not a polynomials of degree $n-1$.
(iii) $D^{n-\alpha}_{C,a^+}u(x) = I^{\alpha}_{0+}u^{(n)}(x) = 0$ for a.e. $x \in [0,1]$.

Proof. (i) By Lemma 2.3, we have for $i \in \mathbb{N}_{n-2}$,

$$u^{(i)}(x) = I^{n-1-i}_{0^+}f(x) \quad \text{for each } x \in [0,1] \tag{5.5}$$

and $u^{(i)}(0) = 0$ for $i \in \mathbb{N}_{n-2}$. By (5.5) with $i = n-2$, we have

$$u^{(n-2)}(x) = I^1_{a^+}f(x) \quad \text{for each } x \in [0,1].$$

Since f is a continuous increasing nonconstant function on $[0,1]$, by (2.2), we have

$$u^{(n-1)}(x) = f(x) \quad \text{for each } x \in [0,1]. \tag{5.6}$$

Since $f \notin AC[0,1]$, by (5.6), we have $u^{(n-1)} \notin AC[0,1]$. Since $f(0) = 0$, by (5.6) we have $u^{(n-1)}(0) = 0$. Hence, result (i) holds.

(ii) Since f is a continuous increasing nonconstant function on $[0,1]$ and $f(0) = 0$, by (5.3), $u(x) \not\equiv 0$ on $[0,1]$. Since f is a nonconstant function on $[0,1]$, by (5.6), u is not a polynomial of degree $n-1$ since the $(n-1)$th order derivative of a polynomial of degree $n-1$ must be a constant. Hence, the result (ii) holds.

(iii) Since $f'(x) = 0$ for a.e. $x \in [0,1]$, By (5.6), we have

$$u^{(n)}(x) = f'(x) = 0 \quad \text{for a.e. } x \in [0,1].$$

This implies that the result (iii) holds. □

A sufficient condition for u to satisfy $u^{(n)} \in L^1(a,b)$ is $u^{(n-1)} \in AC[a,b]$. When studying the equivalence between (5.1)-(5.2) and the integral equation (4.3), the condition $u^{(n-1)} \in AC[a,b]$ is needed to ensure the equivalence, see Theorem 5.6 below. Following [21, Section 9], the condition $u^{(n-1)} \in AC[a,b]$ would be used as a necessary condition for the definition of a solution u of (5.1) to prevent having functions such as Lebesgue's singular function like the function u given in Example 5.2 as solutions of (5.1).

We first give the following result on the nonexistence of solutions of (5.1).

Theorem 5.3. *If $v \in L^1(a,b) \setminus \mathscr{R}_\alpha(a,b)$, then (5.1) has no solutions.*

Proof. By Proposition 4.4, $C[a,b] \setminus \mathscr{R}_\alpha(a,b) \neq \emptyset$. Assume that (5.1) has a solution $u : [a,b] \to \mathbb{R}$. By Definition 5.1, $u^{(n)} \in L^1(a,b)$ and u satisfies (5.1). By (5.1) and (4.6), we have

$$v(x) = D^{n-\alpha}_{C,a^+}u(x) = I^{\alpha}_{a^+}u^{(n)} \in \mathscr{R}_\alpha(a,b), \tag{5.7}$$

a contradiction. □

The following result shows that under suitable conditions on u, the Caputo fractional derivative $D^{n-\alpha}_{C,a^+}u$ is equal to the fractional derivative $D^{n-\alpha}_{p,a^+}u$.

Lemma 5.4. *Let $u_i \in \mathbb{R}$ be given for each $i \in \mathbb{N}_{0,n-1}$. Assume that $u^{(n-1)} \in AC[a,b]$ and u satisfies (5.2). Then the following assertions hold.*

(i)
$$I^n_{a^+}u^{(n)}(x) = (u - P_{n-1})(x) \quad \textit{for each } x \in [a,b] \tag{5.8}$$
where $P_{n-1}(x) = \sum_{i=0}^{n-1} \frac{u_i}{i!}(x-a)^i$ for each $x \in [a,b]$.

(ii) *$(I^{\alpha}_{a^+}(u - P_{n-1}))^{(n-1)} \in AC[a,b]$ and $(I^{\alpha}_{a^+}(u - P_{n-1}))^{(i)}(a) = 0$ for $i \in \mathbb{N}_{0,n-1}$.*

(iii) *$D^{n-\alpha}_{C,a^+}u(x) = D^{n-\alpha}_{p,a^+}u(x)$ for a.e. $x \in [a,b]$.*

(iv) $D^{n-\alpha}_{C,a^+}u = D^{n-\alpha}_{p,a^+}u \in \mathscr{R}_\alpha(a,b)$.

Proof. (i) Because $u^{(n-1)} \in AC[a,b]$ and u satisfies (5.2), we have

$$I^1_{a^+}u^{(n)}(x) = u^{(n-1)}(x) - u^{(n-1)}(a) = u^{(n-1)}(x) - u_{n-1} \quad \text{for each } x \in [a,b].$$

Since $u^{(n-2)} \in AC[a,b]$, integrating the above equation from a to x implies

$$\begin{aligned} I^2_{a^+}u^{(n)}(x) &= u^{(n-2)}(x) - u^{(n-2)}(a) - u_{n-1}(x-a) \\ &= u^{(n-2)}(x) - u_{n-2} - u_{n-1}(x-a) \quad \text{for each } x \in [a,b]. \end{aligned}$$

Repeating the process implies (5.8).

(ii) Applying $I^\alpha_{a^+}$ to (5.8), we have

$$I^n_{a^+}I^\alpha_{a^+}u^{(n)}(x) = I^\alpha_{a^+}I^n_{a^+}u^{(n)}(x) = I^\alpha_{a^+}(u - P_{n-1})(x) \quad \text{for each } x \in [a,b].$$

Differentiating the above equation i and using Lemma 2.3 (2) imply

$$I^{n-i}_{a^+}I^\alpha_{a^+}u^{(n)}(x) = (I^\alpha_{a^+}(u - P_{n-1}))^{(i)}(x) \quad \text{for each } x \in [a,b]. \tag{5.9}$$

By (2.3), we have $I^{n-i}_{a^+}I^\alpha_{a^+}u^{(n)}(a) = 0$ for each $i \in \mathbb{N}_{n-1}$. It follows from (5.9) that

$$(I^\alpha_{a^+}(u - P_{n-1}))^{(i)}(a) = 0 \quad \text{for each } i \in \mathbb{N}_{0,n-1}.$$

By (5.9) with $i = n-1$, we obtain

$$I^1_{a^+}I^\alpha_{a^+}u^{(n)}(x) = (I^\alpha_{a^+}(u - P_{n-1}))^{(n-1)}(x) \quad \text{for each } x \in [a,b]. \tag{5.10}$$

Since $I^1_{a^+}I^\alpha_{a^+}u^{(n)} \in AC[a,b]$, we have $(I^\alpha_{a^+}(u - P_{n-1}))^{(n-1)} \in AC[a,b]$.

(iii) Differentiating both sides of (5.10) implies (iii).

(iv) The result follows from (5.7) and (iii). □

When u_i is replaced by $u^{(i)}(a)$ for each $i \in \mathbb{N}_{0,n-1}$, Lemma 5.4 *(iii)* is proved in [5, Theorem 3.1, p.50], [15, Theorem 2.1, p.92] and [29, Lemma 4.12] with different proofs. Other results of Lemma 5.4 are new.

The following result provides the conditions which ensure that (5.1)-(5.2) and (4.1)-(5.2) are equivalent.

Theorem 5.5. *Assume that $v \in L^1(a,b)$. Then the following assertions hold.*

1. *If $u^{(n-1)} \in AC[a,b]$ and u is a solution of (5.1)-(5.2), then $v \in \mathscr{R}_\alpha(a,b)$, $(I^\alpha_{a^+}(u - P_{n-1}))^{(n-1)} \in AC[a,b]$ and u is a solution of (4.1)-(5.2).*
2. *Let $u^{(n-1)} \in AC[a,b]$ and $v \in \mathscr{R}_\alpha(a,b)$. Then u is a solution of (5.1)-(5.2) if and only if u is a solution of (4.1)-(5.2).*

Proof. (1) Assume that u is a solution of (5.1)-(5.2). Since $u^{(n-1)} \in AC[a,b]$, by Lemma 5.4 *(ii)*, $(I^\alpha_{a^+}(u - P_{n-1}))^{(n-1)} \in AC[a,b]$. By Lemma 5.4 *(iii)* and u is a solution of (5.1)-(5.2), we see that u is a solution of (4.1)-(5.2).

(2) Let $u^{(n-1)} \in AC[a,b]$ and (5.2) holds. By Lemma 5.4 *(iii)*, we see that u is a solution of (5.1) if and only if u is a solution of (4.1). □

The following result provides conditions which ensure that (5.1)-(5.2) and (4.3) are equivalent.

Theorem 5.6. *The following assertions are equivalent.*

1. *$v \in L^1(a,b)$, $u^{(n-1)} \in AC[a,b]$ and u is a solution of (5.1)-(5.2).*
2. *$v \in \mathscr{R}_\alpha(a,b)$ and u is a solution of (4.3).*
3. *$v \in L^1(a,b)$, $I^{1-\alpha}_{a^+}v(a) = 0$, $u^{(n-1)} \in AC[a,b]$ and u is a solution of (4.3).*

(4) $v \in H_B^\alpha(a,b) \cap H_0^\alpha(a,b)$, $u^{(n-1)} \in AC[a,b]$ *and* u *is a solution of* (4.3).

Proof. (1) implies (2). Assume that (1) holds. It follows from Theorem 5.5(1) that $(I_{a^+}^\alpha(u - P_{n-1}))^{(n-1)} \in AC[a,b]$ and u is a solution of (4.1)-(5.2). Since $u^{(n-1)} \in AC[a,b]$ and (5.2) holds, by Lemma 5.4 (*iv*) and (5.1), we have

$$v = D_{C,a^+}^{n-\alpha} u \in \mathscr{R}_\alpha(a,b) \subset H_B^\alpha(a,b).$$

By Theorem 4.7(2), u is a solution of (4.3).

(2) implies (3). Assume that (2) holds. By Proposition 4.3(1), $\mathscr{R}_\alpha(a,b) \subset C_0^\alpha(a,b)$. This with (4) implying (1) of Theorem 4.9 implies the result (3).

(3) implies (1). By (1) implying (3) of Theorem 4.9, u is a solution of (4.1)-(5.2). By Lemma 5.4 (iii), u is a solution of (5.1)-(5.2).

(4) and (2). By Proposition 4.5(3), we see that (4) and (2) are equivalent. □

Theorem 5.7. *If* $u \in C[a,b]$ *satisfies* (5.2) *and* $u^{(n-1)} \in AC[a,b]$*, then*

$$I_{a^+}^{n-\alpha} D_{C,a^+}^{n-\alpha} u(x) = u(x) - P_{n-1}(x) \quad \text{for each } x \in [a,b]. \tag{5.11}$$

Proof. Let $v(x) = D_{C,a^+}^{n-\alpha} u(x)$ for a.e. $x \in [a,b]$. Then $v \in L^1(a,b)$. By hypotheses, $u^{(n-1)} \in AC[a,b]$ and u is a solution of (5.1)-(5.2). By (1) implying (2) of Theorem 5.6, $v \in \mathscr{R}_\alpha(a,b)$ and u is a solution of (4.3). Substituting v in (5.1) into (4.3) implies (5.11). □

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