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ASYMPTOTIC STABILITY OF A STOCHASTIC AGE-STRUCTURED COOPERATIVE LOTKA-VOLTERRA SYSTEM WITH POISSON JUMPS

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Abstract. In this article, we study a stochastic age-structured cooperative Lotka-Volterra system with Poisson jumps. Applying the M-matrix theory, we prove the existence and uniqueness of a global solution for the system. Then we use an optimized Euler-Maruyama numerical scheme to approximate the solution. We obtain second-moment boundedness and convergence rate of the numerical solutions. The numerical solutions illustrate the theoretical results.

1. Introduction

The determined cooperative Lotka-Volterra system [13, 20] is described as

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)(-\alpha_{11}x(t) + \alpha_{12}y(t) + r_1), \\ \frac{dy(t)}{dt} &= y(t)(\alpha_{21}x(t) - \alpha_{22}y(t) + r_2), \end{aligned} \tag{1.1}$$

where $x(t), y(t)$ are the densities of the two cooperative species, r_1, r_2 present the intrinsic growth rates of the two species, α_{11}, α_{22} are the intraspecific competition rates, and α_{12}, α_{21} denote the interspecific cooperation rates. This system has been widely studied and has many applications [8, 24]. In particular, it can be used to describe the cooperative relationship between multiple species, such as bees and flowers, algal-fungal associations of lichens, etc. Besides the studies on the deterministic properties of cooperative Lotka-Volterra systems [4, 16, 21], there are important studies on stochastic Lotka-Volterra systems [1, 9, 17, 18]. Especially, noises with jumps are necessary to describe phenomena that burst out in nature. For example, the impact of sudden pesticide spraying or tsunami on the population can not be ignored. There are some works on the stochastic differential equations with Poisson jumps[3, 7, 22] to consider the discontinuous random effects. One example is system

$$\begin{aligned} dx(t) &= x(-\alpha_{11}(a)x + \alpha_{12}(a)y + r_1(a))dt + \sigma_1 x dw(t) + h_1 x dN(t), \\ dy(t) &= y(\alpha_{21}(a)x - \alpha_{22}(a)y + r_2(a))dt + \sigma_2 y dw(t) + h_2 y dN(t). \end{aligned} \tag{1.2}$$

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Considering the actual biological background of system (1.2), population reproduction is determined by the fertility of female individuals which is related to their age. According to [2, 5], the birth rate is sensitive to age. Therefore, we need to include an age-structured fertility rate into system (1.2). The age-structured theory was first introduced by Lotka [13] to describe the fertility of species and latter been applied in many systems [11, 19, 26, 29, 30, 31]. For example, Zhang and Liu [29] focused on the age-structured predator-prey model and discussed the non-trivial periodic oscillation phenomenon. Solis and Carrillo [19] investigated the type of theoretical predation of the predator-prey Lotka-Volterra model with age-structured. However, very few studies are on the cooperative Lotka-Volterra system. To fill this gap, we establish the stochastic age-structured cooperative Lotka-Volterra system with Poisson jumps,

$$
\begin{gathered}
d_tX = -\frac{\partial X}{\partial a}dt + X(-\alpha_{11}(a)X + \alpha_{12}(a)Y + r_1(a))dt + \sigma_1 X dw(t) + h_1 X dN(t),\\
d_tY = -\frac{\partial Y}{\partial a}dt + Y(\alpha_{21}(a)X - \alpha_{22}(a)Y + r_2(a))dt + \sigma_2 Y dw(t) + h_2 Y dN(t),\\
X(t,0) = \int_0^A \gamma(t,a)X(t,a)da, \quad t \in [0,T],\\
Y(t,0) = \int_0^A \beta(t,a)Y(t,a)da, \quad t \in [0,T],\\
X(0,a) = X_a(0),\ Y(0,a) = Y_a(0), \quad a \in [0,A],
\end{gathered}
\tag{1.3}
$$

where $X(t,a)$ and $Y(t,a)$ denote the densities of the two species at time t, age a. d_tX and d_tY denote the differential of $X(t,a)$ and $Y(t,a)$ relative to t. $w(t)$ is a standard Brownian motion, $N(t)$ is a scalar Poisson process independent of $w(t)$. The compensated Poisson process is $\tilde{N}(t) = N(t) - \lambda_1 t$, where the parameter λ_1 is the jump intensity. In the first two equations $(t,a) \in Q$ and $Q = (0,T) \times (0,A)$. Features of the parameters in system (1.3) are showed in [27].

It is extremely hard to find the explicit form of the exact solution to (1.3), so we target on finding the numerical approximation the exact solution. So we can describe the changes of number of species and predict the species size of the system. However, the Euler-Maruyama (EM) method cannot discretize the age-structured cooperative Lotka-Volterra system directly because the system does not satisfy the Lipschitz continuous condition and the linear growth condition on the drift coefficients. There are many numerical approximate methods to deal with the different algebraic structure of the model, such as the positive preserving method, the truncated EM method, the tamed EM method and Runge-Kutta-Fehlberg method, etc. [6, 12, 14, 28]. Zhang et al. [28] constructed a positive preserving numerical method for the stochastic R&D model and proved that the convergence order is $\frac{1}{2}(1-\frac{1}{p})$. Yang et al. [23] constructed the truncated EM method for the stochastic differential equations to deal with the blasting phenomenon, while, the method is hard to guarantee the positive numerical solution. Liu et al. [12] established the tamed EM approximation for McKean-Vlasov stochastic differential equations with superlinear drift and Hölder diffusion coefficients. The convergence rate was obtained by replacing the one-sided local Lipschitz condition with the global condition. However, most of the above mentioned methods focused on the stochastic differential equations but not involving the age-structured. In this study, we use an optimized Euler-Maruyama scheme to find a reasonable and effective numerical solution for system (1.3). Using the theory of equilibrium point, we have successfully avoided the restrictions on the coefficients.

Two main contributions in this are the following:

• The existence and uniqueness of the positive global solution of the stochastic age-structured cooperative Lotka-Volterra system with Poisson jumps (this is proved for the first time).

• The finite-time convergence and boundedness of the numerical approximation to solution.

This article is arranged as follows. In section 2, we prove the existence and uniqueness of the global solution for the stochastic age-structured cooperative Lotka-Volterra system with Poisson jumps. In section 3, we introduce the EM approximation method to the system and obtain the 2nd-moment estimations of this algorithm. The convergence of the algorithm is presented in section 4. We find a strong convergence order. The numerical simulation is reported in section 5.

2. Existence and uniqueness of a solution

2.1. **Preliminaries.** Throughout the paper, let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space. Here, the filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfies the usual conditions (that is, it is increasing and right continuous with $\mathfrak{F}_0$ containing all $\mathbb{P}$-null sets), $\mathbb{E}$ denotes the expectation corresponding to $\mathbb{P}$. We denote by $\mathbb{R}^n_+$ the positive cone in $\mathbb{R}^n$, that is $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } 1 \leq i \leq n\}$. $\check{\gamma}(t,a)$ and $\hat{\gamma}(t,a)$ are the maximum and minimum value of the continuous function $\gamma(t,a)$, respectively. For a pair of real numbers, a and b, we have $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$. For a set $\mathbb{A}$, its indicator function is denoted by

$$\mathbf{1}_{\mathbb{A}} = \begin{cases} 1, & x \in \mathbb{A}, \\ 0, & x \notin \mathbb{A}. \end{cases}$$

For $x \in \mathbb{R}^n$, its norms is denoted by

$$|x|_\iota = \begin{cases} |x_1| + |x_2| + \cdots + |x_n|, & \iota = 1, \\ \big(\sum_{i=1}^n x_i^2\big)^{1/2}, & \iota = 2. \end{cases}$$

Let $V = \big\{\varphi \in L^2([0,A]) : \frac{\partial \varphi}{\partial a} \in L^2([0,A])$, where $\frac{\partial \varphi}{\partial a}$ is the generalized partial derivatives with respect to age a$\big\}$. $H = L^2([0,A])$ and $V \hookrightarrow H \equiv H' \hookrightarrow V'$, where V' is the dual space of V and H' is the dual space of H. The norms in V, H, V' are denoted as $\|\|$, $|\cdot|$, and $\|\|_*$, respectively. The duality product between V, V' is written as $\langle \cdot, \rangle$, the scalar product in H is denoted by $(\cdot,\cdot)$. $C = C([0,T];H)$ is the space of all continuous functions from $[0,T]$ into H. $I^2([0,T];V)$ denotes the space of all V-valued processes $(P_t)_{t\in[0,T]}$, $L^2_V = L^2([0,T];V)$. $W := (I^2([0,T];V) \bigcap L^2(\Omega; C([0,T];H))) \times (I^2([0,T];V) \bigcap L^2(\Omega; C([0,T];H)))$.

To discuss the order of convergence of the numerical methods, we have the following assumptions.

- (A1) $\lim_{\alpha\to A^-} \int_\alpha^A \gamma(t,a)X(t,a)da = \lambda_x > 0$; $\lim_{\alpha\to A^-} \int_\alpha^A \beta(t,a)Y(t,a)da = \lambda_y > 0$.
- (A2) The fertility rate of females $\gamma(t,a), \beta(t,a) \in C([0,T]\times[0,A];H)$.
- (A3) $\alpha_{ij}(a) \in C([0,A];\mathbb{R}_+), i,j \in \{1,2\}$ and satisfy $\sup_{a\in[0,A]} \alpha_{12}(a)\alpha_{21}(a) < \inf_{a\in[0,A]} \alpha_{11}(a)\alpha_{22}(a)$.

(A4) $r_i(a) \in C\big([0,A];\mathbb{R}\big)$ and $r_i(a)$ is nondecreasing with $-\infty < r_i(0) < 0 < r_i(A) < \infty$, for $i = 1, 2$, and $h_i(t)$ is the continuous function on [0, T] for $i = 1, 2$.

(A5) $\sup_{a\in[0,A]} \alpha_{12}(a)\alpha_{21}(a) < 1$ and $r_1(0)(1-\alpha_{12}\alpha_{21})+\alpha_{12}(r_2(A)+\alpha_{21}r_1(A)) < 0$; $r_2(0)(1-\alpha_{12}\alpha_{21}) + \alpha_{21}(r_1(A) + \alpha_{12}r_2(A)) < 0$.

Theorem 2.1. *Under Assumptions* (A1)–(A5)*, for each initial value* $(X_{a_0}(0), Y_{a_0}(0)) \in \mathbb{R}^2_+$*, there exists at most one solution* $(X(t), Y(t))$ *of system* (1.3) *on* W*.*

Proof. Assume $\Psi_1(t) := (X_1(t), Y_1(t))$ and $\Psi_2(t) := (X_2(t), Y_2(t))$ are two solutions of (1.3). Applying the Itô formula to $|X_1(t) - X_2(t)|^2$, we have

$$
\begin{aligned}
|X_1(t) - X_2(t)|^2 &= 2\int_0^t \langle -\frac{\partial X_1(s)}{\partial a} + \frac{\partial X_2(s)}{\partial a}, X_1(s) - X_2(s)\rangle ds \\
&\quad + 2\int_0^t (H_{1x}(\Psi_1(s)) - H_{1x}(\Psi_2(s)), X_1(s) - X_2(s))ds \\
&\quad + 2\int_0^t (H_{2x}(\Psi_1(s)) - H_{2x}(\Psi_2(s)), X_1(s) - X_2(s))ds \\
&\quad + \lambda_1\int_0^t |J_{1x}(\Psi_1(s)) - J_{1x}(\Psi_2(s))|^2 ds \\
&\quad + \int_0^t |G_{1x}(\Psi_1(s)) - G_{1x}(\Psi_2(s))|^2 ds \\
&\quad + 2\int_0^t (X_1(s) - X_2(s), (J_{1x}(\Psi_1(s)) - J_{1x}(\Psi_2(s)))dN(s)) \\
&\quad + 2\int_0^t (X_1(s) - X_2(s), (G_{1x}(\Psi_1(s)) - G_{1x}(\Psi_2(s)))dw(s)).
\end{aligned}
$$

Since

$$
\langle \frac{\partial (X_1(s) - X_2(s))}{\partial a}, X_1(s) - X_2(s)\rangle \le \frac{1}{2}A^2\check{\gamma}^2|X_1(s) - X_2(s)|^2,
$$

and by Assumptions (A1-A5), we have

$$
\begin{aligned}
|X_1(t) - X_2(t)|^2 &\le (A^2\check{\gamma}^2 + \lambda_1 + 2)\int_0^t |X_1(s) - X_2(s)|^2 ds \\
&\quad + 2(L_1^2 + 2\rho_1^2 + \lambda_1 L_1^2)\int_0^t |\Psi_1(s) - \Psi_2(s)|^2 ds \\
&\quad + 2\int_0^t (X_1(s) - X_2(s), (G_{1x}(\Psi_1(s)) - G_{1x}(\Psi_2(s)))dw(s)) \\
&\quad + 2\int_0^t (X_1(s) - X_2(s), (J_{1x}(\Psi_1(s)) - J_{1x}(\Psi_2(s)))d\tilde{N}(s)).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}|X_1(s)-X_2(s)|^2\\
&\le (A^2\check{\gamma}^2+\lambda_1+2)\int_0^t \mathbb{E}|X_1(s)-X_2(s)|^2ds\\
&\quad +2(L_1^2+2\rho_1^2+\lambda_1 L_1^2)\int_0^t \mathbb{E}|\Psi_1(s)-\Psi_2(s)|^2ds\\
&\quad +2\mathbb{E}\sup_{0\le s\le t}\int_0^s (X_1(r)-X_2(r),(G_{1x}(\Psi_1(r))-G_{1x}(\Psi_2(r)))dw(r))\\
&\quad +2\mathbb{E}\sup_{0\le s\le t}\int_0^s (X_1(r)-X_2(r),(J_{1x}(\Psi_1(r))-J_{1x}(\Psi_2(r)))d\tilde{N}(r)).
\end{aligned}
\tag{2.1}
$$

Applying the Burkholder-Davis-Gundy (BDG) inequality, we have

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}\int_0^s (X_1(r)-X_2(r),(G_{1x}(\Psi_1(r))-G_{1x}(\Psi_2(r)))dw(r))\\
&\le \frac{1}{8}\mathbb{E}[\sup_{0\le s\le t}|X_1(s)-X_2(s)|^2]+k_1L_1^2\int_0^t \mathbb{E}|\Psi_1(s)-\Psi_2(s)|^2ds,
\end{aligned}
\tag{2.2}
$$

and

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}\int_0^s (X_1(r)-X_2(r),(J_{1x}(\Psi_1(r))-J_{1x}(\Psi_2(r)))d\tilde{N}(r))\\
&\le \frac{1}{8}\mathbb{E}[\sup_{0\le s\le t}|X_1(s)-X_2(s)|^2]+k_2L_1^2\int_0^t \mathbb{E}|\Psi_1(s)-\Psi_2(s)|^2ds,
\end{aligned}
\tag{2.3}
$$

where k_1 and k_2 are determined by the BDG inequality. Substituting (2.2) and (2.3) into (2.1), we have

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}|X_1(s)-X_2(s)|^2\\
&\le \frac{1}{2}\mathbb{E}[\sup_{0\le s\le t}|X_1(s)-X_2(s)|^2]+(A^2\check{\gamma}^2+\lambda_1+2)\int_0^t \mathbb{E}\sup_{0\le r\le s}|X_1(s)-X_2(s)|^2ds\\
&\quad +2(L_1^2+2\rho_1^2+\lambda_1L_1^2+k_1L_1^2+k_2L_1^2)\int_0^t \mathbb{E}\sup_{0\le r\le s}|X_1(s)-X_2(s)|^2ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}|X_1(s)-X_2(s)|^2\\
&\le 4(L_1^2+2\rho_1^2+\lambda_1L_1^2+k_1L_1^2+k_2L_1^2)\int_0^t \mathbb{E}\sup_{0\le r\le s}|\Psi_1(s)-\Psi_2(s)|^2ds\\
&\quad +2(A^2\check{\gamma}^2+\lambda_1+2)\int_0^t \mathbb{E}\sup_{0\le r\le s}|X_1(s)-X_2(s)|^2ds.
\end{aligned}
\tag{2.4}
$$

Similarly, we have

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}|Y_1(s)-Y_2(s)|^2\\
&\le 2(A^2\check{\beta}^2+\lambda_1+1+2\rho_2^2))\int_0^t \mathbb{E}\sup_{0\le r\le s}|Y_1(r)-Y_2(r)|^2dr \\
&\quad + 2(3L_2^2+2\rho_2^2+\lambda_1L_2^2+2k_1L_2^2+2k_2L_2^2)\int_0^t \mathbb{E}\sup_{0\le r\le s}|Y_1(r)-Y_2(r)|^2ds.
\end{aligned}
\tag{2.5}
$$

By (2.4) and (2.5), we deduce that

$$
\mathbb{E}\sup_{0\le s\le t}|\Psi_1(s)-\Psi_2(s)|^2 \le 2C\int_0^t \mathbb{E}\sup_{0\le r\le s}|\Psi_1(r)-\Psi_2(r)|^2ds, \forall t\in[0,T],
$$

where $C>0$ is a genetic constant whose values may vary for its different appearance. L_1 is defined in Lemma 3.1. Now, by the Gronwall's lemma, we obtain the uniqueness of the solution. □

Theorem 2.2. *Under Assumptions* (A1)–(A5), *for each given initial value* $(X_{a_0}(0),Y_{a_0}(0))\in\mathbb{R}_+^2$, *there exists a nonnegative global solution* $(X(t),Y(t))$ *of system* (1.3) *on* W.

Proof. It is easy to see that for any given initial value $(X_{a_0}(0),Y_{a_0}(0))\in\mathbb{R}_+^2$, there exists a local solution $(X(t,a),Y(t,a))\in W$ when $t\in[0,\tau_e)$, where τ_e is the explosion time of system (1.3). Next, we need to show that $\tau_e=\infty$, a.s.

We choose a positive constant $k_0>1$ such that for any initial value

$$
(X_{a_0}(0),Y_{a_0}(0))\in[\frac{1}{k_0},k_0]\times[\frac{1}{k_0},k_0].
$$

Then, for each $k\ge k_0$, the stopping time τ_k is defined by

$$
\tau_k=\inf\{t\wedge a\in[0,\tau_e):(X,Y)\notin(\frac{1}{k},k)\times(\frac{1}{k},k)\}.
$$

Next, we should prove that τ_k is an increasing function and $\lim_{k\to\infty}\tau_k=\tau_\infty$, where $\tau_\infty\le\tau_e$. Assuming that there are two constants $T>0$ and $\varepsilon\in(0,1)$ such that $\mathbb{P}\{\tau_\infty\le T\}>\varepsilon$. Then, for the constant k_0, there is an integer k_1 satisfying

$$
\mathbb{P}\{\tau_k\le T\}\ge\varepsilon, \forall k\ge k_1. \tag{2.6}
$$

Now, for any constants $c_x,c_y>0$, we define a function $V:W\to\mathbb{R}^+$ by

$$
V(X,Y)=c_x(X-\log X-1)+c_y(Y-\log Y-1).
$$

Using the Itô formula [15], we have

$$
\begin{aligned}
&LV(X,Y)\\
&=c_x\Big\langle 1-\frac{1}{X},-\frac{\partial X}{\partial a}+X(-\alpha_{11}X+\alpha_{12}Y+r_1)\Big\rangle+\frac{1}{2}(c_x\sigma_1^2+c_y\sigma_2^2)\\
&\quad +c_y\Big\langle 1-\frac{1}{Y},-\frac{\partial Y}{\partial a}+Y(\alpha_{21}X-\alpha_{22}Y+r_2)\Big\rangle+\frac{1}{2}\lambda_1(c_xh_1^2+c_yh_2^2)\\
&\le\Psi^\top\operatorname{diag}(c_x,c_y)\bar{R}-\bar{C}(\bar{A}\Psi+\bar{R})+\frac{1}{2}\Psi^\top(\operatorname{diag}(c_x,c_y)\bar{A}+\bar{A}^\top\operatorname{diag}(c_x,c_y))\Psi\\
&\quad +c_x\int_0^A\frac{1}{X}d_aX+c_y\int_0^A\frac{1}{Y}d_aY+\frac{1}{2}(c_x\sigma_1^2+c_y\sigma_2^2)+\frac{1}{2}\lambda_1(c_xh_1^2+c_yh_2^2)
\end{aligned}
$$

$$
\begin{aligned}
&\le c_x \log \lambda_x + c_y \log \lambda_y - \Big[c_x \log \big| \int_0^A \gamma(t,a) X(t,a) da \big| \\
&\quad + c_y \log \big| \int_0^A \beta(t,a) Y(t,a) da \big| \Big] + \frac{1}{2}(c_x \sigma_1^2 + c_y \sigma_2^2) + \Psi^\top \operatorname{diag}(c_x, c_y) \bar{R} \\
&\quad - \bar{C}(\bar{A}\Psi + \bar{R}) + \frac{1}{2}\lambda_1 (c_x h_1^2 + c_y h_2^2) + \frac{1}{2}\Psi^\top (\operatorname{diag}(c_x, c_y)\bar{A} + \bar{A}^\top \operatorname{diag}(c_x, c_y))\Psi,
\end{aligned}
$$

where

$$
\Psi = \begin{pmatrix} X(t,a) \\ Y(t,a) \end{pmatrix}, \quad \bar{R} = \begin{pmatrix} r_1(a) \\ r_2(a) \end{pmatrix}, \quad \bar{C}^\top = \begin{pmatrix} c_x \\ c_y \end{pmatrix},
$$

$$
\bar{A} = \begin{pmatrix} -\alpha_{11}(a) & \alpha_{12}(a) \\ \alpha_{21}(a) & -\alpha_{22}(a) \end{pmatrix}, \quad \operatorname{diag}(c_x, c_y) = \begin{pmatrix} c_x & 0 \\ 0 & c_y \end{pmatrix}.
$$

By Assumptions (A1)–(A5), since $-\bar{A}$ is a nonsingular M-matrix, we have

$$
\begin{aligned}
&LV(X,Y) \\
&\le c_x \log \lambda_x + c_y \log \lambda_y - \Big[c_x \log \big| \int_0^A \gamma(t,a) X(t,a) da \big| \\
&\quad + c_y \log \big| \int_0^A \beta(t,a) Y(t,a) da \big| \Big] + \frac{1}{2}(c_x \sigma_1^2 + c_y \sigma_2^2) \\
&\quad + \Psi^\top \operatorname{diag}(c_x, c_y)\bar{R} - \bar{C}(\bar{A}\Psi + \bar{R}) + \frac{1}{2}\lambda_1 (c_x h_1^2 + c_y h_2^2) \\
&\le c_x \log \lambda_x + c_y \log \lambda_y - \Big[c_x \log \big| \int_0^A \gamma(t,a) X(t,a) da \big| \\
&\quad + c_y \log \big| \int_0^A \beta(t,a) Y(t,a) da \big| \Big] + \{\bar{R}^\top \operatorname{diag}(c_x, c_y) - \bar{C}\bar{A}\}\Psi \\
&\quad + \frac{1}{2}(c_x \sigma_1^2 + c_y \sigma_2^2) - r_1 c_x - r_2 c_y + \frac{1}{2}\lambda_1 (c_x h_1^2 + c_y h_2^2),
\end{aligned}
$$

and

$$
LV(x,y) \le \big| \bar{R}^\top \operatorname{diag}(c_x, c_y) - \bar{C}\bar{A} \big| \cdot |\Psi| + C \le M_1(1 + |\Psi|). \tag{2.7}
$$

Letting $M_2 = \frac{c_x \vee c_y}{c_x \wedge c_y}$ and $M_3 = c_x \wedge c_y$, we obtain

$$
M_2 V(X,Y) = \frac{c_x \vee c_y}{c_x \wedge c_y} \big[c_x (X - 1 - \log X) + c_y (Y - 1 - \log Y) \big] \ge V(X,Y).
$$

Since $|\Psi| = (X^2 + Y^2)^{1/2} \le X + Y$, it yields

$$
|\Psi| \le 2(X - 1 - \log X) + 2(Y - 1 - \log Y) + 4 \le 4 + \frac{2}{M_3} V(X,Y). \tag{2.8}
$$

By (2.7) and (2.8), we obtain

$$
LV(X,Y) \le M_4(1 + V(X,Y)),
$$

where $M_4 = M_1(5 \vee 2/M_3)$. Then, we have

$$
\begin{aligned}
\mathbb{E}V(X(t\wedge\tau_k), Y(t\wedge\tau_k)) &\le V(X_0, Y_0) + \mathbb{E}\int_0^{t\wedge\tau_k} M_4(1 + V(X(s), Y(s)))ds \\
&\le M_5 + M_4 \int_0^t \mathbb{E}V(X(s\wedge\tau_k), Y(s\wedge\tau_k))ds,
\end{aligned}
$$

where $M_5 = V(X_0, Y_0) + M_4 T$.

By the Gronwall inequality, we have

$$\mathbb{E}V(X(\tau_k \wedge T), Y(\tau_k \wedge T)) \le M_5 e^{M_4 T}.$$

According to (2.6) and for any $k \ge k_1$, $\omega \in \Omega_k, (t,a) \in Q$, $X(\tau_k,\omega)$ and $Y(\tau_k,\omega)$ equal either k or $\frac{1}{k}$, hence we have

$$\begin{aligned} &V(X(\tau_k,\omega), Y(\tau_k,\omega)) \\ &\ge \big[c_x(k-1-\log k) + c_y(k-1-\log k)\big] \wedge \big[c_x(\frac{1}{k}-1+\log k) + c_y(\frac{1}{k}-1+\log k)\big], \end{aligned}$$

and

$$M_5 e^{M_4 T} \ge \varepsilon\big[c_x(k-1-\log k)+c_y(k-1-\log k)\big]\wedge\big[c_x(\frac{1}{k}-1+\log k)+c_y(\frac{1}{k}-1+\log k)\big].$$

Taking $k \to \infty$, we obtain a contradictiion, $\infty > M_5 e^{M_4 T} \ge \infty$. □

3. Euler Maruyama (EM) method

In this section, we apply the EM approximate solution to system (1.3), and discuss the boundedness moments of the scheme. First, we introduce notation.

$$\begin{gathered} H_{1x}(\Psi) := H_{1x}(X,Y) := r_1(a)X(t,a), \\ H_{1y}(\Psi) := H_{1y}(X,Y) := r_2(a)Y(t,a), \\ G_{1x}(\Psi) := G_{1x}(X,Y) := \sigma_1 X(t,a) := \sigma_1 X(t), \\ G_{1y}(\Psi) := G_{1y}(X,Y) := \sigma_2 Y(t,a) := \sigma_2 Y(t), \\ H_{2x}(\Psi) := H_{2x}(X,Y) := X(t,a)[-\alpha_{11}(a)X(t,a) + \alpha_{12}(a)Y(t,a)], \\ H_{2y}(\Psi) := H_{2y}(X,Y) := Y(t,a)[\alpha_{21}(a)X(t,a) - \alpha_{22}(a)Y(t,a)], \\ J_{1x}(\Psi) := J_{1x}(X,Y) := h_1(t)X(t,a) := h_1(t)X(t), \\ J_{1y}(\Psi) := J_{1y}(X,Y) := h_2(t)Y(t,a) := h_2(t)Y(t). \end{gathered}$$

The equilibria of (1.3) are: $E_0(0,0)$, $E_1(\frac{r_1(A)}{\alpha_{11}},0)$, $E_2(0,\frac{r_2(A)}{\alpha_{22}})$, and $E_*(x^*,y^*)$, where

$$x^* = \frac{\alpha_{22}(a)r_1(A) + \alpha_{12}(a)r_2(A)}{\alpha_{11}(a)\alpha_{22}(a) - \alpha_{12}(a)\alpha_{21}(a)}, \quad y^* = \frac{\alpha_{11}(a)r_2(A) + \alpha_{21}(a)r_1(A)}{\alpha_{11}(a)\alpha_{22}(a) - \alpha_{12}(a)\alpha_{21}(a)}.$$

Lemma 3.1. *Under Assumptions* (A1)–(A5)*, for each $\psi_1, \psi_2 \in W$, we have*

$$\begin{gathered} |H_{2x}(\psi_1) - H_{2x}(\psi_2)| \le \rho_1|\psi_1 - \psi_2|_1 \le 2^{1/2}\rho_1|\psi_1 - \psi_2|_2, \\ |H_{2y}(\psi_1) - H_{2y}(\psi_2)| \le \rho_2|\psi_1 - \psi_2|_1 \le 2^{1/2}\rho_2|\psi_1 - \psi_2|_2, \\ |H_{1x}(\psi_1) - H_{1x}(\psi_2)| \vee |G_{1x}(\psi_1) - G_{1x}(\psi_2)| \vee |J_{1x}(\psi_1) - J_{1x}(\psi_2)| \le L_1|\psi_1 - \psi_2|, \\ |H_{1y}(\psi_1) - H_{1y}(\psi_2)| \vee |G_{1y}(\psi_1) - G_{1y}(\psi_2)| \vee |J_{1y}(\psi_1) - J_{1y}(\psi_2)| \le L_2|\psi_1 - \psi_2|, \end{gathered}$$

where L_1, L_2 are constants, $\rho_1 = 2x^* - r_1(0) + \alpha_{12}(x^* + y^*)$, $\rho_2 = 2y^* - r_2(0) + \alpha_{21}(x^* + y^*)$, $\psi_i^\top(t,a) = (x_i(t,a), y_i(t,a))$, $(i = 1,2)$.

The proof of this lemma is similar to that in Yang et al. [25], we omit it here.

Corollary 3.2. *From Lemma 3.1, we can conclude that there exist constants K_1 and K_2 such that*

$$\begin{gathered} |H_{1x}(\psi)| \vee |G_{1x}(\psi)| \vee |J_{1x}(\psi)| \le K_1(1 + |\psi|), \\ |H_{1y}(\psi)| \vee |G_{1y}(\psi)| \vee |J_{1y}(\psi)| \le K_2(1 + |\psi|), \end{gathered}$$

and

$$x \cdot H_{2x}(\psi) \le 2^{1/2}\rho_1(1+|\psi|^2), \quad y \cdot H_{2y}(\psi) \le 2^{1/2}\rho_2(1+|\psi|^2),$$

where $\psi(t,a)$ *is the numerical solution of* (1.3) *and* $\psi(t,a) = (x(t,a), y(t,a))^\top$.

From now on, we fix $T > 0$ and the time step $\Delta \in (0,1)$. Let $t_k = k\Delta$ for $k = 0,1,2,\dots,[T/\Delta]$, where $[T/\Delta]$ denotes the integer part of T/Δ. Then, we form the discrete time EM solutions by setting $X(0) = X_a(0)$, $Y(0) = Y_a(0)$ and computing

$$\begin{aligned} x_{k+1} &= x_k + \big\{ -\frac{\partial x_k}{\partial a} + r_1(a)x_k - \alpha_{11}(a)x_k x_k + \alpha_{12}(a)x_k y_k \big\}\Delta + \sigma_1 x_k \Delta w_k \\ &\quad + h_1 x_k \Delta N_k, \\ y_{k+1} &= y_k + \big\{ -\frac{\partial y_k}{\partial a} + r_2(a)y_k + \alpha_{21}(a)x_k y_k - \alpha_{22}(a)y_k y_k \big\}\Delta + \sigma_2 y_k \Delta w_k \\ &\quad + h_2 y_k \Delta N_k, \end{aligned}$$

where $\Delta w_k = w_{t_{k+1}} - w_{t_k}$, $\Delta N_k = N_{t_{k+1}} - N_{t_k}$.

The continuous time process is defined as

$$\begin{aligned} x(t) &= x_0 - \int_0^t \frac{\partial \bar{x}(s)}{\partial a} ds + \int_0^t \{r_1(a)\bar{x}(s) - \alpha_{11}(a)\bar{x}(s)\bar{x}(s) + \alpha_{12}(a)\bar{x}(s)\bar{y}(s)\} ds \\ &\quad + \int_0^t \sigma_1 \bar{x}(s) dw(s) + \int_0^t h_1 \bar{x}(s) dN(s), \\ y(t) &= y_0 - \int_0^t \frac{\partial \bar{y}(s)}{\partial a} ds + \int_0^t \{r_2(a)\bar{y}(s) + \alpha_{21}(a)\bar{x}(s)\bar{y}(s) - \alpha_{22}(a)\bar{y}(s)\bar{y}(s)\} ds \\ &\quad + \int_0^t \sigma_2 \bar{y}(s) dw(s) + \int_0^t h_2 \bar{y}(s) dN(s). \end{aligned} \tag{3.1}$$

where

$$\bar{x}(t) = \sum_{k=0}^{[\frac{T}{\Delta}]} x_k 1_{[t_k,t_{k+1})}(t), \quad \bar{y}(t) = \sum_{k=0}^{[\frac{T}{\Delta}]} y_k 1_{[t_k,t_{k+1})}(t)$$

are the step processes. We can easily see that $\bar{x}(t) = x_k$ and $\bar{y}(t) = y_k$ for $t \in [t_k, t_{k+1})$ when $k = 0,1,2,\dots,[\frac{T}{\Delta}]$. In the following proof, we use $\psi(t) = (x(t), y(t))$ to represent the numerical solution of the system (1.3).

Theorem 3.3. *Under Assumptions* (A1)–(A5)*, there exists a constant* $C > 0$ *such that*

$$\sup_{0 \le t \le T} \mathbb{E}|\psi(t)|^2 \le C,$$

where C *dependents only on* T.

Proof. Applying the Itô formula to the first equation of (3.1), we have

$$\begin{aligned} |x(t)|^2 &= |x_0|^2 + 2\int_0^t \langle -\frac{\partial x(s)}{\partial a} + H_{1x} + H_{2x}, x(s)\rangle ds + \int_0^t ||G_{1x}||^2 ds \\ &\quad + 2\int_0^t (x(s), G_{1x} dw(s)) + 2\int_0^t (x(s), J_{1x} dN(s)) + \lambda_1 \int_0^t |J_{1x}|^2 ds. \end{aligned}$$

Since

$$\big\langle -\frac{\partial x(s)}{\partial a}, x(s) \big\rangle = -\int_0^A x(s) d_a x(s) = \frac{1}{2}\Big[\int_0^A \gamma(s,a) x(s,a) da\Big]^2$$

$$\begin{aligned} &\le \frac{1}{2}\int_0^A \gamma^2(s,a)da\int_0^A x^2(s,a)da \\ &\le \frac{1}{2}A^2\check{\gamma}^2(s,a)|x(s,a)|^2, \end{aligned}$$

by the assumptions we have

$$\begin{aligned} &|x(t)|^2 \le \\ &|x_0|^2 + A^2\check{\gamma}^2(s,a)\int_0^t |x(s)|^2ds + 2\int_0^t (H_{1x}+H_{2x},x(s))ds + 2\int_0^t (x(s),G_{1x}dw(s)) \\ &+ \int_0^t ||G_{1x}||^2ds + 2\int_0^t (J_{1x}dN(s),x(s)) + \lambda_1\int_0^t |J_{1x}|^2ds \\ &\le |x_0|^2 + A^2\check{\gamma}^2(s,a)\int_0^t |x(s)|^2ds + 2\int_0^t |H_{1x}|^2ds + 2\int_0^t |H_{2x}|^2ds + \int_0^t |x(s)|^2ds \\ &\quad + 2\int_0^t (x(s),G_{1x}dw(s)) + \int_0^t ||G_{1x}||^2ds + 2\int_0^t (J_{1x}dN(s),x(s)) + \lambda_1\int_0^t |J_{1x}|^2ds \\ &\le |x_0|^2 + (A^2\cdot\check{\gamma}^2+1)\int_0^t |x(s)|^2ds + [(3+\lambda_1)L_1^2+4\rho_1^2]\int_0^t |\psi(s)|^2ds \\ &\quad + 2\int_0^t (x(s),G_{1x}dw(s)) + 2\int_0^t (x(s),J_{1x}d\tilde{N}(s)). \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}\sup_{0\le s\le t}|x(s)|^2 &\le (A^2\cdot\check{\gamma}^2+1)\int_0^t \mathbb{E}\sup_{0\le s\le t}|x(s)|^2ds \\ &\quad + [(3+\lambda_1)L_1^2+4\rho_1^2]\int_0^t \mathbb{E}\sup_{0\le s\le t}|\psi(s)|^2ds \\ &\quad + \mathbb{E}|x_0|^2 + 2\mathbb{E}\sup_{0\le s\le t}\int_0^s (x(r),G_{1x}dw(r)) \\ &\quad + 2\mathbb{E}\sup_{0\le s\le t}\int_0^s (x(r),J_{1x}d\tilde{N}(r)). \end{aligned} \tag{3.2}$$

By the BDG inequality, we obtain

$$\mathbb{E}\sup_{0\le s\le t}\int_0^s (x(r),G_{1x}dw(r)) \le \frac{1}{8}\mathbb{E}[\sup_{0\le s\le t}|x(s)|^2] + C_1L_1^2\int_0^t \mathbb{E}|\psi(s)|^2ds, \tag{3.3}$$

$$\mathbb{E}\sup_{0\le s\le t}\int_0^s (x(r),J_{1x}d\tilde{N}(r)) \le \frac{1}{8}\mathbb{E}[\sup_{0\le s\le t}|x(s)|^2] + C_2\int_0^t \mathbb{E}|\psi(s)|^2ds, \tag{3.4}$$

where C_1 and C_2 are determined by the BDG inequality.

According to (3.2), (3.3), and (3.4), we have

$$\begin{aligned} \mathbb{E}\sup_{0\le s\le t}|x(s)|^2 &\le [(3+\lambda_1+2C_1+2C_2)L_1^2+4\rho_1^2]\int_0^t \mathbb{E}\sup_{0\le s\le t}|\psi(s)|^2ds \\ &\quad + 2\mathbb{E}|x_0|^2 + (A^2\cdot\check{\gamma}^2+1)\int_0^t \mathbb{E}\sup_{0\le r\le s}|x(r)|^2ds. \end{aligned} \tag{3.5}$$

Similarly,

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}|y(s)|^2 &\le 2\mathbb{E}|y_0|^2+(A^2\check{\beta}^2+2\lambda_1+1)\int_0^t\mathbb{E}\sup_{0\le s\le t}|y(s)|^2ds\\&\quad+[(3+2\lambda_1+2C_1+2C_2)L_2^2+4\rho_2^2]\int_0^t\mathbb{E}\sup_{0\le s\le t}|\psi(s)|^2ds.\end{aligned}\tag{3.6}$$

By (3.5) and (3.6), we have

$$\begin{aligned}&\mathbb{E}\sup_{0\le s\le t}|\psi(t)|^2\\&\le 2\mathbb{E}|\psi_0|^2+[2A^2(\check{\beta}^2\vee\check{\gamma}^2)+2+4\lambda_1]\int_0^t\mathbb{E}\sup_{0\le r\le s}|\psi(r)|^2ds\\&\quad+[(6+2\lambda_1+2C_1+2C_2)(L_1^2\vee L_2^2)+2(\rho_1^2\vee\rho_2^2)]\int_0^t\mathbb{E}\sup_{0\le s\le t}|\psi(s)|^2ds.\end{aligned}$$

By Gronwall's inequality, $\mathbb{E}\sup_{t\in[0,T]}|\psi(t)|^2\le C$. □

Based on Theorem 3.3, we can prove the convergence of $\psi(t)$ and $\bar{\psi}(t)$.

Theorem 3.4. *Under Assumptions* (A1)–(A5), *we have*

$$\mathbb{E}\sup_{0\le t\le T}|\psi(t)-\bar{\psi}(t)|^2\le C\Delta,$$

where C only dependents on T.

Proof. First, we have

$$x(t)-\bar{x}(t)=-\int_{t_k}^t\frac{\partial x(s)}{\partial a}ds+\int_{t_k}^t[H_{1x}+H_{2x}]ds+\int_{t_k}^tG_{1x}dw(s)+\int_{t_k}^tJ_{1x}dN(s),$$

thus

$$\begin{aligned}&|x(t)-\bar{x}(t)|^2\\&\le 4|\int_{t_k}^t\frac{\partial x(s)}{\partial a}ds|^2+4|\int_{t_k}^t[H_{1x}+H_{2x}]ds|^2+4|\int_{t_k}^tG_{1x}dw(s)|^2+4|\int_{t_k}^tJ_{1x}dN(s)|^2\\&\le 4\Delta\int_{t_k}^t|\frac{\partial x(s)}{\partial a}|^2ds+4\Delta\int_{t_k}^t|H_{1x}+H_{2x}|^2ds+4|\int_{t_k}^tG_{1x}dw(s)|^2+8|\lambda_1\int_{t_k}^tJ_{1x}ds|^2\\&\le 4\Delta\int_{t_k}^t|\frac{\partial x(s)}{\partial a}|^2ds+8\Delta[\int_{t_k}^t(|H_{1x}^2|+|H_{2x}^2|)ds]+4|\int_{t_k}^tG_{1x}dw(s)|^2\\&\quad+8|\int_{t_k}^tJ_{1x}d\bar{N}(s)|^2+8|\lambda_1\int_{t_k}^tJ_{1x}ds|^2.\end{aligned}$$

by Lemma 3.1,

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}|x(t)-\bar{x}(t)|^2&\le 5\mathbb{E}\sup_{t\in[0,T]}\max_{k=0,1,\dots,N-1}\Big|\int_{t_k}^tG_{1x}(\psi)dw(s)\Big|^2\\&\quad+8\mathbb{E}\sup_{t\in[0,T]}\max_{k=0,1,\dots,N-1}\Big|\int_{t_k}^tJ_{1x}(\psi)d\tilde{N}(s)\Big|^2\\&\quad+5\Delta\int_{t_k}^t|\frac{\partial x(s)}{\partial a}|^2ds+8TC[L_1^2+2\rho_1+8\lambda_1^2L_1^2]\Delta.\end{aligned}$$

According to the Doob inequality, we obtain

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}|x(t)-\bar{x}(t)|^2 &\le 5\Delta\int_{t_k}^t|\frac{\partial x(s)}{\partial a}|^2ds+8TC[L_1^2+2\rho_1+8\lambda_1^2L_1^2]\Delta\\ &\quad+5\max_{k=0,1,\dots,N-1}\int_{k\Delta}^{(k+1)\Delta}\mathbb{E}|G_{1x}(x,y)|^2ds\\ &\quad+8\lambda_1\max_{k=0,1,\dots,N-1}\int_{k\Delta}^{(k+1)\Delta}\mathbb{E}|J_{1x}(x,y)|^2ds\\ &\le 5\Delta\int_{t_k}^t|\frac{\partial x(s)}{\partial a}|^2ds+8TC[L_1^2+2\rho_1+8\lambda_1^2L_1^2]\Delta\\ &\quad+5L_1^2C\Delta+8\lambda_1L_1^2C\Delta.\end{aligned}\tag{3.7}$$

Since

$$y(t)-\bar{y}(t)=\int_{t_k}^t\frac{\partial y(s)}{\partial a}ds+\int_{t_k}^t[H_{1y}+H_{2y}]ds+\int_{t_k}^tG_{1y}dw(s)+\int_{t_k}^tJ_{1y}dN(s),$$

we have

$$\begin{aligned}&|y(t)-\bar{y}(t)|^2\\ &\le 4|\int_{t_k}^t\frac{\partial y(s)}{\partial a}ds|^2+4|\int_{t_k}^t[H_{1y}+H_{2y}]ds|^2+4|\int_{t_k}^tG_{1y}dw(s)|^2+4|\int_{t_k}^tJ_{1y}dN(s)|^2\\ &\le 4\Delta\int_{t_k}^t|\frac{\partial y(s)}{\partial a}|^2ds+4\Delta\int_{t_k}^t|H_{1y}+H_{2y}|^2ds+4|\int_{t_k}^tG_{1y}dw(s)|^2\\ &\quad+8|\int_{t_k}^tJ_{1y}d\bar{N}(s)|^2+8|\lambda_1\int_{t_k}^tJ_{1y}ds|^2\\ &\le 4\Delta\int_{t_k}^t|\frac{\partial y(s)}{\partial a}|^2ds+8\Delta\frac{\partial y(s)}{\partial a}[|H_{1y}^2+|H_{2y}|^2]ds+4|\int_{t_k}^tG_{1y}dw(s)|^2\\ &\quad+8|\int_{t_k}^tJ_{1y}d\bar{N}(s)|^2+8|\lambda_1J_{1y}ds|^2.\end{aligned}$$

Using Lemma 3.1, we obtain

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}|y(t)-\bar{y}(t)|^2\\ &\le 4\Delta\int_{t_k}^t|\frac{\partial y(s)}{\partial a}|^2ds+8TC\Delta[L_2^2+2\rho_2]+8\lambda_1^2L_2^2TC\Delta\\ &\quad+4\mathbb{E}\sup_{t\in[0,T]}\max_{0\le k\le N-1}\Big|\int_{k\Delta}^tG_{1y}(\psi)dw(s)\Big|^2\\ &\quad+4\mathbb{E}\sup_{t\in[0,T]}\Big|\int_{k\Delta}^tJ_{1y}(\psi)d\tilde{N}(s)\Big|^2\\ &\le 4\Delta\int_{t_k}^t|\frac{\partial y(s)}{\partial a}|^2ds+C[8T(L_2^2+2\rho_2+\lambda_1^2L_2^2)+4L_2^2+8\lambda_1L_2^2]\Delta.\end{aligned}\tag{3.8}$$

By (3.7) and (3.8), we deduce that

$$\mathbb{E}\sup_{t\in[0,T]}|\psi(t)-\bar{\psi}(t)|^2\le 4\Delta\int_{t_k}^t|\frac{\partial\psi(s)}{\partial a}|^2ds+C'\Delta.$$

Then, we obtain the result. □

Theorem 3.4 shows that the numerical solution converges to the step process. Now, we can discuss the convergence relation between the true and numerical solution of the system (1.3).

4. Convergence rates over the time interval $[0,T]$

Theorem 4.1. *Under Assumptions* (A1)–(A5)*, there exists a constant* C *such that*

$$\mathbb{E}|\Psi(t)-\psi(t)|^2 < C\Delta.$$

Proof. For any $t\in[0,T]$, there exists a positive integer k such that $t\in[t_k,t_{k+1}]$, and

$$\begin{aligned}
X(t)-x(t) &= -\int_0^t \frac{\partial X(s)-x(s)}{\partial a}ds \\
&\quad + \int_0^t [H_{1x}(\Psi(s))+H_{2x}(\Psi(s))-H_{1x}(\bar\psi(s))-H_{2x}(\bar\psi(s))]ds \\
&\quad + \int_0^t (G_{1x}(\Psi(s))-G_{1x}(\psi(s)))dw(s) \\
&\quad + \int_0^t (J_{1x}(\Psi(s))-J_{1x}(\bar\psi(s)))dN(s).
\end{aligned}$$

According to the Itô formula, we have

$$\begin{aligned}
& d|X(t)-x(t)|^2 \\
&= -2\langle X(t)-x(t), \frac{\partial (X(t)-x(t))}{\partial a}\rangle dt + 2(X(t)-x(t), H_{1x}(\Psi(s)-H_{1x}(\bar\psi(s))))dt \\
&\quad + 2(X(t)-x(t), H_{2x}(\Psi(s)-H_{2x}(\bar\psi(s))))dt \\
&\quad + 2(X(t)-x(t),(G_{1x}(\Psi(t))-G_{1x}(\bar\psi(t)))dw(t)) \\
&\quad + |G_{1x}(\Psi(t))-G_{1x}(\bar\psi(t))|^2dt + \lambda_1|J_{1x}(\Psi(t))-J_{1x}(\bar\psi(t))|^2dt \\
&\quad + 2(X(t)-x(t),(J_{1x}(\Psi(t))-J_{1x}(\psi(\bar t)))dN(t)) \\
&\le A^2\tilde\gamma^2|X(t)-x(t)|^2dt + L_1|X(t)-x(t)|^2dt + L_1|\Psi(t)-\bar\psi(t)|^2dt \\
&\quad + |X(t)-x(t)|^2dt + 2\rho_1^2|\Psi(t)-\bar\psi(t)|^2dt \\
&\quad + 2(X(t)-x(t),(G_{1x}(\Psi(t))-G_{1x}(\bar\psi(t)))dw(t)) \\
&\quad + L_1^2|\Psi(t)-\psi(t)|^2dt + \lambda_1L_1^2|\Psi(t)-\bar\psi(t)|^2dt \\
&\quad + 2(X(t)-x(t),(J_{1x}(\Psi(t))-J_{1x}(\bar\psi(t)))d\tilde N(t)) + \lambda_1|X(t)-x(t)|^2 \\
&\quad + \lambda_1|J_{1x}(\Psi(t))-J_{1x}(\bar\psi(t))|^2dt.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}|X(t)-x(t)|^2\\
&\le ((A^2)\check{\gamma}^2+L_1+\lambda_1+1)\int_0^t \mathbb{E}\sup_{s\in[0,t]}|X(r)-x(r)|^2ds\\
&\quad+((2\lambda_1+1)L_1^2+2\rho_1^2+1)\mathbb{E}\int_0^t|\Psi(s)-\bar{\psi}(s)|^2ds\\
&\quad+\mathbb{E}\sup_{s\in[0,t]}\int_0^t 2(X(s)-x(s),(G_{1x}(\Psi(s))-G_{1x}(\bar{\psi}(s)))dw(s))\\
&\quad+2\mathbb{E}\sup_{s\in[0,t]}\int_0^t (X(s)-x(s),(J_{1x}(\Psi(s))-J_{1x}(\bar{\psi}(s)))d\tilde{N}(s)).
\end{aligned} \tag{4.1}$$

By the BDG inequality, we have

$$\begin{aligned}
&E\sup_{s\in[0,t]}\int_0^s (X(s)-x(s),(G_{1x}(\Psi(s))-G_{1x}(\bar{\psi}(s)))dw(s))\\
&\le \frac{1}{8}\mathbb{E}[\sup_{0\le s\le t}|X(s)-x(s)|^2]+k_1\int_0^t E|\Psi(s)-\bar{\psi}(s)|^2ds,
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&E\sup_{s\in[0,t]}\int_0^s (X(s)-x(s),(J_{1x}(\Psi(s))-J_{1x}(\bar{\psi}(s)))d\tilde{N}(s))\\
&\le \frac{1}{8}\mathbb{E}[\sup_{0\le s\le t}|X(s)-x(s)|^2]+k_2\int_0^t E|\Psi(s)-\bar{\psi}(s)|^2ds,
\end{aligned} \tag{4.3}$$

where k_1 and k_2 are two positive constants.

Substituting (4.2) and (4.3) into (4.1), we have

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}|X(s)-x(s)|^2 \le (A^2\check{\gamma}^2+L_1+\lambda_1+1)\int_0^t \mathbb{E}\sup_{r\in[0,s]}|X(s)-x(s)|^2ds\\
&+2(L_1^2+2\lambda_1L_1^2+2\rho_1^2+L_1)\int_0^t \mathbb{E}\sup_{r\in[0,s]}|\Psi(s)-\psi(s)|^2ds\\
&+2(L_1^2+2\lambda_1L_1^2+2\rho_1^2+L_1)\mathbb{E}\sup_{r\in[0,s]}|\psi(s)-\bar{\psi}(s)|^2ds\\
&+\frac{1}{4}\mathbb{E}[\sup_{0\le s\le t}|X(s)-x(s)|^2]+2k_1\int_0^t \mathbb{E}|\Psi(s)-\bar{\psi}(s)|^2ds\\
&+\frac{1}{4}\mathbb{E}[\sup_{0\le s\le t}|X(s)-x(s)|^2]+2k_2\int_0^t \mathbb{E}|\Psi(s)-\bar{\psi}(s)|^2ds\\
&=\frac{1}{2}\mathbb{E}[\sup_{0\le s\le t}|X(s)-x(s)|^2]+(A^2\check{\gamma}^2+L_1+\lambda_1+1)\int_0^t \mathbb{E}\sup_{r\in[0,s]}|X(s)-x(s)|^2ds\\
&\quad+2(L_1^2+2\lambda_1L_1^2+2\rho_1^2+L_1+2k_1+2k_2)\int_0^t \mathbb{E}\sup_{r\in[0,s]}|\Psi(s)-\psi(s)|^2ds\\
&\quad+2(L_1^2+2\lambda_1L_1^2+2\rho_1^2+L_1+2k_2)\int_0^t \mathbb{E}\sup_{s\in[0,t]}|\psi(s)-\bar{\psi}(s)|^2ds.
\end{aligned}$$

We apply the same method to $|Y(t)-y(t)|^2$ and complete the proof. □

Corollary 4.2. *Under Assumptions* (A1)–(A5), *the numerical solution of system* (1.3) *will converge to the true solution in the sense that*

$$\lim_{\Delta t \to 0} \mathbb{E}\big(\sup_{0 \le t \le T} |\Psi(t) - \psi(t)|^2 \big) = 0.$$

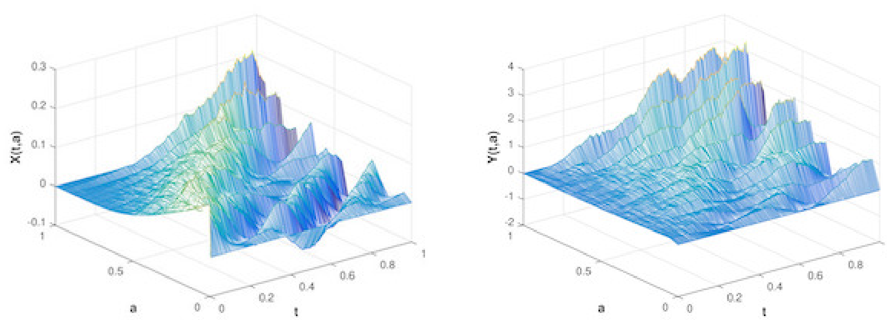
From Theorem 4.1 and Corollary 4.2, we can conclude that the numerical solution $\psi(t)$ and the true solution $\Psi(t)$ are close to each other, which shows that the numerical algorithm constructed in this paper is effective.

5. Numerical experiments

To illustrate the theorems in this paper, we consider the stochastic age-structured cooperative LV system for numerical simulations,

$$\begin{gathered}
d_t X = -\frac{\partial X}{\partial a} dt + X[-\frac{1}{(1-a)^2} X + \cos^2 a Y - 2] dt - \frac{1}{2} dw(t) + \frac{1}{3} x(t^-) dN(t), \\
d_t Y = -\frac{\partial Y}{\partial a} dt + Y[\sin^2 a X - e^{\frac{1}{a}} Y + 3a - 2] dt - \frac{1}{2} dw(t) + \frac{1}{3} y(t^-) dN(t), \\
X(t,0) = \int_0^1 \frac{1}{1-a} X(t,a) da, t \in [0,1], \\
Y(t,0) = \int_0^1 \frac{1}{1-a} Y(t,a) da, t \in [0,1], \\
X(0,a) = e^{\frac{-2}{1-a}}, Y(0,a) = e^{\frac{-2}{1-a}}, a \in [0,1],
\end{gathered} \tag{5.1}$$

where $\alpha_{11}(a) = \frac{1}{(1-a)^2}$, $\alpha_{12}(a) = \cos^2 a$, $\alpha_{21}(a) = \sin^2 a$, $\alpha_{22}(a) = e^{-\frac{1}{a}}$, and $\gamma(t,a) = \beta(t,a) = \frac{1}{1-a}$. In the first two equations, we have $(t,a) \in Q$ We simulate the EM numerical approximate solution of the two species respectively (see Figure 1).

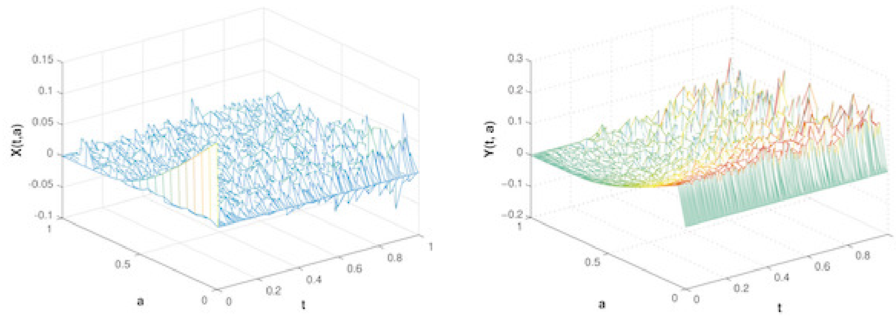


Approximate solution $X(t,a)$ Approximate solution $Y(t,a)$

Figure 1. EM numerical approximate solutions of system (5.1).

Next, we simulate the true solution of the system (5.1) by using the method in literature [10]. The Figure 2 shows that the true solutions of $X(t,a)$ and $Y(t,a)$ with perturbation, respectively.

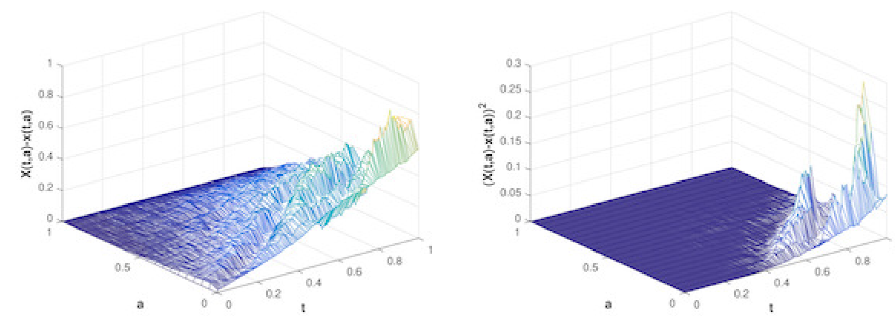
Figure 3 and Figure 4 are the error between the true solution and the numerical approximate solution of system (5.1). We can observe that the maximum value of



True solution $X(t, a)$ True solution $Y(t, a)$

FIGURE 2. True solutions with the perturbation of system (5.1).

the squared error of $X(t, a)$ and $Y(t, a)$ are below 0.3 and 0.15, respectively, which means that the numerical approximate solution convergent to the true solution effectively.



Error beqtween $X(t, a)$ and $x(t, a)$ Error square beqtween $X(t, a)$ and $x(t, a)$

FIGURE 3. Simulation error in species $X(t, a)$ of system (5.1).

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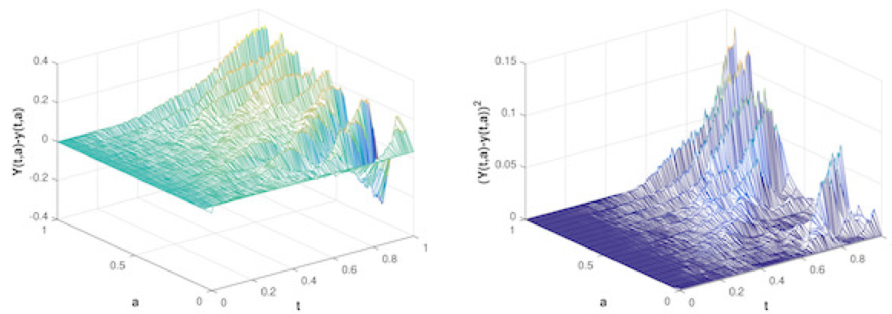

Error between $Y(t,a)$ and $y(t,a)$ Error square between $Y(t,a)$ and $y(t,a)$

FIGURE 4. Simulation error in species $Y(t,a)$ of system (5.1).

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