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MULTIPLICITY RESULTS OF NONLOCAL SINGULAR PDES WITH CRITICAL SOBOLEV-HARDY EXPONENT

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Abstract. In this article we study a nonlocal equation involving singular and critical Hardy-Sobolev non-linearities,

$$\begin{gathered}(-\Delta_p)^s u - \mu\frac{|u|^{p-2}u}{|x|^{sp}} = \lambda u^{-\alpha} + \frac{|u|^{p_s^*(t)-2}u}{|x|^t}, \quad \text{in } \Omega,\\ u>0, \quad \text{in } \Omega,\\ u=0, \quad \text{in } \mathbb{R}^N\setminus\Omega,\end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary and $(-\Delta_p)^s$ is the fractional p-Laplacian operator. We combine some variational techniques with a perturbation method to show the existence of multiple solutions.

1. Introduction

In this work, we consider the singular critical nonlocal problem with critical Hardy-Sobolev non-linearities,

$$\begin{gathered}(-\Delta_p)^s u - \mu\frac{|u|^{p-2}u}{|x|^{sp}} = \lambda u^{-\alpha} + \frac{|u|^{p_s^*(t)-2}u}{|x|^t}, \quad \text{in } \Omega,\\ u>0, \quad \text{in } \Omega,\\ u=0, \quad \text{in } \mathbb{R}^N\setminus\Omega,\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $0 < s < 1$, λ is a positive parameter, $0 \le \mu < \mu_0$ is the sharp constant of the fractional Hardy Sobolev in $\mathbb{R}^N$, $0 < t < sp < N$, $0 < \alpha < 1 < p < p_s^*(t)$ where $p_s^* = \frac{Np}{N-sp}$ and $p_s^*(t) = \frac{p(N-t)}{N-sp}$ are the fractional critical Sobolev and Hardy Sobolev exponents respectively. The fractional p-laplacian non-linear nonlocal operator defined for $s \in (0,1)$ is defined by

$$(-\Delta_p)^s u(x) = 2\lim_{\epsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\epsilon} |\frac{u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}dy, \quad \text{for all } x\in\mathbb{R}^N.$$

Problems of the type (1.1) play an important role in many field of sciences such as: optimization, electromagnetism, astronomy, water waves, fluid dynamics,

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probability theory, phase transitions. etc. For further details on applications, we refer the readers to [2, 3, 22, 23] and references therein.

Before stating our results, let us briefly recall some of the literature concerning problems with Sobolev and Hardy nonlinearities. In the previous years, the study of fractional elliptic equations involving singular nonlinearity attracted lot of attention; see for example [10, 11, 12, 13, 15, 16, 19] and the references therein. The following problem has been study in several works,

$$\begin{gathered}(-\Delta_p)^s u = \lambda a(x) u^{-\alpha} + M f(x,u), \quad \text{in } \Omega,\\ u > 0, \quad \text{in } \Omega,\\ u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,\end{gathered} \tag{1.2}$$

where $N > ps$, $M \geq 0$, $a : \Omega \longrightarrow \mathbb{R}$ is a nonnegative bounded function. When $M = 0$ and $p = 2$ (the purely singular problem), Fang [5] prove the existence and uniqueness of a solution in $C^{2,\alpha}(\Omega)$ for $0 < \alpha < 1$ in (1.2). In [17], the author prove a multiplicity result for (1.2) by converting the nonlocal problem to a local problem. In [7, 14] using a Nehari method combine with fibering map, the authors established the existence and multiplicity of weak solutions to (1.2). In that sense the current problem (1.1) is new, not only because of a nonlocal operator and a singularity, but also because of the Hardy-Sobolev nonlinearities. In a nutshell, we will prove the existence of multiple solutions to (1.1) for sufficiently small λ, μ. Now, we state the main result of this paper.

Theorem 1.1. *There exist λ^* such that for every $\lambda \in (0, \lambda^*)$ problem* (1.1) *has at least two positives solutions u_λ with $E_{\lambda,\mu}(u_\lambda) < 0$, and v_λ with $E_{\lambda,\mu}(v_\lambda) > 0$.*

The outline of this work is as follows. In Section 2 we present notation and basic results. In Section 3, we prove the existence of a solution which is a local minimizer in X_0 of the functional energy $E_{\lambda,\mu}$ associated with (1.1). Section 4 is devoted to study the approximated problem. While, multiplicity of solutions will be presented in Section 5.

2. Functional framework and main results

This section is devoted to recalling a few definitions, notation, and function spaces which will be used later. Let $\Omega \subset \mathbb{R}^N$ and $Q = \mathbb{R}^{2N} \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega))$, then the space $(X, \|\cdot\|_X)$ is defined by

$$X = \big\{u : \mathbb{R}^{\mathbb{N}} \to \mathbb{R} \text{ is measurable}, u|_\Omega \in L^p(\Omega) \text{ and } \frac{|u(x) - u(y)|}{|x-y|^{\frac{N+ps}{p}}} \in L^p(Q)\big\}$$

equipped with the Gagliardo norm

$$\|u\|_X = \|u\|_p + \Big(\int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy\Big)^{1/p}.$$

Here $\|u\|_p$ refers to the L^p-norm of u. We further define the space

$$X_0 = \big\{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\big\}$$

equipped with the norm

$$\|u\| = \Big(\int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy - \mu \int_\Omega \frac{|u|^p}{|x|^{sp}}\, dx\Big)^{1/p}.$$

The best Sobolev constant is defined as

$$S = \inf_{u\in X_0\setminus\{0\}} \frac{\left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \mu\int_\Omega \frac{|u|^p}{|x|^{sp}}dx\right)}{\left(\int_\Omega \frac{|u|^{p_s^*(t)}}{|x|^t}dx\right)^{p/p_s^*(t)}}. \tag{2.1}$$

We now state the following definitions associated with problem (1.1).

Definition 2.1. We say that $u \in X_0$ is a weak solution to (1.1), if (i) $u > 0$, $u^{-\gamma}\phi \in L^1(\Omega)$, and (ii)

$$\begin{aligned}&\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}}\,dx\,dy - \mu\int_\Omega \frac{|u|^{p-2}u}{|x|^{sp}}\phi\,dx\\ &-\int_\Omega \frac{\lambda}{(u^+)^{1-\alpha}}\phi + \int_\Omega \frac{|u|^{p_s^*(t)-2}u}{|x|^t}\phi\,dx = 0\end{aligned}$$

for each $\phi \in X_0$. Here, $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$.

Associated with (1.1) we have the functional energy $E_{\lambda,\mu}\colon X_0 \to \mathbb{R}$ defined as

$$\begin{aligned}E_{\lambda,\mu}(u) = \frac{1}{p}\Big(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \mu\int_\Omega \frac{|u|^p}{|x|^{sp}}dx\Big) - \frac{\lambda}{1-\alpha}\int_\Omega (u^+)^{1-\alpha}dx\\ - \frac{1}{p_s^*(t)}\int_\Omega \frac{|u|^{p_s^*(t)}}{|x|^t}dx.\end{aligned} \tag{2.2}$$

Obviously, every critical point of $E_{\lambda,\mu}$ is a weak solution of the problem (1.1). We now list the embedding results pertaining to the function space X_0 [20, 21].

Lemma 2.2. *The following embedding results holds for the space X_0.*

1. *If Ω has a Lipschitz boundary and $N > ps$, then the embedding $X_0 \hookrightarrow L^q(\Omega)$ for $q \in [1, p_s^*]$ is continuous and is compact for $q \in [1, p_s^*)$, where $p_s^* = \frac{Np}{N-ps}$.*
2. *If Ω has a Lipschitz boundary and $N = ps$, then the embedding $X_0 \hookrightarrow L^q(\Omega)$ for $q \in [1,\infty)$ is both continuous and compact.*
3. *If Ω has a Lipschitz boundary and $N < ps$, then the embedding $X_0 \hookrightarrow C^{0,\beta}(\overline{\Omega})$ where $\beta = \frac{sp-N}{p}$ is both continuous and compact.*

Let us define

$$|u|_{p_s^*(t)} = \Big(\int_\Omega \frac{|u|^{p_s^*(t)}}{|x|^t}dx\Big)^{1/p_s^*(t)}.$$

We now recall the fractional Hardy-Sobolev inequality.

Lemma 2.3.

1. *Fractional Hardy inequality* [6]*: For all $u \in W_0^{s,p}(\mathbb{R}^N)$, we have*
$$\mu_0\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{sp}}\,dx \le \int_{\mathbb{R}^N} |(-\Delta)^{s/p}u|^p\,dx. \tag{2.3}$$
2. *Fractional Hardy Sobolev inequality* [9]*: Assume $0 \le \alpha \le sp < N$. Then, there exist positive constants c and C, such that for all $u \in W_0^{s,p}(\mathbb{R}^N)$,*
$$\Big(\int_{\mathbb{R}^N} \frac{|u|^{p_s^*(t)}}{|x|^t}dx\Big)^{p/p_s^*(t)} \le c\int_{\mathbb{R}^N} |(-\Delta)^{s/p}u|^p\,dx. \tag{2.4}$$

Moreover, if $mu < mu_0$, then

$$\Big(\int_{\mathbb{R}^N} \frac{|u|^{p_s^*(t)}}{|x|^t} dx\Big)^{p/p_s^*(t)} \le C\Big(\int_{\mathbb{R}^N} |(-\Delta)^{s/p} u|^p \, dx - \mu \int_{R^N} \frac{|u|^p}{|x|^{sp}}\Big) dx, \tag{2.5}$$

for all $u \in W_0^{s,p}(\mathbb{R}^N)$.

The following embedding results have been proved in [4].

Lemma 2.4. (1) *The embedding $W_0^{s,p}(\Omega) \to L^q(\Omega, \frac{dx}{|x|^t})$ is continuous for $q \in [1, p_s^*(t)]$ and compact for $q \in [1, p_s^*(t))$.*

(2) *For $p > 1$, $W_0^{s,p}(\Omega)$ and $D^{s,p}(\mathbb{R}^N)$ are separable reflexive Banach space w.r.t the norm $[\cdot]_{s,p}$.*

3. Existence of a weak solution to (1.1)

Besides proving the existence of a weak solution, we show that this solution is a local minimizer of the associated functional $E_{\lambda,\mu}$. Our first result is the following.

Lemma 3.1. *There exists $\lambda_0 > 0$, $R_0 > 0$, and $\rho_0 > 0$ such that $E_{\lambda,\mu}(u) \ge \rho_0 > 0$ for all $\|u\| = R_0$. Also, we have $C = \inf_{u \in B_{R_0}} E_{\lambda,\mu}(u) < 0$.*

Proof. Note that using Hölder's inequality combined with the fractional Sobolev-Hardy inequality, we have

$$\begin{aligned} E_{\lambda,\mu}(u) &= \frac{1}{p}\|u\|_{X_0}^p - \frac{\lambda}{1-\alpha}\|u\|^{1-\alpha} - \frac{C_1}{p_s^*(t)}\|u\|^{p_s^*(t)} \\ &= \|u\|^{1-\alpha}\Big(\frac{1}{p}\|u\|^{p+1+\alpha} - \frac{\lambda}{1-\alpha}C_0 - \frac{C_1}{p_s^*(t)}\|u\|^{p_s^*(t)+\alpha-1}\Big), \end{aligned}$$

where C_0, C_1 are two constants. Put

$$f(x) = \frac{1}{p}x^{p+1+\alpha} - \frac{C_1}{p_s^*(t)}x^{p_s^*(t)+\alpha-1} - \frac{\lambda}{1-\alpha}C_0.$$

Since $1 - \alpha < 1 < p < p_s^*(t)$, we find the existence of a constant

$$R = \Big(\frac{p_s^*(t)(p+\alpha-1)}{\lambda p C_1(p_s^*(t)+\alpha-1}\Big)^{1/(p_s^*(t)-p} > 0$$

such that $f(R) = \max_{x>0} f(x) > 0$. Choosing $\lambda_0 = \frac{(1-\alpha)f(R)}{C_0}$, we deduce the existence of a constant $\delta_0 > 0$ satisfying $E_{\lambda,\mu} \ge \delta_0 > 0$ for all $\lambda \in (0, \lambda_0)$. The proof of Lemma 3.1 is now completed. □

Lemma 3.2. *Problem (1.1) admits a positive solution $u_\lambda \in X_0$ with $E_{\lambda,\mu} < 0$ for all $\lambda \in (0, \lambda_0)$, where λ_0 is defined in Lemma 3.1.*

Proof. Let λ_0, R_0 and ρ_0 be as in Lemma 3.1. Since $1 - \alpha < 1 < p < p_s^*(t)$, noting that for all $\varphi \in X_0, \varphi \ge 0$, $\varphi \not\equiv 0$ and $r > 1$, one has

$$E_{\lambda,\mu}(r\varphi) = \frac{r^p}{p}\|u\|^p - \frac{\lambda r^{1-\alpha}}{1-\alpha}\int_\Omega (\varphi^+)^{1-\alpha}\, dx - \frac{r^{p_s^*(t)}}{p_s^*(t)}\int_\Omega \frac{|\varphi|^{p_s^*(t)}}{|x|^t}\, dx < 0. \tag{3.1}$$

So, for $\|u\|$ sufficiently small, we conclude that

$$C = \inf_{u \in B_{R_0}} E_{\lambda,\mu}(u) < 0. \tag{3.2}$$

Hence, by the definition of the infimum (3.2), we guarantee the existence of a minimizing sequence $\{u_n\}$ for C. Therefore, using the reflexivity of X_0, there exists a subsequence, still denoted by u_n, there exists u_λ, such that

$$\begin{gathered} u_n \to u_\lambda \quad \text{weakly in } X_0, \\ u_n \to u_\lambda \quad \text{strongly in } L^k(\Omega, \frac{dx}{|x|^t}) \text{ for } 1 \le k < p_s^*(t), \\ u_n \to u_\lambda \quad \text{pointwise a.e. in } \Omega. \end{gathered} \tag{3.3}$$

Thus, from the Brezis-Lieb Lemma [4], one has

$$|u_n|_{p_s^*(t)}^{p_s^*(t)} = |u_\lambda|_{p_s^*(t)}^{p_s^*(t)} + |u_n - u_\lambda|_{p_s^*(t)}^{p_s^*(t)} + o(1), \tag{3.4}$$

$$\|u_n\|^p = \|u_\lambda\|^p + \|u_n - u_\lambda\|^p + o(1). \tag{3.5}$$

On the other hand, using Hölder inequality and letting $n \to \infty$, we obtain

$$\begin{aligned} \int_\Omega u_n^{1-\alpha} \mathrm{d}x &\le \int_\Omega u_\lambda^{1-\alpha} \mathrm{d}x + \int_\Omega \mid u_n - u_\lambda \mid^{1-\alpha} \mathrm{d}x \\ &\le \int_\Omega u_\lambda^{1-\alpha} \mathrm{d}x + C \parallel u_n - u_\lambda \parallel_p^{1-\alpha} \\ &= \int_\Omega u_\lambda^{1-\alpha} \mathrm{d}x + o(1). \end{aligned}$$

Similarly

$$\begin{aligned} \int_\Omega u_\lambda^{1-\alpha} \mathrm{d}x &\le \int_\Omega u_n^{1-\alpha} \mathrm{d}x + \int_\Omega \mid u_n - u_\lambda \mid^{1-\alpha} \mathrm{d}x \\ &\le \int_\Omega u_n^{1-\alpha} \mathrm{d}x + C \parallel u_n - u_\lambda \parallel_p^{1-\alpha} \\ &= \int_\Omega u_n^{1-\alpha} \mathrm{d}x + o(1). \end{aligned}$$

Thus,

$$\int_\Omega u_n^{1-\alpha}\, dx = \int_\Omega u_\lambda^{1-\alpha}\, dx + o(1). \tag{3.6}$$

Hence, using (3.4), (3.5), and (3.6), we conclude that

$$E_{\lambda,\mu}(u_n) = E_{\lambda,\mu}(u_\lambda) + \frac{1}{p}\|u_n - u_\lambda\|^p - \frac{1}{p_s^*(t)}|u_n - u_\lambda|_{p_s^*(t)}^{p_s^*(t)} + o(1). \tag{3.7}$$

Moreover, from (3.4)-(3.5) and for n sufficiently large u, $u_n - u_\lambda \in B_r$ and $\frac{1}{p}\|u_n - u_\lambda\|^p - \frac{1}{p_s^*(t)}|u_n - u_\lambda|_{p_s^*(t)}^{p_s^*(t)} \ge o(1)$. Therefore, we deduce that

$$\begin{gathered} \frac{1}{p}\|u_n - u_\lambda\|^p - \frac{1}{p_s^*(t)}|u_n - u_\lambda|_{p_s^*(t)}^{p_s^*(t)} > 0 \text{ on } \partial B_r, \\ \frac{1}{p}\|u_n - u_\lambda\|^p - \frac{1}{p_s^*(t)}|u_n - u_\lambda|_{p_s^*(t)}^{p_s^*(t)} \ge 0 \text{ in } B_r, \end{gathered} \tag{3.8}$$

for $r > 0$ sufficiently small. Hence, we conclude that

$$\frac{1}{p}\|u_n - u_\lambda\|^p - \frac{1}{p_s^*(t)}|u_n - u_\lambda|_{p_s^*(t)}^{p_s^*(t)} \ge o(1). \tag{3.9}$$

Then, using (3.3) and (3.9), we obtain

$$C = E_{\lambda,\mu}(u_n) + \circ(1)$$

$$
\begin{aligned}
&= \frac{1}{p}\|u_n\|^p - \frac{\lambda}{1-\alpha}\int_\Omega (u_n^+)^{1-\alpha}\,dx - \frac{1}{p_s^*(t)}\int_\Omega \frac{(u_n^+)^{p_s^*(t)}}{|x|^t}\,dx + \circ(1) \\
&\geq E_{\lambda,\mu}(u_\lambda) + \frac{1}{p}\|u_n - u_\lambda\|^p - \frac{1}{p_s^*(t)}\int_\Omega \frac{((u_n-u_\lambda)^+)^{p_s^*(t)}}{|x|^t}\,dx \\
&\quad - \frac{\lambda}{1-\alpha}\int_\Omega ((u_n-u_\lambda)^+)^{1-\alpha}\,dx + \circ(1) \\
&\geq E_{\lambda,\mu}(u_\lambda) + \circ(1).
\end{aligned}
$$

Therefore, $C \geq E_{\lambda,\mu}(u_\lambda)$ as $n \to \infty$. Since B_{R_0} is convex and closed, we conculde that $u_\lambda \in B_{R_0}$. Thus, from Eq. (3.2), we deduce that $E_{\lambda,\mu}(u_\lambda) = C < 0$ and we have $u_\lambda \not\equiv 0$ which is a minimizer of $E_{\lambda,\mu}$ over X_0.

Now, we prove that u_λ is a weak solution to(1.1) and $u_\lambda > 0$. Let $\phi \in X_0$ $\phi \geq 0$ and $r > 0$ small enough such that $(u_\lambda + r\phi) \in B_{R_0}$. Since, u_λ is a local minimizer of $E_{\lambda,\mu}$, we have

$$
\begin{aligned}
0 &\leq E_{\lambda,\mu}(u_\lambda + r\phi) - E_{\lambda,\mu}(u_\lambda) \\
&= \frac{1}{p}\Big(\|u_\lambda + r\phi\|^p - \|u_\lambda\|^p\Big) - \frac{\lambda}{1-\alpha}\int_\Omega \Big[((u_\lambda + r\phi)^+)^{1-\alpha} - (u_\lambda^+)^{1-\alpha}\Big]\,dx \\
&\quad - \frac{1}{p_s^*(t)}\int_\Omega \Big[\frac{((u_\lambda + r\phi)^+)^{p_s^*(t)}}{|x|^t} - \frac{(u_\lambda^+)^{p_s^*(t)}}{|x|^t}\Big]\,dx \\
&\leq \frac{1}{p}\Big(\|u_\lambda + r\phi\|^p - \|u_\lambda\|^p\Big).
\end{aligned} \tag{3.10}
$$

Now, we divide (3.10) by $t > 0$ and we take the limit as $r \to 0^+$, we obtain

$$
\begin{aligned}
&\liminf_{r\to 0^+} \frac{\lambda}{1-\alpha}\int_\Omega \frac{((u_\lambda + r\phi)^+)^{1-\alpha} - (u_\lambda^+)^{1-\alpha}}{r}\,dx \\
&\leq \int_{\mathcal{Q}} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad - \mu\int_{\mathcal{Q}} \frac{|u|^{p-2}u\phi}{|x|^{sp}} dx - \int_\Omega \frac{(u_\lambda^+)^{p_s^*(t)-1}\phi}{|x|^t}\,dx.
\end{aligned} \tag{3.11}
$$

Therefore,

$$
\frac{\lambda}{1-\alpha}\int_\Omega \frac{((u_\lambda + r\phi)^+)^{1-\alpha} - (u_\lambda^+)^{1-\alpha}}{r} = ((u_\lambda + \xi r\phi)^+)^{-\alpha}\phi \quad \text{a.e. in } \Omega, \tag{3.12}
$$

with $\xi \in (0,1)$ and $((u_\lambda + \xi r\phi)^+)^{-\alpha}\phi \to (u_\lambda^+)^{-\alpha}\phi$ a.e. in Ω, as $r \to 0^+$. Using, Fatou's Lemma,

$$
\lambda\int_\Omega (u_\lambda^+)^{-\alpha}\phi\,dx \leq \frac{\lambda}{1-\alpha}\liminf_{r\to 0^+}\int_\Omega \frac{((u_\lambda + r\phi)^+)^{1-\alpha} - (u_\lambda^+)^{1-\alpha}}{r}\,dx. \tag{3.13}
$$

Consequently, using (3.11) and (3.13), ones has

$$
\begin{aligned}
&\int_{\mathcal{Q}} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&- \mu\int_{\mathcal{Q}} \frac{|u|^{p-2}u\phi}{|x|^{sp}}\,dx - \lambda\int_\Omega (u_\lambda^+)^{-\alpha}\phi\,dx - \int_\Omega \frac{(u_\lambda^+)^{p_s^*(t)-1}\phi}{|x|^t}\,dx \geq 0
\end{aligned} \tag{3.14}
$$

for $\phi \geq 0$ a.e. in $\mathbb{R}^N$. Since $E_{\lambda,\mu} < 0$ and using Lemma 3.1, we derive that $u_\lambda \in B_{R_0}$. Therefore, there exists $\delta \in (0,1)$ satisfying $(1+r)u_\lambda \in \overline{B}_{R_0}$ $(|r| \leq \delta)$. Then, define the functional $J_{\lambda,\mu}$ by

$$J_{\lambda,\mu}(r) = E_{\lambda,\mu}((1+r)u_\lambda).$$

Hence, the functional $J_{\lambda,\mu}$ attains its minimum at $r = 0$, since u_λ is a local minimizer of $J_{\lambda,\mu}$ in $\overline{B}_{R_0}$. Furthermore,

$$J'_{\lambda,\mu}(r)\backslash_{r=0} = \|u_\lambda\|^p - \lambda\int_\Omega (u_\lambda^+)^{1-\alpha}dx - \int_\Omega \frac{(u_\lambda^+)^{p_s^*(t)}}{|x|^t}\,dx = 0. \tag{3.15}$$

Now, we define $\Psi \in X_0$ by $\Psi := (u_\lambda^+ + \epsilon\phi)^+$, where $(u_\lambda^+ + \epsilon\phi)^+ = \max\{u_\lambda^+ + \epsilon\phi, 0\}$. Let $\Omega_\epsilon = \{u_\lambda^+ + \epsilon\phi \le 0\}$ and $\Omega^\epsilon = \{u_\lambda^+ + \epsilon\phi > 0\}$. Replacing ϕ with Ψ in (3.14) and combining with (3.15), we obtain

$$\begin{aligned}
0 &\le \int_{\mathcal{Q}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(\Psi(x)-\Psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad - \int_\Omega \mu\frac{|u|^{p-2}u\Psi}{|x|^{sp}}dx - \lambda\int_\Omega (u_\lambda^+)^{-\alpha}\Psi\,dx - \int_\Omega \frac{(u_\lambda^+)^{p_s^*-1(t)}\Psi}{|x|^t}\,dx \\
&= \int_{\{(x,y)\in\Omega^\epsilon\times\Omega^\epsilon\}} \Big(|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))((u_\lambda^+ + \epsilon\phi)(x) \\
&\quad - (u_\lambda^+ + \epsilon\phi)(y))\Big)/|x-y|^{N+sp}\,dx\,dy - \mu\int_\Omega \frac{|u_\lambda|^{p-2}u_\lambda}{|x|^{sp}}(u_\lambda^+ + \epsilon\phi)dx \\
&\quad - \int_{\Omega^\epsilon} \Big(\lambda(u_\lambda^+)^{-\alpha}(u_\lambda^+ + \epsilon\phi) + \frac{(u_\lambda^+)^{p_s^*-1(t)}(u_\lambda^+ + \epsilon\phi)}{|x|^t}\Big)\,dx \\
&= \Big(\int_{\mathcal{Q}} - \int_{\Omega^\epsilon\times\Omega^\epsilon}\Big) \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(\Psi(x)-\Psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad - \Big(\int_\Omega - \int_{\Omega_\epsilon}\Big)\Big[\mu\frac{|u_\lambda|^{p-2}u_\lambda}{|x|^{sp}}(u_\lambda^+ + \epsilon\phi) + \lambda(u_\lambda^+)^{-\alpha}(u_\lambda^+ + \epsilon\phi) \\
&\quad + \frac{(u_\lambda^+)^{p_s^*(t)-1}(u_\lambda^+ + \epsilon\phi)}{|x|^t}\Big]\,dx \\
&\le \int_{\mathcal{Q}} \frac{|u_\lambda(x)-u_\lambda(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \mu\int_\Omega \frac{|u_\lambda|^p}{|x|^{sp}} - \int_\Omega \Big[\lambda(u_\lambda^+)^{1-\gamma} + \frac{(u_\lambda^+)^{p_s^*(t)}}{|x|^t}\Big]\,dx \\
&\quad + \epsilon\Big(\int_{\mathcal{Q}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad - \mu\int_\Omega \frac{|u_\lambda|^{p-2}u_\lambda}{|x|^{sp}}\phi\,dx\Big) - \epsilon\int_\Omega \Big[\lambda(u_\lambda^+)^{-\alpha}\phi + \frac{(u_\lambda^+)^{p_s^*(t)-1}\phi}{|x|^t}\Big]\,dx \\
&\quad - \int_{\{(x,y)\in\Omega_\epsilon\times\Omega_\epsilon\}} \Big(|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))((u_\lambda^+ + \epsilon\phi)(x) \\
&\quad - (u_\lambda^+ + \epsilon\phi)(y))\Big)/|x-y|^{N+sp}\,dx\,dy \\
&\quad + \mu\int_\Omega \frac{|u_\lambda|^{p-2}u_\lambda}{|x|^{sp}}(u_\lambda^+ + \epsilon\phi)dx \\
&\quad + \int_{\Omega_\epsilon} \Big[\lambda(u_\lambda^+)^{-\alpha}(u_\lambda^+ + \epsilon\phi) + \frac{(u_\lambda^+)^{p_s^*(t)-1}(u_\lambda^+ + \epsilon\phi)}{|x|^t}\Big]\,dx \\
&\le \epsilon\Big(\int_{\mathcal{Q}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\,dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&-\mu \int_\Omega \frac{|u_\lambda|^{p-2}u_\lambda}{|x|^{sp}}\phi\,dx\Big) - \epsilon \int_\Omega \Big[\lambda(u_\lambda^+)^{-\alpha}\phi + \frac{(u_\lambda^+)^{p_s^*(t)-1}\phi}{|x|^t}\Big]\,dx \\
&-\epsilon \int_{\{(x,y)\in\Omega_\epsilon\times\Omega_\epsilon\}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\,dx\,dy.
\end{aligned}
$$

Since the measure of the domain of integration Ω_ϵ tends to zero as $\epsilon \to 0^+$, we deduce as $\epsilon \to 0^+$, that

$$
\int_{\{(x,y)\in\Omega_\epsilon\times\Omega_\epsilon\}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\,dx\,dy \to 0.
$$

Dividing by ϵ and letting $\epsilon \to 0^+$, we obtain

$$
\begin{aligned}
&\int_{\mathcal{Q}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\,dx\,dy - \mu\int_\Omega \frac{|u_\lambda|^{p-2}u_\lambda}{|x|^{sp}}\phi\,dx \\
&-\int_\Omega \Big[\lambda(u_\lambda^+)^{-\gamma}\phi + \frac{(u_\lambda^+)^{p_s^*(t)-1}\phi}{|x|^t}\Big]\,dx \geq 0.
\end{aligned}
$$

Since ϕ is an arbitrary test function, we obtain the equality if we change ϕ by $-\phi$. Hence, u_λ is a weak solution to the problem (1.1). Finally, putting $\phi = u_\lambda^-$ in (2.1), we obtain that u_λ is nonnegative. Moreover, since $I_\lambda = C < 0$, then $u_\lambda \not\equiv 0$. Therefore, using the maximum principle, we conclude that u_λ is a positive solution to (1.1). The proof is complete. □

4. Existence of a solution of the perturbed problem

Note that $E_{\lambda,\mu}$ is not differentiable because of the singular term in it. Hence, the classical approach of min-max methods fails. Therefore, to show the existence of a second solution to (P), we introduce the following auxillary perturbed problem

$$
\begin{gathered}
(-\Delta_p)^s u - \mu\frac{|u|p-2u}{|x|^{sp}} = \frac{\lambda}{(u^+ + \frac{1}{n})^\alpha} + \frac{(u^+)^{p_s^*(t)-2}u^+}{|x|^t}, \quad \text{in } \Omega, \\
u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{gathered} \tag{4.1}
$$

The functional energy $E_{n,\lambda,\mu} : X_0 \to \mathbb{R}$ associated with (4.1), is defined by

$$
\begin{aligned}
&E_{n,\lambda,\mu}(u) \\
&= \frac{1}{p}\|u\|^p - \frac{\lambda}{1-\gamma}\int_\Omega \Big((u^+ + \frac{1}{n})^{1-\gamma} - (\frac{1}{n})^{1-\gamma}\Big)\,dx - \frac{1}{p_s^*(t)}\int_\Omega \frac{(u^+)^{p_s^*(t)}}{|x|^t}\,dx.
\end{aligned}
$$

From the definition of the functional energy $E_{n,\lambda,\mu}$, it is easy to see that $E_{n,\lambda,\mu}$is Fréchet differentiable, for all $\phi \in X_0$, ones has

$$
\begin{aligned}
&\langle E'_{n,\lambda,\mu}(u), \phi\rangle \\
&= \int_{\mathcal{Q}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad -\mu\int_\Omega \frac{|u|^{p-2}u}{|x|^{sp}}\phi\,dx - \lambda\int_\Omega \frac{\phi}{(u^+ + \frac{1}{n})^{1-\alpha}}\,dx - \int_\Omega \frac{(u^+)^{p_s^*(t)-2}u\phi}{|x|^\alpha}\,dx.
\end{aligned} \tag{4.2}
$$

It is easy to see that any critical points of the functional energy $E_{n,\lambda,\mu}$, are exactly the solutions of (4.1).

Lemma 4.1. *Let $R_0 \in (0, 1]$, λ_0 and ρ_0 be the constants given by Lemma 3.1. Then for any $\lambda \in (0, \lambda_0]$, $E_{n,\lambda,\mu}$ satisfies the following properties:*

(1) *$E_{n,\lambda,\mu}(u) \geq \rho_0$, for all $u \in X_0$ with $\|u\| \leq R_0$.*
(2) *There exists $v_\lambda \in X_0$, with $\|v_\lambda\| > R_0$ and $E_{n,\lambda,\mu}(v_\lambda) < \rho_0$.*

Proof. From the subadditivity of $r^{1-\alpha}$, ones has

$$(u^+ + \frac{1}{n})^{1-\alpha} - (\frac{1}{n})^{1-\alpha} \leq (u^+)^{1-\alpha}.$$

Therefore,

$$E_{n,\lambda,\mu}(u) \geq E_{\lambda,\mu}(u).$$

So, from Lemma 3.1, we deduce the first part of the Lemma 4.1.

Now, let $u \in X_0$ such that $u^+ \not\equiv 0$ and $r > 0$, then

$$\begin{aligned} E_{n,\lambda,\mu}(ru) = \frac{r^p}{p}\|u\|^p - \frac{\lambda r^{1-\alpha}}{1-\alpha}\int_\Omega \Big((u^+ + \frac{1}{n})^{1-\alpha} - (\frac{1}{n})^{1-\alpha}\Big)\,dx \\ - \frac{r^{p_s^*(t)}}{p_s^*(t)}\int_\Omega \frac{(u^+)^{p_s^*(t)}}{|x|^t}\,dx \to -\infty \quad \text{as } r \to +\infty, \end{aligned}$$

since $1-\gamma < 1 < p < p_s^*(t)$. Therefore, we obtain the existence of $v_\lambda \in X_0$, satisfying $\|v_\lambda\| > R_0$ and $E_{n,\lambda,\mu}(v_\lambda) < \rho_0$. The proof of the second part of Lemma 4.1 is complete. □

Now, we prove the compactness property for the functional energy $E_{n,\lambda,\mu}$.

Lemma 4.2. *Suppose that $0 < \alpha < 1$. So, the functional energy $E_{n,\lambda,\mu}$ satisfies the (PS) condition at any level $c \in \mathbb{R}$ with $c < \frac{(sp-t)}{p(N-t)}S^{\frac{N-t}{sp-t}} - C_\lambda$ for any $\lambda > 0$, where*

$$C_\lambda = \frac{1+\alpha}{p}\Big[\lambda\Big(\frac{(sp-t)}{(N-t)(1-\alpha)}\Big)^{\frac{\alpha-1}{p}}\Big(\frac{1}{1-\alpha} + \frac{1}{p_s^*(t)}\Big)|\Omega|^{\frac{p_s^*(t)-1+\alpha}{p_s^*(t)}} S^{-\frac{1-\alpha}{p}}\Big]^{\frac{p}{1+\alpha}}.$$

Proof. Consider $\{u_k\} \subset X_0$ be a (PS) minimizing sequence for the functional energy $E_{n,\lambda,\mu}$ at level $c \in \mathbb{R}$, with c satisfying

$$E_{n,\lambda,\mu}(u_k) \to c \text{ and } E'_{n,\lambda,\mu}(u_k) \to 0 \quad \text{as } k \to \infty. \tag{4.3}$$

Therefore, using the Hölder inequality and the Sobolev embedding, there exists $\epsilon > 0$ and $C > 0$ such that

$$\begin{aligned} c + \epsilon\|u_k\| + \circ(1) &\geq E_{n,\lambda,\mu}(u_k) - \frac{1}{p_s^*(t)}\langle E'_{n,\lambda,\mu}(u_k), u_k\rangle \\ &= \Big(\frac{1}{p} - \frac{1}{p_s^*(t)}\Big)\|u_k\|^p + \frac{\lambda}{p_s^*(t)}\int_\Omega \Big(u_k^+ + \frac{1}{n}\Big)^{-\alpha} u_k\,dx \\ &\quad - \frac{\lambda}{1-\alpha}\int_\Omega \Big((u_k^+ + \frac{1}{n})^{1-\alpha} - (\frac{1}{n})^{1-\alpha}\Big)\,dx \\ &\geq \Big(\frac{1}{p} - \frac{1}{p_s^*(t)}\Big)\|u_k\|^p - \lambda\Big(\frac{1}{1-\alpha} + \frac{1}{p_s^*(t)}\Big)\int_\Omega |u_k|^{1-\alpha}\,dx \\ &\geq \Big(\frac{1}{p} - \frac{1}{p_s^*(t)}\Big)\|u_k\|^p - \lambda C\Big(\frac{1}{1-\alpha} + \frac{1}{p_s^*(t)}\Big)|\Omega|^{\frac{p_s^*(t)+1-\alpha}{p_s^*(t)}}\|u_k\|^{1-\alpha}. \end{aligned}$$

So, $\{u_k\}$ is bounded, since $1-\gamma < 1 < p < p_s^*(t)$. Moreover, $\{u_k^-\}$ is bounded in X_0, therefore using (4.3), ones has

$$\lim_{k\to\infty}\langle E'_{n,\lambda,\mu}(u_k), u_k\rangle = \lim_{k\to\infty}\langle u_k, -u_k^-\rangle.$$

Now, we recall the following elementary inequality.

$$(a-b)(a^- - b^-) \le -(a^- - b^-)^2. \tag{4.4}$$

Then, using (4.4), we obtain

$$\begin{aligned} 0 &\le \iint_{\mathbb{R}^{2N}} \frac{|(u(x)-u(y)|^{p-2}(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &\le -\iint_{\mathbb{R}^{2N}} \frac{|(u(x)-u(y)|^{p-2}(u^-(x)-u^-(y))^2}{|x-y|^{N+sp}}\,dx\,dy. \end{aligned} \tag{4.5}$$

So, using (4.5), we can conclude that $\|u_k^-\| \to 0$ as k tends to infinity. Hence, for k large enough, ones has

$$E_{n,\lambda,\mu}(u_k) = E_{n,\lambda,\mu}(u_k^+) + o(1) \quad \text{and} \quad E'_{n,\lambda,\mu}(u_k) = E'_{n,\lambda,\mu}(u_k^+) + o.$$

Therefore, $\{u_k\}$ is a sequence of positive functions.

Now, since $\{u_k\}$ is bounded, up to a subsequence, using [1, 21], there exists $\{u_k\} \subset X_0$, v_λ in X_0 and a non-negative numbers l, μ such that

$$\begin{gathered} u_k \rightharpoonup v_\lambda \quad \text{weakly in } X_0, \\ u_k \rightharpoonup v_\lambda \quad \text{weakly in } L^{p_s^*(t)}(\Omega), \\ u_k \to v_\lambda \quad \text{strongly in } L^k(\Omega, \frac{dx}{|x|^t}) \quad \text{for } k \in [1, p_s^*(t)), \\ u_k \to v_\lambda \quad \text{a.e. in } \Omega, \\ |u_k(x)| \le h(x) \quad \text{a.e. in } \Omega \quad \text{for all } n \text{ with } h(x) \in L^1(\Omega), \end{gathered} \tag{4.6}$$

and

$$\begin{gathered} \|u_k\| \to \mu, \\ \|u_k - v_\lambda\|_{p_s^*(t)} \to l. \end{gathered} \tag{4.7}$$

It is easy to see that if $\mu = 0$, then $u_k \to 0$ in X_0. Therefore, we suppose that $\mu > 0$. Using the above assertion, we obtain

$$\Big|\frac{u_k - v_\lambda}{(u_k^+ + \frac{1}{n})^\alpha}\Big| \le n^\alpha (h + |v_\lambda|).$$

Now, applying the dominated convergence theorem, we obtain

$$\lim_{k\to\infty} \int_\Omega \frac{u_k - v_\lambda}{(u_k^+ + \frac{1}{n})^\alpha}\,dx = 0. \tag{4.8}$$

Hence, we conclude that

$$\lim_{k\to\infty} \int_\Omega \frac{u_k}{(u_k^+ + \frac{1}{n})^\alpha}\,dx = \int_\Omega \frac{v_\lambda}{(v_\lambda^+ + \frac{1}{n})^\alpha}\,dx. \tag{4.9}$$

Then, we demonstrate that $u_k \to v_\lambda$ strongly in X_0. Since, $E'_{n,\lambda,\mu}(u_k) \to 0$, we obtain

$$\|u_k\|^p - \lambda \int_\Omega \frac{u_k}{\big(u_k^+ + \frac{1}{n})^\alpha}\,dx - \int_\Omega \frac{(u_k^+)^{p_s^*(t)-1} u_k}{|x|^t}\,dx = o(1).$$

Therefore, using Brezis-Lieb Lemma [4], we obtain

$$\begin{gathered} \|u_k\|^p = \|u_k - v_\lambda\|^p + \|u\|^p + o(1), \\ \|u_k\|_{p_s^*(t)}^{p_s^*(t)} = \|u_k - v_\lambda\|_{p_s^*(t)}^{p_s^*(t)} + \|u\|_{p_s^*(t)}^{p_s^*(t)} + o(1). \end{gathered} \tag{4.10}$$

So, using (4.3), (4.8), and (4.10), we deduce that

$$\begin{aligned} o(1) &= \langle E'_{n,\lambda,\mu}(u_k), u_k - v_\lambda\rangle \\ &= \int_{\mathcal{Q}} \frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))((u_k-v_\lambda)(x)-(u_k-v_\lambda)(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &\quad - \mu\int_\Omega \frac{|u|^{p-2}u}{|x|^{sp}}(u_k-v_\lambda)\,dx - \lambda\int_\Omega \frac{u_k-v_\lambda}{(u_k+\frac{1}{n})^\alpha}\,dx - \int_\Omega \frac{u_k^{p_s^*(t)-1}(u_k-v_\lambda)}{|x|^t}\,dx \\ &= (\|u_k\|^p - \|v_\lambda\|^p) - \|u_k\|_{p_s^*(t)}^{p_s^*(t)} + \|v_\lambda\|_{p_s^*(t)}^{p_s^*(t)} + o(1) \\ &= \|u_k - v_\lambda\|^p - \|u_k - v_\lambda\|_{p_s^*(t)}^{p_s^*(t)} + o(1). \end{aligned}$$

Therefore,

$$\begin{gathered} \lim_{k\to\infty}\|u_k - v_\lambda\|^p = \lim_{k\to\infty}\int_\Omega \frac{((u_k-v_\lambda)^+)^{p_s^*(t)-1}(u_k-v_\lambda)}{|x|^t}\,dx = l, \\ \int_\Omega \frac{|u_k-v_\lambda|^{p_s^*(t)}}{|x|^t}\,dx \geq \int_\Omega \frac{((u_k-v_\lambda)^+)^{p_s^*(t)-1}(u_k-v_\lambda)}{|x|^t}\,dx. \end{gathered}$$

Then, using Sobolev's inequality, we deduce that

$$\|u_k - v_\lambda\|^p \geq S\Big(\int_\Omega \frac{|u_k-v_\lambda|^{p_s^*(t)}}{|x|^t}\,dx\Big)^{\frac{p}{p_s^*(t)}}.$$

Hence, we conclude that

$$Sl^p \leq l^{p_s^*(t)}. \tag{4.11}$$

We guarantee that $l=0$. We obtain that $u_k \to v_\lambda$ in X_0 and the proof is complete. Otherwise, we suppose that

$$S^{\frac{N-t}{sp-t}} \leq l. \tag{4.12}$$

Therefore, using (4.10) and (4.12), the Hölder inequality and the Young inequality, if k tends to infinity, we obtain

$$\begin{aligned} c &= E_{n,\lambda,\mu}(u_k) - \frac{1}{p_s^*(t)}\langle E'_{n,\lambda,\mu}(u_k), u_k\rangle + o(1) \\ &= \frac{(sp-t)}{p(N-t)}\|u_k\|^p - \lambda\int_\Omega \Big[(u_k^+ + \frac{1}{n})^{1-\alpha} - (\frac{1}{n})^{-\alpha}\Big]\,dx \\ &\quad + \frac{\lambda}{p_s^*(t)}\int_\Omega (u_k^+ + \frac{1}{n})^{-\alpha}u_k\,dx + o(1) \\ &\geq \frac{(sp-t)}{p(N-t)}S^{\frac{N-t}{sp-t}} + \frac{(sp-t)}{p(N-t)}\|v_\lambda\|^p - \lambda\Big(\frac{1}{1-\alpha} + \frac{1}{p_s^*(t)}\Big)\int_\Omega |u_k|^{1-\alpha}\ dx + o(1) \\ &\geq \frac{(sp-t)}{p(N-t)}S^{\frac{N-t}{sp-t}} + \frac{(sp-t)}{p(N-t)}\|v_\lambda\|^p - \lambda\Big(\frac{1}{1-\alpha} + \frac{1}{p_s^*(t)}\Big)\int_\Omega v_\lambda^{1-\alpha}\ dx + o(1) \\ &\geq \frac{(sp-t)}{p(N-t)}S^{\frac{N-t}{sp-t}} + \frac{(sp-t)}{p(N-t)}\|v_\lambda\|^p - \lambda\Big(\frac{1}{1-\alpha} + \frac{1}{p_s^*(t)}\Big)|\Omega|^{\frac{p_s^*(t)-1+\alpha}{p_s^*(t)}} \\ &\quad \times S^{-\frac{1-\alpha}{p}}\|v_\lambda\|^{1-\alpha} + o(1) \\ &\geq \frac{(sp-t)}{p(N-t)}S^{\frac{N-t}{sp-t}} + \frac{(sp-t)}{p(N-t)}\|v_\lambda\|^p - \lambda\Big(\frac{p}{(N-t)(1-\alpha)}\Big)^{\frac{\alpha-1}{p}} \\ &\quad \times\Big(\frac{1}{1-\alpha} + \frac{1}{p_s^*(t)}\Big)|\Omega|^{\frac{p_s^*(t)-1+\alpha}{p_s^*(t)}} S^{-\frac{1-\alpha}{p}}\Big(\frac{p}{(N-t)(1-\alpha)}\Big)^{\frac{1-\alpha}{p}}\|v_\lambda\|^{1-\alpha} + o(1) \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{(sp-t)}{p(N-t)}S^{\frac{N-t}{sp-t}} + \frac{(sp-t)}{p(N-t)}\|v_\lambda\|^p - \frac{1-\alpha}{p}\Big[\Big(\frac{(sp-t)}{(N-t)(1-\alpha)}\Big)^{\frac{1-\alpha}{p}} \\
&\quad \times \|v_\lambda\|^{1-\alpha}\Big]^{\frac{p}{1-\alpha}} - \frac{1+\alpha}{p}\Big[\lambda\Big(\frac{(sp-t)}{(N-t)(1-\alpha)}\Big)^{\frac{\alpha-1}{p}}\Big(\frac{1}{1-\alpha} + \frac{1}{p_s^*(t)}\Big) \\
&\quad \times |\Omega|^{\frac{p_s^*(t)-1+\alpha}{p_s^*(t)}} S^{-\frac{1-\alpha}{p}}\Big]^{\frac{p}{1+\alpha}} \\
&= \frac{(sp-t)}{p(N-t)}S^{\frac{N-t}{sp-ta}} - C_\lambda,
\end{aligned}
$$

which is a contradiction. Therefore, $l = 0$ and $u_k \to v_\lambda$. The proof of Lemma 4.2 is complete. □

To state a control from above for the functional $E_{n,\lambda,\mu}$, we recall some necessary tools (for more details, see [4]). Let $1 < p < N$, $0 \leq t < p$, and $o \leq \mu < \mu_0$. Then, the limiting problem

$$
\begin{gathered}
(-\Delta_p)^s u - \mu\frac{|u|p-2u}{|x|^{sp}} = \frac{(u^+)^{p_s^*(t)-2}u^+}{|x|^t}, \quad \text{in } \Omega, \\
u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{gathered} \tag{4.13}
$$

has a positive radial solution

$$
U_{t,\epsilon}(x) = \epsilon^{-\frac{N-sp}{p}}U_{p,\mu}(\frac{|x|}{\epsilon}),
$$

where $\epsilon > 0,\ x \in \mathbb{R}^N$. Note that $U_{t,\epsilon}(x)$ is a minimizer for S satisfying

$$
\int_Q \frac{|U_{t,\epsilon}(x) - U_{t,\epsilon}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \mu\int_\Omega \frac{|U_{t,\epsilon}|^p}{|x|^{sp}}\,dx = \int_\Omega \frac{U_{t,\epsilon}^{p_s^*(t)}}{|x|^t}dx = S^{\frac{N-t}{sp-t}} \tag{4.14}
$$

where the function $U_{p,\mu}(x) = U_{p,\mu}(|x|)$ is the unique radial solution of (4.13).

Now, we define

$$
m_{\epsilon,\delta} = \frac{U_{t,\epsilon}(\delta)}{U_{t,\epsilon}(\delta) - U_{t,\epsilon}(\theta\delta)},
$$

where $\epsilon, \delta > 0$, and $\theta > 1$. For fixed $\epsilon, \delta > 0$, we set

$$
g_{\epsilon,\delta}(k) = \begin{cases} 0 & \text{if } 0 \leq k \leq U_{t,\epsilon}(\theta\delta), \\ m_{\epsilon,\delta}^p(k - U_{t,\epsilon}(\theta\delta)) & \text{if } U_{t,\epsilon}(\theta\delta) \leq k \leq U_{t,\epsilon}(\delta), \\ k + U_{t,\epsilon}(\delta)(m_{\epsilon,\delta}^{p-1} - 1) & \text{if } k \geq U_\epsilon(\delta), \end{cases}
$$

and define

$$
G_{\epsilon,\delta}(k) = \int_0^k g'_{\epsilon,\delta}(\tau)\,d\tau = \begin{cases} 0 & \text{if } 0 \leq k \leq U_{t,\epsilon}(\theta\delta), \\ m_{\epsilon,\delta}(k - U_{t,\epsilon}(\theta\delta)) & \text{if } U_{t,\epsilon}(\theta\delta) \leq k \leq U_{t,\epsilon}(\delta), \\ k & \text{if } k \geq U_{t,\epsilon}(\delta). \end{cases}
$$

Note that the functions $g_{\epsilon,\delta}$ and $G_{\epsilon,\delta}$ are nondecreasing and absolutely continuous. We define the radially symmetric non-increasing function

$$
u_{t,\epsilon,\delta}(r) = G_{\epsilon,\delta}(U_{t,\epsilon}(r)),
$$

which satisfies

$$
u_{t,\epsilon,\delta}(r) = \begin{cases} U_{t,\epsilon}(r) & \text{if } r \leq \delta, \\ 0 & \text{if } r \geq \theta\delta, \end{cases}
$$

for all $r \geq 1$. We follow here the arguments of [4, Lemma 2.10]. For each sufficiently small $\epsilon, \delta > 0$, we have the following estimates for $u_{t,\epsilon,\delta}$.

Lemma 4.3. *There exists a constant $C = C(N,s) > 0$ such that for any $0 < p\epsilon \leq \delta < \theta^{-1} dist(0, \partial\Omega)$, it holds*

$$\|u_{t,\epsilon,\delta}\|^p \leq S^{\frac{N-t}{sp-t}} + C(\frac{\epsilon}{\delta})^{(N-sp)}, \tag{4.15}$$

$$\int_\Omega \frac{u_{t,\epsilon,\delta}^{p_s^*(t)}}{|x|^t}(x)\,dx \geq S^{\frac{N-t}{sp-t}} - C(\frac{\epsilon}{\delta})^{b(\mu)p_s^*(t)-N+t}. \tag{4.16}$$

where $b(\mu)$ is the solution of $f(r) = (p-1)r^p - (N-p)r^{p-1} + \mu$, $r \geq 0$. On the other hand, for any $\beta > 0$, there exists C_β such that

$$\int_{\mathbb{R}^N} u_{t,\epsilon,\delta}(x)^\beta \geq C_\beta \begin{cases} \epsilon^{N-\frac{N-sp}{p}\beta}|\log(\frac{\epsilon}{\delta})| & \text{if } \beta = \frac{p_s^*(t)}{p}, \\ \epsilon^{\frac{N-sp}{p}\beta}\delta^{N-\frac{N-sp}{p}\beta} & \text{if } \beta < \frac{p_s^*(t)}{p}, \\ \epsilon^{N-\frac{N-sp}{p}\beta} & \text{if } \beta > \frac{p_s^*(t)}{p}. \end{cases} \tag{4.17}$$

Now, we have the following result for the functional energy $E_{n,\lambda,\mu}$.

Lemma 4.4. *There exits $\lambda_1 > 0$ and $\psi \in X_0$ satisfying*

$$\sup_{t>0} E_{n,\lambda,\mu}(r\psi) < \frac{(sp-t)}{p(N-t)} S^{\frac{N-t}{sp-t}} - C_\lambda$$

for all $\lambda \in (0, \lambda_1)$, where C_λ is defined in Lemma 4.2.

Proof. Firstly, using (3.1), we have

$$E_{n,\lambda,\mu}(ru_{t,\epsilon,\delta}) \underset{r\to\infty}{\longrightarrow} -\infty \quad \forall(\epsilon,\delta) \in (0,\epsilon_0) \times (0,\delta_0),$$

and

$$E_{n,\lambda,\mu}(ru_{t,\epsilon,\delta}) \underset{r\to 0}{\longrightarrow} 0 \quad \forall(\epsilon,\delta) \in (0,\epsilon_0) \times (0,\delta_0).$$

Now, setting

$$A_{\epsilon,\delta}(r) = \frac{r^p}{p}\|u_{t,\epsilon,\delta}\|^p - \frac{r^{p_s^*(t)}}{p_s^*(t)} \int_\Omega \frac{(u_{t,\epsilon,\delta}^+)^{p_s^*(t)}}{|x|^t}\,dx,$$

$$B_{\epsilon,\delta}(r) = -\frac{1}{\gamma-1}\int_\Omega \Big[\Big(ru_{t,\epsilon,\delta}^+ + \frac{1}{n}\Big)^{1-\alpha} - (\frac{1}{n})^{1-\alpha}\Big]\,dx.$$

It is very easy to see that

$$\lim_{r\to\infty} A_{\epsilon,\delta}(r) = -\infty, \quad A_{\epsilon,\delta}(0) = 0, \quad \lim_{r\to 0^+} A_{\epsilon,\delta}(r) > 0.$$

Therefore, $A_{\epsilon,\delta}$ attains its maximum at some $T_{\epsilon,\delta} > 0$. Indeed,

$$A'_{\epsilon,\delta}(r) = r\|u_{t,\epsilon,\delta}\|^p - r^{p_s^*(t)-1}\int_\Omega \frac{(u_{t,\epsilon,\delta}^+)^{p_s^*(t)}}{|x|^t}\,dx = 0,$$

hence

$$T_{\epsilon,\delta} = \Bigg(\frac{\|u_{t,\epsilon,\delta}\|^p}{\int_\Omega \frac{(u_{t,\epsilon,\delta}^+)^{p_s^*(t)}}{|x|^t}\,dx}\Bigg)^{\frac{1}{p_s^*(t)-2}}.$$

So, $A'_{\epsilon,\delta}(r) > 0$ for $0 < t < T_{\epsilon,\delta}$ and $A'_{\epsilon,\delta}(r) < 0$ for $t > T_{\epsilon,\delta}$. Therefore, there exists $t_{\epsilon,\delta} > 0$, satisfying

$$E_{n,\lambda,\mu}(r_{\epsilon,\delta}u_{t,\epsilon,\delta}) = \max_{r\geq 0} E_{n,\lambda,\mu}(ru_{t,\epsilon,\delta}).$$

Then, using (4.15) and (4.16), we obtain

$$\begin{aligned} A_{\epsilon,\delta}(T_{\epsilon,\delta}) &= \frac{1}{p}\Big(\frac{\|u_{t,\epsilon,\delta}\|^p}{\int_\Omega \frac{(u^+_{t,\epsilon,\delta})^{p^*_s(t)}}{|x|^t}\,dx}\Big)^{\frac{p}{p^*_s(t)-2}}\|u_{t,\epsilon,\delta}\|^p \\ &\quad - \frac{1}{p^*_s(t)}\Big(\frac{\|u_{t,\epsilon,\delta}\|^p}{\int_\Omega \frac{(u^+_{t,\epsilon,\delta})^{p^*_s(t)}}{|x|^t}\,dx}\Big)^{\frac{p^*_s(t)}{p^*_s(t)-2}}\int_\Omega \frac{(u^+_{t,\epsilon,\delta})^{p^*_s(t)}}{|x|^t}\,dx \\ &= \frac{(sp-t)}{p(N-t)}\Big(\frac{\|u_{t,\epsilon,\delta}\|^p}{\int_\Omega \frac{(u^+_{t,\epsilon,\delta})^{p^*_s(t)}}{|x|^t}\,dx}\Big)^{\frac{p}{p^*_s(t)-2}}\|u_{t,\epsilon,\delta}\|^p \\ &\le \frac{(sp-t)}{p(N-t)}S^{\frac{N-t}{sp-t}} + C(\frac{\epsilon}{\delta})^{(N-sp)}. \end{aligned} \tag{4.18}$$

Then, we use the following elementary inequality to estimate $B_{\epsilon,\delta}$,

$$a^{1-\alpha} - (a+b)^{1-\alpha} \le -(1-\alpha)b^{\frac{1-\alpha}{p}}a^{\frac{(1-\alpha)(p-1)}{p}} \tag{4.19}$$

for any $a>0$, $b>0$ large enough, $p>1$. Therefore, from (4.19), with $q=\frac{p^*_s(t)}{p}$ and setting $\epsilon < r^{\frac{1}{q}}$ for all $q>0$ small enough, we deduce the existence of $c_1, c_2 > 0$ independent of ϵ, such that

$$\begin{aligned} B_\epsilon(r_{\epsilon,\delta}) &\le \frac{1}{1-\alpha}\int_{\{x\in\Omega:|x|\le\epsilon\}}\Big((\frac{1}{n})^{1-\alpha} - (r_{\epsilon,\delta}u_{t,\epsilon,\delta}+\frac{1}{n})^{1-\alpha}\Big)\,dx \\ &\le -c_1(1-\alpha)\epsilon^{\frac{(N-sp)(1-\alpha)}{p^*_s(t)}}\int_{\{x\in\Omega:|x|\le\epsilon^{q'}\}}\Big(\frac{1}{(\frac{1}{|x|^{p'}+\epsilon^{p'}})^{\frac{N-sp}{p}}}\Big)^{\frac{p(1-\alpha)}{p^*_s(t)}}dx \\ &\le -c_2(1-\alpha)\epsilon^{\frac{(N-sp)(1-\alpha)}{p^*_s(t)}}\int_{\{x\in\Omega:|x|\le\epsilon^{q'}\}}\Big(r_{\epsilon,\delta}U_{p,\mu}(\frac{|x|}{\epsilon}\Big)^{\frac{p(1-\alpha)}{p^*_s(t)}}dx \\ &\le -c_3(1-\alpha)\epsilon^{\frac{(N-sp)(1-\alpha)}{p^*_s(t)}}\int_0^{\epsilon^{q'}}\Big(U_{p,\mu}(\frac{|x|}{\epsilon}\Big)^{\frac{p(1-\alpha)}{p^*_s(t)}}y^{N-1}\epsilon^N dy \\ &\le -c_4(1-\alpha)\epsilon^{\frac{(N-sp)(1-\alpha)+N}{p^*_s(t)}}\int_0^{\epsilon^{q'}} y^{-b(\mu)p+N-1}dy \\ &\le \begin{cases} (1-\alpha)\epsilon^{\frac{(N-sp)(1-\alpha)+N}{p^*_s(t)}} & b(\mu) > \frac{N}{p}, \\ (1-\alpha)\epsilon^{\frac{(N-sp)(1-\alpha)+N}{p^*_s(t)}}|\ln\epsilon| & b(\mu) = \frac{N}{p}, \\ (1-\alpha)\epsilon^{\frac{(N-sp)(1-\alpha)-2q(N-sp)(1-\alpha)+p^*_s(t)qN}{p^*_s(t)}} & b(\mu) < \frac{N}{p}. \end{cases} \end{aligned} \tag{4.20}$$

Hence, using (4.18) and (4.20), we deduce the existence of a positive constant λ_1 such that, for every $\lambda\in(0,\lambda_1)$, we obtain

$$\begin{aligned} &E_{n,\lambda,\mu}(u_{t,\epsilon,\delta}) \\ &= A_{\epsilon,\delta}(u_{t,\epsilon,\delta}) + \lambda B_{\epsilon,\delta}(u_{t,\epsilon,\delta}) \\ &\le \frac{(sp-t)}{p(N-t)}S^{\frac{N-t}{sp-t}} + C(\frac{\epsilon}{\delta})^{(N-sp)} - c_2(1-\alpha)\epsilon^{\frac{(N-sp)(1-\alpha)-2q(N-sp)(1-\alpha)+p^*_s(t)qN}{p^*_s(t)}} \\ &\quad\times\epsilon^{\frac{(N-2s)(1-\gamma)-2q(N-2s)(1-\gamma)+2^*_\alpha qN}{2^*_\alpha}} \end{aligned}$$

$$< \frac{(sp-t)}{p(N-t)} S^{\frac{N-t}{sp-t}} - C_\lambda.$$

This completes the proof. □

Lemma 4.5. *Suppose* $0 < \alpha < 1$. *Then, there exists* $\lambda^* = \min(\lambda_0, \lambda_1)$, *such that* (4.1) *has a positive solution* $v_n \in X_0$ *satisfying*

$$\rho_0 < E_{n,\lambda,\mu}(v_n) < \frac{sp-t}{p(N-t)} S^{\frac{N-t}{sp-t}} - C_\lambda,$$

where ρ_0 *is given in Lemma 3.1 and* C_λ *in Lemma 4.2.*

Proof. Consider $\lambda^* = \min(\lambda_0, \lambda_1)$. Therefore, the results in Lemmas 4.1-4.4 holds for all $\lambda \in (0, \lambda^*)$. Now, using Lemma 3.1 we deduce that the functional $E_{n,\lambda,\mu}$ satisfies the geometry of the mountain pass Lemma. Hence, we introduce the mountain pass level

$$c_{n,\lambda,\mu} = \inf_{g\in\Gamma} \max_{r\in[0,1]} E_{n,\lambda,\mu}(g(r)),$$

where

$$\Gamma = \{g \in C([0,1], X_0) : g(0) = 0, E_{n,\lambda}(g(1)) < 0\}.$$

Moreover,

$$0 < \rho_0 < c_{n,\lambda} \le \sup_{t\ge 0} E_{n,\lambda,\mu}(t\psi) < c_{n,\lambda}.$$

Therefore, according to Lemmas 4.1-4.4, $E_{n,\lambda,\mu}$ satisfies the (PS) condition at the level $c_{n,\lambda,\mu}$. Thn there exists a non-regular point v_n for $I_{n,\lambda,\mu}$ at the level $c_{n,\lambda,\mu}$. Moreover, $E_{n,\lambda,\mu}(v_n) = c_{n,\lambda,\mu} > \rho_0 > 0$. We deduce that v_n is a non-trivial critical point of the functional energy $E_{n,\lambda,\mu}$ and also a solution to the problem (4.1). Now, if we replace ϕ by v_n^- in (4.2) and using (4.5), it follows that $\|v_n\| = 0$. Therefore, v_n is positive. At the end, we apply the strong maximum principle (see [18]), we deduce that v_n is a non-negative solution to the problem (4.1). The proof of Lemma 4.5 is now complete. □

5. Multiple solutions to (1.1)

In this section we show the existence of a second solution to (1.1), as a limit of solutions of the perturbed problem (4.1). To do this, we consider $\{v_n\}_n$ be a family of positive function given by Lemma 4.5. Then, using Lemma 4.5 combineed with Hölder's inequality, we have

$$\begin{aligned}
&\frac{sp-t}{p(N-t)} S^{\frac{N-t}{sp-t}} - C_\lambda \\
&> E_{n,\lambda,\mu} - \frac{1}{p_s^*(t)} \langle E'_{n,\lambda,\mu}(v_n), v_n\rangle \\
&= \Big(\frac{1}{p} - \frac{1}{p_s^*(t)}\Big)\|v_n\|^p - \frac{\lambda}{1-\alpha}\int_\Omega \Big((v_n + \frac{1}{n})^{1-\alpha} - (\frac{1}{n})^{1-\alpha}\Big)\,dx \\
&\quad + \frac{\lambda}{p_s^*(t)}\int_\Omega (v_n + \frac{1}{n})^{-\gamma} v_n\,dx \\
&\ge \Big(\frac{1}{p} - \frac{1}{p_s^*(t)}\Big)\|v_n\|^p - \frac{\lambda}{1-\alpha}\int_\Omega v_n^{1-\alpha}\,dx \\
&\ge \Big(\frac{1}{p} - \frac{1}{p_s^*(t)}\Big)\|v_n\|^p - \frac{\lambda}{1-\alpha}|\Omega|^{\frac{p_s^*(t)-1+\alpha}{p_s^*(t)}} S^{-\frac{1-\alpha}{p}}\|v_n\|^{1-\alpha}
\end{aligned}$$

since $\alpha \in (0,1)$, so $\{v_n\}$ is bounded in X_0. Therefore, applying the reflexivity of X_0, we obtain the existence a subsequence, still denoted by $\{v_n\}$ and a function v_λ, satisfying

$$\begin{gathered} v_n \rightharpoonup v_\lambda \quad \text{weakly in } X_0, \\ v_n \rightharpoonup v_\lambda \quad \text{weakly in } L^{p_s^*(t)}(\Omega), \\ v_n \to v_\lambda \quad \text{strongly in } L^k(\Omega, \frac{dx}{|x|^t}) \text{ for } k \in [1, p_s^*(t)), \\ v_n \to v_\lambda \quad \text{a.e. in } \Omega, \end{gathered} \tag{5.1}$$

and

$$\begin{gathered} \|v_n\| \to \mu, \\ \|v_n - v_\lambda\|_{p_s^*(t)} \to l. \end{gathered} \tag{5.2}$$

Then, we want to show that $v_n \to v_\lambda$ strongly in X_0. This means that $\|v_n - v_\lambda\| \to 0$ as $n \to \infty$.

Firstly, using (5.1) and if $\mu = 0$, we obtain $\|v_n\| \to 0$ as $n \to \infty$. Now, we assume that $\mu > 0$. Therefore, since

$$0 \leq \frac{v_n}{(v_n + \frac{1}{n})^\alpha} \leq v_n^{1-\alpha} \text{ a.e. in } \Omega.$$

Therefore, from Hölder inequality and (5.1), we obtain

$$\begin{aligned} \int_\Omega \frac{v_n}{(v_n + \frac{1}{n})^\alpha}\, dx &\leq \int_\Omega v_n^{1-\alpha}\, dx \\ &\leq \int_\Omega |v_n - v_\lambda|^{1-\alpha}\, dx + \int_\Omega v_\lambda^{1-\alpha}\, dx \\ &= |v_n - v_\lambda|_p^{1-\alpha} |\Omega|^{\frac{1+\alpha}{p}} + \int_\Omega v_\lambda^{1-\alpha}\, dx \\ &\leq \int_\Omega v_\lambda^{1-\alpha}\, dx + o(1). \end{aligned} \tag{5.3}$$

In the same way, we have

$$\int_\Omega v_\lambda^{1-\alpha}\, dx \leq \int_\Omega \frac{v_n}{(v_n + \frac{1}{n})^\alpha}\, dx + o(1). \tag{5.4}$$

Hence, using (5.3)-(5.4), we obtain

$$\lim_{n\to\infty} \int_\Omega \frac{v_n}{(v_n + \frac{1}{n})^\gamma}\, dx = \int_\Omega v_\lambda^{1-\gamma}\, dx.$$

Therefore, if we replace both u and ϕ by v_n in (4.2), and using (5.1)-(5.2) as $n \to \infty$, we obtain

$$\|v_n - v_\lambda\|^p + \|v_\lambda\|^p - \int_\Omega \frac{|v_n - v_\lambda|^{p_s^*(t)}}{|x|^t}\, dx - \int_\Omega \frac{v_\lambda^{p_s^*(t)}}{|x|^t}\, dx - \lambda \int_\Omega v_\lambda^{1-\alpha}\, dx \to 0. \tag{5.5}$$

Then, since $\{v_n\}_n$ is bounded in X_0 and from the strong maximum principle (see [18]), we obtain the existence $\widetilde{\Omega} \subset \Omega$ and $\widetilde{c} > 0$ such that

$$v_n \geq \widetilde{c} > 0, \quad \text{a.e. in } \Omega, \tag{5.6}$$

for any integer n. Let us define $\phi \in C_0^\infty(\Omega)$ such that $\operatorname{supp}(\varphi) = \widetilde{\Omega} \subset \Omega$. Then, using (5.6), ones has

$$0 \le \Big|\frac{\varphi}{(v_n + \frac{1}{n})^\gamma}\Big| \le \frac{|\varphi|}{\widetilde{c}^\gamma}, \quad \text{a.e. in } \Omega.$$

Hence, from (5.1) and by using the dominated convergence Theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega \frac{\varphi}{(v_n + \frac{1}{n})^\gamma}\,dx = \int_\Omega v_\lambda^{-\gamma}\varphi\,dx.$$

Thus, replacing u by v_n in (4.2), letting $n \to \infty$, and using (5.1) with the above equality, we have

$$\begin{aligned} &\int_Q \frac{|v_\lambda(x) - v_\lambda(y)|^{p-2}(v_\lambda(x) - v_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &- \lambda \int_\Omega v_\lambda^{-\alpha}\phi\,dx - \int_\Omega \frac{v_\lambda^{p_s^*(t)-2} v_\lambda \phi}{|x|^t}\,dx = 0. \end{aligned} \tag{5.7}$$

Moreover, since $\partial\Omega$ is continuous, the space $C_0^\infty(\Omega)$ is dense in X_0. Therefore, (5.7) holds for any $\phi \in X_0$. Thus, if we replace ϕ by v_λ in (5.7) and combining this with (4.2), we obtain

$$\|v_\lambda\|^p - \lambda \int_\Omega v_\lambda^{1-\alpha} - \int_\Omega \frac{v_\lambda^{p_s^*(t)}}{|x|^t}\,dx = 0. \tag{5.8}$$

Consequently, from (5.7), we have

$$\|v_n - v_\lambda\|^p - \int_\Omega \frac{|v_n - v_\lambda|^{p_s^*(t)}}{|x|^t}\,dx = o(1). \tag{5.9}$$

So, we obtain

$$\lim_{n\to\infty} \|v_n - v_\lambda\|^2 = \lim_{n\to\infty} \int_\Omega \frac{|v_n - v_\lambda|^{p_s^*(t)}}{|x|^t}\,dx = l > 0. \tag{5.10}$$

Now, since

$$\int_\Omega \frac{|v_n - v_\lambda|^{p_s^*(t)}}{|x|^t}\,dx \ge \int_\Omega \frac{((v_n - v_\lambda)^+)^{p_s^*(t)}}{|x|^t}\,dx.$$

It follows that $l \ge S^{\frac{N-t}{sp-t}}$. Therefore, using (5.5), we obtain

$$\begin{aligned} &E_{n,\lambda,\mu}(v_\lambda) \\ &= \frac{1}{p}\|v_\lambda\|^p - \frac{\lambda}{1-\alpha}\int_\Omega v_\lambda^{1-\alpha}\,dx - \frac{1}{p_s^*(t)}\int_\Omega \frac{v_\lambda^{p_s^*(t)}}{|x|^t}\,dx \\ &= \frac{sp-t}{p(N-t)}\|v_\lambda\|^p - \lambda\Big(\frac{1}{1-\alpha} - \frac{1}{p_s^*(t)}\Big)\int_\Omega v_\lambda^{1-\alpha}\,dx \\ &\ge \frac{sp-t}{p(N-t)}\|v_\lambda\|^p - \lambda\Big(\frac{1}{1-\alpha} - \frac{1}{p_s^*(t)}\Big)|\Omega|^{\frac{p_s^*(t)-1+\gamma}{p_s^*(t)}} S^{-\frac{1-\alpha}{p}}\|v_\lambda\|^{1-\alpha} \\ &> -C_\lambda. \end{aligned} \tag{5.11}$$

Therefore, using (5.5)-(5.9), we have

$$\begin{aligned} E_{\lambda,\mu}(v_n) &= E_{n,\lambda,\mu}(v_\lambda) - \frac{sp-t}{p(N-t)}\|v_n - v_\lambda\|^p + o(1) \\ &< \frac{sp-t}{p(N-t)}\Big(S^{\frac{N-t}{sp-t}} - l\Big) - C_\lambda \\ &\le -C_\lambda \end{aligned} \tag{5.12}$$

which contradicts (5.11). Therefore, $E_{\lambda,\mu}(v_\lambda) = \lim_{n\to\infty} E_{n,\lambda,\mu}(v_n)$. Hence, it is very easy to see that v_λ is a solution of (1.1). Moreover, using Lemma 4.5, we obtain $E_{\lambda,\mu}(v_\lambda) \ge \alpha > 0$, that is, v_λ is nontrivial solution. Also, we proceed as in the proof of Lemma 4.5 to conclude that v_λ is a non-negative solution of problem (1.1). At the end, $u_\lambda \not\equiv v_\lambda$ since $E_{\lambda,\mu}(u_\lambda) < 0 < E_{\lambda,\mu}(v_\lambda)$. Therefore, the proof is complete.

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