Electronic Journal of Differential Equations, Vol. 2023 (2023), No. 14, pp. 1–10.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu or https://ejde.math.unt.edu

EXISTENCE AND MULTIPLICITY RESULTS FOR SUPERCRITICAL NONLOCAL KIRCHHOFF PROBLEMS

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Abstract. We study the existence and multiplicity of solutions for the nonlocal perturbed Kirchhoff problem

$$-\Big(a+b\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = \lambda g(x,u)+f(x,u), \quad \text{in } \Omega,$$
$$u=0, \quad \text{on } \partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N>4$, $a,b,\lambda>0$, and $f,g:\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions, with f subcritical, and g of arbitrary growth. This paper is motivated by a recent results by Faraci and Silva [4] where existence and multiplicity results were obtained when g is subcritical and f is a power-type function with critical exponent.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, with $N>4$, and let $a,b,\lambda>0$. In this article, we study the nonlocal Kirchhoff problem

$$\begin{gathered}-\Big(a+b\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = \lambda g(x,u)+f(x,u), \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $f,g:\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions satisfying

$$\sigma_f := \operatorname{ess\,sup}_{(x,t)\in\Omega\times\mathbb{R}} \frac{|f(x,t)|}{1+|t|^{p-1}} < +\infty, \tag{1.2}$$

for some $p\in(2,2^*)$, where $2^*=\frac{2N}{N-2}$;

$$\rho_g(C) := \operatorname{ess\,sup}_{(x,t)\in\Omega\times[-C,C]} |g(x,t)| < +\infty, \quad \text{for each } C>0. \tag{1.3}$$

Recall that 2^* is the critical Sobolev exponent for the embedding $L^m(\Omega)\hookrightarrow W_0^{1,2}(\Omega)$. Since we are assuming $N>4$, one has $2^*<4$.

Our aim is to establish some existence and multiplicity results for problem (1.1) without assuming any other conditions on g, except the summability condition (1.3). This paper is motivated by the results recently obtained by Faraci and Silva

2020 *Mathematics Subject Classification.* 35J20, 35J25.
Key words and phrases. Nonlocal problem; Kirchhoff equation; weak solution; supercritical growth; variational methods.

Submitted March 20, 2022. Published February 15, 2023.

[4] on the existence and multiplicity of solutions to the problem

$$
\begin{gathered}
-\Big(a+b\int_\Omega |\nabla u|^2\,dx\Big)\Delta u=\lambda g(x,u)+|u|^{2^*-2}u, \quad \text{in } \Omega,\\
u=0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.4}
$$

where $a,b,\lambda>0$ and g satisfy the following conditions:

- (A1) $\operatorname{ess\,sup}_{(x,t)\in\Omega\times\mathbb{R}} \frac{|g(x,t)|}{1+|t|^{p-1}}<+\infty$, for some $p\in(2,2^*)$;
- (A2) $\lim_{t\to 0}\frac{g(x,t)}{t}=0$, uniformly for a.a. $x\in\Omega$;
- (A3) $g(x,t)t>0$, for all $t\in\mathbb{R}\setminus\{0\}$ and for a.a. $x\in\Omega$;
- (A4) $\operatorname{ess\,inf}_{(x,t)\in\Omega\times A} g(x,t)>0$, for some nonempty open set $A\subset(0,\infty)$.

In particular, under assumptions (A1)–(A4), Faraci and Silva proved in [4] that problem (1.4) admits at least a nonzero solution if one of the following conditions holds

- $a^{\frac{N-4}{2}}b>C_1(N):=\frac{4(N-4)^{\frac{N-4}{2}}}{N^{\frac{N-2}{2}}c_{2^*}^N}$ and λ is large,
- $a^{\frac{N-4}{2}}b=C_1(N)$.

Here, c_{2^*} is the best constant for the embedding $L^{2^*}(\Omega)\hookrightarrow W_0^{1,2}(\Omega)$. Moreover, a second solution is proved to exist for λ large, under the following more strict condition on a,b

$$
a^{\frac{N-4}{2}}b\ge\Big(\frac{N}{N-2}\Big)^{\frac{N-2}{N}}C_1(N). \tag{1.5}
$$

Problem (1.1) is associated with the stationary version of the well known equation proposed by Kirchhoff to describe the transversal oscillations of a stretched string. For more details, we refer the reader to [4] or [7] and references therein. To the best of our knowledge, the case in which problem (1.1) involves nonlinearities of arbitrary growth has been addressed in few papers. Among them, we can cite [1, 2, 3, 6]. However, in these papers only existence results were established.

We stress out that variational methods are not directly applicable when supercritical nonlinearities are involved. Usually, in this case, an auxiliary problem involving a suitable truncation of the supercritical nonlinearity is introduced. After that, one shows, by using L^∞-norm estimates, that the solutions of the auxiliary problem are also solutions of the original problem. We will make use of this technique to prove our main results.

Now we recall some basic concepts of variational methods. Let $h:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function and let $H:\Omega\times\mathbb{R}\to\mathbb{R}$ be the primitive of h, defined by

$$
H(x,\xi)=\int_0^\xi h(x,t)dt, \quad \text{for all } (x,\xi)\in\Omega\times\mathbb{R}. \tag{1.6}
$$

Consider the set $X_h\subseteq W_0^{1,2}(\Omega)$ given by

$$
X_h=\big\{u\in W_0^{1,2}(\Omega): x\in\Omega\to H(x,u(x)) \text{ is summable in } \Omega\big\}
$$

By Sobolev embeddings, the set X_h is the whole $W_0^{1,2}(\Omega)$ whenever

$$
\operatorname{ess\,sup}_{(x,t)\in\Omega\times\mathbb{R}}\frac{|H(x,t)|}{1+|t|^{2^*}}<+\infty.
$$

Throughout the paper, if $h : \Omega \times \mathbb{R} \to \mathbb{R}$ and X_h are as above, we denote by $I_h : X_h \to \mathbb{R}$ the energy functional associated with the problem

$$\begin{gathered} -\Big(a + b\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = h(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.7}$$

which is defined by

$$I_h(u) = \frac{a}{2}\int_\Omega |\nabla u(x)|^2\,dx + \frac{b}{4}\Big(\int_\Omega |\nabla u(x)|^2\,dx\Big)^2 - \int_\Omega H(x,u(x))\,dx,$$

for all $u \in X_h$. By a solution of problem (1.7) we mean any function $u \in W_0^{1,2}(\Omega)$ satisfying, for each $v \in W_0^{1,2}(\Omega)$, the following conditions: the function $x \in \Omega \to h(x,u(x))v(x)$ belongs to $L^1(\Omega)$, and

$$\Big(a + b\int_\Omega |\nabla u(x)|^2\,dx\Big)\int_\Omega \nabla u(x)\nabla v(x) = \int_\Omega h(x,u(x))v(x)\,dx.$$

When $X_h = W_0^{1,2}(\Omega)$ and I_h is differentiable in $W_0^{1,2}(\Omega)$, the solutions of (1.7) are exactly the critical points of I_h. We denote by $\tilde{I}_\lambda$ the energy functional associated with problem (1.4), that is

$$\tilde{I}_\lambda := I_h, \quad \text{where } h(x,t) = \lambda g(x,t) + |t|^{2^*-2}t.$$

A key ingredient in the proofs of the results in [4] is the sequential weak lower semicontinuity of the functional

$$\Phi(u) = \frac{a}{2}\int_\Omega |\nabla u(x)|^2\,dx + \frac{b}{4}\Big(\int_\Omega |\nabla u(x)|^2\,dx\Big)^2 - \frac{1}{q}\int_\Omega |u(x)|^q\,dx, \quad u \in W_0^{1,2}(\Omega),$$

when $q = 2^*$. It is well known that Φ is sequentially weakly lower semicontinuous for $0 < q < 2^*$, but this is not true, in general, if $q = 2^*$. In [4] the condition $a^{\frac{N-4}{2}}b \geq C_1(N)$ assumes a key role since it just ensures the sequential weak lower semicontinuity of Φ in the case $q = 2^*$. Thus, if one assumes $a^{\frac{N-4}{2}}b \geq C_1(N)$ and the subcritical growth condition $i)$ on g, one gets the sequential weak lower semicontinuity of $\tilde{I}_\lambda$. When $q > 2^*$, the set X_h corresponding to $h(x,t) = \lambda g(x,t) + |t|^{q-2}t$ is strictly contained in $W_0^{1,2}(\Omega)$ and, moreover, the functional Φ (and therefore also the functional $\tilde{I}_\lambda$) is never sequentially weakly lower semicontinuous in X_h. Thus, the arguments used in [4] cannot be applied when $q > 2^*$ and, in general, when a nonlinearity of arbitrary growth is involved.

As said above, in the present paper, we address the question of the existence and multiplicity of solutions to problem (1.1) in the case g has an arbitrary growth. We will establish existence and multiplicity results by assuming only condition $\alpha_g)$ on g, and imposing (as in [4]) some constrains on a, b. However, differently to the problem (1.4) considered in [4], where the parameter λ is multiplied by the subcritical nonlinearity, in our case the parameter λ is multiplied by the nonlinearity of arbitrary growth. This allows to deduce that the solutions of the auxiliary truncated problem are also solutions of the original problem, for λ small enough.

Besides (A1), we assume on the nonlinearity f the following two additional conditions:

- (A5) $\limsup_{\xi\to 0} \frac{1}{\xi^2}\int_0^\xi f(x,t)dt < a\lambda_1/2$, uniformly for a.a. $x \in \Omega$;
- (A6) $\liminf_{|\xi|\to+\infty} \frac{1}{\xi^2}\int_0^\xi f(x,t)dt > a\lambda_1/2$, uniformly for a.a. $x \in \Omega$.

Here,

$$\lambda_1 := \inf_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(x)|^2\,dx}{\int_\Omega |u(x)|^2\,dx}$$

is the first eigenvalue of the Laplacian on Ω.

Under (A1), (A2), (A5), and (A6), we will be able to prove a multiplicity result for problem (1.1) for all $a, b > 0$, with $b \le \beta(a)$, where $\beta(a)$ is a suitable number depending on a), and for all λ small enough. We will also show that, if conditions (A5) and (A6) are replaced by

(A7) $\liminf_{\xi\to 0} \frac{1}{\xi^2}\int_0^\xi f(x,t)dt > \frac{a\lambda_1}{2}$, uniformly for a.a. $x \in \Omega$,

an existence result can be proved, again for λ small, without imposing any constrain on a, b.

Note that the constrain $0 < b \le \beta(a)$ is a sort of opposite condition to (1.5) assumed in [4] (indeed, observe that (1.5) can be rewritten $b \ge \beta(a) := a^{-\frac{N-2}{4}} C_1(N)$). Our main results read as follows:

Theorem 1.1. *Let $a > 0$. Assume f satisfying* (A1), (A5), (A6)*, and g satisfying* (A2)*. Then, there exists $\beta(a) > 0$ with the following property: for each $b \in (0, \beta(a)]$, there exists $\lambda(a,b) > 0$, such that, for each $\lambda \in [0, \lambda(a,b)]$, problem* (1.1) *admits at least three distinct solutions.*

Theorem 1.2. *Let $a, b > 0$. Assume f satisfying* (A1) *and* (A7) *and g satisfying* (A2)*. Then, there exists $\lambda(a,b) > 0$ such that, for all $\lambda \in [0, \lambda(a,b)]$, problem* (1.1) *admits at least a nonzero solution.*

2. Notation and preliminary lemmas

Throughout this paper, we use of the following notation:

(1) for each $u \in W_0^{1,2}(\Omega)$, $\|u\| := \big(\int_\Omega |\nabla u(x)|^2\,dx\big)^{1/2}$ denotes the Poincaré norm of u;
(2) for each $m \in [1, +\infty[$ and $u \in L^m(\Omega)$, $\|u\|_m := \big(\int_\Omega |u(x)|^m\,dx\big)^{1/m}$ denotes the norm of u in the space $L^m(\Omega)$;
(3) for each $u \in L^\infty(\Omega)$, $\|u\|_\infty := \operatorname{ess\,sup}_{x\in\Omega} |u(x)|$ denotes the norm of u in the space $L^\infty(\Omega)$;
(4) for each $m \in [1, 2^*]$,

$$c_m := \sup_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\|u\|_m}{\|u\|}$$

denotes the best constant for the Sobolev embedding $L^m(\Omega) \hookrightarrow W_0^{1,2}(\Omega)$. Note that $\lambda_1 := c_2^{-2}$.
(5) for each $\lambda \in \mathbb{R}$ and $C > 0$, $g_C : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h_{\lambda,C} : \Omega \times \mathbb{R} \to \mathbb{R}$ are the functions defined by

$$g_C(x,t) = \begin{cases} g(x,t) & \text{if } (x,t) \in \Omega \times [-C, C], \\ g(x,C) & \text{if } (x,t) \in \Omega \times (C, +\infty), \\ g(x,-C) & \text{if } (x,t) \in \Omega \times (-\infty, -C). \end{cases}$$

$$h_{\lambda,C}(x,t) = \lambda g_C(x,t) + f(x,t), \quad \text{for each } (x,t) \in \Omega \times \mathbb{R}. \tag{2.1}$$

The next lemmas provide regularity estimates for the solutions of the problem

$$\begin{gathered} -\Big(a+b\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = h_{\lambda,C}(x,u), \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$

In particular, by these estimates we will infer that, for certain values of C and λ, every solution of (2.2) is also a solution of (1.1).

Lemma 2.1. *It holds*

$$t^x \le \gamma^x + \gamma^{x-y}t^y, \quad \textit{for each } t,\gamma,x,y>0, \ \textit{with } x<y, \tag{2.3}$$

Proof. One has

$$\big(\frac{t}{\gamma}\big)^x \le 1 \text{ if } t\le\gamma, \quad \big(\frac{t}{\gamma}\big)^x \le \big(\frac{t}{\gamma}\big)^y \text{ if } t\ge\gamma.$$

Hence,

$$\big(\frac{t}{\gamma}\big)^x \le 1+\big(\frac{t}{\gamma}\big)^y$$

from which (2.3) follows. □

The following two regularity lemmas are well known (see for instance [8], Appendix B, and Theorem 8.16 of [5]).

Lemma 2.2. *Let $p\in[2,2^*)$, $K>0$, and let $l:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that $|l(x,t)|\le K(1+|t|^{p-1})$, for each $t\in\mathbb{R}$ and for a.a. $x\in\Omega$. Moreover, let $u\in W_0^{1,2}(\Omega)$ satisfying*

$$\int_\Omega \nabla u(x)\nabla v(x)\,dx = \int_\Omega l(x,u(x))v(x)\,dx, \quad \textit{for each } v\in W_0^{1,2}(\Omega).$$

Then, $u\in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha\in(0,1)$.

Lemma 2.3. *Let $s>N/2$ and $l\in L^s(\Omega)$. Assume that $u\in W_0^{1,2}(\Omega)$ satisfies*

$$\int_\Omega \nabla u(x)\nabla v(x)\,dx = \int_\Omega l(x)v(x)\,dx, \quad \textit{for each } v\in W_0^{1,2}(\Omega).$$

Then, $u\in L^\infty(\Omega)$, and there exists a constant $K_s>0$, independent of u,l, such that $\|u\|_\infty\le K_s\|l\|_s$.

Lemma 2.4. *Let $a,b,C,\lambda>0$ and let $f,g:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfying conditions* (A1) *and* (A2)*, respectively. Then, there exists a constant $\gamma>0$, independent of a,b,C,λ, such that, for every solution u of problem* (2.2) *one has*

$$\|u\|_{2^*} \le \gamma\big[\lambda b^{-1}\rho_g(C)+b^{-1}+b^{\frac{3}{p-4}}\big]^{1/3} \tag{2.4}$$

Proof. Let $u\in W_0^{1,2}(\Omega)$ be a solution of (2.2). Then

$$0=I'_{h_{\lambda,C}}(u)(u)=(a+b\|u\|^2)\|u\|^2-\int_\Omega h_{\lambda,C}(x,u(x))u(x)\,dx. \tag{2.5}$$

Moreover, one has

$$(a+b\|u\|^2)\|u\|^2 \ge b\|u\|^4 \ge bc_{2^*}^{-4}\|u\|_{2^*}^4$$

and, by (2.5) and conditions (A1) and (A2),

$$(a+b\|u\|^2)\|u\|^2=\int_\Omega h_{\lambda,C}(x,u(x))u(x)\,dx \le \lambda\rho_g(C)\|u\|_1+\sigma_f\|u\|_1+\sigma_f\|u\|_p^p.$$

Consequently,
$$b\|u\|_{2^*}^3 \le \sigma_1(\lambda\rho_g(C) + 1 + \|u\|_{2^*}^{p-1}), \tag{2.6}$$
for some constant $\sigma_1 > 0$ independent of a, b, λ, C.

Recall that, since $N > 4$, one has $2^* < 4$. Therefore, one has $p-1 < 2^*-1 < 3$. Then applying Lemma 2.1 with $t = \|u\|_{2^*}$, $\gamma = (2^{-1}\sigma_1^{-1}b)^{\frac{1}{p-4}}$, $x = p-1$, and $y = 3$, one obtains
$$\|u\|_{2^*}^{p-1} \le \left(2^{-1}\sigma_1^{-1}b\right)^{\frac{p-1}{p-4}} + 2^{-1}\sigma_1^{-1}b\|u\|_{2^*}^3. \tag{2.7}$$
By (2.6) and (2.7), one has
$$\|u\|_{2^*}^3 \le 2\sigma_1 b^{-1}\Big(\lambda\rho_g(C) + 1 + (2^{-1}\sigma_1^{-1}b)^{\frac{p-1}{p-4}}\Big)$$
from which (2.4) easily follows. □

Lemma 2.5. *Let $a, b > 0$ and let $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying conditions* (A1) *and* (A2)*, respectively. Then, there exists $C = C(a, b)$, such that for every $\lambda \in (0, \rho_g(C)^{-1})$ and every solution u of* (2.2)*, one has $\|u\|_\infty \le C$.*

Proof. Since $2 < p < 2^*$ and $\frac{N}{2} = \frac{2^*}{2^*-2}$, we can fix $s \in \mathbb{R}$ such that
$$\max\big\{\frac{N}{2}, \frac{2^*}{p-1}\big\} < s < \frac{2^*}{p-2}.$$
Then
$$0 < p - 1 - \frac{2^*}{s} < 1, \quad 0 < 2 - p + \frac{2^*}{s} < 1, \quad s > \frac{N}{2}. \tag{2.8}$$
Now, let $C > 0$, $\lambda \in (0, \rho_g(C)^{-1})$, and let $u \in W_0^{1,2}(\Omega)$ be a solution of (2.2). By Lemma 2.2 we known that $u \in C^1(\overline{\Omega})$. Hence, the function
$$x \in \Omega \to h_{\lambda,C}(x, u(x))$$
belongs to $L^\infty(\Omega)$. By Lemma 2.3, Lemma 2.4, conditions (A1), (A2), and (2.8), we infer, recalling $\lambda\rho_g(C) < 1$, that
$$\begin{aligned}
a\|u_\lambda\|_\infty &\le \sigma_s\Big[\lambda\rho_g(C) + 1 + \Big(\int_\Omega |u_\lambda(x)|^{s(p-1)}\,dx\Big)^{1/s}\Big] \\
&\le \sigma_s\Big[\lambda\rho_g(C) + 1 + \|u_\lambda\|_\infty^{p-1-\frac{2^*}{s}}\|u_\lambda\|_{2^*}^{\frac{2^*}{s}}\Big] \\
&\le \sigma_s'\Big[\lambda\rho_g(C) + 1 + \|u_\lambda\|_\infty^{p-1-\frac{2^*}{s}}\Big(\lambda b^{-1}\rho_g(C) + b^{-1} + b^{\frac{3}{p-4}}\Big)^{2^*/(3s)}\Big] \\
&\le \sigma_s''\Big[1 + \|u_\lambda\|_\infty^{p-1-\frac{2^*}{s}}\Big(b^{-1} + b^{\frac{3}{p-4}}\Big)^{2^*/(3s)}\Big]
\end{aligned}$$
where the constants $\sigma_s, \sigma_s', \sigma_s'' > 0$ are independent of a, b, λ, C. In particular, if $\|u_\lambda\|_\infty \ge 1$, one has (in view of (2.8))
$$a\|u_\lambda\|_\infty^{2-p+\frac{2^*}{s}} \le \sigma_s''\Big[1 + \Big(b^{-1} + b^{\frac{3}{p-4}}\Big)^{2^*/(3s)}\Big]$$
Thus, if C is the constant defined by
$$C^{2-p+\frac{2^*}{s}} = \sigma_s'' a^{-1}\Big[1 + \Big(b^{-1} + b^{\frac{3}{p-4}}\Big)^{2^*/(3s)}\Big] + 1 \tag{2.9}$$
one has in any case $\|u_\lambda\|_\infty \le C$. □

3. Proofs of main results

Proof of Theorem 1.1. Let $a, b > 0$. By conditions (A1) and (A6), we can find two constants $\mu, \tau > 0$ (both depending on a) such that

$$\int_0^\xi f(x,t)dt > \lambda_1\big(\frac{a}{2} + \mu\big)\xi^2 - \tau, \quad \text{for each } \xi \in \mathbb{R} \text{ and a.a. } x \in \Omega.$$

Let ψ be the positive eigenfunction associated with λ_1 and normalized with respect to the norm $\|\cdot\|$. Moreover, put $\theta = \sqrt{\frac{2\tau|\Omega|}{\mu}}$.

By the above inequality, for $b < \beta(a) := \frac{\mu^2}{\tau|\Omega|}$, one gets

$$\begin{aligned}
\frac{a}{2}\|\theta\psi\|^2 + \frac{b}{4}\|\theta\psi\|^4 - \int_\Omega \int_0^{\theta\psi(x)} f(x,t)dt &< \frac{a\theta^2}{2} + \frac{b\theta^4}{4} - \big(\frac{a}{2} + \mu\big)\theta^2 + \tau|\Omega| \\
&= -\mu\theta^2 + \frac{b\theta^4}{4} + \tau|\Omega| \\
&= -\tau|\Omega| + \frac{4b\tau^2|\Omega|^2}{4\mu^2} < 0.
\end{aligned}$$

Thus, if we consider the functional $I_f : W_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$I_f(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega \int_0^{u(x)} f(x,t)dt, \quad \text{for all } u \in W_0^{1,2}(\Omega),$$

we realize that

$$\inf_{W_0^{1,2}(\Omega)} I_f < 0, \quad \text{if } 0 < b \le \beta(a). \tag{3.1}$$

Now, by (A1) and (A5), we can also find two constants $\delta, \eta > 0$ such that

$$\int_0^\xi f(x,t)dt \le \lambda_1\big(\frac{a}{2} - \eta\big)|\xi|^2 + \delta|\xi|^p,$$

for each $\xi \in \mathbb{R}$ and a.a. $x \in \Omega$. Consequently,

$$I_f(u) \ge \frac{a}{2}\|u\|^2 - \big(\frac{a}{2} - \eta\big)\|u\|^2 - \frac{\delta c_p^p}{p}\|u\|^p = \eta\|u\|^2 - \frac{\delta c_p^p}{p}\|u\|^p,$$

for each $u \in W_0^{1,2}(\Omega)$. Therefore, since $p > 2$, if we fix

$$0 < \epsilon < \Big(\frac{\eta p}{\delta c_p^p}\Big)^{\frac{1}{p-2}}$$

and take (3.1) into account, we obtain, for all $b \in (0, \beta(a))$,

$$\inf_{\|u\|=\epsilon} I_f(u) > 0 = I_f(0) = \inf_{\|u\|\le\epsilon} I_f(u) > \inf_{u \in W_0^{1,2}(\Omega)} I_f(u). \tag{3.2}$$

Now, let $\lambda \in \mathbb{R}$, and let $C = C(a,b) > 0$ be the constant defined in (2.9). Moreover, let $h_{\lambda,C}$ be the function defined in (2.1). Since $p < 2^* < 4$, by assumptions (1.2) and (1.3), it easily follows that

$$\lim_{\|u\|\to+\infty} I_{h_{\lambda,C}}(u) = +\infty. \tag{3.3}$$

Since, by standard results, $I_{h_{\lambda,C}}$ is sequentially weakly lower semicontinuous in $W_0^{1,2}(\Omega)$, we infer that $I_{h_{\lambda,C}}$ is bounded below on $W_0^{1,2}(\Omega)$ as well. Consequently, we can consider the functions $\omega, \omega_1 : \mathbb{R} \to \mathbb{R}$ defined by

$$\omega(\lambda) = \inf_{\|u\|=\epsilon} I_{h_{\lambda,C}}(u) - \inf_{\|u\|\le\epsilon} I_{h_{\lambda,C}}(u),$$
$$\omega_1(\lambda) = \inf_{\|u\|\le\epsilon} I_{h_{\lambda,C}}(u) - \inf_{u\in W_0^{1,2}(\Omega)} I_{h_{\lambda,C}}(u)$$

for each $\lambda \in \mathbb{R}$. Since $\lambda \in \mathbb{R} \to I_{h_{\lambda,C}}(u)$ is an affine function for each $u \in W_0^{1,2}(\Omega)$, we have that the functions ω, ω_1 are both the difference of two concave functions, and so they are continuous in $\mathbb{R}$. By (3.2), we also have

$$\omega(0) = \inf_{\|u\|=\epsilon} I_f(u) - \inf_{\|u\|\le\epsilon} I_f(u) > 0$$
$$\omega_1(0) = \inf_{\|u\|\le\epsilon} I_f(u) - \inf_{u\in W_0^{1,2}(\Omega)} I_f(u) > 0$$

Thus, by the continuity of ω and ω_1, we can find $\lambda(a, b) \in (0, \rho_g(C)^{-1})$ such that

$$\omega(\lambda) = \inf_{\|u\|=\epsilon} I_{h_{\lambda,C}}(u) - \inf_{\|u\|\le\epsilon} I_{h_{\lambda,C}}(u) > 0$$
$$\omega_1(\lambda) = \inf_{\|u\|\le\epsilon} I_{h_{\lambda,C}}(u) - \inf_{u\in W_0^{1,2}(\Omega)} I_{h_{\lambda,C}}(u) > 0$$

for each $\lambda \in [0, \lambda(a, b)]$. Fix $\lambda \in [0, \lambda(a, b)]$. By the above two inequalities and by the sequential weak lower semicontinuity of $I_{h_{\lambda,C}}$, one infers that

- $I_{h_{\lambda,C}}$ admits a local minimum point $u_\lambda \in W_0^{1,2}(\Omega)$, such that $\|u_\lambda\| < \epsilon$;
- $I_{h_{\lambda,C}}$ admits a global minimum point $v_\lambda \in W_0^{1,2}(\Omega)$,

with

$$I_{h_{\lambda,C}}(v_\lambda) < I_{h_{\lambda,C}}(u_\lambda) = \inf_{\|u\|\le\epsilon} I_{h_{\lambda,C}}(u) < \inf_{\|u\|=\epsilon} I_{h_{\lambda,C}}(u).$$

Of course, u_λ, v_λ are critical points of $I_{h_{\lambda,C}}$. Observe also that the inequality $I_{h_{\lambda,C}}(v_\lambda) < \inf_{\|u\|\le\epsilon} I_{h_{\lambda,C}}(u)$ implies $\|v_\lambda\| > \epsilon$. Hence, in particular, the functional $I_{h_{\lambda,C}}$ turns out to have the mountain pass geometry. In addiction, we know, again by standard results, that:

- the functional
$$u \in W_0^{1,2}(\Omega) \to \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4$$
is differentiable in $W_0^{1,2}(\Omega)$ with continuously invertible derivative;
- the functional
$$u \in W_0^{1,2}(\Omega) \to \int_\Omega \Big(\int_0^{u(x)} h_{\lambda,C}(x, t)dt \Big)\, dx$$
is differentiable in $W_0^{1,2}(\Omega)$ with compact derivative.

Therefore, taking (3.3) into account, we infer that $I_{h_{\lambda,C}}$ satisfies the Palais-Smale condition (see, for instance, Example [9, 38.25]). By applying the classical Mountain Pass Theorem by Ambrosetti-Rabinowitz, we derive the existence of a third critical point w_λ for $I_{h_{\lambda,C}}$, which is of mountain pass type. Finally, since $\lambda \in (0, \rho_g(C))$ and C is as in (2.9), by Lemma 2.5 we conclude that $u_\lambda, v_\lambda, w_\lambda$ are three distinct solutions of (1.1). $\square$

Proof of Theorem 1.2. Let $a, b > 0$. By conditions (A1) and (A7), we can find two constants μ, τ (depending on a) such that

$$\int_0^\xi f(x,t)dt > \big(\frac{a}{2} + \mu\big)\xi^2 - \tau|\xi|^p, \quad \text{for each } \xi \in \mathbb{R} \text{ and a.a. } x \in \Omega.$$

Let ψ be the positive eigenfunction associated with λ_1 and normalized with respect to the norm $\|\cdot\|$. Since $p > 2$, one has

$$-\mu\theta^2\|\psi\|^2 + \frac{b\theta^4}{4}\|\psi\|^4 + \tau\theta^p\|\psi\|_p^p = \theta^2\Big(- \mu + \frac{b\theta^2}{4} + \tau\theta^{p-2}\|\psi\|_p^p\Big) < 0.$$

for $\theta > 0$ small enough.

Fix such a θ and let $C = C(a, b)$ be the constant defined in (2.9). By the previous inequality, we can find $\lambda(a, b) \in (0, \rho_g(C))$ such that, for $\lambda \in [0, \lambda(a, b)]$ and $h_{\lambda,C}$ as in (2.1), one has

$$\begin{aligned} I_{h_{\lambda,C}}(\theta\psi) &= \frac{a\theta^2}{2} + \frac{b\theta^4}{4} - \int_\Omega \Big(\int_0^{\theta\psi} h_{\lambda,C}(x,t)dt\Big)\, dx \\ &\le \frac{a\theta^2}{2} + \frac{b\theta^4}{4} - \big(\frac{a}{2} + \mu\big)\theta^2 + \tau\theta^p\|\psi\|_p^p + \lambda\theta\rho_g(C)\|\psi\|_1 \\ &\le \theta^2\Big(- \mu + \frac{b\theta^2}{4} + \tau\theta^{p-2}\|\psi\|_p^p\Big) + \lambda\theta\rho_g(C)\|\psi\|_1 < 0 \end{aligned}$$

This means that

$$\inf_{W_0^{1,2}(\Omega)} I_{h_{\lambda,C}} < 0.$$

Since $I_{h_{\lambda,C}}$ also satisfies the coercivity condition (3.3), then $I_{h_{\lambda,C}}$ admits a nonzero global minimum point u_λ, which is a solution of (1.1) in view of the condition $\lambda < \rho_g(C)^{-1}$ and Lemma 2.5. □

4. Conclusion

In this paper, we have considered a supercritical non local problem of Kirchhoff type and we have proved, via variational methods and truncation arguments, both existence and multiplicity results. The main feature of these results is that the presence of the nonlocal term allows to obtain the multiplicity of solutions even in the supercritical case. We point out that we found very few results where the multiplicity of solutions is established for critical or supercritical problems. Among them, we have mentioned the interesting paper [4]. In [4], the right hand-side in the problem considered there is a sum of a subcritical nonlinearity multiplied by a parameter λ and a critical nonlinearity (of power-type). Therefore, the problem considered in [4] is different from problem (1.1) considered here, where, instead, the parameter λ multiplies the supercritical term. We think that an interesting question is to investigate, by the approach used in present paper, the possible extension to the supercritical case of the multiplicity result obtained in [4].

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