Electronic Journal of Differential Equations, Vol. 2023 (2023), No. 21, pp. 1–15.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu or https://ejde.math.unt.edu

HYERS-ULAM STABILITY OF LINEAR QUATERNION-VALUED DIFFERENTIAL EQUATIONS

JIAOJIAO LV, JINRONG WANG, RUI LIU

ABSTRACT. In this article, we study the Hyers-Ulam stability of the first-order linear quaternion-valued differential equations. We transfer a linear quaternion-valued differential equation into a real differential system. The Hyers-Ulam stability results for the linear quaternion-valued differential equations are obtained according to the equivalent relationship between the vector 2-norm and the quaternion module.

1. INTRODUCTION

Quaternion is a noncommutative algebra that extend the field of complex numbers. Quaternion-valued differential equations (QDEs) are widely used in real life, such as life science [7, 21], neural networks [17, 20, 26] and quantum mechanics [2, 3, 14].

In recent years, the basic theory of QDEs has been developed by many researchers. For example, Kou and Xia [15] introduced the definition of Liouville formula and Wronskian in the sense of quaternion, studied the solution of linear QDEs, and developed two methods to calculate the fundamental matrix. Kou et al. [16] proposed a method to find the fundamental matrix of linear systems with multiple eigenvalues. Zhang [28] studied the global structure of the quaternion Bernoulli equation. Xia et al. [24, 25] gave the stability results of quaternion periodic systems and the variation of constants formula in the sense of quaternion, and presented an algorithm for solving linear non-homogeneous QDEs. Chen et al. [4] derived an explicit quaternion norm estimation in the sense of quaternion, proved that the first-order linear QDEs is asymptotically stable and Hyers-Ulam stable and the nth-order linear QDEs is generalized Hyers-Ulam stable. Meanwhile, Suo et al. [22, 23] gave the expression of solutions for linear quaternion-valued impulsive differential equations in the sense of complex numbers and quaternion. Further, the periodic solutions of linear homogeneous and nonhomogeneous quaternion-valued impulsive differential equations were considered. Chen et al. [5, 6] used Laplace transform to derive the Hyers-Ulam stability of linear QDEs, and utilized a new method to study the controllability and observability of linear quaternion-valued systems from the perspective of complex valued systems. Fu et al. [8] derived the solutions of homogeneous and nonhomogeneous linear QDEs under the permutation matrix hypothesis based on delayed quaternion matrix exponential and variation of

2020 *Mathematics Subject Classification.* 34D20.

Key words and phrases. Hyers-Ulam stability; linear quaternion-valued differential equation.

Submitted January 4, 2023. Published February 27, 2023.

constants. Lv et al. [18] studied the Hyers-Ulam stability of linear QDEs by Fourier transform. Zou et al. [27] considered the Hyers-Ulam stability of linear recurrence equations with constant coefficients in the sense of quaternion. In addition, Huang et al. [9] studied the stability of QDEs by means of the second Lyapunov method in the sense of quaternion and Zahid et al. [29] derived the exponential matrix of QDEs.

It is remarkable that Jung [11] proved the Hyers-Ulam stability of first-order differential systems with constant coefficients by matrix method. Further, Jung [12, 13] also proved the generalized Hyers-Ulam stability of differential equations and first-order matrix differential equations. Motivated by [11, 12, 13], we study the Hyers-Ulam stability of first-order linear QDEs via the different approach used in [4, 5]. In the current paper, we consider the Hyers-Ulam stability of quaternion homogeneous differential equation

$$f'(t) = \lambda f(t),\ t \in I, \tag{1.1}$$

and quaternion nonhomogeneous differential equation

$$f'(t) = \lambda f(t) + u(t),\ t \in I, \tag{1.2}$$

where $I = [0, s) \subseteq [0, +\infty)$, $f : I \to \mathbb{H}$ is continuously differentiable function, $u : I \to \mathbb{H}$ is continuous function, and $\lambda = a + bi + cj + dk \in \mathbb{H}$ is quaternion constant. Meanwhile, we consider the generalized Hyers-Ulam stability of QDE (1.1) and (1.2) when $\lambda : I \to \mathbb{H}$ is continuous function.

To achieve our aim, we transfer the desired linear QDEs into 4-dimensional real differential systems. By developing the approach in [11, 12, 13] via the equivalence between the module of a quaternion and the 2-norm of the corresponding 4-dimensional real vector, we provide a new framework to show that linear QDEs are Hyers-Ulam type stable.

Note that Chen et al. [4, 5] studied the Hyers-Ulam stability of first-order matrix differential equations by using the norm estimation of exponential functions of quaternion matrices and derived the Hyers-Ulam stability of linear quaternion-valued differential equations by using the Laplace transform. However, we adopt a different idea to complete the study. Compared with [4, 5], we adopt a different approach to deal with the same issue. We state our contribution as follows. We first transfer the desired differential equation into a suitable differential system. Then, we apply the knowledge of real differential system via the equivalent relationship between the vector 2-norm and the quaternion module to obtain the Hyers-Ulam stability of the original equation. In addition, it is worth to point that the norm estimation of matrix exponential function can be directly obtained by using the matrix 2-norm, which is much different from [4, 5].

The organizational structure of this article is as follows. In Section 2, we provide some necessary preparations and give 2-norm estimation of matrix function e^{At} ($t \in I$). In Section 3, we give the Hyers-Ulam stability results of $f'(t) = f(t)$, $f'(t) = \lambda f(t)$, $f'(t) = \lambda f(t) + u(t)$, $f'(t) = \lambda(t) f(t)$ and $f'(t) = \lambda(t) f(t) + u(t)$ for $t \in I$. In Section 4, we give two examples to illustrate the validity of these results.

2. Preliminaries

This part introduces some basic symbols, definitions, and concepts of quaternion algebra [15, 24, 25]. Let $\mathbb{H}$ stand for the quaternion set, $\mathbb{R}$ represent the field of real numbers and $\mathbb{C}$ represent the field of complex numbers. If a quaternion $q \in \mathbb{H}$, then

$q = q_0 + q_1 i + q_2 j + q_3 k$, where $q_0,\ q_1,\ q_2,\ q_3 \in \mathbb{R}$ and i, j, k are imaginary units that satisfy $ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,\ i^2 = j^2 = k^2 = ijk = -1$.

A quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ can be denote by a 4-dimensional real vector $p = (q_0, q_1, q_2, q_3)^T \in \mathbb{R}^4$, the vector norm of p can be defined as $\|p\|_2 = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$. In addition, the module of any quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ can be expressed as $|q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$. It is remarkable that the module of a quaternion q is equivalent to the norm of the corresponding 4-dimensional real vector of q, i.e.,

$$\|p\|_2 = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = |q|. \tag{2.1}$$

Let $f : I \to \mathbb{H}$ be a quaternion-valued function, where $I = [0, s) \subseteq [0, +\infty)$. Denote the set of quaternion-valued functions by $\mathbb{H} \otimes \mathbb{R}$, and the derivative of $f \in \mathbb{H} \otimes \mathbb{R}$ is

$$f'(t) = f_1'(t) + f_2'(t)i + f_3'(t)j + f_4'(t)k. \tag{2.2}$$

We denote a 4×4 real matrix A by

$$A = aE + B, \tag{2.3}$$

where

$$B = \begin{pmatrix} 0 & -b & -c & -d \\ b & 0 & -d & c \\ c & d & 0 & -b \\ d & -c & b & 0 \end{pmatrix},$$

where E is a 4×4 unit matrix and $b, c, d \in \mathbb{R}$ are not all zero.

Definition 2.1. The QDE (1.1) is called Hyers-Ulam stable on I if there exists a constant $M > 0$ such that for every $\epsilon > 0$ and every continuously differentiable $f : I \to \mathbb{H}$ satisfying

$$|f'(t) - \lambda f(t)| \le \epsilon,\ t \in I,$$

there exists a solution f_0 of QDE (1.1) such that

$$|f(t) - f_0(t)| \le M\epsilon,\ t \in I.$$

Definition 2.2. Let $\varphi : I \to [0, \infty)$ and $\lambda : I \to \mathbb{H}$ be a continuous function. The QDE (1.2) is called generalized Hyers-Ulam stable on I if for every continuously differential function $f : I \to \mathbb{H}$ satisfying the inequality

$$|f'(t) - \lambda(t)f(t) - u(t)| \le \varphi(t),\ t \in I,$$

there exists a solution $f_0 : I \to \mathbb{H}$ of (1.2) such that

$$|f(t) - f_0(t)| \le \|Y(t)\|_2 \int_0^t \|Y(\tau)^{-1}\|_2 \varphi(\tau) d\tau, \quad t \in I,$$

where $Y(t)_{4\times 4}$ is a fundamental matrix of $y'(t) = A(t)y(t)$.

Lemma 2.3 ([19]). *The matrix B in* (2.3) *is diagonalizable.*

Lemma 2.4 ([19]). *The eigenvalues of the matrix B in* (2.3) *are*

$$\lambda_1 = i\sqrt{b^2 + c^2 + d^2}, \quad \lambda_2 = i\sqrt{b^2 + c^2 + d^2},$$
$$\lambda_3 = -i\sqrt{b^2 + c^2 + d^2}, \quad \lambda_4 = -i\sqrt{b^2 + c^2 + d^2},$$

where i is an imaginary unit.

Lemma 2.5 ([10, Theorem 1.3.3.]). *Let $A, B \in \mathbb{C}^{n\times n}$. If B is similar to A, then A and B have the same characteristic polynomial.*

Theorem 2.6. *The eigenvalues of matrix A in* (2.3) *are*

$$\lambda_{A1} = a + \lambda_1, \quad \lambda_{A2} = a + \lambda_2,$$
$$\lambda_{A3} = a + \lambda_3, \quad \lambda_{A4} = a + \lambda_4.$$

Proof. According to Lemma 2.3, there exists a unitary matrix P such that $B = PDP^{-1}$, where $D = diag\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. Since

$$A = aE + B = PaEP^{-1} + PDP^{-1} = P(aE + D)P^{-1},$$

matrix A is similar to matrix $(aE + D)$. According to Lemma 2.4 and Lemma 2.5, it can be concluded that the eigenvalues of matrix A are $\lambda_{A1}, \lambda_{A2}, \lambda_{A3}, \lambda_{A4}$. The proof is complete. $\square$

Definition 2.7 ([10, Definition 5.7.12.]). A norm $\|\cdot\|$ on $\mathbb{C}^n$ and a matrix norm $\|\cdot\|_m$ on $\mathbb{C}^{n\times n}$ are compatible if $\|Ax\| \le \|A\|_m \|x\|$ for all $A \in \mathbb{C}^{n\times n}$ and $x \in \mathbb{C}^n$.

In this article, we use the 2-norm of matrix $A \in \mathbb{C}^{n\times n}$ that is defined as

$$\beta = \|A\|_2 = \max\{\sqrt{\lambda} : \lambda \text{ are eigenvalues of matrix } A^*A\}, \tag{2.4}$$

where $A^* = \bar{A}^T = \overline{A^T}$ represents the conjugate transpose of matrix A. Now we have the following 2-norm estimation of matrix function $e^{A\cdot}$.

Lemma 2.8. *For a matrix $A \in \mathbb{R}^{n\times n}$, one has $\|e^{At}\|_2 \le e^{\beta t}$ for $t \in I$, where β is defined in* (2.4).

Proof. In fact,

$$\begin{aligned}
\|e^{At}\|_2 &= \Big\| \sum_{n=0}^{+\infty} \frac{A^n t^n}{n!} \Big\|_2 \\
&\le \sum_{n=0}^{+\infty} \frac{t^n}{n!} \|A^n\|_2 \\
&\le \sum_{n=0}^{+\infty} \frac{t^n \|A\|_2^n}{n!} \\
&\le \sum_{n=0}^{+\infty} \frac{t^n \beta^n}{n!} = e^{\beta t}.
\end{aligned}$$

The proof is complete. $\square$

3. Main results

In this section, we study the Hyers-Ulam stability and generalized Hyers-Ulam stability of first-order linear QDEs.

3.1. **Hyers-Ulam stability of $f'(t) = f(t)$, $t \in I$.** In this subsection, we consider the Hyers-Ulam stability of QDE (1.1) when $\lambda = 1$.

Theorem 3.1. *If $\lambda = 1$, then for any $\epsilon > 0$ and every continuously differentiable $f : I \to \mathbb{H}$ satisfying*

$$|f'(t) - f(t)| \le \epsilon,$$

there exists a solution $f_0 : I \to \mathbb{H}$ of (1.1) such that

$$|f(t) - f_0(t)| \le \epsilon,$$

that is, QDE (1.1) is Hyers-Ulam stable.

Proof. Let $v(t) = f'(t) - f(t)$, then $|v(t)| \le \epsilon$ and

$$f'(t) = f(t) + v(t). \tag{3.1}$$

By (2.2), we have

$$\begin{aligned} f'(t) &= f_1'(t) + f_2'(t)i + f_3'(t)j + f_4'(t)k \\ &= f_1(t) + f_2(t)i + f_3(t)j + f_4(t)k + v_1(t) + v_2(t)i + v_3(t)j + v_4(t)k \\ &= (f_1(t) + v_1(t)) + (f_2(t) + v_2(t))i + (f_3(t) + v_3(t))j + (f_4(t) + v_4(t))k. \end{aligned}$$

If two quaternion-valued function are equal, then their corresponding real and imaginary parts are the same. Therefore,

$$f_l'(t) = f_l(t) + v_l(t), \ l = 1, 2, 3, 4, \tag{3.2}$$

and the solution of nonhomogeneous ordinary differential equation (3.2) is

$$f_l(t) = C_l e^t + e^t \int_0^t e^{-\tau} v_l(\tau) d\tau, \quad l = 1, 2, 3, 4,$$

where $C_l = f_l(0) \in \mathbb{R}$.

Furthermore, we can derive the solution of QDE (3.1) is

$$\begin{aligned} f(t) &= f_1(t) + f_2(t)i + f_3(t)j + f_4(t)k \\ &= C_1 e^t + e^t \int_0^t v_1(\tau) e^{-\tau} d\tau + \Big(C_2 e^t + e^t \int_0^t v_2(\tau) e^{-\tau} d\tau\Big) i \\ &\quad + \Big(C_3 e^t + e^t \int_0^t v_3(\tau) e^{-\tau} d\tau\Big) j + \Big(C_4 e^t + e^t \int_0^t v_4(\tau) e^{-\tau} d\tau\Big) k \\ &= e^t (C_1 + C_2 i + C_3 j + C_4 k) \\ &\quad + \int_0^t e^{(t-\tau)} (v_1(\tau) + v_2(\tau) i + v_3(\tau) j + v_4(\tau) k) d\tau \\ &= e^t q + \int_0^t e^{(t-\tau)} v(\tau) d\tau, \end{aligned} \tag{3.3}$$

where $q = C_1 + C_2 i + C_3 j + C_4 k = f(0) \in \mathbb{H}$. Notice that

$$\lim_{t \to +\infty} \frac{f(t)}{e^t} = f(0) + \int_0^{+\infty} e^{-\tau} v(\tau) d\tau$$

exists, since $|v(\tau)| \le \epsilon$. Let

$$f_0(t) = e^t \Big(f(0) + \int_0^{+\infty} e^{-\tau} v(\tau) d\tau\Big). \tag{3.4}$$

Obviously, when $\lambda = 1$, $f_0(t)$ is a solution to QDE (1.1).

Next, combining (3.3) and (3.4), we have

$$\begin{aligned}|f(t)-f_0(t)| &= \Big|\int_t^{+\infty} e^{t-\tau}v(\tau)d\tau\Big| \\ &\le \int_t^{+\infty} |e^{t-\tau}v(\tau)|d\tau \\ &\le \epsilon e^t\int_t^{+\infty} e^{-\tau}d\tau \le \epsilon.\end{aligned}$$

The proof is complete. □

Remark 3.2. When $\lambda = 1$, which is the special case of QDE (1.1), we take the approach that two quaternions are equal then their real and imaginary parts correspond to each other.

3.2. **Hyers-Ulam stability of $f'(t) = \lambda f(t),\ t \in I$.** In this subsection, we consider the Hyers-Ulam stability of QDE (1.1) when $\lambda \neq 1$.

Theorem 3.3. *There exists $M > 0$ such that for any $\epsilon > 0$ and every continuously differentiable $f : I \to \mathbb{H}$ satisfying*

$$|f'(t) - \lambda f(t)| \le \epsilon,$$

there exists a solution $f_0 : I \to \mathbb{H}$ of (1.1) such that

$$|f(t) - f_0(t)| \le M\epsilon,$$

that is, QDE (1.1) is Hyers-Ulam stable.

Proof. Let $v(t) = f'(t) - \lambda f(t)$, then $|v(t)| \le \epsilon$ and $f'(t) = \lambda f(t) + v(t)$. By (2.2), we have

$$\begin{aligned}f'(t) &= f_1'(t) + f_2'(t)i + f_3'(t)j + f_4'(t)k \\ &= (a + bi + cj + dk)(f_1(t) + f_2(t)i + f_3(t)j + f_4(t)k) \\ &\quad + v_1(t) + v_2(t)i + v_3(t)j + v_4(t)k \\ &= af_1(t) - bf_2(t) - cf_3(t) - df_4(t) + (bf_1(t) + af_2(t) - df_3(t) \\ &\quad + cf_4(t))i + (cf_1(t) + df_2(t) + af_3(t) - bf_4(t))j + (df_1(t) - cf_2(t) \\ &\quad + bf_3(t) + af_4(t))k + v_1(t) + v_2(t)i + v_3(t)j + v_4(t)k.\end{aligned} \tag{3.5}$$

Obviously, equation (3.5) is equivalent to the following real linear differential system

$$\begin{pmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}. \tag{3.6}$$

Let $y = (f_1, f_2, f_3, f_4)^T$, $v = (v_1, v_2, v_3, v_4)^T$, and

$$A = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}.$$

Then system (3.6) can be written as $y'(t) = Ay(t) + v(t)$, and the solution for system (3.6) is

$$y(t) = e^{At}y(0) + e^{At}\int_0^t e^{-A\tau}v(\tau)d\tau, \tag{3.7}$$

where $y(0) = (f_1(0), f_2(0), f_3(0), f_4(0))^T \in \mathbb{R}^4$.

In addition, QDE (1.1) can be represented as $y'(t) = Ay(t)$, $t \in I$. Let

$$y_0(t) = e^{At}y(0), \quad t \in I. \tag{3.8}$$

Obviously, $y_0 = (f_1, f_2, f_3, f_4)^T$ is a solution to QDE (1.1).

By (2.4) and Theorem 2.6, the 2-norm of A is $\|A\|_2 = \sqrt{a^2+b^2+c^2+d^2}$. Next, set $M = \frac{e^{\sqrt{a^2+b^2+c^2+d^2}s}-1}{\sqrt{a^2+b^2+c^2+d^2}}$. Then by (3.7), (3.8) and Definition 2.7, using Lemma 2.8, we have

$$\begin{aligned}
\|y(t) - y_0(t)\|_2 &\le \int_0^t \|e^{A(t-\tau)}v(\tau)\|_2 d\tau \\
&\le \int_0^t \|e^{A(t-\tau)}\|_2 \|v(\tau)\|_2 d\tau \\
&\le \epsilon \int_0^t e^{\sqrt{a^2+b^2+c^2+d^2}(t-\tau)} d\tau \\
&= \frac{\epsilon}{\sqrt{a^2+b^2+c^2+d^2}}(e^{\sqrt{a^2+b^2+c^2+d^2}t} - 1) \\
&\le \frac{\epsilon}{\sqrt{a^2+b^2+c^2+d^2}}(e^{\sqrt{a^2+b^2+c^2+d^2}s} - 1) = M\epsilon.
\end{aligned}$$

Finally, by (2.1), we obtain $|f - f_0| \le M\epsilon$. The proof is complete. □

Remark 3.4. When $s < +\infty$, QDE (1.1) is Hyers-Ulam stable.

3.3. **Hyers-Ulam stability of $f'(t) = \lambda f(t) + u(t)$, $t \in I$.** In this subsection, we consider the Hyers-Ulam stability of QDE (1.2).

Theorem 3.5. *There exists $M > 0$ such that for any $\epsilon > 0$ and every continuously differentiable $f : I \to \mathbb{H}$ satisfying*

$$|f'(t) - \lambda f(t) - u(t)| \le \epsilon,$$

there exists a solution $f_0 : I \to \mathbb{H}$ of (1.2) such that

$$|f(t) - f_0(t)| \le M\epsilon,$$

that is, QDE (1.2) is Hyers-Ulam stable.

Proof. Let $v(t) = f'(t) - \lambda f(t) - u(t)$, then $|v(t)| \le \epsilon$ and $f'(t) = \lambda f(t) + u(t) + v(t)$. By (2.2), we have

$$\begin{aligned}
f'(t) &= f_1'(t) + f_2'(t)i + f_3'(t)j + f_4'(t)k \\
&= af_1(t) - bf_2(t) - cf_3(t) - df_4(t) + (bf_1(t) + af_2(t) - df_3(t) \\
&\quad + cf_4(t))i + (cf_1(t) + df_2(t) + af_3(t) - bf_4(t))j + (df_1(t) - cf_2(t) \\
&\quad + bf_3(t) + af_4(t))k + (v_1(t) + u_1(t)) + (v_2(t) + u_1(t))i + (v_3(t) \\
&\quad + u_3(t))j + (v_4(t) + u_3(t))k.
\end{aligned} \tag{3.9}$$

Obviously, equation (3.9) is equivalent to the following real linear differential system

$$\begin{pmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} + \begin{pmatrix} v_1+u_1 \\ v_2+u_2 \\ v_3+u_3 \\ v_4+u_4 \end{pmatrix}. \tag{3.10}$$

Let $y = (f_1, f_2, f_3, f_4)^T$, $w = (v_1+u_1, v_2+u_2, v_3+u_3, v_4+u_4)^T$, and

$$A = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}.$$

Then system (3.10) can be written as $y'(t) = Ay(t) + w(t)$, and the solution for system (3.10) is

$$y(t) = e^{At}y(0) + e^{At}\int_0^t e^{-A\tau}w(\tau)d\tau. \tag{3.11}$$

In addition, QDE (1.2) can be represented as $y'(t) = Ay(t) + u(t)$. Let

$$y_0(t) = e^{At}y(0) + e^{At}\int_0^t e^{-A\tau}u(\tau)d\tau, \tag{3.12}$$

where $y(0) = (f_1(0), f_2(0), f_3(0), f_4(0))^T \in \mathbb{R}^4$. Obviously, $y_0 = (f_1, f_2, f_3, f_4)^T$ is a solution to (1.2).

By (3.11), (3.12) and Definition 2.7, using Lemma 2.8, we have

$$\begin{aligned} \|y(t) - y_0(t)\|_2 &\le \int_0^t \|e^{A(t-\tau)}v(\tau)\|_2 d\tau \\ &\le \int_0^t \|e^{A(t-\tau)}\|_2 \|v(\tau)\|_2 d\tau \\ &\le \epsilon \int_0^t e^{\sqrt{a^2+b^2+c^2+d^2}(t-\tau)} d\tau \\ &= \frac{\epsilon}{\sqrt{a^2+b^2+c^2+d^2}}(e^{\sqrt{a^2+b^2+c^2+d^2}t} - 1) \\ &\le \frac{e^{\sqrt{a^2+b^2+c^2+d^2}s} - 1}{\sqrt{a^2+b^2+c^2+d^2}}\epsilon = M\epsilon, \end{aligned}$$

where $t \in I = [0, s) \subseteq [0, +\infty)$ and $M = \frac{e^{\sqrt{a^2+b^2+c^2+d^2}s} - 1}{\sqrt{a^2+b^2+c^2+d^2}}$.

Finally, by (2.1), we can get $|f - f_0| \le M\epsilon$. The proof is complete. $\square$

Remark 3.6. Notice that QDE (1.2) is Hyers-Ulam stable when $s < +\infty$.

3.4. **Generalized Hyers-Ulam stability of $f'(t) = \lambda(t)f(t)$, $t \in I$.** In this subsection, we consider the Hyers-Ulam stability of QDE (1.1) when $\lambda : I \to \mathbb{H}$ is a continuous function.

Theorem 3.7. *Let $\varphi : I \to [0, \infty)$ and $\lambda : I \to \mathbb{H}$ be a continuous function. QDE (1.1) is generalized Hyers-Ulam stable that is for every continuously differentiable function $f : I \to \mathbb{H}$ satisfying the inequality*

$$|f'(t) - \lambda(t)f(t)| \le \varphi(t), \quad t \in I,$$

there exists a solution $f_0 : I \to \mathbb{H}$ of (1.1) *such that*

$$|f(t) - f_0(t)| \le \|Y(t)\|_2 \int_0^t \|Y(\tau)^{-1}\|_2 \varphi(\tau) d\tau, \tag{3.13}$$

where $Y(t)_{4\times 4}$ is a fundamental matrix of $y'(t) = A(t)y(t)$.

Proof. Let $v(t) = f'(t) - \lambda(t)f(t)$, then $|v(t)| \le \varphi(t)$ and $f'(t) = \lambda(t)f(t) + v(t)$. By (2.2), we have

$$\begin{aligned} f'(t) &= f_1'(t) + f_2'(t)i + f_3'(t)j + f_4'(t)k \\ &= a(t)f_1(t) - b(t)f_2(t) - c(t)f_3(t) - d(t)f_4(t) + (b(t)f_1(t) \\ &\quad + a(t)f_2(t) - d(t)f_3(t) + c(t)f_4(t))i \\ &\quad + (c(t)f_1(t) + d(t)f_2(t) + a(t)f_3(t) - b(t)f_4(t))j \\ &\quad + (d(t)f_1(t) - c(t)f_2(t) + b(t)f_3(t) + a(t)f_4(t))k \\ &\quad + v_1(t) + v_2(t)i + v_3(t)j + v_4(t)k. \end{aligned} \tag{3.14}$$

Obviously, equation (3.14) is equivalent to the following real linear differential system

$$\begin{pmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} a(t) & -b(t) & -c(t) & -d(t) \\ b(t) & a(t) & -d(t) & c(t) \\ c(t) & d(t) & a(t) & -b(t) \\ d(t) & -c(t) & b(t) & a(t) \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}. \tag{3.15}$$

Let $y = (f_1, f_2, f_3, f_4)^T$, $v = (v_1, v_2, v_3, v_4)^T$, and

$$A(t) = \begin{pmatrix} a(t) & -b(t) & -c(t) & -d(t) \\ b(t) & a(t) & -d(t) & c(t) \\ c(t) & d(t) & a(t) & -b(t) \\ d(t) & -c(t) & b(t) & a(t) \end{pmatrix}.$$

Then system (3.15) can be written as $y'(t) = A(t)y(t) + v(t)$, and the solution for system (3.15) is

$$y(t) = Y(t)\eta + Y(t) \int_0^t Y(\tau)^{-1} v(\tau) d\tau, \tag{3.16}$$

where $\eta \in \mathbb{R}^4$.

In addition, when $\lambda : I \to \mathbb{H}$ is a continuous function, QDE (1.1) can be represented as $y'(t) = A(t)y(t)$. Let

$$y_0(t) = Y(t)\eta, \tag{3.17}$$

where $\eta \in \mathbb{R}^4$. Obviously, $y_0 = (f_1, f_2, f_3, f_4)^T$ is a solution to QDE (1.1).

Next, by (3.16), (3.17) and Definition 2.7, we have

$$\begin{aligned} \|y(t) - y_0(t)\|_2 &= \|Y(t) \int_0^t Y(\tau)^{-1} v(\tau) d\tau\|_2 \\ &\le \|Y(t)\|_2 \| \int_0^t Y(\tau)^{-1} v(\tau) d\tau\|_2 \\ &\le \|Y(t)\|_2 \int_0^t \|Y(\tau)^{-1}\|_2 \|v(\tau)\|_2 d\tau \\ &\le \|Y(t)\|_2 \int_0^t \|Y(\tau)^{-1}\|_2 \varphi(\tau) d\tau. \end{aligned}$$

Finally, by (2.1), we obtain (3.13). The proof is complete. □

Remark 3.8. When $\lambda : I \to \mathbb{H}$ is a continuous function, QDE (1.1) is generalized Hyers-Ulam stable.

3.5. **Generalized Hyers-Ulam stability of** $f'(t) = \lambda(t)f(t) + u(t)$**,** $t \in I$**.** In this subsection, we consider the Hyers-Ulam stability of QDE (1.2) when $\lambda : I \to \mathbb{H}$ is a continuous function.

Theorem 3.9. *Let $\varphi : I \to [0, \infty)$ and $\lambda : I \to \mathbb{H}$ be a continuous function. QDE* (1.2) *is generalized Hyers-Ulam stable that is for every continuously differentiable function $f : I \to \mathbb{H}$ satisfying the inequality*

$$|f'(t) - \lambda(t)f(t) - u(t)| \le \varphi(t), \quad t \in I,$$

there exists a solution $f_0 : I \to \mathbb{H}$ of (1.2) *such that*

$$|f(t) - f_0(t)| \le \|Y(t)\|_2 \int_0^t \|Y(\tau)^{-1}\|_2 \varphi(\tau) d\tau, \tag{3.18}$$

where $Y(t)_{4\times 4}$ is a fundamental matrix of $y'(t) = A(t)y(t)$.

Proof. Let $v(t) = f'(t) - \lambda(t)f(t) - u(t)$, then $|v(t)| \le \varphi(t)$ and $f'(t) = \lambda(t)f(t) + u(t) + v(t)$. By (2.2), we have

$$\begin{aligned} f'(t) &= f_1'(t) + f_2'(t)i + f_3'(t)j + f_4'(t)k \\ &= a(t)f_1(t) - b(t)f_2(t) - c(t)f_3(t) - d(t)f_4(t) \\ &\quad + (b(t)f_1(t) + a(t)f_2(t) - d(t)f_3(t) + c(t)f_4(t))i \\ &\quad + (c(t)f_1(t) + d(t)f_2(t) + a(t)f_3(t) - b(t)f_4(t))j \\ &\quad + (d(t)f_1(t) - c(t)f_2(t) + b(t)f_3(t) + a(t)f_4(t))k \\ &\quad + (v_1(t) + u_1(t)) + (v_2(t) + u_1(t))i + (v_3(t) + u_3(t))j \\ &\quad + (v_4(t) + u_3(t))k. \end{aligned} \tag{3.19}$$

Obviously, equation (3.19) is equivalent to the following real linear differential system

$$\begin{pmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} a(t) & -b(t) & -c(t) & -d(t) \\ b(t) & a(t) & -d(t) & c(t) \\ c(t) & d(t) & a(t) & -b(t) \\ d(t) & -c(t) & b(t) & a(t) \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} + \begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \\ v_3 + u_3 \\ v_4 + u_4 \end{pmatrix}. \tag{3.20}$$

Let $y = (f_1, f_2, f_3, f_4)^T$, $w = (v_1 + u_1, v_2 + u_2, v_3 + u_3, v_4 + u_4)^T$, and

$$A(t) = \begin{pmatrix} a(t) & -b(t) & -c(t) & -d(t) \\ b(t) & a(t) & -d(t) & c(t) \\ c(t) & d(t) & a(t) & -b(t) \\ d(t) & -c(t) & b(t) & a(t) \end{pmatrix}.$$

System (3.20) can be written as $y'(t) = A(t)y(t) + w(t)$, and the solution for system (3.20) is

$$y(t) = Y(t)\eta + Y(t) \int_0^t Y(\tau)^{-1} w(\tau) d\tau, \tag{3.21}$$

where $\eta \in \mathbb{R}^4$.

In addition, when $\lambda : I \to \mathbb{H}$ is a continuous function QDE (1.2) can be represented as $y'(t) = A(t)y(t) + u(t)$. Let

$$y_0 = Y(t)\eta + Y(t)\int_0^t Y(\tau)^{-1}u(\tau)d\tau. \tag{3.22}$$

Obviously, $y_0 = (f_1, f_2, f_3, f_4)^T$ is a solution to QDE (1.2).

Next, by (3.21), (3.22) and Definition 2.7, we have

$$\begin{aligned} \|y(t) - y_0(t)\|_2 &= \|Y(t)\int_0^t Y(\tau)^{-1}v(\tau)d\tau\|_2 \\ &\le \|Y(t)\|_2\|\int_0^t Y(\tau)^{-1}(w(\tau) - u(\tau))d\tau\|_2 \\ &\le \|Y(t)\|_2\int_0^t \|Y(\tau)^{-1}\|_2\|v(\tau)\|_2 d\tau \\ &\le \|Y(t)\|_2\int_0^t \|Y(\tau)^{-1}\|_2\varphi(\tau)d\tau. \end{aligned}$$

Finally, by (2.1), we obtain (3.18). The proof is complete. □

Remark 3.10. When $\lambda : I \to \mathbb{H}$ is a continuous function, QDE (1.2) is generalized Hyers-Ulam stable.

4. Examples

Example 4.1. Consider the quaternion-valued differential equation

$$f'(t) = (-i - j - k)f(t), \quad f(0) = i + j, \quad t \in I. \tag{4.1}$$

Let $v(t) = f'(t) - (-i-j-k)f(t)$, then $|v(t)| \le \epsilon$ and $f'(t) = v(t) + (-i-j-k)f(t)$. Equation (4.1) can be written in the form

$$\begin{pmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}. \tag{4.2}$$

Let $y = (f_1, f_2, f_3, f_4)^T$, $v = (v_1, v_2, v_3, v_4)^T$, and

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}.$$

We can obtain that the eigenvalues of A are $\lambda_1 = \sqrt{3}i$, $\lambda_2 = -\sqrt{3}i$, $\lambda_3 = \sqrt{3}i$, and $\lambda_4 = -\sqrt{3}i$. The solution for system (4.2) is

$$y(t) = e^{At}(0, 1, 1, 0)^T + e^{At}\int_0^t e^{-Ax}v(x)dx. \tag{4.3}$$

In addition, QDE (4.1) can be represented as $y'(t) = Ay(t)$. Let

$$y_0(t) = e^{At}(0, 1, 1, 0)^T. \tag{4.4}$$

Obviously, $y_0 = (f_1, f_2, f_3, f_4)^T$ is a solution to QDE (4.1). We can find the 2-norm of A is $\|A\|_2 = \sqrt{3}$. Next, combining (4.3) and (4.4), we have

$$\begin{aligned}
\|y(t) - y_0(t)\|_2 &= \|e^{At} f(0) + e^{At} \int_0^t e^{-Ax} v(x)dx - e^{At} y(0)\|_2 \\
&\le \int_0^t \|e^{A(t-x)} v(x)\|_2 dx \\
&\le \int_0^t \|e^{A(t-x)}\|_2 \|v(x)\|_2 dx \\
&\le \epsilon \int_0^t e^{\sqrt{3}(t-x)} dx \\
&= \frac{\epsilon}{\sqrt{3}} (e^{\sqrt{3}t} - 1) \\
&\le \frac{\epsilon}{\sqrt{3}} (e^{\sqrt{3}s} - 1),
\end{aligned}$$

where $t \in I = [0, s) \subseteq [0, +\infty)$.

Finally, by (2.1), we obtain $|f(t) - f_0(t)| \le \frac{\epsilon}{\sqrt{3}} (e^{\sqrt{3}s} - 1)$. Notice that (4.1) is Hyers-Ulam stable when $s < +\infty$.

Example 4.2. Consider the quaternion-valued differential equation

$$f'(t) = (1 + i + k)f(t) + (i + k)t, \quad f(0) = i + j, \quad t \in I. \tag{4.5}$$

Let $v(t) = f'(t) - (1 + i + k)f(t) - (i + k)t$, then $|v(t)| \le \epsilon$ and $f'(t) = v(t) + (1 + i + k)f(t) + (i + k)t$. Equation (4.5) can be written in the form

$$\begin{pmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 + t \\ v_3 \\ v_4 + t \end{pmatrix}. \tag{4.6}$$

Let $y = (f_1, f_2, f_3, f_4)^T$, $w = (v_1, v_2 + t, v_3, v_4 + t)^T$, and

$$A = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

We can show that the eigenvalues of A are $\lambda_1 = 1+\sqrt{2}i$, $\lambda_2 = 1-\sqrt{2}i$, $\lambda_3 = 1+\sqrt{2}i$, and $\lambda_4 = 1 - \sqrt{2}i$. Also we can show that the solution for system (4.6) is

$$y(t) = e^{At}(0, 1, 1, 0)^T + e^{At} \int_0^t e^{-Ax} w(x)dx. \tag{4.7}$$

In addition, QDE (4.5) can be represented as $y'(t) = A(t)y(t) + u(t)$. Let

$$y_0(t) = e^{At}(0, 1, 1, 0)^T + e^{At} \int_0^t e^{-Ax} u(x)dx. \tag{4.8}$$

Obviously, $y_0 = (f_1, f_2, f_3, f_4)^T$ is a solution to QDE (4.5). We can find the 2-norm of A is $\|A\|_2 = \sqrt{3}$. Next, combining (4.7) and (4.8),we have

$$\|y(t) - y_0(t)\|_2 = \|e^{At} \int_0^t e^{-Ax}(w(t) - u(x))dx\|_2$$

$$
\begin{aligned}
&\le \int_0^t \|e^{A(t-x)}v(x)\|_2 dx \\
&\le \int_0^t \|e^{A(t-x)}\|_2 \|v(x)\|_2 dx \\
&\le \epsilon \int_0^t e^{\sqrt{3}(t-x)} dx \\
&= \frac{\epsilon}{\sqrt{3}}(e^{\sqrt{3}t} - 1) \\
&\le \frac{\epsilon}{\sqrt{3}}(e^{\sqrt{3}s} - 1),
\end{aligned}
$$

where $t \in I = [0, s) \subseteq [0, +\infty)$. Finally, by (2.1), we obtain $|f(t) - f_0(t)| \le \frac{\epsilon}{\sqrt{3}}(e^{\sqrt{3}s} - 1)$. Notice that (4.5) is Hyers-Ulam stable when $s < +\infty$.

5. Conclusion

We presented the Hyers-Ulam stability and general Hyers-Ulam stability results for first-order linear quaternion-valued differential equations. We transfer the original quaternion problem into a real 4-dimensional matrix problem and develop the classical approach to derive the main theorems. Recently, Anderson and Onitsuka [1] studied the Hyers-Ulam stability of perturbations for a homogeneous linear differential system with 2×2 constant coefficient and obtained some new necessary and sufficient conditions for the linear system, which have been updated the results in [11, 12, 13].

Note that the quaternion module is equivalent to the vector 2-norm, not ∞-norm. If one consider to transfer the idea in [1] for ∞-norm to the same issue over the quaternions for 2-norm, then it will bring a new and interesting problem: whether the equivalence of norms always hold over the quaternions? If yes, then one can try to develop the approach in [1] for linear differential system with 4×4 constant coefficient and discuss the possible real or complex nonzero eigenvalues, which is associated with the original quaternion-valued problem. In future work, we will study such problems.

Acknowledgments. This work was supported by the National Natural Science Foundation of China (12161015), by the Qian Ke He Ping Tai Ren Cai-YSZ[2022] 002, Guizhou Data Driven Modeling Learning and Optimization Innovation Team ([2020] 5016), and by the Major Project of Guizhou Postgraduate Education and Teaching Reform (YJSJGKT[2021]041). The authors are grateful to the referees for their careful reading of the manuscript and valuable comments. Also the authors want to thank the journal editor for his help.

References

[1] D. R. Anderson, M. Onitsuka; *Hyers-Ulam stability for differential systems with* 2×2 *constant coefficient matrix*, Results in Mathematics, 77 (2022), Art. 136.

[2] D. C. Brody, E. M. Graefe; *On complexified mechanics and coquaternions*, Journal of Physics A: Mathematical and Theoretical, 44 (2011), 072001.

[3] L. Borsten, D. Dahanayake, M. J. Duff, et al.; *Black holes, qubits and octonions*, Physics Reports, 471(2009), 113–219.

[4] D. Chen, M. Fečkan, J. Wang; *On the stability of linear quaternion-valued differential equations*, Qualitative Theory of Dynamical Systems, 21 (2022), Art. 9.

[5] D. Chen, M. Fečkan, J. Wang; *Hyers-Ulam stability for linear quaternion-valued differential equations with constant coefficient*, Rocky Mountain Journal of Mathematics, 52 (2022), 1237–1250.
[6] D. Chen, M. Fečkan, J. Wang; *Investigation of controllability and observability for linear quaternion-valued systems from its complex-valued systems*, Qualitative Theory of Dynamical Systems, 21 (2022), Art. 66.
[7] C. M. Carlevaro, R. M. Irastorza, F. Vericat; *Quaternionic representation of the genetic code*, Biosystems, 141 (2016), 10–19.
[8] T. Fu, K. Kou, J. Wang; *Representation of solutions to linear quaternion differential equations with delay*, Qualitative Theory of Dynamical Systems, 21 (2022), Art. 118.
[9] S. Huang, F. He; *On the second Lyapunov method for quaternionic differential equations*, Qualitative Theory of Dynamical Systems, 20 (2021), Art. 41.
[10] R. A. Horn, C. R. Johnson; *Matrix Analysis*, Cambridge University Press, New York, 2012, ISBN 978-0-521-83940-2.
[11] S. M. Jung; *Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients*, Journal of Mathematical Analysis and Applications, 320 (2006), 549–561.
[12] S. M. Jung; *Hyers-Ulam stability of the first-order matrix differential equations*, Journal of Function Spaces, 2015 (2015), 614745.
[13] S. M. Jung; *Hyers-Ulam stability of linear differential equations of first order*, Applied Mathematics Letters, 17 (2004), 1135–1140.
[14] T. Jiang, Z. Jiang, S. Ling; *An algebraic method for quaternion and complex least squares coneigen-problem in quantum mechanics*, Applied Mathematics and Computation, 249 (2014), 222–228.
[15] K. Kou, Y. Xia; *Linear quaternion differential equations: Basic theory and fundamental results*, Studies in Applied Mathematics, 141 (2018), 3–45.
[16] K. Kou, W. Liu, Y. Xia; *Solve the linear quaternion-valued differential equations having multiple eigenvalues*, Journal of Mathematical Physics, 60 (2019), 023510.
[17] M. Kobayashi; *Split quaternion-valued twin-multistate Hopfield neural networks*, Advances in Applied Clifford Algebras, 30 (2020), Art. 30.
[18] J. Lv, K. Kou, J. Wang; *Hyers-Ulam stability of linear quaternion-valued differential equations with constant coefficients via Fourier transform*, Qualitative Theory of Dynamical Systems, 21 (2022), Art. 116.
[19] T. Politi; *A formula for the exponential of a real skew-symmetric matrix of order 4*, BIT Numerical Mathematics, 41 (2001), 842–845.
[20] A. Pratap, R. Raja, J. Alzabut, et al.; *Mittag-Leffler stability and adaptive impulsive synchronization of fractional order neural networks in quaternion field*, Mathematical Methods in the Applied Sciences, 43 (2020), 6223–6253.
[21] J. Prošková; *Description of protein secondary structure using dual quaternions*, Journal of Molecular Structure, 1076 (2014), 89–93.
[22] L. Suo, M. Fečkan, J. Wang; *Quaternion-valued linear impulsive differential equations*, Qualitative Theory of Dynamical Systems, 20 (2021), Art. 33.
[23] L. Suo, M. Fečkan, J. Wang; *Existence of periodic solutions to quaternion-valued impulsive differential equations*, Qualitative Theory of Dynamical Systems, 22 (2023), Art. 1.
[24] Y. Xia, K. Kou, Y. Liu; *Theory and applications of quaternion-valued differential equations*, Science Press, Beijing, 2021, ISBN 978-7-03-069056-2.
[25] Y. Xia, H. Huang, K. Kou; *An algorithm for solving linear nonhomogeneous quaternion-valued differential equations and some open problems*, Discrete and Continuous Dynamical Systems-Series S, 15 (2022), 1685–1697.
[26] M. Yoshida, Y. Kuroe, T. Mori; *Models of hopfield-type quaternion neural networks and their energy functions*, International Journal of Neural Systems, 15 (2005), 129–135.
[27] Y. Zou, M. Fečkan, J. Wang; *Hyers-Ulam stability of linear recurrence with constant coefficients over the quaternion skew yield*, Qualitative Theory of Dynamical Systems, 22 (2023), Art. 3.
[28] X. Zhang; *Global structure of quaternion polynomial differential equations*, Communications in Mathematical Physics, 303 (2011), 301–316.
[29] M. Zahid, A. Younus, M. E. Ghoneim, et al.; *Quaternion-valued exponential matrices and its fundamental properties*, International Journal of Modern Physics B, 37 (2022), 2350027.

Jiaojiao Lv
Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, China
Email address: `ljj990110@163.com`

Jinrong Wang (corresponding author)
Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, China
Email address: `jrwang@gzu.edu.cn`

Rui Liu
Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, China
Email address: `gzliuruiha@163.com`