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PRESCRIBED ENERGY SADDLE-POINT SOLUTIONS OF NONLINEAR INDEFINITE PROBLEMS

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Abstract. A minimax variational method for finding mountain pass-type solutions with prescribed energy levels is introduced. The method is based on application of the Linking Theorem to the energy-level nonlinear Rayleigh quotients which critical points correspond to the solutions of the equation with prescribed energy. An application of the method to nonlinear indefinite elliptic problems with nonlinearities that does not satisfy the Ambrosetti-Rabinowitz growth conditions is also presented.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 1$ and consider

$$\begin{gathered} -\Delta u - \lambda u = \mu |u|^{q-1} u + g(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $\mu > 0$, $1 < q < 2$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with primitive $G(x,u)$.

The problem has a variational structure and under some assumptions (see below (A1)) the associated *energy functional* $E_{\lambda,\mu} \in C^1(\mathring{W}_2^1(\Omega), \mathbb{R})$ is

$$E_{\lambda,\mu}(u) = \frac{1}{2}\Big(\int |\nabla u|^2 dx - \lambda \int |u|^2 dx \Big) - \frac{\mu}{q} \int |u|^q dx - \int G(x,u) dx.$$

By definition, the critical point $u \in \mathring{W}_2^1(\Omega)$ of $E_{\lambda,\mu}(u)$ is a weak solution to (1.1). The problem with $\lambda > \lambda_1$, where λ_1 is the principal eigenvalue of the operator $(-\Delta)$ in $\mathring{W}_2^1(\Omega)$ is called indefinite due to the fact that the linear part of (1.1) is indefinite (see [5, 25]). Equation (1.1) is related to finding the amplitude function u of the standing waves $\psi = e^{i\lambda t} u$ to the nonlinear Schrödinger (NLS) equation

$$i\psi_t = \Delta \psi + \mu |\psi|^{q-2} \psi + g(x,\psi), \quad (t,x) \in \mathbb{R}^+ \times \Omega, \tag{1.2}$$

where ψ is a complex-valued function of (t,x), and it is supposed that $g(x, \rho e^{i\theta}) = g(x,\rho) e^{i\theta}$ a.e. Ω, for all $\rho, \theta \in \mathbb{R}$. The Cauchy problem for (1.2) with the initial value $\psi_0 \in \mathring{W}_2^1(\Omega)$ is locally well posed and for some $T(\psi_0) > 0$ has a unique

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local solution $\psi \in C([0,T(\psi_0)),\mathring{W}_2^1(\Omega)) \cap C^1([0,T(\psi_0)),\mathring{W}_2^{-1}(\Omega))$ (see, e.g, [10]). Moreover, it holds *energy and mass conservation laws*:

$$H_\mu(\psi(t)) := \int \Big(\frac{1}{2}|\nabla\psi|^2 - \frac{\mu}{q}|\psi|^q - G(x,\psi)\Big)dx = \text{const},$$

$$Q(\psi(t)) := \frac{1}{2}\int |\psi|^2 dx = \text{const}.$$

As a result, the energy functional (action) $E_{\lambda,\mu}(\psi(t)) := H_\mu(\psi) - \lambda Q(\psi) = \text{const}$, $\lambda \in \mathbb{R}$ is also conserved.

This article focuses on the existence of the so-called *prescribed energy solution* of (1.1), i.e., which for a given energy $E \in \mathbb{R}$ satisfies

$$E_{\lambda,\mu}(u^E) = E, \quad DE_{\lambda,\mu}(u^E) = 0,$$

where "$D(\cdot)$" denotes the Fréchet derivative.

In the literature, solutions to the Schrödinger equations having a *prescribed frequency* λ and unknowns energy E and mass $\alpha = Q(u)$ are commonly studied (see, e.g, [10, 26]). An alternative formulation which has also been actively investigated over the last decades consists of finding the solution u to (1.2) having *prescribed mass* α, while λ and E are unknown (see, e.g., [4, 11, 22, 27]). Mathematically, all three approaches, namely, prescribed frequency, prescribed energy, and prescribed mass, are equally valid. Moreover, all of these approaches evidently are relevant from the physical point of view. In particular, the approach with prescribed energy arises in the study of inverse problems and the spectral and scattering control problems (see, e.g., [2, 12, 20, 23, 24, 29]).

The prescribed energy solutions of nonlinear problems was studied recently in [7, 18, 19] by using the nonlinear Rayleigh quotients [17]. The nonlinear Rayleigh quotients have the remarkable property that the critical points of these functionals correspond to the solutions of the equations while having a simpler structure than the corresponding energy functionals (see, e.g., [17]). They were particularly useful (see, e.g., [17, 19]) for finding nonnegative solutions to zero-mass problems [6] and detecting S-shaped bifurcations of nonlinear partial differential equations [7]. The nonlinear Rayleigh quotients method and solutions with prescribed energies were used to introduce a generalization of the Poincaré and Courant-Fischer-Weil minimization principles to nonlinear problems [18], as well as to study the orbital stability for ground states of the NLS equations [7].

There are at least two motivations to study prescribed energy solutions of (1.1), apart from the fact that it appears in some physical models. First, we develop the nonlinear Rayleigh quotient method for new classes of problems, in particular for equations with inhomogeneous and general forms of nonlinearities. Second, we develop the Mountain Pass methods in order to capture qualitative properties of the solutions that it generates.

The Mountain Pass Theorem introduced by Ambrosetti and Rabinowitz [1] and its generalization as the Benci-Rabinowitz Linking Theorem [5] is a powerful tool to establish the existence of solutions for nonlinear problems of the variational form. The solutions obtained by this method usually correspond to saddle critical points of the energy functional and are often referred to as mountain pass-type solutions or saddle-point solutions. In essence, this method is topological, which makes it possible to use it for solving problems of very general forms. On the other hand, this generality often makes it difficult to find out detailed information about the obtained solutions. The aim of this work is to show that the nonlinear Rayleigh quotient method can be applied to generate saddle-point solutions with prescribed energy within the framework of the Linking Theorem.

Let us state our main result. We seek for prescribed energy solutions using the *energy level nonlinear Rayleigh quotient* [7, 17, 19]:

$$\mathcal{R}_\lambda^E(u) := \frac{\frac{1}{2}\left(\int |\nabla u|^2\,dx - \lambda\int |u|^2\,dx\right) - \int G(x,u)\,dx - E}{\frac{1}{q}\int |u|^q dx},$$

for $u \in \mathring{W}_2^1(\Omega)\setminus\{0\}$ and $E \in \mathbb{R}$.

Notice that for $u \in \mathring{W}_2^1(\Omega)\setminus\{0\}$, $\lambda\in\mathbb{R}$, and $E\in\mathbb{R}$, we have

$$\begin{gathered}\mu = \mathcal{R}_\lambda^E(u) \;\Leftrightarrow\; E_{\lambda,\mu}(u) = E,\\ \mu = \mathcal{R}_\lambda^E(u),\; DR^E(u) = 0 \;\;\Leftrightarrow\;\; E_{\lambda,\mu}(u) = E,\; DE_{\lambda,\mu}(u) = 0.\end{gathered} \tag{1.3}$$

We assume that

(A1) there exist $\gamma_1,\gamma_2 \in (2,2^*)$, $C>0$ such that $0 \le g(x,u) \le C(|u|^{\gamma_1-1} + |u|^{\gamma_2-1})$ a.e. Ω, $u\in\mathbb{R}$,

(A2) there exist $\alpha>2$, $R_0>0$ such that $\alpha G(x,u) \le g(x,u)u$ a.e. Ω, $|u|\ge R_0$, where $2^* = 2N/(N-2)$ if $N>2$, $2^*=+\infty$ if $N\le 2$.

The operator $(-\Delta)$ with Dirichlet boundary conditions defines a self-adjoint operator in $L^2(\Omega)$ (see, e.g., [14]) and its spectrum consists of an infinite sequence ordered $0<\lambda_1<\lambda_2\le\dots$ of eigenvalues repeated according to their finite multiplicity. Now, with the convention that $\lambda_0=-\infty$, our main result is as follows.

Theorem 1.1. *Assume that $1<q<2<\gamma<2^*$, $\lambda\in(\lambda_k,\lambda_{k+1})$, $k=0,\dots$, and (A1)-(A2) hold. Then there exists $E_\lambda^k>0$ such that for any given $E\in(0,E_\lambda^k)$ corresponds $\mu_\lambda^k(E)\in(0,+\infty)$ such that (1.1) with $\mu=\mu_\lambda^k(E)$ possesses a non-zero weak solution $u_{\mu_\lambda^k(E)}\in C^{1,\alpha}(\overline{\Omega})$, $\alpha\in(0,1)$ with energy value E, i.e., $DE_{\mu_\lambda^k(E)}(u_{\mu_\lambda^k(E)})=0$, $E_{\mu_\lambda^k(E)}(u_{\mu_\lambda^k(E)})=E$. Furthermore,*

(i) *$\mu_\lambda^k(\cdot)$ is a non-increasing function in $(0,E_\lambda^k)$;*

(ii) *if $\lambda<\lambda_1$, then there exists $\lim_{E\to0}\mu_\lambda^0(E)=\bar\mu_\lambda(0)\in(0,+\infty)$ such that (1.1) possesses a non-zero weak solution $u_{\bar\mu_\lambda(0)}\in C^{1,\alpha}(\overline{\Omega})$, $\alpha\in(0,1)$ with zero energy value $E=0$ and $\mu=\bar\mu_\lambda(0)$.*

To find solutions of (1.1), we apply the Benci-Rabinowitz Linking Theorem [5] to the energy level nonlinear Rayleigh quotient $\mathcal{R}_\lambda^E(u)$. Note that $\mathcal{R}_\lambda^E\in C^1(\mathring{W}_2^1(\Omega)\setminus\{0\},\mathbb{R})$, while for the application of the linking theorem, in general, it is required that the functional belongs to $C^1(\mathring{W}_2^1(\Omega),\mathbb{R})$. Below we overcome this difficulty by using an appropriate truncation function for $\mathcal{R}_\lambda^E$ which can be properly introduced in the case $E>0$. In the zero-energy case $E=0$, the solution is obtained by passing to the limit $E\to0$.

Remark 1.2. Condition (A2) is the well-known Ambrosetti-Rabinovich (AR) condition [1, 15] and implies the superquadratic behavior of $G(\cdot,s)$, $G(x,s)\ge C|s|^\alpha$ for some $C>0$ and $|s|$ large. However, the complete nonlinearity $\mu|u|^{q-1}u+g(x,u)$ of equation (1.1) does not satisfy the (AR) condition.

Remark 1.3. The zero-energy case $E=0$ is particularly interesting, since the value $\bar\mu_\lambda(0)$ in this case resembles linear eigenvalue. Indeed, for linear problems such as $Lu=\lambda u$, where L is a self-adjoint linear operator on a Hilbert space H, any isolated eigenvalue λ_n, $n = 1, 2, \dots$, corresponds to an eigenfunction ϕ_n of the zero-energy level, i. e., $E = \frac{1}{2}\langle \phi_n, L\phi_n\rangle - \lambda_n \frac{1}{2}\langle \phi_n, \phi_n\rangle = 0$.

We shall use the following notation:

- $W := \overset{\circ}{W}{}^1_2(\Omega)$ denotes the standard Sobolev space with the norm $\|u\|_W = (\int_\Omega |\nabla u|^2\, dx)^{1/2}$;
- $|u|_{L^r} := (\int_\Omega |u|^r\, dx)^{1/r}$, $1 \le r < +\infty$ denotes the norm on the Lebesgue space $L^r := L^r(\Omega)$, $(\cdot, \cdot)$ denotes the scalar product in L^2;
- S_p is the best Sobolev constant for the embedding $W_0^{1,2}(\Omega) \subset L^p(\Omega)$, $1 \le p \le 2^*$;
- $\|\cdot\|_*$ denotes the norm in the dual space W^*;
- $d(A, B) := \min\{\|u - v\|_W :\ u \in A,\ v \in B\}$ denotes the distance between sets $A, B \subset W$.

This work is organized as follows. In Section 2, we introduce the nonlinear Rayleigh quotient together with its appropriate truncation function. In Section 3, we derive some properties of the nonlinear Rayleigh quotient $\mathcal{R}^E_\lambda$ and prove that the Cerami condition for $\mathcal{R}^E_\lambda$ is satisfied. In Section 4, we prove our main result. Conclusions are drawn in Section 5. In the Appendix, we state the Benci-Rabinowitz Linking Theorem and corresponding definitions.

2. Preliminaries

Let $(e_k) \subset W$ be the orthogonal basis in L^2 of the eigenfunctions of $(-\Delta)$ with zero Dirichlet conditions satisfying $\|e_k\|^2_W = \lambda_k$ and $|e_k|^2_{L^2} = 1$, for each $k \in \mathbb{N}$. Let $\lambda \in (\lambda_k, \lambda_{k+1})$ be fixed for some $k \in \mathbb{N}$. We write

$$W = W^+ \oplus W^-,\ \text{where } W^- = \operatorname{span}\{e_1, e_2, \dots, e_k\},\ W^+ = \overline{\operatorname{span}\{e_{k+1}, e_{k+2}, \dots\}}.$$

Then one can introduce the following equivalent norm $\|\cdot\|_1$ for $\|\cdot\|_W$ in W

$$\|u\|_1^2 = \sum_{i=k+1}^{\infty} (\lambda_i - \lambda)u_i^2 + \sum_{i=1}^{k} (\lambda - \lambda_i)u_1^2 := \|u^+\|_1^2 + \|u^-\|_1^2,$$

where $u_i = (u, e_i)$, $i = 1, \dots$. Then $u = (u^+ + u^-) \in W$, $u^\pm \in W^\pm$, and $c_0\|u\|_1^2 \le \|u\|^2_W \le c_1\|u\|_1^2$, $\forall u \in W$, where $0 < c_0, c_1 < +\infty$ do not depend on $u \in W$. In addition,

$$H_\lambda(u) := \|u\|^2_W - \lambda|u|^2_{L^2} = H_\lambda(u^+) + H_\lambda(u^-) = \|u^+\|_1^2 - \|u^-\|_1^2, \quad u \in W.$$

Notice that $H_\lambda(u) = -\|u\|_1^2 < 0$ if $u \in W^- \setminus \{0\}$, and $H_\lambda(u) = \|u\|_1^2 > 0$ if $u \in W^+ \setminus \{0\}$, for $\lambda \in (\lambda_k, , \lambda_{k+1})$.

With this notation, we have

$$E_{\lambda,\mu}(u) = \frac{1}{2}H_\lambda(u) - \frac{\mu}{q}|u|^q_{L^q} - \int G(x, u)dx,$$

$$\mathcal{R}^E_\lambda(u) = \frac{\frac{1}{2}H_\lambda(u) - \int G(x, u)dx - E}{\frac{1}{q}|u|^q_{L^q}}, \quad u \in W \setminus \{0\}.$$

Obviously, $\mathcal{R}^E \in C^1(W \setminus \{0\}, \mathbb{R})$ and

$$DE_{\lambda,\mu}(u) = 0,\ E_{\lambda,\mu}(u) = E \ \Leftrightarrow\ D\mathcal{R}^E_\lambda(u) = 0,\ \mu = \mathcal{R}^E_\lambda(u),\ u \in W \setminus 0. \tag{2.1}$$

To avoid the singularity at origin of $\mathcal{R}^E$, we define $\phi_\rho \in C^\infty(\mathbb{R})$, for $\rho > 0$ such that

$$\phi_\rho(s) = \begin{cases} 0 & \text{if } |s| < \rho/2, \\ = 1 & \text{if } |s| > \rho, \end{cases}$$

and introduce

$$\mathcal{R}^E_\rho(u) = \begin{cases} \phi_\rho(\|u\|_1)\mathcal{R}^E(u), & u \in W \setminus 0, \\ 0, & u = 0. \end{cases}$$

Thus, $\mathcal{R}^E_\rho(u) \in C^1(W)$ for any $\rho > 0$.

We define $B_r := \{u \in W : \|u\|_1 \le r\}$, $r > 0$. We need the following result.

Lemma 2.1. *Assume that $E > 0$ and $\lambda \in (\lambda_k, , \lambda_{k+1})$. Then there exists $\rho(E) > 0$ such that $\mathcal{R}^E(u) < 0$ for any $u \in B_\rho$ with $0 < \rho < \rho(E)$.*

Proof. Since $G(x,u) \ge 0$ a.e. Ω, $u \in \mathbb{R}$,

$$\mathcal{R}^E_\lambda(u) < q\frac{1}{|u|^q_{L^q}}\Big(\frac{1}{2}H_\lambda(u) - E\Big) < \frac{q}{|u|^q_{L^q}}\Big(\frac{1}{2}\|u\|_1^2 - E\Big), \quad u \in W \setminus \{0\}.$$

Hence, setting $\rho(E) := \sqrt{2E}$ we obtain the proof. □

Corollary 2.2. *Assume that $\rho < \rho(E)$. If $\hat{u}$ is a critical point of $\mathcal{R}^E_\rho(u)$ such that $\mathcal{R}^E_\rho(\hat{u}) > 0$, then u is a critical point of $\mathcal{R}^E(u)$ as well.*

Proof. By Lemma 2.1, $\mathcal{R}^E_\rho(\hat{u}) > 0$ implies that $\|\hat{u}\|_W \ge \rho$. Therefore $\mathcal{R}^E_\rho(\hat{u}) = \mathcal{R}^E(\hat{u}) = \mu$ and $D\mathcal{R}^E(\hat{u}) = 0$. □

We say that $(u_n) \subset W$ is a Cerami sequence at the level $c \in \mathbb{R}$ of $\mathcal{R}^E$, in short (Ce) sequence, whenever $\mathcal{R}^E(u_n) \to c$ and $(1+\|u_n\|_W)\|D\mathcal{R}^E(u_n)\| \to 0$ as $n \to \infty$. The functional $\mathcal{R}^E$ satisfies the Cerami condition at the level $c \in \mathbb{R}$, in short (Ce) condition, whenever any (Ce) sequence possesses a convergent subsequence. The definitions of the (Ce) sequence and the (Ce) condition for $\mathcal{R}^E_\rho$ are similar.

Corollary 2.3. *If $\rho < \rho(E)$, then $\mathcal{R}^E_\rho(u)$ satisfies the (Ce) condition at level $c > 0$ if and only if $\mathcal{R}^E(u)$ does.*

Proof. Assume $\mathcal{R}^E(u)$ satisfies the (Ce) condition at $c > 0$. Let (u_n) be a (Ce) sequence for $\mathcal{R}^E_\rho$ at c, i.e., $\mathcal{R}^E_\rho(u_n) \to c$ and $(1+\|u_n\|_1)\|D\mathcal{R}^E_\rho(u_n)\|_* \to 0$ as $n \to \infty$. By Lemma 2.1, $\mathcal{R}^E_\rho(u) \le 0$ as $u \in B_\rho$, whereas $\mathcal{R}^E_\rho(u) = \mathcal{R}^E(u)$ for $u \in W \setminus B_\rho$. Thus, $\mathcal{R}^E_\rho(u) > 0$ implies $\mathcal{R}^E(u) = \mathcal{R}^E_\rho(u)$ and $D\mathcal{R}^E(u) = D\mathcal{R}^E_\rho(u)$. Therefore (u_n) is also (Ce) sequence for $\mathcal{R}^E$ and consequently, (u_n) possesses a convergent subsequence in W. The proof of opposite statement is similar. □

3. Properties of $\mathcal{R}^E_\rho$

Consider the sphere $S^\pm_r := \{u \in W^\pm : \|u\|_1 = r\}$, $r > 0$. Recall that by the assumption $g(x,u) \le C(|u|^{\gamma_1 - 1} + |u|^{\gamma_2 - 1})$ a.e. Ω, $u \in \mathbb{R}$, for some $\gamma_1, \gamma_2 \in (2, 2^*)$, $C > 0$. Hence, for any given $\epsilon > 0$, there exist $C(\epsilon) > 0$ such that

$$G(x,s) \le \frac{\epsilon}{2}|s|^2 + C(\epsilon)|s|^\gamma, \quad \forall s \in \mathbb{R}, \text{ a.e. } \Omega,$$

where $\gamma := \max\{\gamma_1, \gamma_2\}$. This by the Sobolev inequalities implies

$$\int G(x,u)dx \le \frac{\epsilon}{2}C_1\|u\|_1^2 + C_2(\epsilon)\|u\|_1^\gamma, \; u \in W, \tag{3.1}$$

where $C_1, C_2(\epsilon) \in (0, +\infty)$ do not depend on $u \in W$ and C_1 does not depend on $\epsilon > 0$.

Proposition 3.1. *For any $\lambda \in (\lambda_k, \lambda_{k+1})$, there exist $E^k_\lambda > 0$ and $r^k_\lambda > 0$ such that $\inf_{w \in S^+_{r^k_\lambda}} \mathcal{R}^E_\rho(w) > 0$, for any $E \in [0, E^k_\lambda)$, for all $\rho \in (0, r^k_\lambda)$.*

Proof. Note that $H_\lambda(w) = \|w\|_1^2$, $\forall w \in W^+$. Take $\epsilon \in (0, 1/C_1)$. Then by (3.1) we have

$$\mathcal{R}^E(w) \geq q \frac{\frac{1}{2}(1 - C_1\epsilon)\|w\|_1^2 - C_2(\epsilon)\|w\|_1^\gamma - E}{|w|^q_{L^q}} = q \frac{f(\|w\|_1) - E}{|w|^q_{L^q}}, \quad \forall w \in W^+,$$

where $f(r) := \frac{1}{2}(1 - C_1\epsilon) r^2 - C_2(\epsilon) r^\gamma$. Observe that $f(r)$ attains its global maximum $E^k_\lambda := f(r^k_\lambda)$ at

$$r^k_\lambda := \left[(1 - C_1\epsilon)/(\gamma C_2(\epsilon))\right]^{1/(\gamma - 2)}.$$

Thus, for any $E \in [0, E^k_\lambda)$,

$$\inf_{w \in S^+_{r^k_\lambda}} \mathcal{R}^E(w) \geq \inf_{w \in S^+_{r^k_\lambda}} q \frac{f(r^k_\lambda) - E}{|w|^q_{L^q}} \geq q \frac{E^k_\lambda - E}{S^q_q (r^k_\lambda)^q} =: \delta_E > 0,$$

which implies the proof, since $\mathcal{R}^E_\rho(w) = \mathcal{R}^E(w)$, $w \in S^+_{r^k_\lambda}$ if $\rho \in (0, r^k_\lambda)$. □

Proposition 3.2. *For any $u \in W \setminus 0$ and $r > 0$, it holds $\mathcal{R}^E(tu + v) \to -\infty$ as $t \to +\infty$ uniformly for $v \in B_r$.*

Proof. Observe that (A2) implies

$$u|u|^\alpha \frac{d}{du}(|u|^{-\alpha} G(x, u)) \geq 0, \quad \text{for } |u| \geq R_0.$$

Integrating this yields $G(x, u) \geq c(x)|u|^\alpha > 0$ a.e. Ω, for $|u| \geq R_0$, with some Lesbegue-measurable function $c(x) \geq 0$. Since (A2), $c(x) \in L^\infty(\Omega)$ and $G(x, s) \geq c(x)|s|^\alpha - C_0$, for all $s \in \mathbb{R}$, a.e. Ω with some constant $C_0 \in \mathbb{R}$. Note that (A2) implies $\alpha < \gamma < 2^*$. Observe that $c^{1/\alpha}(x)\big(u(x) + v(x)/t\big) \to c^{1/\alpha}(x) u(x)$ in $L^\alpha(\Omega)$ uniformly in $v \in B_r$ as $t \to +\infty$. Indeed, using the Sobolev inequality we have

$$\Big| \int c(x) \big| u + \frac{v}{t} \big|^\alpha \, dx - \int c(x)|u|^\alpha \, dx \Big| \leq \frac{1}{t} \int c(x)|v|^\alpha \, dx \leq \frac{1}{t} C r^\alpha, \quad v \in B_r$$

for some constant C which does not depend on $v \in B_r$. Thus, uniformly in $v \in B_r$,

$$\lim_{t \to \infty} \frac{1}{t^\alpha} \int G(x, tu + v) \, dx \geq \lim_{t \to \infty} \Big(\int c(x) |u + \frac{v}{t}|^\alpha dx - \frac{C_0 |\Omega|}{t^\alpha} \Big) = \int c(x)|u|^\alpha dx.$$

This implies that

$$\lim_{t \to \infty} \frac{1}{t^\alpha} \Big(\frac{1}{2} \|tu + v\|^2_W - \int G(x, tu + v) \, dx \Big) \leq - \int c(x)|u|^\alpha dx < 0,$$

uniformly in $v \in B_r$. Since $|u + v/t|^q_{L^q} \leq C\left(\|u\|^q_q + S^q_q r^q / t^q \right)$ and $\alpha > 2 > q$, we conclude that uniformly for $v \in B_r$ it holds

$$\lim_{t \to \infty} \mathcal{R}^E(tu + v) \leq \lim_{t \to \infty} \frac{t^{\alpha - q}}{|u + \frac{v}{t}|^q_{L^q}} \Big[\frac{\|tu + v\|^2_W}{2t^\alpha} - \int \frac{G(x, tu + v)}{t^\alpha} \, dx \Big] = -\infty.$$

□

Proposition 3.3. *Assume that $E \in (0, E^k_\lambda)$, $0 < \rho < r^k_\lambda$. The functional $\mathcal{R}^E_\rho$ satisfies the (Ce) condition at any level $\mu > 0$.*

Proof. By Corollary 2.3, it is sufficient to prove that the functional $\mathcal{R}^E$ satisfies the (Ce) condition at any $\mu > 0$. Assume that (u_m) is a (Ce) sequence for $\mathcal{R}^E$, i.e., $\mu_m := \mathcal{R}^E(u_m) \to \mu > 0$ and $\|D\mathcal{R}^E(u_m)\|_*(1+\|u_m\|_1) \to 0$ as $m \to +\infty$. Then

$$\begin{aligned}
&\alpha\mu_m + o(1)\|u_m\|_1(1+\|u_m\|_1)^{-1}\\
&= \alpha\mathcal{R}^E(u_m) - D\mathcal{R}^E(u_m)(u_m)\\
&= \frac{q}{|u_m|^q_{L^q}}\Big(\frac{\alpha-2}{2}H_\lambda(u_m) + \int (g(x,u_m)u_m - \alpha G(x,u_m))\,dx + \mu_m|u_m|^q_{L^q} - qE\Big)\\
&\geq \frac{q}{|u_m|^q_{L^q}}\Big(\frac{\alpha-2}{2}H_\lambda(u_m) + \mathcal{L}^n(\Omega)\mathrm{essinf}_{x\in\Omega,s\in\mathbb{R}}\,(g(x,s)s - \alpha G(x,s)) - qE\Big)
\end{aligned}$$

By (A2), $\operatorname{ess\,inf}_{x\in\Omega,s\in\mathbb{R}}\Big(g(x,s)s-\alpha G(x,s)\Big) =: c_0 > -\infty$. Hence $H_\lambda(u_m) \leq c_1\,(1+|u_m|^q_{L^q})$, where $0 < c_1 < +\infty$ does not depend on $m = 1,2,\dots$, and therefore

$$\|u_m\|^2_W \leq \lambda|u_m|^2_{L^2} + c_1\,(1+|u_m|^q_{L^q}), \quad m = 1,2,\dots. \tag{3.2}$$

Thus, if $|u_m|_{L^2}$ is bounded, then $\|u_m\|_W$ is also bounded. If $|u_{m_j}|_{L^2} \to \infty$, for some subsequence (m_j) such that $m_j \to +\infty$ as $j \to +\infty$, then by (3.2) $\lim_{j\to\infty}\frac{H_\lambda(u_{m_j})}{|u_{m_j}|^2_{L^2}} \leq 0$, and consequently, we obtain a contradiction:

$$\begin{aligned}
0 < \mu &= \lim_{j\to\infty}\mathcal{R}^E(u_{m_j})\\
&= \lim_{j\to\infty} q\frac{|u_{m_j}|^2_{L^2}}{|u_{m_j}|^q_{L^q}}\Big[\frac{1}{2}\frac{H_\lambda(u_{m_j})}{|u_{m_j}|^2_{L^2}} - \int\frac{G(x,u_{m_j})}{|u_{m_j}|^2_{L^2}}\,dx - \frac{E}{|u_{m_j}|^2_{L^2}}\Big] \leq 0.
\end{aligned}$$

Thus, (u_m) is bounded and we may assume that $u_m \rightharpoonup u$ weakly in W and $u_m \to u$ strongly in $L^r(\Omega)$, $r \in [1, 2^*)$, as $m \to \infty$. In particular, this gives

$$\int G(x,u_m)dx \to \int G(x,u)dx, \quad \|u_m\|^q_q \to \|u\|^q_q \quad \text{as } m \to +\infty. \tag{3.3}$$

By the convergence $\|D\mathcal{R}^E(u_m)\|_* \to 0$ we obtain $D\mathcal{R}^E(u_m)(u-u_m) \to 0$ as $m \to +\infty$. Hence by (3.3), we obtain that $\langle -\Delta u_m, u-u_m\rangle \to 0$. Thus by the S^+ property of the Laplace operator (see [13]) we derive that $u_m \to u$ strongly in $W^{1,2}(\Omega)$. □

Remark 3.4. Since $\|u_m\|_1 > \rho$, for all m and $\mathcal{R}^E_\rho(u_m) \to \mu \in (0,+\infty)$, it follows that $|u_m|_{L^q}$ dos not approach 0.

4. Proof of Theorem 1.1

Let $\lambda \in (\lambda_k, \lambda_{k+1})$, $E \in (0, E^k_\lambda)$ and $0 < \rho < r^k_\lambda$. For $T > r^k_\lambda$, take $\bar{u}^+ \in S^+_1$, and define

$$\begin{gathered}
B_0 = B_0(T) := \big\{u = t\bar{u}^+ + sv : v \in S^-_1, (0 < t < T, s = T) \text{ or}\\
(t \in \{0,T\}, 0 \leq s \leq T)\big\},\\
B = B(T) := \{u = t\bar{u}^+ + sv : v \in S^-_1,\ 0 < t < T,\ 0 \leq s \leq T\},\\
B^c_0 = B^c_0(T) := \{u = t\bar{u}^+ + sv : v \in S^-_1,\ (0 < t < T,\ s = T)\},\\
B^d_0 = B^d_0(T) := \{u = t\bar{u}^+ + sv :\ v \in S^-_1,\ t \in \{0,T\},\ 0 \leq s \leq T\}.
\end{gathered}$$

Observe that if $u = t\bar{u}^+ + Tv$, $u^+ \in S_1^+$, $v \in S_1^-$, then $H_\lambda(u) = t^2\|\bar{u}^+\|_1^2 - T^2\|v\|_1^2 = (t^2 - T^2) < 0$ for $T > t$. This implies

$$b^c(T) := \sup_{u \in B_0^c} \mathcal{R}_\rho^E(u) = \sup_{u \in B_0^c} \phi_\rho(\|u\|_1) \frac{\frac{1}{2}H_\lambda(u) - \int G(x,u)\,dx - E}{\frac{1}{q}|u|_{L^q}^q} \le 0,$$

for $\rho > 0$. Note that $H_\lambda(v) < 0$, for $v \in S_1^-$. This by Proposition 3.2 implies that

$$b^d(T) := \sup_{u \in B_0^d} \mathcal{R}_\rho^E(u) \le 0,$$

for sufficiently large T. Thus, by Proposition 3.1, for $\rho \in (0, r_\lambda^k)$ and sufficiently large $T > r_\lambda^k$, it holds

$$b := \sup_{u \in B_0} \mathcal{R}_\rho^E(u) \le 0 < \inf_{u \in S^+_{r_\lambda^k}} \mathcal{R}_\rho^E(u) =: a.$$

Let $\lambda \in (\lambda_k, \lambda_{k+1})$, $E \in (0, E_\lambda^k)$ and $0 < \rho < \min\{r_\lambda^k, \rho(E)\}$. Consider

$$\mu_\lambda^k(E) := \inf_{h \in \Gamma} \max_{u \in B} \mathcal{R}_\rho^E(h(u)), \tag{4.1}$$

where $\Gamma = \{h \in C(B; W) : h|_{B_0} = id_{B_0}\}$. By Propositions 3.3 and Corollary 2.3 the functional $\mathcal{R}_\rho^E$ satisfies the (Ce) condition at the level $c = \mu_\lambda^k(E) > 0$. Note that, for $T > r_\lambda^k$, $B_0 \cap S^+_{r_\lambda^k} = \emptyset$, $d(B_0, S^+_{r_\lambda^k}) > 0$, and $S^+_{r_\lambda^k}$ is closed in W. Furthermore, it can be shown in a standard way (see [25, P. 156]) that $\{B_0, B\}$ links $S^+_{r_\lambda^k}$ in W. Hence, by the Benci-Rabinowitz Linking Theorem [5] for functionals satisfying (Ce) condition (see [25, Theorem 5.39] and Appendix below), there exists a nonzero critical point $u_{\mu_\lambda^k(E)} \in W \setminus \{0\}$ of the functional $\mathcal{R}_\rho^E$ such that $\mathcal{R}_\rho^E(u_{\mu_\lambda^k(E)}) = \mu_\lambda^k(E) \ge a > 0$. Consequently, Corollary 2.2 and (1.3) yields that $u_{\mu_\lambda^k(E)}$ is a weak solution of (1.1) with $\mu = \mu_\lambda^k(E)$ and energy value E.

Standard bootstrap arguments and Sobolev's embedding theorem (see, e.g., [28]) entail that $u_{\mu_\lambda^k(E)} \in L^\infty(\Omega)$. Therefore, by the L^p-regularity results in [16], $u_{\mu_\lambda^k(E)} \in W^{2,p}(\Omega)$ for any $1 < p < \infty$ and thus, by Sobolev's embedding theorem, $u_{\mu_\lambda^k(E)} \in C^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$. This completes the proof of the first part of the theorem.

(i) Take $E_1 > E_0 > 0$. It is not hard to see that the sets $\{B_0, B\}$ and the path sets Γ in (4.1) can be taken the same for E_1, E_0 if $|E_1 - E_0|$ is sufficiently small. Note that

$$\mathcal{R}_\rho^{E_1}(u) = \mathcal{R}_\rho^{E_0}(u) - \phi_\rho(\|u\|_1)\frac{E_1 - E_0}{\int G(x,u)\,dx}, \ \forall u \in W \setminus 0,$$

and thus

$$\begin{aligned}\max_{u \in B} \mathcal{R}_\rho^{E_1}(h(u)) &= \max_{u \in B}\Big(\mathcal{R}_\rho^{E_0}(h(u)) - \phi_\rho(\|h(u)\|_1)\frac{E_1 - E_0}{\int G(x,h(u))\,dx}\Big)\\ &\le \max_{u \in B} \mathcal{R}_\rho^{E_0}(h(u)), \quad \forall h \in \Gamma,\end{aligned}$$

and therefore, for sufficiently small $|E_1 - E_0|$, we have

$$\mu_\lambda^k(E_1) = \inf_{h \in \Gamma} \max_{u \in B} \mathcal{R}_\rho^{E_1}(h(u)) \le \inf_{h \in \Gamma} \max_{u \in B} \mathcal{R}_\rho^{E_0}(h(u)) = \mu_\lambda^k(E_0).$$

(ii) Let $E = 0$. Consider $\mathcal{R}^0(u) \equiv \mathcal{R}^E(u)|_{E=0}$. Using (3.1) and $1 < q < 2$ it is not hard to show that $\mathcal{R}^0(u) \to 0$ as $\|u\|_1 \to 0$. Consequently, by the continuation we can set that $\mathcal{R}^0(0) = 0$.

Assume that $\lambda < \lambda^1$. Then $W^- = \emptyset$ and $W^+ \equiv W$. By Proposition 3.2, one can find $u_1 \in W$ such that $\mathcal{R}^E(u_1) < 0$. Let $E \in [0, E^0_\lambda)$ and $0 < \rho < \min\{r^\lambda_0, \rho(E)\}$. Observe that (4.1) can be rewritten as follows

$$\mu^0_\lambda(E) := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{R}^E_\rho(\gamma(t)), \tag{4.2}$$

where $\Gamma = \{\gamma \in C([0, 1]; W) : \gamma(0) = 0, \gamma(1) = u_1\}$, $0 < \rho < \rho(E)$. Here we set $\mathcal{R}^0_\rho(u) := \mathcal{R}^0(u)$, $\rho > 0$.

Note that by the above, for any $E \in (0, E^0_\lambda)$ and $0 < \rho < \min\{r^\lambda_0, \rho(E)\}$, $\mu^0_\lambda(E) > 0$ and there exists a critical point $u_{\mu^0_\lambda(E)} \in W \setminus 0$ of $\mathcal{R}^E(u)$ such that $DE_{\mu^0_\lambda(E)}(u_{\mu^0_\lambda(E)}) = 0$ and $E_{\mu^0_\lambda(E)}(u_{\mu^0_\lambda(E)}) = E$. As in the proof of (i), from (4.2) it follows that $\mu^0_\lambda(E)$ is a non-increasing function on $E \in [0, E^0_\lambda)$. Moreover, $\mu^0_\lambda(E) \le \mu^0_\lambda(0) < +\infty$, for any $E \in (0, E^0_\lambda)$. Hence there exists $\lim_{E\to 0} \mu^0_\lambda(E) = \bar\mu_\lambda(0) \le \mu^0_\lambda(0)$. Furthermore, $\bar\mu_\lambda(0) > 0$ since $\mu^0_\lambda(E) > 0$, $E \in (0, E^k_\lambda)$ and $\mu^0_\lambda(E)$ is a non-increasing function.

Since $D\mathcal{R}^E(u_{\mu^0_\lambda(E)}) = 0$ and $\mu^0_\lambda(E) \equiv \mathcal{R}^E(u_{\mu^0_\lambda(E)}) \to \bar\mu_\lambda(0) > 0$, any countable subset of $(u_{\mu^0_\lambda(E)})_{E\in(0,E^0_\lambda)}$ is a (Ce) sequence. Hence Proposition 3.3 implies that there exists a sequences $u_{\mu_\lambda(E_m)}$, $m = 1, 2, \dots$, such that $\lim_{m\to+\infty} E_m = 0$ and $u_{\mu_\lambda(E_m)}$ convergences in W to some point $u_{\bar\mu_\lambda(0)} \in W$ as $m \to +\infty$. Note that $u_{\bar\mu_\lambda(0)} \ne 0$ (see Remark 3.4), and therefore $u_{\bar\mu_\lambda(0)}$ is a weak solution of (1.1) with $\mu = \bar\mu_{\lambda(0)}$. Moreover,

$$0 = \lim_{m\to+\infty} E_m = \lim_{m\to+\infty} E_{\lambda,\mu}(u_{\mu_\lambda(E_m)}) = E_{\lambda,\mu}(u_{\bar\mu_\lambda(0)}).$$

Thus $u_{\bar\mu_\lambda(0)}$ is a solution with zero energy. As above it can be shown that $u_{\bar\mu_\lambda(0)} \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$.

5. Conclusions and discussion

In this paper, we develop the mountain pass methods applicable to a new class of problems. In particular, an approach to finding mountain pass-type solutions with prescribed energy for indefinite elliptic problems with nonlinearities which does not satisfy the Ambrosetti-Rabinowitz growth conditions is introduced. Furthermore, the method of nonlinear Rayleigh quotients is used for the first time to solve indefinite elliptic equations with general forms of non-linearities.

A valuable property of the nonlinear Rayleigh quotients method is that it simplifies the complexity problem in a sense by reducing degree of degeneracy of the system (see [3, 8]). However, applicability of general theories like the Mountain Pass Theorem, Index Theory, and Ljusternik-Schnirelman's Theory, etc. to nonlinear generalized Rayleigh quotients is limited in light of prohibitive regularity and non-degeneracy conditions for variational functionals. Indeed, the energy-level nonlinear Rayleigh quotient $\mathcal{R}^E_\lambda(u)$ corresponding to problem (1.1) is not regular at zero. Hence, direct applying the Mountain Pass Theorem in this case is impossible. We have overcome this difficulty in the present work by introducing an appropriate truncation function. We believe, however, there are other ways for overcoming this obstacle. For instance, one might try to answer the question: Is it possible to develop general methods, like Mountain Pass Theorem, etc, applicable to the Rayleigh quotient type function? The answer to this question would help apparently resolve a number of open problems.

6. Appendix

We use a generalized version of the Benci-Rabinowitz Linking Theorem [5] for functionals satisfying $(Ce)_c$ condition. this was developed by D. Motreanu, V. Motreanu, N. Papageorgiou [25]. Let $(W, \|\cdot\|_W)$ be a Banach space, $B_0 \subset B$, C be nonempty sets in W, and id_{B_0} is an identity map in B_0. The pair $\{B_0, B\}$ is said to be links C in W if the following conditions hold: (a) $B_0 \cap C = \emptyset$; (b) for any $h \in C(B; W)$ with $h|_{B_0} = id_{B_0}$ it holds $h(B) \cap C \neq \emptyset$. The following result follows from [25, Theorem 5.39].

Theorem 6.1. *Let $\{B_0, B\}$ links C in W, C closed, $d(B_0, C) > 0$. Let $\Gamma = \{h \in C(B; W) : \ h|_{B_0} = id_{B_0}\}$ and $\phi \in C^1(W, \mathbb{R})$ be such that $b := \sup_{u \in B_0} \phi(u) \leq \inf_{u \in S_\rho^+} \phi(u) =: a$. Let*

$$c := \inf_{h \in \Gamma} \max_{u \in B} \phi(h(u)), \tag{6.1}$$

and assume that ϕ satisfying the (C_e)-condition at c. Then $c \geq a$ and c is a critical value of ϕ, i.e., there exists $u \in W \setminus 0$ such that $D\phi(u) = 0$ and $\phi(u) = c$.

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