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BLOW-UP CRITERIA AND INSTABILITY OF STANDING WAVES FOR THE FRACTIONAL SCHRÖDINGER POISSON EQUATION

YICHUN MO, MIN ZHU, BINHUA FENG

Abstract. In this article, we consider blow-up criteria and instability of standing waves for the fractional Schrödinger-Poisson equation. By using the localized virial estimates, we establish the blow-up criteria for non-radial solutions in both mass-critical and mass-supercritical cases. Based on these blow-up criteria and three variational characterizations of the ground state, we prove that the standing waves are strongly unstable. These obtained results extend the corresponding ones presented in the literature.

1. Introduction

In recent years, there has been a great deal of interest in using fractional Laplacians to model physical phenomena. By extending the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths, Laskin [23, 24] used the theory of functionals over functional measure generated by the Lévy stochastic process to introduce the fractional nonlinear Schrödinger equation (NLS)

$$i\partial_t\psi - (-\Delta)^s\psi + f(\psi) = 0, \tag{1.1}$$

where $i^2 = -1$, $0 < s < 1$ and $f(\psi)$ is the nonlinearity. The fractional differential operator $(-\Delta)^s$ is defined by $(-\Delta)^s\psi = \mathcal{F}^{-1}[|\xi|^{2s}\mathcal{F}(\psi)]$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and inverse Fourier transform, respectively. The fractional NLS also appears in the continuum limit of discrete models with long-range interactions (see [22]) and in the description of Bonson stars as well as in water wave dynamics (see [15]). Recently, an optical realization of the fractional Schrödinger equation was proposed by Longhi [28].

In this article, we consider the blow-up criteria and the strong instability of standing waves for the fractional Schrödinger-Poisson equation

$$\begin{gathered} i\partial_t\psi - (-\Delta)^s\psi - \phi\psi + |\psi|^p\psi = 0, \quad (t,x) \in [0,T^*) \times \mathbb{R}^3, \\ (-\Delta)^r\phi = |\psi|^2, \end{gathered} \tag{1.2}$$

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where $\psi : [0,T^*) \times \mathbb{R}^3 \to \mathbb{C}$ is the complex valued function, $s, r \in (0,1)$, and $0 < T^* \le \infty$, $0 < p < \frac{4s}{3-2s}$. Under this assumption, ϕ can be expressed as

$$\phi(x) = c_r \int_{\mathbb{R}^3} \frac{|\psi(y)|^2}{|x-y|^{3-2r}} dy, \tag{1.3}$$

which is called the r-Riesz potential, where

$$c_r = \pi^{-3/2} 2^{-2r} \frac{\Gamma(\frac{3}{2}-2r)}{\Gamma(r)}.$$

In (1.3), and in the sequel, in we often omit the constant c_r for convenience of notation. Substituting ϕ into (1.2) leads to the fractional Schrödinger equation

$$\begin{gathered} i\partial_t \psi - (-\Delta)^s \psi - (|x|^{-(3-2r)} * |\psi|^2)\psi + |\psi|^p \psi = 0, \quad (t,x) \in [0,T^*) \times \mathbb{R}^3, \\ \psi(0,x) = \psi_0(x), \end{gathered} \tag{1.4}$$

where $\psi_0 \in H^s$.

For the classical NLS, i.e., $s = 1$, we have the Variance-Virial Law

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} |x|^2 |\psi(t,x)|^2 \, dx = 2 \operatorname{Im} \int_{\mathbb{R}^3} \bar{\psi}(t,x) x \cdot \nabla \psi(t,x) \, dx, \tag{1.5}$$

provided that $\psi_0 \in \Sigma := \{v \in H^1 : xv \in L^2\}$, where Im denotes the imaginary part. By using (1.5) and the virial identity, we can obtain the blow-up results for the classical NLS with negative energy $E(\psi_0) < 0$ and finite variance [5]. However, this argument breaks down for $0 < s < 1$, since identity (1.5) fails in this case by the dimensional analysis. It turns out that the suitable generalization of the variance for the fractional NLS is

$$\mathcal{V}^{(s)}[\psi(t)] := \int_{\mathbb{R}^3} \bar{\psi}(t,x) x \cdot (-\Delta)^{1-s} x \psi(t,x) \, dx = \|x(-\Delta)^{\frac{1-s}{2}} \psi(t)\|_{L^2}^2. \tag{1.6}$$

Given any sufficiently regular and spatially localized solution $\psi(t)$ of the free fractional Schrödinger equation $i\partial_t \psi = (-\Delta)^s \psi$, a calculation yields

$$\frac{1}{2}\frac{d}{dt} \mathcal{V}^{(s)}[\psi(t)] := 2 \operatorname{Im} \int_{\mathbb{R}^3} \bar{\psi}(t,x) x \cdot \nabla \psi(t,x) \, dx. \tag{1.7}$$

This idea has been successfully applied to prove the blow-up results for (1.1) with radial solutions and the Hartree-type nonlinearity $(|x|^{-\gamma} * |\psi|^2)\psi$ with $\gamma \ge 1$ in [6, 43]. But this method can not work due to the nontrivial error terms which seem very hard to control for the local nonlinearity $|\psi|^p \psi$. Boulenger et al. [3] applied the Balakrishman's formula

$$(-\Delta)^s = \frac{\sin \pi s}{\pi} \int_0^\infty m^{s-1} \frac{-\Delta}{-\Delta + m} \, dm, \tag{1.8}$$

and obtained the differential estimate

$$\begin{aligned} &\frac{d}{dt}\Big(\operatorname{Im} \int_{\mathbb{R}^3} \bar{\psi}(t) \nabla \varphi_R \cdot \nabla \psi(t) \, dx \Big) \\ &\le 12pE(\psi_0) - 2\delta \|(-\Delta)^{s/2} \psi(t)\|_{L^2}^2 + \circ_R(1)(1 + \|(-\Delta)^{s/2}\psi(t)\|_{L^2}^{p/s+}), \end{aligned}$$

where $\delta = 3p - 2s$. With the help of this key estimate, they proved the existence of radial blow-up H^s solutions by applying the comparison theory.

However, to the best of our knowledge, there are no any blow-up results for (1.4) so far. In particular, equation (1.4) includes two classical nonlinearities, i.e., power-type $|\psi|^p \psi$ and Hartree-type $(|x|^{-(3-2r)} * |\psi|^2)\psi$. The study of blow-up solutions for (1.4) is of particular challenge, because the methods for proving blow-up results of (1.1) with power-type $|\psi|^p\psi$ or Hartree-type $(|x|^{-(3-2r)} * |\psi|^2)\psi$ are usually different, so we should develop a new method when both nonlinearities appear simultaneously.

Inspired by the ideas in [9], we study the blow-up criteria for (1.4). The difficulty is the presence of the fractional order Laplacian $(-\Delta)^s$. When $s = 1$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \varphi(x)|\psi(t,x)|^2\,dx = 2\,\mathrm{Im}\int_{\mathbb{R}^3} \bar{\psi}(t,x)\nabla\varphi(x)\cdot\nabla\psi(t,x)\,dx. \tag{1.9}$$

Using this identity, Du et al. [9] derived an L^2-estimate in the exterior ball. Thanks to this L^2-estimate and the virial estimates, they established the blow-up criteria for the classical NLS. In the case $s \in (\frac{1}{2}, 1)$, the identity (1.9) does not hold. However, by exploiting the idea in [3] and the use of the Balakrishman's formula (1.8), we can obtain the time derivative of the virial action. Thus, we can obtain the blow-up criteria for (1.4).

Theorem 1.1. *Let $s \in (1/2, 1)$ and $\psi_0 \in H^s$ be the corresponding (not necessary radial) solution to (1.4) on the maximal time interval $[0, T^*)$. If there exists $\delta > 0$ such that*

$$\sup_{t\in[0,T^*)} Q(\psi(t)) \le -\delta < 0, \tag{1.10}$$

where $Q(\psi(t))$ is defined by (1.14). Then one of the following statements is true:

- *$\psi(t)$ blows up in finite time, i.e. $T^* < +\infty$; or*
- *$\psi(t)$ blows up in infinite time and there exists a time sequence $(t_n)_{n\ge1}$ such that $t_n \to +\infty$ and*

$$\lim_{n\to\infty} \|(-\Delta)^{s/2}\psi(t_n)\|_{L^2} = \infty. \tag{1.11}$$

Based on the blow-up criterion (1.10), we will study the strong instability of standing waves of (1.4). The standing waves of (1.4) are solutions of the form $e^{i\omega t}u$, where $\omega \in \mathbb{R}$ is a frequency and $u \in H^s\backslash\{0\}$ is a nontrivial solution to the elliptic equation

$$(-\Delta)^s u + \omega u + (|x|^{-(3-2r)} * |u|^2)u - |u|^p u = 0. \tag{1.12}$$

Note that (1.12) can be written as $S'_\omega(u) = 0$, where

$$\begin{aligned} S_\omega(u) := \frac{1}{2}\|u\|^2_{\dot{H}^s} + \frac{\omega}{2}\|u\|^2_{L^2} + \frac{1}{4}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |u|^2)|u|^2 dx \\ - \frac{1}{p+2}\|u\|^{p+2}_{L^{p+2}} \end{aligned} \tag{1.13}$$

is the action functional. Then we define

$$\begin{aligned} Q(u) &:= \partial_\lambda S_\omega(u^\lambda)|_{\lambda=1} \\ &= s\|u\|^2_{\dot{H}^s} + \frac{3-2r}{4}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |u|^2)|u|^2 dx - \frac{3p}{2(p+2)}\|u\|^{p+2}_{L^{p+2}} \end{aligned} \tag{1.14}$$

with $u^\lambda(x) := \lambda^{3/2}u(\lambda x)$ and

$$\begin{aligned} K_\omega(u) &:= (s+r)\langle S'_\omega(u), u\rangle - I_\omega(u) \\ &= \frac{4s+2r-3}{2}\|u\|^2_{\dot H^s} + \frac{\omega(2s+2r-3)}{2}\|u\|^2_{L^2} \\ &\quad + \frac{4s+2r-3}{4}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |u|^2)|u|^2 dx \\ &\quad - \frac{(s+r)(p+2)-3}{p+2}\|u\|^{p+2}_{L^{p+2}}, \end{aligned} \tag{1.15}$$

where $I_\omega(u)$ denotes the Pohozaev identity related to (1.12), see (2.3).

The usual strategy to show the strong instability of standing waves of the classical NLS (s=1) is to establish the finite time blow-up by using the variational characterization of ground states as minimizers of the action functional and the virial identity. More specifically, the variational characterization of ground states by the manifold $\mathcal{N} := \{v \in H^1\backslash\{0\},\ Q(v) = 0\}$ can imply the key estimate $Q(\psi(t)) \le 2(S_\omega(\psi_0) - S_\omega(u))$, where u is the ground state solution. Then, it follows from the virial identity and the choice of initial data ψ_0 that

$$\frac{d^2}{dt^2}\|x\psi(t)\|^2_{L^2} = 8Q(\psi(t)) \le 16(S_\omega(\psi_0) - S_\omega(u)) < 0,$$

for $t \in [0, T^*)$. This implies that the solution $\psi(t)$ blows up in a finite time. Thus, we can prove the strong instability of ground state standing waves [5, 26, 31, 32, 33, 37]. However, in many cases, it is hard to obtain the variational characterization of ground states by the manifold $\mathcal{N}$. But we can obtain the variational characterization of ground states by the Nehari manifold and obtain the key estimate $Q(\psi(t)) \le 2(S_\omega(\psi_0) - S_\omega(u))$ [16, 17, 18, 27, 19, 29, 30, 34, 38, 41].

When $s = r = 1$ and $p \in \{2/3\} \cup (1, 4/3)$, Bellazzini and Siciliano [1] proved the existence of orbitally stable standing waves for (1.4). Kikuchi [25] showed the existence of standing waves for (1.4) with $s = r = 1$ and $0 < p < 4$ and proved that the standing wave $e^{i\omega t}u$ is strongly unstable for all $\omega > 0$ when $2 \le p < 4$. When $4/3 < p < 2$, it shows that there exists $\bar\omega > 0$ such that the standing wave $e^{i\omega t}u$ is strongly unstable for all $\omega > \bar\omega$. In the L^2-supercritical case, i.e., $4/3 < p < 4$, Bellazzini et al. [2] improved the result of Kikuchi and proved that the standing wave $e^{i\omega t}u$ is strongly unstable for all $\omega > 0$. When $4/3 \le p < 4$, Feng et al. [13] proved that the standing wave $e^{i\omega t}u$ is strongly unstable for all $\omega > 0$.

Equation (1.4) with $p = 4s/3$ is a class of nonlinear Schrödinger equations with combined L^2-critical and L^2-subcritical nonlinearities. When we try to study the variational characterization of ground states by the manifold $\mathcal{N}$, it is hard to obtain $S'_\omega(u) = 0$, see Lemma 5.4. Moreover, we find that the usual Nehari manifold is not a good choice in this case. Fortunately, we can obtain the variational characterization of ground states by the Nehari-Pohozaev manifold $\mathcal{N}_1 := \{u \in H^s\backslash\{0\},\ K_\omega(u) = 0\}$. Based on this variational characterization and a theoretical analysis, we can obtain the strong instability of standing waves for (1.4).

Theorem 1.2. *Let $\omega > 0$, $s \in (1/2, 1)$, $2s + 2r > 3$, $4s/3 \le p < 4s/(3-2s)$ and u be the ground state related to* (1.12). *Then the standing wave $\psi(t,x) = e^{i\omega t}u(x)$ is strongly unstable in the following sense: there exists $\{\psi_{0,n}\} \subset H^s$ such that $\psi_{0,n} \to u$ in H^s as $n \to \infty$ and the corresponding solution ψ_n of* (1.4) *with initial data $\psi_{0,n}$ blows up in finite time or infinite time for any $n \ge 1$.*

This article is organized as follows. In Section 2, we present some useful lemmas such as the local well-posedness theory of (1.4), Brezis-Lieb's lemma, and the compactness lemma. In Section 3, we prove the localized virial estimates related to (1.4). In Sections 4 and 5, we prove Theorems 1.1 and 1.2, respectively.

2. Preliminary lemmas

In this section, we recall some preliminary results that will be used later. Firstly, let us recall the local theory for the Cauchy problem (1.4). The local well-posedness for the fractional NLS in the energy space H^s was studied by Hong and Sire in [20]. The proof is based on Strichartz's estimates and the contraction mapping argument. Note that for non-radial data, Strichartz's estimates have a loss of derivatives. Fortunately, this loss of derivatives can be compensated by using the Sobolev embedding. However, it leads to a weak local well-posedness in the energy space compared to the classical nonlinear Schrödinger equation. We refer the reader to [7, 20] for more details. We can remove the loss of derivatives in Strichartz's estimates by considering radially symmetric data. However, it needs a restriction on the validity of s, namely $\frac{3}{5} \le s < 1$.

Proposition 2.1. *Let $3/5 \le s < 1$, $0 < p < \frac{4s}{3-2s}$ and $\psi_0 \in H^s$ be radial. Then there exists $T = T(\|\psi_0\|_{H^s})$ such that (1.4) admits a unique solution $\psi \in C([0,T], H^s)$. Let $[0,T^*)$ be the maximal time interval on which the solution ψ is well-defined. If $T^* < \infty$, then $\|\psi(t)\|_{\dot{H}^s} \to \infty$ as $t \uparrow T^*$. Moreover, for all $0 \le t < T^*$, the solution $\psi(t)$ satisfies the following conservation of mass and energy*

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2},$$
$$E(\psi(t)) = E(\psi_0),$$

where

$$\begin{aligned} E(\psi(t)) = \frac{1}{2}\|\psi(t)\|^2_{\dot{H}^s} + \frac{1}{4}\int_{\mathbb{R}^3} (|x|^{-(3-2r)} * |\psi(t)|^2)(x)|\psi(t,x)|^2 dx \\ - \frac{1}{p+2}\|\psi(t)\|^{p+2}_{L^{p+2}}. \end{aligned} \tag{2.1}$$

In this article, we use the so called Brezis-Lieb's lemma [4].

Lemma 2.2. *Let $0 < p < \infty$. Suppose that $u_n \to u$ almost everywhere and $\{u_n\}$ is a bounded sequence in L^p. Then*

$$\lim_{n\to\infty} (\|u_n\|^p_{L^p} - \|u_n - u\|^p_{L^p} - \|u\|^p_{L^p}) = 0.$$

Lemma 2.3 ([40]). *Let $u \in H^s$ and $2s + 2r > 3$. Suppose that $u_n \rightharpoonup u$ in H^s and $u_n \to u$ a.e. in $\mathbb{R}^3$. Then*

$$\begin{aligned} \int_{\mathbb{R}^3} (|x|^{-(3-2r)} * |u_n|^2)|u_n|^2 dx = \int_{\mathbb{R}^3} (|x|^{-(3-2r)} * |u_n - u|^2)|u_n - u|^2 dx \\ + \int_{\mathbb{R}^3} (|x|^{-(3-2r)} * |u|^2)|u|^2 dx + \circ(1). \end{aligned}$$

The following compactness lemma is vital in our discussions [8, 10].

Lemma 2.4. *Let $0 < p < \frac{4s}{3-2s}$ and $\{u_n\}$ be a bounded sequence in H^s such that*
$$\limsup_{n\to\infty} \|u_n\|_{\dot{H}^s} \le M, \quad \liminf_{n\to\infty} \|u_n\|_{L^{p+2}} \ge m.$$
Then there exist a sequence $(x_n)_{n\ge1}$ in $\mathbb{R}^3$ and $u \in H^s \setminus \{0\}$ such that up to a subsequence,
$$u_n(\cdot + x_n) \rightharpoonup u \quad \text{weakly in } H^s.$$

Finally, we recall the Pohozaev identity related to (1.12) [40].

Lemma 2.5. *If $u \in H^s$ satisfies equation (1.12), then it holds*
$$\|u\|_{\dot{H}^s}^2 + \omega\|u\|_{L^2}^2 + \int_{\mathbb{R}^3} (|x|^{-(3-2r)} * |u|^2)|u|^2 dx - \|u\|_{L^{p+2}}^{p+2} = 0 \tag{2.2}$$
and
$$\begin{aligned} I_\omega(u) := \frac{3-2s}{2}\|u\|_{\dot{H}^s}^2 + \frac{3\omega}{2}\|u\|_{L^2}^2 + \frac{2r+3}{4}\int_{\mathbb{R}^3} (|x|^{-(3-2r)} * |u|^2)|u|^2 dx \\ - \frac{3}{p+2}\|u\|_{L^{p+2}}^{p+2} = 0. \end{aligned} \tag{2.3}$$

3. Localized virial estimates

In this section, we prove some localized virial estimates related to (1.4). Let us recall some useful results in [3].

Lemma 3.1 ([3]). *Suppose $\varphi : \mathbb{R}^3 \to \mathbb{R}$ is such that $\nabla\varphi \in W^{1,\infty}$. Then for all $u \in H^{1/2}$, it holds*
$$\Big|\int_{\mathbb{R}^3} \overline{u}(x)\nabla\varphi(x)\cdot\nabla u(x)dx\Big| \le C\|\nabla\varphi\|_{W^{1,\infty}}\Big(\||\nabla|^{1/2}u\|_{L^2}^2 + \|u\|_{L^2}\||\nabla|^{1/2}u\|_{L^2}\Big)$$
for some constant $C > 0$.

To study the localized virial estimates for (1.4), we introduce the auxiliary function
$$u_m(x) := c_s\frac{1}{-\Delta + m}u(x) = c_s\mathcal{F}^{-1}\Big(\frac{\hat{u}(\xi)}{|\xi|^2 + m}\Big), \quad m > 0, \tag{3.1}$$
where
$$c_s := \sqrt{\frac{\sin \pi s}{\pi}}.$$

Lemma 3.2 ([3]). *Let $s \in (0, 1)$ and $\varphi : \mathbb{R}^3 \to \mathbb{R}$ with $\Delta\varphi \in W^{2,\infty}$. Then for all $u \in L^2$ it holds*
$$\Big|\int_0^\infty m^s \int_{\mathbb{R}^3} (\Delta^2\varphi)|u_m|^2\, dx\, dm\Big| \le C\|\Delta^2\varphi\|_{L^\infty}^s \|\Delta\varphi\|_{L^\infty}^{1-s}\|u\|_{L^2}^2$$
for some constant $C > 0$ dependent only on s.

We refer the reader to [3, Appendix A] for the proof of Lemmas 3.1 and 3.2. Given that
$$\frac{\sin \pi s}{\pi}\int_0^\infty \frac{m^s}{(|\xi|^2 + m)^2}\, dm = s|\xi|^{2s-2},$$

Plancherel's and Fubini's theorems imply that

$$\begin{aligned}\int_0^\infty m^s \int_{\mathbb{R}^3} |\nabla u_m|^2\,dx\,dm &= \int_{\mathbb{R}^3} \Big(\frac{\sin \pi s}{\pi} \int_0^\infty \frac{m^s\,dm}{(|\xi|^2+m)^2}\Big)|\xi|^2|\hat{u}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^3} (s|\xi|^{2s-2})|\xi|^2|\hat{u}(\xi)|^2 d\xi = s\|(-\Delta)^{s/2}u\|_{L^2}^2\end{aligned} \tag{3.2}$$

for any $u \in \dot{H}^s$.

Lemma 3.3. *Let $s \in (1/2,1)$ and $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be such that $\nabla\varphi \in W^{1,\infty}$. Then for any $u \in L^2$ it holds*

$$\Big|\int_0^\infty m^s \int_{\mathbb{R}^3} (\Delta\varphi)|u_m|^2\,dx\,dm\Big| \le C\|\Delta\varphi\|_{L^\infty}^{2s-1}\|\nabla\varphi\|_{L^\infty}^{2-2s}\|u\|_{L^2}^2$$

for some constant $C > 0$ dependent only on s.

Proof. The idea is essentially similar to [3, Lemma A.2]. For the reader's convenience, we just present the outline of our proof. Splitting m-integral into $\int_0^\rho \dots$ and $\int_\rho^\infty \dots$ with $\rho > 0$ to be chosen later. For the first term, we use integration by parts and Hölder's inequality to have

$$\begin{aligned}\Big|\int_0^\rho m^s \int_{\mathbb{R}^3} (\Delta\varphi)|u_m|^2\,dx\,dm\Big| &= \Big|\int_0^\rho m^s \int_{\mathbb{R}^3} \nabla\varphi\cdot(\nabla u_m \overline{u}_m + u_m\nabla\overline{u}_m)\,dx\,dm\Big| \\ &= \|\nabla\varphi\|_{L^\infty}\int_0^\rho m^s\|\nabla u_m\|_{L^2}\|u_m\|_{L^2}\,dm \\ &= \|\nabla\varphi\|_{L^\infty}\|u\|_{L^2}^2\Big(\int_0^\rho m^{s-3/2}\,dm\Big) \\ &\le C\rho^{s-1/2}\|\nabla\varphi\|_{L^\infty}\|u\|_{L^2}^2.\end{aligned}$$

Here we use the fact $\|\nabla u_m\|_{L^2} \le Cm^{-1/2}\|u\|_{L^2}$ and $\|u_m\|_{L^2} \le Cm^{-1}\|u\|_{L^2}$ which follows from the definition of u_m. For the second term, we have

$$\begin{aligned}\Big|\int_\rho^\infty m^s \int_{\mathbb{R}^3} (\Delta\varphi)|u_m|^2\,dx\,dm\Big| &\le C\|\Delta\varphi\|_{L^\infty}\Big(\int_\rho^\infty m^s\|u_m\|_{L^2}^2\,dm\Big) \\ &\le C\|\Delta\varphi\|_{L^\infty}\|u\|_{L^2}^2\Big(\int_\rho^\infty m^{s-2}\,dm\Big) \\ &\le C\rho^{s-1}\|\Delta\varphi\|_{L^\infty}\|u\|_{L^2}^2.\end{aligned}$$

Combining two terms yields

$$\Big|\int_0^\infty \int_{\mathbb{R}^3} (\Delta\varphi)|u_m|^2\,dx\,dm\Big| \le C\Big(\rho^{s-1/2}\|\nabla\varphi\|_{L^\infty} + \rho^{s-1}\|\Delta\varphi\|_{L^\infty}\Big)\|u\|_{L^2}^2$$

for arbitrary $\rho > 0$. Minimizing the right hand side with respect to ρ, i.e. choosing $\rho = \big(\frac{(1-s)\|\Delta\varphi\|_{L^\infty}}{(s-1/2)\|\nabla\varphi\|_{L^\infty}}\big)^2$, we obtain the desired result. $\square$

By the same argument as in Lemma 3.3 and Lemma 3.1, we obtain the following result.

Lemma 3.4. *Let $s \in (1/2,1)$ and $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be such that $\nabla\varphi \in W^{1,\infty}$. Then for any $u \in H^{1/2}$ we have*

$$\Big|\int_0^\infty \int_{\mathbb{R}^3} \overline{u}_m \nabla\varphi\cdot\nabla u_m\,dx\,dm\Big| \le C\|\nabla\varphi\|_{W^{1,\infty}}\|u\|_{H^{1/2}}^2$$

for some constant $C > 0$.

Let $1/2 < s < 1$ and $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be such that $\varphi \in W^{2,\infty}$. Assume that $\psi \in C([0,T^*), H^s)$ is a solution to (1.4). We define the localized virial action of ψ associated to φ by

$$\mathcal{V}_\varphi[\psi(t)] := \int_{\mathbb{R}^3} \varphi(x)|\psi(t,x)|^2\,dx.$$

Lemma 3.5 (Virial identity)**.** *Let $s \in (1/2, 1)$ and $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be such that $\varphi \in W^{2,\infty}$. Assume that $\psi \in C([0,T^*), H^s)$ is a solution to* (1.4)*. Then for any $t \in [0, T^*)$ we have*

$$\begin{aligned}
&\frac{d}{dt}\mathcal{V}_\varphi[\psi(t)] \\
&= -i\int_0^\infty m^s \int_{\mathbb{R}^3} (\Delta\varphi)|\psi_m(t)|^2\,dx\,dm - 2i\int_0^\infty m^s \int_{\mathbb{R}^3} \overline{\psi}_m(t)\nabla\varphi\cdot\nabla\psi_m(t)\,dx\,dm,
\end{aligned}$$

where $\psi_m(t) = c_s(-\Delta + m)^{-1}\psi(t)$.

Proof. It suffices to prove Lemma 3.5 for $\psi(t) \in C_0^\infty(\mathbb{R}^3)$. The general case follows by an approximation argument. We write

$$\mathcal{V}_\varphi[\psi(t)] = \langle \psi(t), \varphi\psi(t)\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in L^2. Since $\psi(t)$ satisfies (1.4), it is easy to see that

$$\frac{d}{dt}\mathcal{V}_\varphi[\psi(t)] = i\langle \psi(t), [(-\Delta)^s, \varphi]\psi(t)\rangle,$$

where $[X,Y] = XY - YX$ is the commutator of X and Y. To study $[(-\Delta)^s, \varphi]$, we recall the Balakrishman's formula

$$(-\Delta)^s = \frac{\sin \pi s}{\pi}\int_0^\infty m^{s-1}\frac{-\Delta}{-\Delta + m}\,dm.$$

Using the fact that for operators $A \ge 0$, B with $m > 0$ being any positive real number

$$\big[\frac{A}{A+m}, B\big] = \big[1 - \frac{m}{A+m}, B\big] = -m\big[\frac{1}{A+m}, B\big] = m\frac{1}{A+m}[A,B]\frac{1}{A+m}.$$

and letting $A = (-\Delta)^s, B = \varphi$ and using the Balakrishman's formula, we have

$$\begin{aligned}
[(-\Delta)^s, \varphi] &= \frac{\sin\pi s}{\pi}\int_0^\infty m^s\big[\frac{-\Delta}{-\Delta+m}, \varphi\big]\,dm \\
&= \frac{\sin\pi s}{\pi}\int_0^\infty m^s \frac{1}{-\Delta+m}[-\Delta,\varphi]\frac{1}{-\Delta+m}\,dm.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\langle \psi(t), [(-\Delta)^s, \varphi]\psi(t)\rangle \\
&= \langle \psi(t), \Big(\frac{\sin\pi s}{\pi}\int_0^\infty m^s \frac{1}{-\Delta+m}[-\Delta,\varphi]\frac{1}{-\Delta+m}\,dm\Big)\psi(t)\rangle \\
&= c_s^2\int_0^\infty m^s\langle\psi(t), \frac{1}{-\Delta+m}[-\Delta,\varphi]\frac{1}{-\Delta+m}\psi(t)\rangle\,dm \\
&= \int_0^\infty m^s\langle c_s(-\Delta+m)^{-1}\psi(t), [-\Delta,\varphi]c_s(-\Delta+m)^{-1}\psi(t)\rangle\,dm \\
&= \int_0^\infty m^s\int_{\mathbb{R}^3}\overline{\psi}_m(t)\Big(-\Delta\varphi\psi_m(t) - 2\nabla\varphi\cdot\nabla\psi_m(t)\Big)\,dx\,dm
\end{aligned}$$

$$= \int_0^\infty m^s \int_{\mathbb{R}^3} \Big((-\Delta\varphi)|\psi_m(t)|^2 - 2\overline{\psi}_m(t)\nabla\varphi\cdot\nabla u_m(t)\Big)\,dx\,dm.$$

□

A direct consequence of Lemmas 3.3 and 3.4 is the following estimate.

Lemma 3.6. *Let $s \in (1/2,1)$ and $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be such that $\varphi \in W^{2,\infty}$. Assume that $\psi \in C([0,T^*), H^s)$ is a solution to (1.4). Then for any $t \in [0,T^*)$ we have*

$$\big|\frac{d}{dt}\mathcal{V}_\varphi[\psi(t)]\big| \le C\|\nabla\varphi\|_{W^{1,\infty}}\|\psi(t)\|_{H^s}^2$$

for some constant $C > 0$ dependent only on s.

We next define the localized Morawetz action of ψ associated to φ by

$$\mathcal{M}_\varphi[\psi(t)] := 2\,\mathrm{Im}\int_{\mathbb{R}^3} \bar{\psi}(t,x)\nabla\varphi(x)\cdot\nabla\psi(t,x)dx. \tag{3.3}$$

By Lemma 3.1, we obtain the bound

$$|\mathcal{M}_\varphi[\psi(t)]| \le C\left(\|\nabla\varphi\|_{L^\infty}, \|\Delta\varphi\|_{L^\infty}\right)\|\psi(t)\|_{H^{1/2}}^2.$$

Hence the quantity $\mathcal{M}_\varphi[\psi(t)]$ is well-defined, given $\psi(t) \in H^s(\mathbb{R}^3)$ with $s > 1/2$.

Lemma 3.7 (Morawetz identity)**.** *Let $s \in (1/2,1)$ and $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be such that $\nabla\varphi \in W^{3,\infty}$. Assume that $\psi \in C([0,T^*), H^s)$ is a solution to (1.4). Then for each $t \in [0,T^*)$ we have*

$$\begin{aligned}
&\frac{d}{dt}\mathcal{M}_\varphi[\psi(t)]\\
&= \int_0^\infty m^s \int_{\mathbb{R}^3} \Big\{ 4\overline{\partial_k\psi_m(t)}(\partial^2_{kl}\varphi)\partial_l\psi_m(t) - (\Delta^2\varphi)|\psi_m(t)|^2\Big\}\,dx\,dm\\
&\quad + (3-2r)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(t,x)|^2|u(t,y)|^2(x-y)\cdot(\nabla\varphi(x)-\nabla\varphi(y))}{|x-y|^{5-2r}}\,dx\,dy\\
&\quad - \frac{2p}{p+2}\int_{\mathbb{R}^3}\Delta\varphi|\psi(t)|^{p+2}\,dx,
\end{aligned} \tag{3.4}$$

where $\psi_m(t) = \psi_m(t,x)$ is defined in (3.1).

Proof. Integration by parts yields

$$\begin{aligned}
&\langle u(t), [-(|x|^{-(3-2r)} * |u(t)|^2), i\Gamma_\varphi]u(t)\rangle\\
&= -\langle u(t), [(|x|^{-(3-2r)} * |u(t)|^2), \nabla\varphi\cdot\nabla + \nabla\cdot\nabla\varphi]u(t)\rangle\\
&= 2\int_{\mathbb{R}^3}\nabla\varphi\cdot\nabla(|x|^{-(3-2r)} * |u(t)|^2)|u(t)|^2dx\\
&= -(3-2r)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(t,x)|^2|u(t,y)|^2(x-y)\cdot(\nabla\varphi(x)-\nabla\varphi(y))}{|x-y|^{5-2r}}\,dx\,dy.
\end{aligned}$$

The rest proof is similar to [3, Lemma 2.1], so we omit the details. □

4. Blow-up criteria for (1.4)

Lemma 4.1. *Let $\eta > 0$, $R > 1$ and the solution $\psi(t)$ of (1.4) satisfy*

$$C_1 := \sup_{t\in[0,+\infty)} \|\psi(t)\|_{H^s} < \infty. \tag{4.1}$$

Then there exists a constant $C > 0$ independent of R and C_1 such that

$$\int_{|x|\geq R} |\psi(t,x)|^2\,dx \leq \eta + o_R(1)$$

for all $t \in [0, T_0]$ with $T_0 := \frac{\eta R}{CC_1^2}$.

Proof. Let us now introduce $\theta : [0,\infty) \to [0,1]$ a smooth function satisfying

$$\theta(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq 1/2, \\ 1 & \text{if } r \geq 1. \end{cases}$$

For $R > 1$, we denote the radial function

$$\phi_R(x) := \theta(r/R), \quad r = |x|.$$

We have

$$\nabla\phi_R(x) = \frac{x}{rR}\theta'(r/R), \quad \Delta\phi_R(x) = \frac{1}{R^2}\theta''(r/R) + \frac{2}{rR}\theta'(r/R).$$

In particular, we have

$$\|\nabla\phi_R\|_{W^{1,\infty}} \sim \|\nabla\phi_R\|_{L^\infty} + \|\Delta\phi_R\|_{L^\infty} \leq CR^{-1}. \tag{4.2}$$

We define the localized virial potential as

$$\mathcal{V}_{\phi_R}[\psi(t)] := \int_{\mathbb{R}^3} \phi_R(x)|\psi(t,x)|^2\,dx.$$

We have

$$\begin{aligned} \mathcal{V}_{\phi_R}[\psi(t)] &= \mathcal{V}_{\phi_R}[\psi_0] + \int_0^t \frac{d}{d\tau}\mathcal{V}_{\phi_R}[\psi(\tau)]d\tau \\ &\leq \mathcal{V}_{\phi_R}[\psi_0] + \Big(\sup_{\tau\in[0,t]} \big|\frac{d}{d\tau}\mathcal{V}_{\phi_R}[\psi(\tau)]\big|\Big)t. \end{aligned}$$

By Lemma 3.6, (4.1) and (4.2), we obtain

$$\sup_{\tau\in[0,t]} \big|\frac{d}{d\tau}\mathcal{V}_{\phi_R}[\psi(\tau)]\big| \leq C\|\nabla\phi_R\|_{W^{1,\infty}} \sup_{\tau\in[0,t]} \|\psi(\tau)\|_{H^s}^2 \leq CC_1^2R^{-1},$$

for some constant $C > 0$ independent of R and C_1. Therefore,

$$\mathcal{V}_{\phi_R}[\psi(t)] \leq \mathcal{V}_{\phi_R}[\psi_0] + CC_1^2R^{-1}t,$$

for all $t \geq 0$. By the choice of θ and the conservation of mass, we have

$$\mathcal{V}_{\phi_R}[\psi_0] = \int_{\mathbb{R}^3} \phi_R(x)|\psi_0(x)|^2\,dx \leq \int_{|x|>R/2} |\psi_0(x)|^2\,dx \to 0,$$

as $R \to \infty$ or $\mathcal{V}_{\phi_R}[\psi_0] = o_R(1)$. On the other hand,

$$\int_{|x|\geq R} |\psi(t,x)|^2\,dx \leq \mathcal{V}_{\phi_R}[\psi(t)].$$

Combing the above estimates, we arrive at the desired result. $\square$

Proof of Theorem 1.1. If $T^* < +\infty$, then the proof is done. If $T^* = +\infty$, then we need to show (1.11). Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be such that $\nabla\varphi \in W^{3,\infty}$. In addition, we assume that $\varphi = \varphi(r)$ is radial and satisfies

$$\varphi(r) = \begin{cases} \frac{r^2}{2} & \text{for } r \le 1, \\ \text{const.} & \text{for } r \ge 10, \end{cases}$$

and $\varphi''(r) \le 1$ for $r \ge 0$. Given $R > 0$, we define the rescaled function $\varphi_R : \mathbb{R}^3 \to \mathbb{R}$ by

$$\varphi_R(r) := R^2 \varphi\Big(\frac{x}{R}\Big). \tag{4.3}$$

We readily verify the inequalities

$$1 - \varphi_R''(r) \ge 0, \quad 1 - \frac{\varphi_R'(r)}{r} \ge 0, \quad 3 - \Delta\varphi_R(x) \ge 0,$$

for all $r \ge 0$ and all $x \in \mathbb{R}^3$. It is easy to see that

$$\|\nabla^k \varphi_R\|_{L^\infty} \le R^{2-k}, \quad k = 0, \dots, 4,$$

and

$$\text{supt}(\nabla^k \varphi_R) \subset \begin{cases} \{|x| \le 10R\} & \text{for } k = 1, 2, \\ \{R \le |x| \le 10R\} & \text{for } k = 3, 4. \end{cases}$$

Applying Lemma 3.7, we have

$$\begin{aligned} &\frac{d}{dt}\mathcal{M}_{\varphi_R}[\psi(t)] \\ &= \int_0^\infty m^s \int_{\mathbb{R}^3} \Big\{ 4\overline{\partial_k \psi_m(t)} (\partial_{kl}^2 \varphi_R) \partial_l \psi_m(t) - (\Delta^2 \varphi_R)|\psi_m(t)|^2 \Big\}\, dx\, dm \\ &\quad - \frac{2p}{p+2} \int_{\mathbb{R}^3} \Delta\varphi_R |\psi(t)|^{p+2}\, dx \\ &\quad + (3-2r) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\psi(t,x)|^2 |\psi(t,y)|^2 (x-y)\cdot(\nabla\varphi_R(x) - \nabla\varphi_R(y))}{|x-y|^{5-2r}}\, dx\, dy, \end{aligned} \tag{4.4}$$

where $\psi_m(t) = \psi_m(t,x)$ is defined in (3.1). Since $\text{supt}(\Delta^2\varphi_R) \subset \{|x| \ge R\}$, by Lemma 3.2, we have

$$\begin{aligned} \Big| \int_0^\infty m^s \int_{\mathbb{R}^3} (\Delta^2 \varphi_R)|\psi_m(t)|^2\, dx\, dm \Big| &\le C \|\Delta^2 \varphi_R\|_{L^\infty}^s \|\Delta\varphi_R\|_{L^\infty}^{1-s} \|\psi(t)\|_{L^2(|x|\ge R)}^2 \\ &\le C R^{-2s} \|\psi(t)\|_{L^2(|x|\ge R)}^2. \end{aligned} \tag{4.5}$$

Since φ_R is radial, we use

$$\partial_{jk}^2 = \Big(\frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3}\Big)\partial_r + \frac{x_j x_k}{r^2}\partial_r^2$$

to deduce

$$\begin{aligned} &\int_0^\infty m^s \int_{\mathbb{R}^3} \overline{\partial_k \psi_m(t)} (\partial_{jk}^2 \varphi_R) \partial_l \psi_m(t)\, dx\, dm \\ &= \int_0^\infty m^s \int_{\mathbb{R}^3} \frac{\varphi_R'}{r} |\nabla\psi_m(t)|^2\, dx\, dm \\ &\quad + \int_0^\infty m^s \int_{\mathbb{R}^3} \Big(\frac{\varphi_R''}{r^2} - \frac{\varphi_R'}{r^3}\Big)|x\cdot\nabla\psi_m(t)|^2\, dx\, dm. \end{aligned}$$

Using (3.2) leads to

$$\begin{aligned}&\int_0^\infty m^s\int_{\mathbb{R}^3}\frac{\varphi_R'}{r}|\nabla\psi_m(t)|^2\,dx\,dm\\&=s\|(-\Delta)^{s/2}\psi(t)\|_{L^2}^2+\int_0^\infty m^s\int_{\mathbb{R}^3}\Big(\frac{\varphi_R'}{r}-1\Big)|\nabla\psi_m(t)|^2\,dx\,dm.\end{aligned}$$

Since $\varphi_R''\leq 1$, the Cauchy-Schwarz inequality implies

$$\begin{aligned}&\int_0^\infty m^s\int_{\mathbb{R}^3}\Big(\frac{\varphi_R'}{r}-1\Big)|\nabla\psi_m(t)|^2\,dx\,dm\\&+\int_0^\infty m^s\int_{\mathbb{R}^3}\Big(\varphi_R''-\frac{\varphi_R'}{r}\Big)\frac{|x\cdot\nabla\psi_m(t)|^2}{r^2}\,dx\,dm\leq 0.\end{aligned}$$

Thus, we have

$$4\int_0^\infty m^s\int_{\mathbb{R}^3}\overline{\partial_k\psi_m(t)}(\partial_{jk}^2\varphi_R)\partial_l\psi_m(t)\,dx\,dm\leq 4s\|(-\Delta)^{s/2}\psi(t)\|_{L^2}^2. \tag{4.6}$$

Note that

$$-\frac{2p}{p+2}\int_{\mathbb{R}^3}\Delta\varphi_R|\psi(t)|^{p+2}\,dx=-\frac{6p}{p+2}\|\psi(t)\|_{L^{p+2}}^{p+2}+\frac{2p}{p+2}\int_{\mathbb{R}^3}(3-\Delta\varphi_R)|\psi(t)|^{p+2}\,dx.$$

Since $\operatorname{supt}(3-\Delta\varphi_R)\subset\{|x|\geq R\}$ and $\|3-\Delta\varphi_R\|_{L^\infty}\leq C$, we have

$$\begin{aligned}\int_{\mathbb{R}^3}(3-\Delta\varphi_R)|\psi(t)|^{p+2}\,dx&\leq C\int_{|x|\geq R}|\psi(t)|^{p+2}\,dx\\&\leq C\|\psi(t)\|_{L^{\frac{6}{3-2s}}(|x|\geq R)}^{\frac{3p}{2s}}\|\psi\|_{L^2(|x|\geq R)}^{\frac{4s-(3-2s)p}{2s}}\\&\leq C\|\psi(t)\|_{H^s}^{\frac{3p}{2s}}\|\psi(t)\|_{L^2(|x|\geq R)}^{\frac{4s-(3-2s)p}{2s}}\\&\leq CC_1^{\frac{3p}{2s}}\|\psi(t)\|_{L^2(|x|\geq R)}^{\frac{4s-(3-2s)p}{2s}}.\end{aligned}$$

Thus we obtain

$$-\frac{2p}{p+2}\int_{\mathbb{R}^3}\Delta\varphi_R|\psi(t)|^{p+2}\,dx\leq-\frac{6p}{p+2}\|\psi(t)\|_{L^{p+2}}^{p+2}+CC_1^{\frac{3p}{2s}}\|\psi(t)\|_{L^2(|x|\geq R)}^{\frac{4s-(3-2s)p}{2s}}. \tag{4.7}$$

We denote the last term in (4.4) by $\mathcal{T}$. We have

$$\mathcal{T}=(3-2r)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(x-y)\cdot(\nabla\varphi_R(x)-\nabla\varphi_R(y))\frac{|\psi(t,x)|^2|\psi(t,y)|^2}{|x-y|^{5-2r}}\,dx\,dy.$$

By using

$$\operatorname{supt}(|x-y|^2-(x-y)\cdot(\nabla\varphi_R(x)-\nabla\varphi_R(y)))\subset\{|x|\geq R\}\cup\{|y|\geq R\},$$

in the region $\{|x|\geq R\}$ we obtain

$$\big|\,|x-y|^2-(x-y)\cdot(\nabla\varphi_R(x)-\nabla\varphi_R(y))\big|\leq C|x-y|^2.$$

Thus, we obtain

$$\begin{aligned}&\Big|\int_{|x|\geq R}\int_{\mathbb{R}^3}[|x-y|^2-(x-y)\cdot(\nabla\varphi_R(x)-\nabla\varphi_R(y))]\frac{|u(t,x)|^2|\psi(t,y)|^{2p_2}}{|x-y|^{5-2r}}\,dx\,dy\Big|\\&\leq C\int_{|x|\geq R}(|x|^{-(3-2r)}*|\psi(t)|^2)|\psi(t)|^2\,dx.\end{aligned}$$

To estimate this term, we deduce from the Sobolev embedding that

$$\|\psi(t)\|^2_{L^{\frac{12}{3+2r}}} \le C\|\psi(t)\|_{L^2}^{\frac{4s+2r-3}{2s}}\|\psi(t)\|^{\frac{3-2r}{2s}}_{L^{\frac{6}{3-2s}}} \le C\|(-\Delta)^{s/2}\psi(t)\|_{L^2}^{\frac{3-2r}{2s}}. \tag{4.8}$$

Thus, it follows from the Hardy-Littlewood-Sobolev inequality and the conservation of mass that

$$\begin{aligned}
&\int_{|x|\ge R}(|x|^{-(3-2r)}*|\psi(t)|^2)|\psi(t)|^2\,dx\\
&\le C\||x|^{-(3-2r)}*|\psi(t)|^2\|_{L^{\frac{6}{3-2r}}(|x|\ge R)}\||\psi(t)|^2\|_{L^{\frac{6}{3+2r}}(|x|\ge R)}\\
&\le C\|\psi(t)\|^2_{L^{\frac{12}{3+2r}}}\|\psi(t)\|^2_{L^{\frac{12}{3+2r}}(|x|\ge R)}\\
&\le C\|\psi(t)\|_{H^s}^{\frac{3-2r}{s}}\|\psi(t)\|_{L^2(|x|\ge R)}^{\frac{4s+2r-3}{2s}}\\
&\le CC_1^{\frac{3-2r}{s}}\|\psi(t)\|_{L^2(|x|\ge R)}^{\frac{4s+2r-3}{2s}}.
\end{aligned}$$

We can derive an estimate in the region $\{|y|\ge R\}$ too. Similarly, we can obtain

$$\mathcal{T}\le (3-2r)\int_{\mathbb{R}^3}(|x|^{-(3-2r)}*|\psi(t)|^2)|\psi(t)|^2\,dx + CC_1^{\frac{3-2r}{s}}\|\psi(t)\|_{L^2(|x|\ge R)}^{\frac{4s+2r-3}{2s}}. \tag{4.9}$$

By using (4.5)–(4.9), we obtain

$$\begin{aligned}
&\frac{d}{dt}\mathcal{M}_{\varphi_R}[\psi(t)]\\
&\le 4s\|(-\Delta)^{s/2}\psi(t)\|^2_{L^2} + CR^{-2s}\|\psi(t)\|^2_{L^2(|x|\ge R)}\\
&\quad + (3-2r)\int_{\mathbb{R}^3}(|x|^{-(3-2r)}*|\psi(t)|^2)|u(t)|^2\,dx - \frac{6p}{p+2}\|\psi(t)\|^{p+2}_{L^{p+2}}\\
&\quad + CC_1^{\frac{3p}{2s}}\|\psi(t)\|_{L^2(|x|\ge R)}^{\frac{4s-(3-2s)p}{2s}} + CC_1^{\frac{3-2r}{s}}\|\psi(t)\|_{L^2(|x|\ge R)}^{\frac{4s+2r-3}{2s}}\\
&\le 4Q(\psi(t)) + CR^{-2s}\|\psi(t)\|^2_{L^2(|x|\ge R)} + CC_1^{\frac{3p}{2s}}\|\psi(t)\|_{L^2(|x|\ge R)}^{\frac{4s-(3-2s)p}{2s}}\\
&\quad + CC_1^{\frac{3-2r}{s}}\|\psi(t)\|_{L^2(|x|\ge R)}^{\frac{4s+2r-3}{2s}}.
\end{aligned} \tag{4.10}$$

By Lemma 4.1, we see that for any $\eta>0$ and any $R>1$, there exists $C>0$ independent of R and C_1 such that for any $t\in[0,T_0]$ with $T_0=\frac{\eta R}{CC_1^2}$, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{M}_{\varphi_R}[\psi(t)] &\le 4Q(\psi(t)) + CR^{-2s}(\eta+o_R(1))^2 + CC_1^{\frac{3p}{2s}}(\eta+o_R(1))^{\frac{4s-p(3-2s)}{2s}}\\
&\quad + CC_1^{\frac{3-2r}{s}}(\eta+o_R(1))^{\frac{4s+2r-3}{2s}}\\
&\le -4\delta + CR^{-2s}(\eta^2+o_R(1)) + CC_1^{\frac{3p}{2s}}(\eta^{\frac{4s-p(3-2s)}{2s}}+o_R(1))\\
&\quad + CC_1^{\frac{3-2r}{s}}(\eta^{\frac{4s+2r-3}{2s}}+o_R(1)).
\end{aligned}$$

We first choose $\eta>0$ small enough so that

$$CC_1^{\frac{3p}{2s}}\eta^{\frac{4s-p(3-2s)}{2s}} + CC_1^{\frac{3-2r}{s}}\eta^{\frac{4s+2r-3}{2s}} = -3\delta > 0.$$

We next choose $R>1$ large enough so that

$$\frac{d}{dt}\mathcal{M}_{\varphi_R}[\psi(t)] \le -\delta < 0 \tag{4.11}$$

for any $t \in [0, T_0]$ with $T_0 = \frac{\eta R}{CC_1^2}$. Note that $\eta > 0$ is fixed, so we can choose $R > 1$ large enough such that T_0 is as large as we want. From (4.11) it follows that

$$\mathcal{M}_{\varphi_R}[\psi(t)] \leq -\delta t,$$

for all $t \in [t_0, T_0]$ with some sufficiently large $t_0 \in [0, T_0]$. On the other hand, by Lemma 3.1 and the conservation of mass, we have for any $t \in [0, +\infty)$,

$$\begin{aligned}|\mathcal{M}_{\varphi_R}[\psi(t)]| &\leq CC(\varphi_R)\Big(\||\nabla|^{1/2}\psi(t)\|_{L^2}^2 + \|\psi(t)\|_{L^2}\||\nabla|^{1/2}\psi(t)\|_{L^2}\Big)\\ &\leq CC(\varphi_R)\Big(\||\nabla|^{1/2}\psi(t)\|_{L^2}^2 + \|\psi(t)\|_{L^2}^2\Big)\\ &\leq CC(\varphi_R)\Big(\||\nabla|^{1/2}\psi(t)\|_{L^2}^2 + 1\Big).\end{aligned}$$

By interpolating between L^2 and $\dot{H}^s$, we obtain for any $t \in [t_0, T_0]$

$$\delta t \leq -\mathcal{M}_{\varphi_R}[\psi(t)] = |\mathcal{M}_{\varphi_R}[\psi(t)]| \leq CC(\varphi_R)\Big(\|(-\Delta)^{s/2}\psi(t)\|_{L^2}^{\frac{1}{s}} + 1\Big).$$

This implies that

$$\|(-\Delta)^{s/2}\psi(t)\|_{L^2} \geq Ct^s \tag{4.12}$$

for all $t \in [t_1, T_0]$ with some sufficiently large $t_1 \in [t_0, T_0]$. Taking t close to $T_0 = \frac{\eta R}{CC_1^2}$, we see that $\|(-\Delta)^{s/2}\psi(t)\|_{L^2} \to \infty$ as $R \to \infty$. Taking $R > 1$ sufficiently large, we have a contradiction with (4.1). The proof is complete. □

5. Strong instability of standing waves

In this section, we prove Theorem 1.2. Let us start with the following characterization of the ground state related to (1.12).

Proposition 5.1. *Let $\omega > 0$, $2s + 2r > 3$ and $\frac{4s}{3} \leq p < \frac{4s}{3-2s}$. Then u is the ground state related to* (1.12) *if and only if u solves the minimization problem*

$$d(\omega) = \inf\{S_\omega(v) : v \in H^s\backslash\{0\}, K_\omega(v) = 0\}. \tag{5.1}$$

To solve this minimization problem, we consider the minimization problem

$$\widetilde{d}(\omega) = \inf\{\widetilde{S}_\omega(v) : v \in H^s\backslash\{0\}, K_\omega(v) \leq 0\}, \tag{5.2}$$

where

$$\begin{aligned}\widetilde{S}_\omega(v) &:= S_\omega(v) - \frac{K_\omega(v)}{4s + 2r - 3}\\ &= \frac{\omega s}{4s + 2r - 3}\|v\|_{L^2}^2 + \frac{p(s + r) - 2s}{(p + 2)(4s + 2r - 3)}\|v\|_{L^{p+2}}^{p+2}.\end{aligned} \tag{5.3}$$

If $K_\omega(v) < 0$, then

$$\begin{aligned}K_\omega(\lambda v) &= \frac{4s + 2r - 3}{2}\lambda^2\|v\|_{\dot{H}^s}^2 + \frac{4s + 2r - 3}{4}\lambda^4\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |v|^2)|v|^2dx\\ &\quad + \frac{\omega(2s + 2r - 3)}{2}\lambda^2\|v\|_{L^2}^2 - \frac{(s + r)(p + 2) - 3}{p + 2}\lambda^{p+2}\|v\|_{L^{p+2}}^{p+2} > 0,\end{aligned}$$

for sufficiently small $\lambda > 0$. Thus, there exists $\lambda_0 \in (0, 1)$ such that $K_\omega(\lambda_0 v) = 0$. It follows that

$$\widetilde{S}_\omega(\lambda_0 v) = \frac{\omega s}{4s + 2r - 3}\lambda_0^2\|v\|_{L^2}^2 + \frac{p(s + r) - 2s}{(p + 2)(4s + 2r - 3)}\lambda_0^{p+2}\|v\|_{L^{p+2}}^{p+2} < \widetilde{S}_\omega(v),$$

This implies that

$$\widetilde{d}(\omega) = \inf\{\widetilde{S}_\omega(v) : v \in H^s\backslash\{0\}, K_\omega(v) = 0\}. \tag{5.4}$$

In following lemma, we will solve the minimizing problem (5.2).

Lemma 5.2. *Let $\omega > 0$, $2s + 2r > 3$ and $\frac{4s}{3} \le p < \frac{4s}{3-2s}$. Then there exists $u \in H^s\backslash\{0\}$, such that $K_\omega(u) = 0$ and $\widetilde{S}_\omega(u) = \widetilde{d}(\omega)$.*

Proof. We first show that $\widetilde{d}(\omega) > 0$. From $K_\omega(v) \le 0$, we have

$$\begin{aligned}&\frac{4s+2r-3}{2}\|v\|^2_{\dot{H}^s} + \frac{\omega(2s+2r-3)}{2}\|v\|^2_{L^2} + \frac{4s+2r-3}{4}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |v|^2)|v|^2dx\\ &\le \frac{(s+r)(p+2)-3}{p+2}\|v\|^{p+2}_{L^{p+2}},\end{aligned}$$

which implies

$$\frac{1}{2}H_\omega(v) \le \frac{(s+r)(p+2)-3}{(p+2)(2s+2r-3)}H_\omega(v)^{\frac{p}{2}+1},$$

where $H_\omega(v) = \|v\|^2_{\dot{H}^s} + \omega\|v\|^2_{L^2}$. Thus, there exists $C_0 > 0$ such that $H_\omega(v) > C_0$ for all $K_\omega(v) \le 0$. This implies that there exists $C_1 > 0$ such that

$$\begin{aligned}\widetilde{S}_\omega(v) &\ge \frac{p(s+r)-2s}{2(4s+2r-3)}\|v\|^{p+2}_{L^{p+2}}\\ &\ge \frac{p(s+r)-2s}{(p+2)(4s+2r-3)}\frac{(2s+2r-3)}{(s+r)(p+2)-3}H_\omega(v) \ge C_1.\end{aligned}$$

Taking the infimum over v, we obtain $\widetilde{d}(\omega) > 0$.

We now show that the minimizing problem (5.2) attains its minimum. Let $\{v_n\}$ be a minimizing sequence for (5.2), i.e., $\{v_n\} \subseteq H^s\backslash\{0\}$, $K_\omega(v_n) \le 0$ and $\widetilde{S}_\omega(v_n) \to \widetilde{d}(\omega)$ as $n \to \infty$. Thus, there exists $C > 0$ such that

$$\|v_n\|^2_{L^2} + \|v_n\|^{p+2}_{L^{p+2}} \le C. \tag{5.5}$$

This, together with $K_\omega(v_n) \le 0$ implies that $\{v_n\}$ is bounded in H^s. It follows from $\widetilde{d}(\omega) > 0$ that $\liminf_{n\to\infty} \|v_n\|^{p+2}_{L^{p+2}} > 0$. Therefore, applying Lemma 2.4, there exists a subsequence, still denoted by $\{v_n\}$ and $u \in H^s\backslash\{0\}$ such that

$$u_n := \tau_{x_n} v_n \rightharpoonup u \neq 0 \textit{ weakly in } H^s$$

for some $\{x_n\} \subseteq \mathbb{R}^3$. We deduce from Brezis-Lieb's lemma (Lemma 2.2) and Lemma 2.3 that

$$K_\omega(u_n) - K_\omega(u_n - u) - K_\omega(u) \to 0, \tag{5.6}$$

$$\widetilde{S}_\omega(u_n) - \widetilde{S}_\omega(u_n - u) - \widetilde{S}_\omega(u) \to 0. \tag{5.7}$$

Now, we claim that $K_\omega(u) \le 0$. If not, it follows from (5.6) and $K_\omega(u_n) \le 0$ that $K_\omega(u_n - u) \le 0$ for sufficiently large n. Thus, by the definition of $\widetilde{d}(\omega)$, it follows that

$$\widetilde{S}_\omega(u_n - u) \ge \widetilde{d}(\omega),$$

which, together with $\widetilde{S}_\omega(u_n) \to \widetilde{d}(\omega)$, implies that $\widetilde{S}_\omega(u) \le 0$, which is a contradiction with $\widetilde{S}_\omega(u) > 0$. We thus obtain $K_\omega(u) \le 0$.

Furthermore, we deduce from the definition of $\widetilde{d}(\omega)$ and the weak lower semi-continuity of norm that

$$\widetilde{d}(\omega) \le \widetilde{S}_\omega(u) \le \liminf_{n\to\infty} \widetilde{S}_\omega(u_n) = \widetilde{d}(\omega).$$

This yields $\widetilde{S}_\omega(u) = \widetilde{d}(\omega)$.

Finally, we show that $K_\omega(u) = 0$. Suppose that $K_\omega(u) < 0$ and set

$$\begin{aligned} K_\omega(u^\lambda) &= \frac{4s+2r-3}{2}\lambda^{2s}\|u\|_{\dot H^s}^2 + \frac{\omega(2s+2r-3)}{2}\|u\|_{L^2}^2 \\ &\quad + \frac{4s+2r-3}{4}\lambda^{3-2r}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |u|^2)|u|^2dx \\ &\quad - \frac{(s+r)(p+2)-3}{p+2}\lambda^{\frac{3p}{2}}\|u\|_{L^{p+2}}^{p+2} > 0 \end{aligned}$$

for sufficiently small $\lambda > 0$. Then there exists $\lambda_0 \in (0,1)$ such that $K_\omega(u^{\lambda_0}) = 0$. It follows that

$$\begin{aligned} \widetilde{S}_\omega(u^{\lambda_0}) &= \frac{\omega s}{4s+2r-3}\|u\|_{L^2}^2 + \frac{p(s+r)-2s}{(p+2)(4s+2r-3)}\lambda_0^{\frac{3p}{2}}\|u\|_{L^{p+2}}^{p+2} \\ &< \widetilde{S}_\omega(u) = \widetilde{d}(\omega), \end{aligned}$$

which contradicts the definition of $\widetilde{d}(\omega)$. Hence, we have $K_\omega(u) = 0$. □

From $d(\omega) = \widetilde{d}(\omega)$ and the above lemma, we can obtain the existence of minimization problem (5.1).

Lemma 5.3. *Let $\omega > 0$, $2s+2r > 3$ and $\frac{4s}{3} \le p < \frac{4s}{3-2s}$. Then there exists $u \in H^s\backslash\{0\}$ such that $K_\omega(u) = 0$ and $S_\omega(u) = d(\omega)$.*

Lemma 5.4. *Let $\omega > 0$, $2s+2r > 3$, and $\frac{4s}{3} \le p < \frac{4s}{3-2s}$. Assume that $u \in H^s\backslash\{0\}$ is a solution of the minimizing problem* (5.1)*, i.e., such that $K_\omega(u) = 0$ and $S_\omega(u) = d(\omega)$. Then $S'_\omega(u) = 0$.*

Proof. We firstly prove $K'_\omega(u) \neq 0$. If $K'_\omega(u) = 0$, then we have

$$\begin{aligned} &(4s+2r-3)(-\Delta)^s u + \omega(2s+2r-3)u + (4s+2r-3)(|x|^{-(3-2r)} * |u|^2)u \\ &- ((s+r)(p+2)-3)|u|^p u = 0. \end{aligned} \tag{5.8}$$

Then

$$\begin{gathered} A + B + C - D = d(\omega), \\ (4s+2r-3)A + (2s+2r-3)B + (4s+2r-3)C \\ - ((s+r)(p+2)-3)D = 0, \\ 2(4s+2r-3)A + 2(2s+2r-3)B + 4(4s+2r-3)C \\ - (p+2)((s+r)(p+2)-3)D = 0, \\ (3-2s)(4s+2r-3)A + 3(2s+2r-3)B + (4s+2r-3)(3+2r)C \\ - 3((s+r)(p+2)-3)D = 0, \end{gathered} \tag{5.9}$$

where

$$A = \frac{1}{2}\|u\|_{\dot H^s}^2, \quad B = \frac{\omega}{2}\|u\|_{L^2}^2,$$
$$C = \frac{1}{4}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |u|^2)(x)|u(x)|^2dx, \quad D = \frac{1}{p+2}\|u\|_{L^{p+2}}^{p+2}.$$

The first equation comes from the fact that $S_\omega(u) = d(\omega)$. The second one holds since $K_\omega(u) = 0$. The third one follows by multiplying (5.8) by u and integrating both sides. The fourth one is derived by applying the Pohozaev equality to (5.8).

After a direct calculations, we have

$$sA = tC, \quad C = \frac{p((s+r)(p+2)-3)D}{2(4s+2r-3)},$$
$$(2s+2r-3)B + \frac{(p(s+r)-2s)((s+r)(p+2)-3)}{2s}D = 0.$$

These $A = B = C = D = 0$ which is a contradiction with $A, B, C, D > 0$. Thus, $K'_\omega(u) \neq 0$.

Next, applying the Lagrange multiplier rule, there exists $\mu \in \mathbb{R}$ such that $S'_\omega(u) + \mu K'_\omega(u) = 0$. We claim that $\mu = 0$. As above, the equation $S'_\omega(u) + \mu K'_\omega(u) = 0$ can be written as

$$\begin{aligned}
&(-\Delta)^s u + \omega u + (|x|^{-(3-2r)} * |u|^2)u - |u|^p u \\
&+ \mu\big[(4s+2r-3)(|x|^{-(3-2r)} * |u|^2)u + \omega(2s+2r-3)u \\
&+ (4s+2r-3)(-\Delta)^s u - ((s+r)(p+2)-3)|u|^p u\big] = 0.
\end{aligned} \tag{5.10}$$

By the same argument as in (5.9), we have

$$\begin{gathered}
A + B + C - D = d(\omega), \\
(4s+2r-3)A + (2s+2r-3)B + (4s+2r-3)C - ((s+r)(p+2)-3)D = 0, \\
2(\mu(4s+2r-3)+1)A + 2(\mu(2s+2r-3)+1)B \\
+4(\mu(4s+2r-3)+1)C - (p+2)(\mu((s+r)(p+2)-3)+1)D = 0, \\
(3-2s)(\mu(4s+2r-3)+1)A + 3(\mu(2s+2r-3)+1)B \\
+ (\mu(4s+2r-3)+1)(3+2r)C - 3(\mu((s+r)(p+2)-3)+1)D = 0.
\end{gathered}$$

We now deal with the above system. Consider A, B, C, D as unknown quantities, and denote the coefficient matrix by M. Computing its determinant, we have

$$\det M = -4s\mu p(s+r)(1+\mu(4s+2r-3))((p-2)s+pr).$$

Note that

$$\det M = 0 \Longleftrightarrow \mu = 0,\ p = 0,\ \mu = -\frac{1}{4s+2r-3},\ (p-2)s + pr = 0.$$

Because of $2s + 2r > 3$ and $\frac{4s}{3} \leq p < \frac{4s}{3-2s}$, it follows that $(p-2)s + pr > 0$. We will show that μ must be equal to zero by excluding the other possibilities:

(1) If $\mu \neq 0$, $\mu \neq -\frac{1}{4s+2r-3}$, then $\det M \neq 0$, and hence the linear system has a unique solution (depending on the parameters $\mu, p, d(\omega)$). Applying Cramer's rule, we obtain

$$D = -\frac{d(\omega)(4s+2r-3)(2s+2r-3)}{p(s+r)((p-2)s+pr)} < 0,$$

which contradicts $D > 0$.

(2) If $\mu = -\frac{1}{4s+2r-3}$, then the third equation reads

$$4sB + (p+2)(s(p-2)+pr)D = 0,$$

which contradicts $B, D > 0$. Thus, $\mu = 0$ and $S'_\omega(u) = 0$. □

We now denote the set of all minimizers of (5.1) by

$$\mathcal{M}_\omega = \{u \in H^s \backslash \{0\} : S_\omega(u) = d(\omega),\ K_\omega(u) = 0\}.$$

Lemma 5.5. $\mathcal{M}_\omega \subseteq \mathcal{G}_\omega$.

Proof. Let $u \in \mathcal{M}_\omega$. It follows from Lemma 5.4 that $S'_\omega(u) = 0$. In particular, we have $u \in \mathcal{A}_\omega$. To prove $u \in \mathcal{G}_\omega$, it remains to show that $S_\omega(u) \le S_\omega(v)$ for all $v \in \mathcal{A}_\omega$. To see this, we notice that

$$K_\omega(v) = (s + r)\langle S'_\omega(v), v\rangle - I_\omega(v) = 0$$

for all $v \in \mathcal{A}_\omega$, where $I_\omega(v)$ is defined by (2.3). By definition of $d(\omega)$, we have $S_\omega(u) \le S_\omega(v)$. Thus, $u \in \mathcal{G}_\omega$. □

Lemma 5.6. $\mathcal{G}_\omega \subset \mathcal{M}_\omega$.

Proof. Let $u \in \mathcal{G}_\omega$. Since $\mathcal{M}_\omega$ is not empty, we take $v \in \mathcal{M}_\omega$. By Lemma 5.5, $v \in \mathcal{G}_\omega$. In particular, $S_\omega(u) = S_\omega(v)$. Since $v \in \mathcal{M}_\omega$, we obtain

$$S_\omega(u) = S_\omega(v) = d(\omega).$$

It remains to show that $K_\omega(u) = 0$. Since $u \in \mathcal{A}_\omega$, we have $S'_\omega(u) = 0$ and $I_\omega(u) = 0$, hence $K_\omega(u) = (s + r)\langle S'_\omega(u), u\rangle - I_\omega(u) = 0$. Therefore, $u \in \mathcal{M}_\omega$. □

Proof of Proposition 5.1. It follows immediately from Lemmas 5.3, 5.5, and 5.6. □

When $\frac{4s}{3} \le p < \frac{4s}{3-2s}$, to study the strong instability of standing waves for (1.4), we need to establish the following characterization of the ground state related to (1.12).

Lemma 5.7. *Let $\omega > 0$, $2s + 2r > 3$, $\frac{4s}{3} \le p < \frac{4s}{3-2s}$, and u be the ground state related to* (1.12). *Then*

$$S_\omega(u) = \inf\{S_\omega(v) : v \in H^s \backslash \{0\},\ Q(v) = 0\}. \tag{5.11}$$

Proof. Firstly, we claim that the minimizing problem in (5.11) is well-defined. Let $v \in H^s \backslash \{0\}$ and $Q(v) = 0$. If $p = \frac{4s}{3}$, then

$$\begin{aligned} S_\omega(v) &= S_\omega(v) - \frac{1}{2s} Q(v) \\ &= \frac{\omega}{2}\|v\|_{L^2}^2 + \frac{2s + 2r - 3}{8s}\int_{\mathbb{R}^3} (|x|^{-(3-2r)} * |v|^2)|v|^2 dx > 0. \end{aligned} \tag{5.12}$$

And if $\frac{4s}{3} < p < \frac{4s}{3-2s}$, then

$$\begin{aligned} &S_\omega(v) \\ &= S_\omega(v) - \frac{2}{3p} Q(v) \\ &= \frac{3p - 4s}{6p}\|v\|_{\dot H^s}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 + \frac{3p + 4t - 6}{12p}\int_{\mathbb{R}^3} (|x|^{-(3-2r)} * |v|^2)|v|^2 dx > 0. \end{aligned} \tag{5.13}$$

Thus we denote $d := \inf\{S_\omega(v) : v \in H^s \backslash \{0\},\ Q(v) = 0\}$. Firstly, we deduce from (2.2) and (2.3) that

$$K_\omega(u) = Q(u) = 0.$$

By the definition of d, we have

$$S_\omega(u) \ge d.$$

Let $v \in H^s\backslash\{0\}$ be such that $Q(v) = 0$. If $K_\omega(v) = 0$, then it follows from Proposition 5.1 that

$$S_\omega(v) \geq S_\omega(u).$$

If $K_\omega(v) \neq 0$, we notice that

$$\begin{aligned} K_\omega(v^\lambda) = &\frac{4s+2r-3}{2}\lambda^{2s}\|v\|^2_{\dot{H}^s} + \frac{\omega(2s+2r-3)}{2}\|v\|^2_{L^2} \\ &+ \frac{4s+2r-3}{4}\lambda^{3-2r}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |v|^2)|v|^2 dx \\ &- \frac{(s+r)(p+2)-3}{p+2}\lambda^{\frac{3p}{2}}\|v\|^{p+2}_{L^{p+2}}, \end{aligned}$$

where $v^\lambda(x) := \lambda^{3/2}v(\lambda x)$. When $\frac{4s}{3} < p < \frac{4s}{3-2s}$, we have

$$\lim_{\lambda\to 0} K_\omega(v^\lambda) = \frac{\omega(2s+2r-3)}{2}\|v\|^2_{L^2} > 0, \quad \text{and} \quad \lim_{\lambda\to\infty} K_\omega(v^\lambda) < 0. \tag{5.14}$$

When $p = 4s/3$, it follows from $Q(v) = 0$ that

$$s\|v\|^2_{\dot{H}^s} < \frac{3p}{2(p+2)}\|v\|^{p+2}_{L^{p+2}},$$

which implies that (5.14) holds. Thus, there exists $\lambda_0 > 0$ such that $K_\omega(v^{\lambda_0}) = 0$. This implies that

$$S_\omega(v^{\lambda_0}) \geq S_\omega(u).$$

On the other hand, by some basic calculations, we have

$$\begin{aligned} \partial_\lambda S_\omega(v^\lambda) &= s\lambda^{2s-1}\|v\|^2_{\dot{H}^s} + \frac{3-2r}{4}\lambda^{2-2r}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |v|^2)|v|^2 dx \\ &\quad - \frac{\lambda^{\frac{3p}{2}-1}}{p+2}\frac{3p}{2}\|v\|^{p+2}_{L^{p+2}} \\ &= \frac{Q(v^\lambda)}{\lambda}. \end{aligned}$$

Next, we define

$$\begin{aligned} f(\lambda) &:= Q(v^\lambda) \\ &= s\lambda^{2s}\|v\|^2_{\dot{H}^s} + \frac{3-2r}{4}\lambda^{3-2r}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |v|^2)|v|^2 dx - \frac{\lambda^{\frac{3p}{2}}}{p+2}\frac{3p}{2}\|v\|^{p+2}_{L^{p+2}}. \end{aligned}$$

When $p = \frac{4s}{3}$, it follows from $Q(v) = 0$ that $s\|v\|^2_{\dot{H}^s} < \frac{3p}{2(p+2)}\|v\|^{p+2}_{L^{p+2}}$. Thus, it is easy to see that the equation $f(\lambda) = 0$ admits a unique positive solution $\lambda = 1$.

When $\frac{4s}{3} < p < \frac{4s}{3-2s}$, assume that there exists $\lambda_1 \neq 1$ such that $f(\lambda_1) = 0$. It easily follows that

$$\frac{3-2r}{4}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |v|^2)|v|^2 dx(\lambda_1^{2s} - \lambda_1^{3-2r}) = \frac{\|v\|^{p+2}_{L^{p+2}}}{p+2}\frac{3p}{2}(\lambda_1^{2s} - \lambda_1^{\frac{3p}{2}}).$$

If $\lambda_1 > 1$, then $\lambda_1^{2s} - \lambda_1^{3-2r} > 0$ and $\lambda_1^{2s} - \lambda_1^{\frac{3p}{2}} < 0$, which is a contradiction. If $\lambda_1 < 1$, then $\lambda_1^{2s} - \lambda_1^{3-2r} < 0$ and $\lambda_1^{2s} - \lambda_1^{\frac{3p}{2}} > 0$, which is a contradiction. Therefore, the equation $f(\lambda) = 0$ admits a unique positive solution $\lambda = 1$. Therefore,

$$\begin{aligned} &\partial_\lambda S_\omega(v^\lambda) > 0, \quad \text{for all } \lambda \in (0,1), \\ &\partial_\lambda S_\omega(v^\lambda) < 0, \quad \text{for all } \lambda \in (1,\infty). \end{aligned}$$

We thus obtain that $S_\omega(v^\lambda) < S_\omega(v)$ for any $\lambda > 0$ and $\lambda \neq 1$. In particular, we have $S_\omega(v^{\lambda_0}) \le S_\omega(v)$. Thus, $S_\omega(u) \le S_\omega(v^{\lambda_0}) \le S_\omega(v)$ for all $v \in H^s\backslash\{0\}$ and $Q(v) = 0$. Taking the infimum over v, we have $S_\omega(u) \le d$. This completes the proof. $\square$

To obtain the key estimate (5.19), we need establish the following variational characterization of the ground states to (1.12). Firstly, when $p = \frac{4s}{3}$, we define

$$\begin{aligned} S^1_\omega(v) &:= S_\omega(v) - \frac{1}{2s}Q(v) \\ &= \frac{\omega}{2}\|v\|^2_{L^2} + \frac{2s+2r-3}{8s}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |v|^2)|v|^2dx. \end{aligned} \tag{5.15}$$

When $\frac{4s}{3} < p < \frac{4s}{3-2s}$, we define

$$\begin{aligned} S^2_\omega(v) &:= S_\omega(v) - \frac{2}{3p}Q(v) \\ &= \frac{3p-4s}{6p}\|v\|^2_{\dot H^s} + \frac{\omega}{2}\|v\|^2_{L^2} \\ &\quad + \frac{3p+4t-6}{12p}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |v|^2)|v|^2dx. \end{aligned} \tag{5.16}$$

Lemma 5.8. *Let $\omega > 0$, $2s + 2r > 3$, $\frac{4s}{3} \le p < \frac{4s}{3-2s}$, and u be the ground state related to* (1.12)*. Then for $k = 1, 2$ we have*

$$S_\omega(u) = S^k_\omega(u) = \inf\{S^k_\omega(v) : v \in H^s\backslash\{0\},\ Q(v) \le 0\}. \tag{5.17}$$

Proof. We only prove the case $k = 1$. The proof of the case $k = 2$ is similar. We denote

$$d^1(\omega) = \inf\{S^k_\omega(v) :\ v \in H^s\backslash\{0\},\ Q(v) \le 0\}.$$

Since u is the ground state related to (1.12), $Q(u) = 0$. It follows from the definition of $d^1(\omega)$ that

$$S^1_\omega(u) \ge d^1(\omega). \tag{5.18}$$

Let $v \in H^s\backslash\{0\}$ and $Q(v) \le 0$. If $Q(v) = 0$, then from Lemma 5.7 it follows that

$$S^1_\omega(v) = S_\omega(v) - \frac{1}{2s}Q(v) = S_\omega(v) \ge S_\omega(u) = S^1_\omega(u).$$

If $Q(v) < 0$, we note that

$$Q(v^\lambda) = \lambda^{2s}\|v\|^2_{\dot H^s} + \frac{\lambda^{3-2r}}{4}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |v|^2)|v|^2dx - \frac{\lambda^{\frac{3p}{2}}}{p+2}\frac{3p}{2}\|v\|^{p+2}_{L^{p+2}} > 0$$

for sufficiently small $\lambda > 0$, so there exists $\lambda_0 \in (0, 1)$ such that $Q(v^{\lambda_0}) = 0$. We thus have

$$\begin{aligned} S^1_\omega(v) &= \frac{\omega}{2}\|v\|^2_{L^2} + \frac{2s+2r-3}{8s}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |v|^2)|v|^2dx \\ &\ge \frac{\omega}{2}\|v\|^2_{L^2} + \frac{2s+2r-3}{8s}\lambda_0^{3-2r}\int_{\mathbb{R}^3}(|x|^{-(3-2r)} * |v|^2)|v|^2dx \\ &= S^1_\omega(v^{\lambda_0}) = S_\omega(v^{\lambda_0}) \ge S_\omega(u) = S^1_\omega(u). \end{aligned}$$

This implies that $d^1(\omega) \ge S^1_\omega(u)$. This, together with (5.18) implies that $S^1_\omega(u) = d^1(\omega)$. $\square$

Let u be the ground state related to (1.12). We define

$$\mathcal{B}_\omega = \{v \in H^s \setminus \{0\} : S_\omega(v) < S_\omega(u),\ Q(v) < 0\}.$$

Lemma 5.9. *Let $\omega > 0$, $2s + 2r > 3$, and u be the ground state related to* (1.12). *If $\frac{4s}{3} \le p < \frac{4s}{3-2s}$, then the set $\mathcal{B}_\omega$ is invariant under the flow of* (1.4). *That is, if $\psi_0 \in \mathcal{B}_\omega$, then the solution $\psi(t)$ to* (1.4) *with initial data ψ_0 belongs to $\mathcal{B}_\omega$ and*

$$Q(\psi(t)) \le 2s(S(\psi_0) - S(u)) \tag{5.19}$$

for any $t \in [0, T^*)$.

Proof. Let $\psi_0 \in \mathcal{B}_\omega$, by Proposition 2.1, we see that there exists a unique solution $\psi \in C([0, T^*), H^s)$ with initial data ψ_0. We deduce from the conservations of mass and energy that

$$S_\omega(\psi(t)) = S_\omega(\psi_0) < S_\omega(u) \tag{5.20}$$

for any $t \in [0, T^*)$. In addition, by the continuity of the function $t \mapsto Q(\psi(t))$ and Lemma 5.7, if there exists $t_0 \in [0, T^*)$ such that $Q(\psi(t_0)) = 0$, then $S_\omega(\psi(t_0)) \ge S_\omega(u)$, which contradicts (5.20). Therefore, we have $Q(\psi(t)) < 0$ for any $t \in [0, T^*)$. This, together with Lemma 5.8 implies that

$$\begin{gathered}
S_\omega(u) \le S^1_\omega(\psi(t)) = S_\omega(\psi(t)) - \frac{1}{2s}Q(\psi(t)) = S_\omega(\psi_0) - \frac{Q(\psi(t))}{2s}, \\
S_\omega(u) \le S^2_\omega(\psi(t)) = S_\omega(\psi(t)) - \frac{2}{3p}Q(\psi(t)) < S_\omega(\psi_0) - \frac{Q(\psi(t))}{2s}
\end{gathered}$$

for all $t \in [0, T^*)$. This completes the proof. □

Proof of Theorem 1.2. Let u be the ground state related to (1.12) and $\{\lambda_n\} \subseteq \mathbb{R}^+$ be such that $\lambda_n > 1$ and $\lim_{n\to\infty} \lambda_n = 1$. We take the initial data

$$\psi_{0,n}(x) := \lambda_n^{3/2} u(\lambda_n x).$$

Therefore,

$$\begin{gathered}
\lim_{n\to\infty} \|\psi_{0,n}\|_{L^2} = \lim_{n\to\infty} \|u\|_{L^2} = \|u\|_{L^2}, \\
\lim_{n\to\infty} \|\psi_{0,n}\|_{\dot{H}^s} = \lim_{n\to\infty} \lambda_n^s \|u\|_{\dot{H}^s} = \|u\|_{\dot{H}^s}.
\end{gathered}$$

Thus, we deduce from Brezis-Lieb's lemma (Lemma 2.2) that $\psi_{0,n} \to u$ in H^s as $n \to \infty$. By Lemma 5.7, we have

$$S_\omega(\psi_{0,n}) < S_\omega(u),\ Q(\psi_{0,n}) < 0$$

for all $n \ge 1$. Thus, $\psi_{0,n} \in \mathcal{B}_\omega$. Let ψ_n be the maximal solution of (1.4) with the initial data $\psi_{0,n}$. We deduce from Lemma 5.9 that $\psi_n(t) \in \mathcal{B}_\omega$ for all $t \in [0, T^*)$ and

$$Q(\psi_n(t)) \le 2s(S(\psi_{0,n}) - S(u)) < 0.$$

Thus, applying Theorem 1.1, we obtain that the solution $\psi_n(t)$ of (1.4) with initial data $\psi_{0,n}$ blows up in finite or infinite time for any $n \ge 1$. □

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Yichun Mo
Department of Mathematics, Northwest Normal University, Lanzhou 730070, China
Email address: `sg25888@163.com`

Min Zhu (corresponding author)
Department of Mathematics, Nanjing Forestry University, Nanjing, Jiangsu 210037, China
Email address: `zhumin@njfu.edu.cn`

Binhua Feng (corresponding author)
Department of Mathematics, Northwest Normal University, Lanzhou 730070, China
Email address: `binhuaf@nwnu.edu.cn`