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PYRAMIDAL TRAVELING FRONTS OF A TIME PERIODIC DIFFUSION EQUATION WITH DEGENERATE MONOSTABLE NONLINEARITY

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ABSTRACT. This article focuses on the nonplanar traveling fronts of degenerate monostable time periodic reaction-diffusion equations in $\mathbb{R}^n$ with $n \geq 3$. By constructing a couple of proper supersolution and subsolution, we prove the existence of periodic pyramidal traveling front in $\mathbb{R}^3$ and then in $\mathbb{R}^n$ with $n > 3$.

1. INTRODUCTION

In this article, we investigate nonplanar traveling fronts of the equation

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = \Delta u(\mathbf{x},t) + f(u(\mathbf{x},t),t), \quad \mathbf{x} \in \mathbb{R}^n,\ t \in (0,+\infty), \tag{1.1}$$

where $n \geq 3$ is an integer, Δ is the Laplace operator and the nonlinear reaction term f is degenerate monostable satisfying the hypotheses:

(H1) $f(u,t) \in C^{1+\iota,\iota/2}([0,1]\times\mathbb{R},\mathbb{R})$ is T-periodic in t, where $\iota \in (0,1)$ and $T > 0$;

(H2) $f(0,t) = f(1,t) = 0$ with $t \in \mathbb{R}$ and $f(u,t) > 0$ in $(0,1)\times\mathbb{R}$; $f_u(0,t) = 0$ and $f_u(1,t) < 0$ for all $t \in \mathbb{R}$, where

$$f_u(0,t) = \lim_{u\to 0^+} \frac{f(u,t)}{u}, \quad f_u(1,t) = \lim_{u\to 1^-} \frac{f(u,t)}{u-1}.$$

Many diffusion phenomena in nature can be portrayed by reaction-diffusion equations such as the movement of populations, propagation of burning flame and the spread of diseases in the air [5, 11]. As the special solutions of reaction-diffusion equations on an unbounded region, the traveling fronts can describe the propagation phenomenon of reaction-diffusion equations well. According to whether the level set of traveling front is a hyperplane, the traveling fronts are classified into planar traveling fronts and nonplanar traveling fronts.

Planar traveling fronts have been well studied in arbitrary dimensional space because their simple form and good geometric properties [10, 13, 16, 17, 30]. However, owing to the effects of curvature and spatial dimension, many reaction phenomena cannot be accurately described by planar traveling fronts in $\mathbb{R}^n$ with $n > 1$, such as the conical premixed Bunsen flames [4] and the fertilization Ca^{2+} waves in mature Xenopus laevis eggs [26]. Therefore, the multidimensional nonplanar traveling

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fronts of reaction-diffusion equations have attracted more and more scholars' attention.

For the combustion case, Bonnet and Hamel [4] established the existence of two-dimensional V-shaped traveling fronts. Then, Hamel and Monneau [12] investigated conical traveling fronts in $\mathbb{R}^n$ with $n \geq 3$. For the Fisher-KPP monostable case, Hamel and Nadirashvili [13] showed the existence of an infinite-dimensional manifold of solutions. For the bistable case, using the method of the comparison principle coupled with the supersolution and subsolution technique, Ninomiya and Taniguchi [18, 23] obtained the existence of the two-dimensional V-shaped traveling fronts and the three-dimensional pyramidal traveling fronts. Kurokawa and Taniguchi [15] further considered the existence of n-dimensional pyramidal traveling fronts with $n \geq 4$. For more information on the higher dimensions of this case, one can refer to the literature of Taniguchi [24, 25]. Recently, the first author of this paper and Wang [7, 28] investigated the existence and stability of the three-dimensional pyramidal traveling fronts to the reaction-diffusion equations with combustion and degenerate Fisher-KPP nonlinearities without periodicity.

To simulate real natural phenomena (e.g. seasonal cycles), the influence of time periodicity has been considered by researchers recently. Wang and Wu [29] and Sheng et al. [21] studied the existence and stability of two-dimensional periodic V-shaped and three-dimensional periodic pyramidal traveling fronts for reaction-diffusion equations with bistable time-periodic nonlinearity, respectively. El Smaily et al. [22] explored the traveling fronts of Fisher-KPP monostable reaction-diffusion equations with periodic advection in $\mathbb{R}^2$. Subsequently, Bu and Wang [6] studied the traveling fronts of reaction-advection-diffusion equations in space-time periodic medium in $\mathbb{R}^n$ ($n \geq 3$). Then Zhang et al. [31] concerned the existence, uniqueness and stability of V-shaped traveling fronts to reaction-diffusion equations with ignition time-periodic nonlinearity. For more results about time-periodic nonplanar traveling fronts, we refer to [2, 20] and the references therein.

In this article, we study the nonplanar traveling fronts of degenerate monostable time periodic reaction-diffusion equation (1.1) in $\mathbb{R}^n$ with $n \geq 3$. Inspired by Wang and Bu [28] and Zhang et al. [31], we will use the super-sub solution method combined with the comparison principle. The sign of derivative of nonlinear term f at the equilibrium points 0 and 1 plays a key role in constructing the supersolution. In contrast to bistable and combustion cases, the derivative of the degenerate monostable case that satisfies the hypotheses (H1) and (H2) at the equilibrium point 0 is zero and $f(t+T,u) = f(t,u) > 0$ in $\mathbb{R} \times (0,1)$. Thus there will be some difficulties in constructing the supersolution. To overcome these difficulties, we will adopt the method of adding small perturbation to the planar traveling front to construct supersolution.

From [3], we know that under the assumptions on f, equation (1.1) has a periodic planar traveling front $\Psi(\xi,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with the wave speed $c_* > 0$ satisfying

$$\begin{gathered}
\Psi_t + c_*\Psi_\xi - \Psi_{\xi\xi} - f(\Psi,t) = 0, \quad \Psi_\xi(\xi,t) > 0,\ (\xi,t) \in \mathbb{R}^2, \\
\Psi(-\infty,t) = 0, \Psi(+\infty,t) = 1 \quad \text{uniformly in } t \in \mathbb{R}, \\
\Psi(\xi,t+T) = \Psi(\xi,t), \quad (\xi,t) \in \mathbb{R}^2
\end{gathered} \tag{1.2}$$

and

$$\lim_{\xi\to-\infty} \frac{\Psi_\xi(\xi,t)}{\Psi(\xi,t)} = \Lambda_2 = c_* > \Lambda_1 = 0, \quad \lim_{\xi\to-\infty} \frac{\Psi_{\xi\xi}(\xi,t)}{\Psi(\xi,t)} = \Lambda_2^2 = c_*^2 \tag{1.3}$$

uniformly in $t \in \mathbb{R}$, where Λ_1 and Λ_2 are roots of the equation $\lambda^2 - c_*\lambda = 0$. In fact, $c_* > 0$ is the critical speed of the periodic planar traveling fronts to (1.1). For any $\beta \in (0,1)$, we can easily obtain $\Pi(\beta\Lambda_2) = (\beta\Lambda_2)^2 - c_*(\beta\Lambda_2) < 0$. In addition, there exist positive constants L_1, L_2, L_3, β_1 such that

$$L_1 e^{\Lambda_2\xi} \le \Psi(\xi,t),\ \Psi_\xi(\xi,t),\ |\Psi_{\xi\xi}(\xi,t)| \le L_2 e^{\Lambda_2\xi}, \quad \forall \xi < 0,\ t \in \mathbb{R}, \tag{1.4}$$

$$|\Psi(\xi,t) - 1|,\ \Psi_\xi(\xi,t),\ |\Psi_{\xi\xi}(\xi,t)| \le L_3 e^{-\beta_1\xi}, \quad \forall \xi > 0,\ t \in \mathbb{R}. \tag{1.5}$$

By the super-sub solution method, this paper firstly studies the existence of the three-dimensional periodic pyramidal traveling fronts of (1.1). That is, we investigate

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = \Delta u(\mathbf{x},t) + f(u(\mathbf{x},t),t), \quad \mathbf{x} \in \mathbb{R}^3,\ t > 0. \tag{1.6}$$

Then we establish the existence of n-dimensional periodic pyramidal traveling fronts of (1.1) with $n \ge 4$.

Assume $c > c_*$ and $m_* = \frac{\sqrt{c^2 - c_*^2}}{c_*}$. Denote $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. We assume that the traveling fronts travel towards $-x_3$ direction with the speed of $c > c_*$. Let

$$u(x_1,x_2,x_3,t) = v(x_1,x_2,x_3+ct,t) = v(x_1,x_2,w,t).$$

We still express $v(x_1,x_2,w,t)$ as $v(x_1,x_2,x_3,t)$ for convenience. By substituting v into (1.6), it follows that

$$\begin{gathered} v_t = \Delta v - c v_{x_3} + f(v,t), \quad \mathbf{x} \in \mathbb{R}^3,\ t > 0, \\ v(\mathbf{x},0) = v_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \end{gathered} \tag{1.7}$$

One of the purposes of this paper is to find the solution $V(\mathbf{x},t)$ satisfying

$$V_t - V_{x_1x_1} - V_{x_2x_2} - V_{x_3x_3} + cV_{x_3} - f(V,t) = 0, \quad \mathbf{x} \in \mathbb{R}^3,\ t \in \mathbb{R}, \tag{1.8}$$

$$V(\cdot,\cdot,\cdot,\cdot) = V(\cdot,\cdot,\cdot,\cdot + T), \quad \mathbf{x} \in \mathbb{R}^3,\ t \in \mathbb{R}. \tag{1.9}$$

Let $l \ge 3$, and $\{(A_j,B_j)\}_{1\le j\le l}$ be a set of unit vectors in $\mathbb{R}^2$ such that

$$A_jB_{j+1} - A_{j+1}B_j > 0,\ j = 1,2,\dots,l-1; \quad A_lB_1 - A_1B_l > 0. \tag{1.10}$$

For each $(x_1,x_2) \in \mathbb{R}^2$, let

$$\begin{gathered} h_j(x_1,x_2) = m_*(x_1A_j + x_2B_j), \quad 1 \le j \le l, \\ h(x_1,x_2) = \max_{1\le j\le l} h_j(x_1,x_2) = m_* \max_{1\le j\le l}(x_1A_j + x_2B_j), \end{gathered}$$

then $\{\mathbf{x} \in \mathbb{R}^3 | -x_3 = h(x_1,x_2)\}$ is a pyramid in $\mathbb{R}^3$. Clearly, for any $(x_1,x_2) \in \mathbb{R}^2$, we have

$$h(x_1,x_2) \ge 0 \quad \text{and} \quad \lim_{R\to\infty} \inf_{x_1^2+x_2^2\ge R^2} h(x_1,x_2) = \infty.$$

Set

$$\Omega_j = \{(x_1,x_2) \in \mathbb{R}^2 : h(x_1,x_2) = h_j(x_1,x_2)\}, \quad j = 1,2,\dots,l,$$

then $\mathbb{R}^2 = \cup_{j=1}^l \Omega_j$. From (1.10), the planes $\Omega_1, \Omega_2, \dots, \Omega_l$ are arranged in a counterclockwise direction. Let $\partial\Omega_j$ be the boundary of Ω_j. Denote $E = \cup_{j=1}^l \partial\Omega_j$. Each side of the pyramid can be represented as

$$G_j = \{\mathbf{x} \in \mathbb{R}^3 : -x_3 = h_j(x_1,x_2), (x_1,x_2) \in \Omega_j\}, \quad j = 1,2,\dots,l.$$

We denote

$$\Gamma_j = \begin{cases} G_j \cap G_{j+1}, & 1 < j < l-1, \\ G_l \cap G_1, & j = l. \end{cases}$$

Then $\Gamma = \cup_{j=1}^{l}\Gamma_j$ represents the set of all edges of a pyramid, and the lateral surfaces of the pyramid consist of $\cup_{j=1}^{l} G_j \subset \mathbb{R}^3$. For each $\bar{\gamma} \geq 0$, we define

$$\mathcal{D}(\bar{\gamma}) = \{\mathbf{x} \in \mathbb{R}^3 : \mathrm{dist}(\mathbf{x}, \Gamma) \geq \bar{\gamma}\}.$$

Note that the above setting on a pyramid comes from Taniguchi [23].

For any $1 \leq j \leq l$, it is obvious that $\Psi(\frac{c_*}{c}(x_3 + h_j(x_1, x_2)), t)$ is the solution of (1.8). We define

$$\underline{\psi}(\mathbf{x}, t) = \Psi(\frac{c_*}{c}(x_3 + h(x_1, x_2)), t) = \max_{1\leq j\leq l} \Psi(\frac{c_*}{c}(x_3 + h_j(x_1, x_2)), t). \tag{1.11}$$

Then $\underline{\psi}(\mathbf{x}, t)$ is a subsolution to (1.8). Furthermore, we have $\underline{\psi}_{x_3}(\mathbf{x}, t) > 0$.

Now we state the main result of this article in $\mathbb{R}^3$. The generalized result in $\mathbb{R}^n$ with $n \geq 4$ will be given in Section 4.

Theorem 1.1. *Assume that* (H1) *and* (H2) *hold. For each $c > c_*$, equation* (1.6) *has a periodic nonplanar traveling front $V(\mathbf{x}, t)$ satisfying* (1.8)-(1.9)*. Moreover,*

$$\lim_{\bar{\gamma}\to\infty} \sup_{\mathbf{x}\in\mathcal{D}(\bar{\gamma}), t\in[0,T]} \frac{|V(\mathbf{x}, t) - \underline{\psi}(\mathbf{x}, t)|}{(\underline{\psi}(\mathbf{x}, t))^{\beta}} = 0, \quad \forall \beta \in (0, 1),$$

and

$$V_{x_3}(\mathbf{x}, t) > 0, \quad (\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}.$$

The rest of this article is organized as follows. Some preliminaries are given in Section 2. In Section 3, we construct the supersolution, and then prove the existence of periodic pyramidal traveling fronts in $\mathbb{R}^3$. That is, we give the proof of Theorem 1.1. We establish the existence of n-dimensional periodic pyramidal traveling fronts with $n \geq 4$ in Section 4. In Section 5, the article ends with a short conclusion.

2. Preliminaries

In this section, we give some preliminaries which are useful in the proof of the existence of three-dimensional periodic pyramidal traveling fronts to (1.6).

Firstly, we mollify the original pyramid $\{\mathbf{x} \in \mathbb{R}^3 | -x_3 = h(x_1, x_2)\}$, see [23]. Let function $\tilde{\rho}(r) \in C^{\infty}[0, \infty)$ satisfy the following properties:

1. $\tilde{\rho}(r) > 0$, $\tilde{\rho}_r(r) \leq 0$, $r \geq 0$;
2. If $r > 0$ is small enough, $\tilde{\rho}(r) = 1$;
3. If $r > 0$ is large enough, say $r > R_0$, $\tilde{\rho}(r) = e^{-r}$, where $R_0 > 0$ is a constant;
4. $\int_{\mathbb{R}^2} \tilde{\rho}(\sqrt{x_1^2 + x_2^2})\, dx_1\, dx_2 = 1$.

It is easy to check that

$$\int_{\mathbb{R}^2} \tilde{\rho}\Big(\sqrt{x_1^2 + x_2^2}\Big) dx_1 dx_2 = 2\pi \int_0^{\infty} r\tilde{\rho}(r) dr = 1.$$

Letting $\rho(x_1, x_2) = \tilde{\rho}(\sqrt{x_1^2 + x_2^2})$, one gets $\rho \in C^{\infty}(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} \rho(x_1, x_2) dx_1 dx_2 = 1$. Set $R_0 > 1$. For all nonnegative integers $i_1 \geq 0$ and $i_2 \geq 0$ with $0 \leq i_1 + i_2 \leq 3$, we have

$$|\mathcal{D}_{x_1}^{i_1} \mathcal{D}_{x_2}^{i_2} \rho(x_1, x_2)| \leq M_* \rho(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2,$$

where $M_* > 0$, $\mathcal{D}^{i_1}_{x_1} = \frac{\partial^{i_1}}{\partial x_1^{i_1}}$ and $\mathcal{D}^{i_2}_{x_2} = \frac{\partial^{i_2}}{\partial x_2^{i_2}}$. Define $\bar\varphi(x_1, x_2) = \rho * h$. That is,

$$\begin{aligned}\bar\varphi(x_1, x_2) &= \int_{\mathbb{R}^2} \rho(x_1 - x_1', x_2 - x_2')h(x_1', x_2')dx_1'dx_2' \\ &= \int_{\mathbb{R}^2} \rho(x_1', x_2')h(x_1 - x_1', x_2 - x_2')dx_1'dx_2'\end{aligned} \tag{2.1}$$

for each $(x_1, x_2) \in \mathbb{R}^2$. The set $\{\mathbf{x} \in \mathbb{R}^3 | -x_3 = \bar\varphi(x_1, x_2)\}$ is called the mollified pyramid of $\{\mathbf{x} \in \mathbb{R}^3 | -x_3 = h(x_1, x_2)\}$. Let

$$G(x_1, x_2) = \frac{c}{\sqrt{1 + |\nabla\bar\varphi(x_1, x_2)|^2}} - c_*, \tag{2.2}$$

where

$$|\nabla\bar\varphi(x_1, x_2)| = \sqrt{\bar\varphi^2_{x_1}(x_1, x_2) + \bar\varphi^2_{x_2}(x_1, x_2)}.$$

The next two lemmas come from [23], which show some properties on the functions $\bar\varphi(x_1, x_2)$ and $G(x_1, x_2)$ on $\mathbb{R}^2$.

Lemma 2.1. *Assume that $\bar\varphi$ and G are defined in* (2.1) *and* (2.2) *respectively. Then*

$$\sup_{(x_1,x_2)\in\mathbb{R}^2} |\mathcal{D}^{i_1}_{x_1}\mathcal{D}^{i_2}_{x_2}\bar\varphi(x_1, x_2)| < \infty,$$

$$h(x_1, x_2) < \bar\varphi(x_1, x_2) \le h(x_1, x_2) + 2\pi m_* \int_0^\infty r^2\tilde\rho(r)dr,$$

$$|\nabla\bar\varphi(x_1, x_2)| < m_*, \quad 0 < G(x_1, x_2) \le c - c_*, \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$\lim_{\lambda\to\infty} \sup\{G(x_1, x_2) | (x_1, x_2) \in \mathbb{R}^2, \operatorname{dist}((x_1, x_2), E) \ge \lambda\} = 0,$$

$$\lim_{\lambda\to\infty} \sup\{\bar\varphi(x_1, x_2) - h(x_1, x_2) | (x_1, x_2) \in \mathbb{R}^2, \operatorname{dist}((x_1, x_2), E) \ge \lambda\} = 0.$$

Lemma 2.2. *There exist two positive constants a_1 and a_2 such that*

$$a_1 = \inf_{(x_1,x_2)\in\mathbb{R}^2} \frac{\bar\varphi(x_1, x_2) - h(x_1, x_2)}{G(x_1, x_2)} \le \sup_{(x_1,x_2)\in\mathbb{R}^2} \frac{\bar\varphi(x_1, x_2) - h(x_1, x_2)}{G(x_1, x_2)} = a_2 < \infty.$$

In addition, for any integers $i_1 \ge 0$ or $i_2 \ge 0$ satisfying $2 \le i_1 + i_2 \le 3$, there exists a constant $\mathcal{K} > 0$ such that

$$\sup_{(x_1,x_2)\in\mathbb{R}^2} \Big|\frac{\mathcal{D}^{i_1}_{x_1}\mathcal{D}^{i_2}_{x_2}\bar\varphi(x_1, x_2)}{G(x_1, x_2)}\Big| < \mathcal{K},$$

and

$$|\bar\varphi_{x_1x_1}(x_1, x_2)|, \ |\bar\varphi_{x_2x_2}(x_1, x_2)| \le m_*M_*, \quad (x_1, x_2) \in \mathbb{R}^2. \tag{2.3}$$

Secondly, we study the eigenfunction at equilibrium point 1. Assume that Λ_0 is the eigenvalue of the linearized periodic system

$$\begin{gathered}\bar\Upsilon'(t) - f_u(1, t)\bar\Upsilon(t) = \Lambda_0\bar\Upsilon(t), \quad t \in \mathbb{R}, \\ \bar\Upsilon(t + T) = \bar\Upsilon(t), \quad t \in \mathbb{R}.\end{gathered} \tag{2.4}$$

By a direct calculation, we have

$$\bar\Upsilon(t) = e^{\Lambda_0 t + \int_0^t f_u(1,s)ds}, \quad \Lambda_0 = -\frac{1}{T}\int_0^t f_u(1, s)ds > 0. \tag{2.5}$$

We define $\nu(t) = k\bar{\Upsilon}(t)$, $P_1 = \min_{t\in[0,T]} \nu(t)$, and $P_2 = \max_{t\in[0,T]} \nu(t)$, where k is a positive constant such that $P_1 > 1$.

Next, we give some properties about the reaction term f. By assumptions (H1) and (H2), we can choose $\varepsilon_1 \in (0,1)$ small enough such that

$$f_u(u,t) < -\frac{1}{16}\Pi(\beta\Lambda_2), \quad -\varepsilon_1 \le u \le \varepsilon_1,\ t > 0, \tag{2.6}$$

$$|f_u(u,t) - f_u(1,t)| \le \frac{1}{2}\Lambda_0, \quad 1-\varepsilon_1 \le u \le 1+\varepsilon_1,\ t > 0. \tag{2.7}$$

Finally, we construct an auxiliary function $\omega(x) \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} \omega(x) = 1, & \text{if } x \ge 1, \\ 0 < \omega(x) < 1,\ 0 < \omega'(x) < 1,\ \omega''(x) < 0, & \text{if } -1 < x < 1, \\ \omega(x) = 0, & \text{if } x \le -1, \end{cases} \tag{2.8}$$

which will be used in constructing the supersolution.

3. Existence of periodic pyramidal traveling fronts in $\mathbb{R}^3$

In this section, we first use the idea of perturbation to construct a suitable supersolution. And then we prove the existence of three-dimensional periodic pyramidal traveling front $V(\mathbf{x},t)$ to (1.6).

Obviously, $\frac{1}{\alpha}h(\alpha x_1, \alpha x_2) = h(x_1,x_2)$ for any $\alpha \in (0,1)$. Let $z_3 = \alpha x_3$, $\mathbf{z}' = (z_1,\ z_2) = (\alpha x_1, \alpha x_2) = \alpha\mathbf{x}'$, $\mathbf{z} = \alpha\mathbf{x}$ and

$$\varpi(\mathbf{x}) = \frac{c_*}{c}\Big(x_3 + \frac{1}{\alpha}\bar{\varphi}(\alpha x_1, \alpha x_2)\Big) = \frac{c_*}{c}\frac{z_3 + \bar{\varphi}(\mathbf{z}')}{\alpha}, \tag{3.1}$$

$$\varrho(\mathbf{x}) = \frac{x_3 + \frac{1}{\alpha}\bar{\varphi}(\alpha x_1, \alpha x_2)}{\sqrt{1 + |\nabla\bar{\varphi}(\alpha x_1, \alpha x_2)|^2}} = \frac{z_3 + \bar{\varphi}(\mathbf{z}')}{\alpha\sqrt{1 + |\nabla\bar{\varphi}(\mathbf{z}')|^2}}. \tag{3.2}$$

Using Lemma 2.1, we can obtain

$$\begin{cases} \frac{c}{c_*}\varpi(\mathbf{x}) < \varrho(\mathbf{x}) < \varpi(\mathbf{x}), & \text{if } \varrho(\mathbf{x}) < 0, \\ \varpi(\mathbf{x}) < \varrho(\mathbf{x}) < \frac{c}{c_*}\varpi(\mathbf{x}), & \text{if } \varrho(\mathbf{x}) > 0. \end{cases} \tag{3.3}$$

By a direct calculation, this indicates

$$\varpi_{x_3} = \frac{c_*}{c}, \quad \varpi_{x_3x_3} = 0, \quad \varpi_{x_i} = \frac{c_*}{c}\bar{\varphi}_{z_i}, \quad \varpi_{x_ix_i} = \alpha\frac{c_*}{c}\bar{\varphi}_{z_iz_i},$$

$$\varrho_{x_3} = \frac{1}{\sqrt{1 + |\nabla\bar{\varphi}(\mathbf{z}')|^2}}, \quad \varrho_{x_3x_3} = 0$$

and

$$\varrho_{x_i} = (\sqrt{1 + |\nabla\bar{\varphi}(\mathbf{z}')|^2})^{-1}\bar{\varphi}_{z_i} - \alpha\varrho C_i(\mathbf{z}'), \quad \varrho_{x_ix_i} = \alpha D_i(\mathbf{z}') - \alpha^2\varrho E_i(\mathbf{z}'),$$

where

$$C_i(\mathbf{z}') = \sqrt{1 + |\nabla\bar{\varphi}(\mathbf{z}')|^2}\,\frac{\partial}{\partial z_i}(\sqrt{1 + |\nabla\bar{\varphi}(\mathbf{z}')|^2})^{-1},$$

$$D_i(\mathbf{z}') = \frac{\partial}{\partial z_i}\Big(\big(\sqrt{1 + |\nabla\bar{\varphi}(\mathbf{z}')|^2}\big)^{-1}\bar{\varphi}_{z_i}\Big) - \frac{C_i(\mathbf{z}')}{\sqrt{1 + |\nabla\bar{\varphi}(\mathbf{z}')|^2}}\bar{\varphi}_{z_i},$$

$$E_i(\mathbf{z}') = \frac{\partial C_i(\mathbf{z}')}{\partial z_i} - C_i^2(\mathbf{z}'),$$

for $i = 1, 2$. Let

$$\sigma(x_1, x_2) = G(\alpha x_1, \alpha x_2) = G(\mathbf{z}'),$$

where $\alpha > 0$ is a constant, which will be determined later. Then

$$\sigma_{x_i}(x_1, x_2) = \alpha G_{z_i}(\mathbf{z}') \quad \text{and} \quad \sigma_{x_i x_i}(x_1, x_2) = \alpha^2 G_{z_i z_i}(\mathbf{z}'), \quad i = 1, 2.$$

3.1. **Construction of the supersolution.** Motivated by Wang and Bu [28] and Zhang et al. [31], we construct an appropriate supersolution in this subsection.

Lemma 3.1. *For each $\beta \in (0, 1)$, there exist positive constants $\varepsilon_0^+(\beta)$ and $\alpha_0^+(\beta, \varepsilon)$ such that, for any $0 < \varepsilon < \varepsilon_0^+(\beta)$ and $0 < \alpha < \alpha_0^+(\beta, \varepsilon)$, the function*

$$\overline{\psi}(\mathbf{x}, t; \beta, \varepsilon, \alpha) = \Psi(\varrho(\mathbf{x}), t) + \varepsilon\sigma(\mathbf{x}')(\omega(\varpi(\mathbf{x}))\nu(t) + (1 - \omega(\varpi(\mathbf{x})))\Psi^\beta(\varpi(\mathbf{x}), t))$$

is a supersolution of (1.8)-(1.9) *on $\mathbb{R}^3 \times (-\infty, +\infty)$. Moreover,*

$$\lim_{\overline{\gamma}\to\infty} \sup_{\mathbf{x}\in\mathcal{D}(\overline{\gamma}), t\in[0,T]} \frac{|\overline{\psi}(\mathbf{x}, t; \beta, \varepsilon, \alpha) - \underline{\psi}(\mathbf{x}, t)|}{\underline{\psi}(\mathbf{x}, t)^\beta} \le 2\varepsilon, \tag{3.4}$$

$$\underline{\psi}(\mathbf{x}, t) < \overline{\psi}(\mathbf{x}, t; \beta, \varepsilon, \alpha), \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times [0, T], \tag{3.5}$$

$$\overline{\psi}_{x_3}(\mathbf{x}, t; \beta, \varepsilon, \alpha) > 0, \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times [0, T]. \tag{3.6}$$

Proof. Firstly, we prove that $\overline{\psi}(\mathbf{x}, t; \beta, \varepsilon, \alpha)$ is the supersolution of (1.8)-(1.9). We always assume $0 < \alpha < \varepsilon < \varepsilon_1$, and denote $\overline{\psi}(\mathbf{x}, t; \beta, \varepsilon, \alpha)$, $\varpi(\mathbf{x})$, $\varrho(\mathbf{x})$ and $\Psi(\varrho(\mathbf{x}), t)$ by $\overline{\psi}(\mathbf{x}, t)$, ϖ, ϱ and $\Psi(\varrho, t)$, respectively.

A direct calculation yields

$$
\begin{aligned}
L(\overline{\psi}) &= \overline{\psi}_t(\mathbf{x}, t) - \overline{\psi}_{x_1x_1}(\mathbf{x}, t) - \overline{\psi}_{x_2x_2}(\mathbf{x}, t) - \overline{\psi}_{x_3x_3}(\mathbf{x}, t) + c\overline{\psi}_{x_3}(\mathbf{x}, t) - f(\overline{\psi}(\mathbf{x}, t), t) \\
&= \Psi_t(\varrho, t) + \varepsilon\sigma(\mathbf{x}')[\omega(\varpi)\nu'(t) + (1 - \omega(\varpi))\beta\Psi^{\beta-1}(\varpi, t)\Psi_t(\varpi, t)] \\
&\quad - \sum_{i=1}^2 \Psi_{\varrho\varrho}(\varrho, t)\varrho_{x_i}^2 - \Psi_{\varrho\varrho}(\varrho, t)\varrho_{x_3}^2 \\
&\quad - \sum_{i=1}^2 \Psi_\varrho(\varrho, t)\varrho_{x_ix_i} - \sum_{i=1}^2 \varepsilon\sigma_{x_ix_i}(\mathbf{x}')[\omega(\varpi)\nu(t) + (1 - \omega(\varpi))\Psi^\beta(\varpi, t)] \\
&\quad - 2\sum_{i=1}^2 \varepsilon\sigma_{x_i}(\mathbf{x}')\big[\omega'(\varpi)\varpi_{x_i}\nu(t) - \omega'(\varpi)\varpi_{x_i}\Psi^\beta(\varpi, t) \\
&\quad + (1 - \omega(\varpi))\beta\Psi^{\beta-1}(\varpi, t)\Psi_\varpi(\varpi, t)\varpi_{x_i}\big] \\
&\quad - \varepsilon\sigma(\mathbf{x}')\Big[\omega''(\varpi)\Big(\sum_{i=1}^2 \varpi_{x_i}^2\Big)\nu(t) + \varpi_{x_3}^2\omega''(\varpi)\nu(t) \\
&\quad - \omega''(\varpi)\Big(\sum_{i=1}^2 \varpi_{x_i}^2\Big)\Psi^\beta(\varpi, t) \\
&\quad - \omega''(\varpi)\varpi_{x_3}^2\Psi^\beta(\varpi, t) + \omega'(\varpi)\Big(\sum_{i=1}^2 \varpi_{x_ix_i}\Big)\nu(t) \\
&\quad - \omega'(\varpi)\Psi^\beta(\varpi, t)\Big(\sum_{i=1}^2 \varpi_{x_ix_i}\Big) - 2\omega'(\varpi)\beta\Psi^{\beta-1}(\varpi, t)\Psi_\varpi(\varpi, t)\sum_{i=1}^2 \varpi_{x_i}^2 \\
&\quad - 2\omega'(\varpi)\beta\Psi^{\beta-1}(\varpi, t)\Psi_\varpi(\varpi, t)\varpi_{x_3}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ (1-\omega(\varpi))\beta(\beta-1)\Psi^{\beta-2}(\varpi,t)\Psi_{\varpi}^2(\varpi,t)\Big(\sum_{i=1}^{2}\varpi_{x_i}^2+\varpi_{x_3}^2\Big)\\
&+ (1-\omega(\varpi))\beta\Psi^{\beta-1}(\varpi,t)\Psi_{\varpi\varpi}(\varpi,t)\Big(\sum_{i=1}^{2}\varpi_{x_i}^2+\varpi_{x_3}^2\Big)\\
&+ (1-\omega(\varpi))\beta\Psi^{\beta-1}(\varpi,t)\Psi_{\varpi}(\varpi,t)\sum_{i=1}^{2}\varpi_{x_ix_i}\Big]\\
&+ c\varrho_{x_3}\Psi_{\varpi}(\varpi,t) + c\varpi_{x_3}\varepsilon\sigma(\mathbf{x}')\omega'(\varpi)(\nu(t)-\Psi^{\beta}(\varpi,t))\\
&+ c\varpi_{x_3}\varepsilon\sigma(\mathbf{x}')(1-\omega(\varpi))\beta\Psi^{\beta-1}(\varpi,t)\Psi_{\varpi}(\varpi,t) - f(\overline{\psi},t)\\
=& \Psi_t(\varrho,t) + \varepsilon\sigma(\mathbf{x}')[\omega(\varpi)\nu'(t) + (1-\omega(\varpi))\beta\Psi^{\beta-1}(\varpi,t)\Psi_t(\varpi,t)]\\
&+ \Big(-\sum_{i=1}^{2}\varrho_{x_i}^2 - \frac{1}{1+|\nabla\bar{\varphi}(\mathbf{z}')|^2}\Big)\Psi_{\varrho\varrho}(\varrho,t) - \Psi_{\varrho}(\varrho,t)\sum_{i=1}^{2}\varrho_{x_ix_i}\\
&- \varepsilon\sigma(\mathbf{x}')\Big\{\frac{\sum_{i=1}^{2}\sigma_{x_ix_i}(\mathbf{x}')}{\sigma(\mathbf{x}')}\omega(\varpi)\nu(t)\\
&+ (1-\omega(\varpi))\Big[\frac{\sum_{i=1}^{2}\sigma_{x_ix_i}(\mathbf{x}')}{\sigma(\mathbf{x}')}\Psi^{\beta}(\varpi,t)\\
&+ 2\frac{\sum_{i=1}^{2}\sigma_{x_i}(\mathbf{x}')}{\sigma(\mathbf{x}')}\beta\Psi^{\beta-1}(\varpi,t)\Psi_{\varpi}(\varpi,t)\frac{c_*}{c}\bar{\varphi}_{z_i}\\
&+ \beta(\beta-1)\Psi^{\beta-2}(\varpi,t)\Psi_{\varpi}^2(\varpi,t)\Big(\sum_{i=1}^{2}\varpi_{x_i}^2+\frac{c_*^2}{c^2}Big)\\
&+ \beta\Psi^{\beta-1}(\varpi,t)\Psi_{\varpi\varpi}\Big(\sum_{i=1}^{2}\varpi_{x_i}^2+\frac{c_*^2}{c^2}\Big)\\
&+ \beta\Psi^{\beta-1}(\varpi,t)\Psi_{\varpi}(\varpi,t)\sum_{i=1}^{2}\varpi_{x_ix_i} - c_*\beta\Psi^{\beta-1}(\varpi,t)\Psi_{\varpi}(\varpi,t)\Big]\Big\}\\
&- \varepsilon\sigma(\mathbf{x}')\Big[\omega''(\varpi)\big(\nu(t)-\Psi^{\beta}(\varpi,t)\big)\Big(\sum_{i=1}^{2}\varpi_{x_i}^2+\frac{c_*^2}{c^2}\Big)\\
&- 2\omega'(\varpi)\beta\Psi^{\beta-1}(\varpi,t)\Psi_{\varpi}(\varpi,t)\Big(\sum_{i=1}^{2}\varpi_{x_i}^2+\frac{c_*^2}{c^2}\Big)\\
&+ \omega'(\varpi)\big(\nu(t)-\Psi^{\beta}(\varpi,t)\big)\Big(2\frac{\sum_{i=1}^{2}\sigma_{x_i}(\mathbf{x}')\varpi_{x_i}}{\sigma(\mathbf{x}')}+\sum_{i=1}^{2}\varpi_{x_ix_i}-c_*\Big)\Big]\\
&- f(\overline{\psi},t)\\
=& \Big(-\sum_{i=1}^{2}\varrho_{x_i}^2 - \frac{1}{1+|\nabla\bar{\varphi}(\mathbf{z}')|^2}+1\Big)\Psi_{\varrho\varrho}(\varrho,t) - \Psi_{\varrho}(\varrho,t)\sum_{i=1}^{2}\varrho_{x_ix_i}\\
&- \varepsilon\sigma(\mathbf{x}')\Big\{\alpha^2\frac{\sum_{i=1}^{2}G_{z_iz_i}(\mathbf{z}')}{\sigma(\mathbf{x}')}\omega(\varpi)\nu(t)
\end{aligned}
$$

$$
\begin{aligned}
&+ (1-\omega(\varpi))\Big[\alpha^2 \frac{\sum_{i=1}^2 G_{z_i z_i}(\mathbf{z}')}{\sigma(\mathbf{x}')}\Psi^\beta(\varpi,t)\\
&+ 2\alpha\beta\frac{c_*}{c}\frac{\sum_{i=1}^2 G_{z_i}(\mathbf{z}')\bar\varphi_{z_i(z_i)}}{\sigma(\mathbf{x}')}\Psi^{\beta-1}(\varpi,t)\Psi_\varpi(\varpi,t)\\
&+ \beta(\beta-1)\Psi^{\beta-2}(\varpi,t)\Psi_\varpi^2(\varpi,t)\frac{c_*^2}{c^2}\Big(|\nabla\bar\varphi(\mathbf{z}')|^2+1\Big)\\
&+ \beta\frac{c_*^2}{c^2}\Psi^{\beta-1}(\varpi,t)\Psi_{\varpi\varpi}(\varpi,t)\Big(|\nabla\bar\varphi(\mathbf{z}')|^2+1\Big)\\
&+ \alpha\beta\frac{c_*}{c}\Psi^{\beta-1}(\varpi,t)\Psi_\varpi(\varpi,t)\Delta\bar\varphi(\mathbf{z}') - c_*\beta\Psi^{\beta-1}(\varpi,t)\Psi_\varpi(\varpi,t)\Big]\Big\}\\
&- \varepsilon\sigma(x')\Big[\omega''(\varpi)\big(\nu(t)-\Psi^\beta(\varpi,t)\big)\Big(|\nabla\bar\varphi(\mathbf{z}')|^2+1\Big)\frac{c_*^2}{c^2}\\
&- 2\omega'(\varpi)\beta\frac{c_*^2}{c^2}\Psi^{\beta-1}(\varpi,t)\Psi_\varpi(\varpi,t)\Big(|\nabla\bar\varphi(\mathbf{z}')|^2+1\Big)\\
&+ \omega'(\varpi)\big(\nu(t)-\Psi^\beta(\varpi,t)\big)\\
&\times\Big(2\alpha\frac{\sum_{i=1}^2 G_{z_i}(\mathbf{z}')\bar\varphi_{z_i(z_i)}}{\sigma(\mathbf{x}')}\frac{c_*}{c} + \alpha\frac{c_*}{c}\Delta\bar\varphi(\mathbf{z}') - c_*\Big)\Big]\\
&+ \Big(\frac{c}{\sqrt{1+|\nabla\bar\varphi(\mathbf{z}')|^2}} - c_*\Big)\Psi_\varrho(\varrho,t)\\
&+ \varepsilon\sigma(\mathbf{x}')[\omega(\varpi)\nu'(t) + (1-\omega(\varpi))\beta\Psi^{\beta-1}(\varpi,t)\Psi_t(\varpi,t)]\\
&+ f(\Psi(\varrho,t),t) - f(\overline{\psi},t).
\end{aligned}
$$

Let

$$
B_1 = \sup_{\mathbf{z}'\in\mathbb{R}^2}\frac{\sum_{i=1}^2 |G_{z_i z_i}(\mathbf{z}')|}{G(\mathbf{z}')},\quad B_2 = \sup_{\mathbf{z}'\in\mathbb{R}^2}\frac{\sum_{i=1}^2 |G_{z_i}(\mathbf{z}')|}{G(\mathbf{z}')}. \tag{3.7}
$$

Lemmas 2.1 and 2.2 imply that there exist constants $B_i > 0$ ($i = 3,4,5,6$) such that

$$
\begin{aligned}
\sum_{i=1}^2 \varrho_{x_i}^2 + \frac{1}{1+|\nabla\bar\varphi(\mathbf{z}')|^2} - 1 &\le \alpha B_3 G(\alpha\mathbf{x}')|\varrho(x)| + \alpha^2 B_4 G(\alpha\mathbf{x}')\varrho^2(x)\\
&= \alpha\sigma(\mathbf{x}')\big(B_3|\varrho(x)| + \alpha B_4\varrho^2(x)\big)\\
&\le \varepsilon\sigma(\mathbf{x}')\big(B_3|\varrho(x)| + \alpha B_4\varrho^2(x)\big),
\end{aligned}\tag{3.8}
$$

and

$$
\begin{aligned}
|\sum_{i=1}^2 \varrho_{x_i x_i}| &\le \alpha B_5 G(\alpha\mathbf{x}') + \alpha B_6 G(\alpha\mathbf{x}')\varrho(x)\\
&= \alpha\sigma(\mathbf{x}')(B_5 + B_6|\varrho(x)|)\\
&\le \varepsilon\sigma(\mathbf{x}')\big(B_5 + B_6|\varrho(x)|\big).
\end{aligned}\tag{3.9}
$$

We divide the remaining part of the proof into three cases: 1. $\varrho < -X'$, 2. $\varrho > X''$, 3. $-X' \le \varrho < X''$, where $X' > 0$ and $X'' > 0$ are sufficiently large constants which will be determined later.

Case 1: $\varrho < -X'$, where $X' > 0$ is a sufficiently large constant. We assume that $\varpi < -1$ without loss of generality, then $\omega = 0$ by the definition of ω, and hence we can obtain

$$
\begin{aligned}
L(\overline{\psi}) &= \Big(1-\sum_{i=1}^{2}\varrho_{x_i}^2-\frac{1}{1+|\nabla\bar{\varphi}(\mathbf{z}')|^2}\Big)\Psi_{\varrho\varrho}(\varrho,t)-\Psi_{\varrho}(\varrho,t)\sum_{i=1}^{2}\varrho_{x_ix_i}\\
&\quad-\varepsilon\sigma(\mathbf{x}')\Psi^{\beta}(\varpi,t)\Big[\alpha^2\frac{\sum_{i=1}^2 G_{z_iz_i}(\mathbf{z}')}{\sigma(\mathbf{x}')}\\
&\quad+2\alpha\beta\frac{c_*}{c}\frac{\sum_{i=1}^2 G_{z_i}(\mathbf{z}')\bar{\varphi}_{z_i}(\mathbf{z}')}{\sigma(\mathbf{x}')}\frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)}\\
&\quad+\beta(\beta-1)\frac{\Psi_{\varpi}^2(\varpi,t)}{\Psi^2(\varpi,t)}\frac{c_*^2}{c^2}\Big(|\nabla\bar{\varphi}(\mathbf{z}')|^2+1\Big)+\frac{c_*}{c}\alpha\beta\frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)}\triangle\bar{\varphi}(\mathbf{z}')\\
&\quad+\beta\frac{\Psi_{\varpi\varpi}(\varpi,t)}{\Psi(\varpi,t)}\frac{c_*^2}{c^2}\Big(|\nabla\bar{\varphi}(\mathbf{z}')|^2+1\Big)-c_*\beta\frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)}\Big]\\
&\quad+\Big(\frac{c}{\sqrt{1+|\nabla\bar{\varphi}(\mathbf{z}')|^2}}-c_*\Big)\Psi_{\varrho}(\varrho,t)+\varepsilon\sigma(\mathbf{x}')\beta\Psi^{\beta-1}(\varpi,t)\Psi_t(\varpi,t)\\
&\quad+f(\Psi(\varrho,t),t)-f(\overline{\psi},t).
\end{aligned}
$$

When $\varrho<0$, inequality (3.3) yields $\varrho<\varpi$. From (3.8) and (3.9), it follows that

$$
\begin{aligned}
L(\overline{\psi}) &\geq -\varepsilon\sigma(\mathbf{x}')\Psi^{\beta}(\varpi,t)(B_3|\varrho|+B_4\varrho^2)\frac{|\Psi_{\varrho\varrho}(\varrho,t)|}{\Psi^{\beta}(\varrho,t)}\\
&\quad-\varepsilon\sigma(\mathbf{x}')\Psi^{\beta}(\varpi,t)(B_5+B_6|\varrho|)\frac{|\Psi_{\varrho}(\varrho,t)|}{\Psi^{\beta}(\varrho,t)}\\
&\quad-\varepsilon\sigma(\mathbf{x}')\Psi^{\beta}(\varpi,t)\Big[\alpha^2B_1+2\alpha B_2m_*\frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)}+\alpha\frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)}\sum_{i=1}^{2}|\bar{\varphi}_{z_iz_i}(\mathbf{z}')|\\
&\quad+\beta\frac{c_*^2}{c^2}|-\Big(\frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)}\Big)^2+\frac{\Psi_{\varpi\varpi}(\varpi,t)}{\Psi(\varpi,t)}|\Big(|\nabla\bar{\varphi}(\mathbf{z}')|^2+1\Big)\\
&\quad+\beta^2\frac{c_*^2}{c^2}\Big(\frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)}\Big)^2\Big(|\nabla\bar{\varphi}(\mathbf{z}')|^2+1\Big)-\beta^2\Big(\frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)}\Big)^2\\
&\quad+\beta^2\Big(\frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)}\Big)^2-c_*\beta\frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)}-\beta\frac{\Psi_t(\varpi,t)}{\Psi(\varpi,t)}\Big]\\
&\quad+(\frac{c}{\sqrt{1+|\nabla\bar{\varphi}(\mathbf{z}')|^2}}-c_*)\Psi_{\varrho}(\varrho,t)-f(\overline{\psi},t)+f(\Psi(\varrho,t),t).
\end{aligned}
$$

Since $\lim_{\vartheta\to-\infty}\Psi(\vartheta,t)=0$ uniformly for $t\in[0,T]$, by (1.3) and Lemma 2.1, we have

$$
\begin{gathered}
\lim_{\varpi\to-\infty}\frac{f(\Psi(\varpi,t),t)}{\Psi(\varpi,t)}=f_u(0,t)=0,\\
\lim_{\varpi\to-\infty}\Big[\Big(-\frac{c_*^2}{c^2}\frac{\Psi_{\varpi}^2(\varpi,t)}{\Psi^2(\varpi,t)}+\frac{c_*^2}{c^2}\frac{\Psi_{\varpi\varpi}(\varpi,t)}{\Psi(\varpi,t)}\Big)(|\nabla\bar{\varphi}(\mathbf{z}')|+1)\Big]=0,\\
\lim_{\varpi\to-\infty}\Big[\beta^2\Big(\frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)}\Big)^2-c_*\beta\frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)}\Big]=\Pi(\beta\Lambda_2),\\
\lim_{\varpi\to-\infty}\Big[\beta c_*\frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)}-\beta\frac{\Psi_{\varpi\varpi}(\varpi,t)}{\Psi(\varpi,t)}\Big]=0,
\end{gathered}
$$

uniformly in $t \in [0,T]$. Thus there exists a sufficiently large constant $X_1 > 0$ such that for any $t \in [0,T]$ and $\varpi < -X_1$,

$$\begin{gathered}
-\beta \frac{f(\Psi(\varpi,t),t)}{\Psi(\varpi,t)} < -\frac{1}{16}\Pi(\beta\Lambda_2), \quad \frac{\Psi_\varpi(\varpi,t)}{\Psi(\varpi,t)} < \frac{3}{2}\Lambda_2, \\
\Big| -\big(\frac{\Psi_\varpi(\varpi,t)}{\Psi(\varpi,t)}\big)^2 + \frac{\Psi_{\varpi\varpi}(\varpi,t)}{\Psi(\varpi,t)}\Big| < -\frac{1}{16}\Pi(\beta\Lambda_2), \\
\beta^2\big(\frac{\Psi_\varpi(\varpi,t)}{\Psi(\varpi,t)}\big)^2 - c_*\beta \frac{\Psi_\varpi(\varpi,t)}{\Psi(\varpi,t)} < \frac{1}{2}\Pi(\beta\Lambda_2), \\
\beta c_* \frac{\Psi_\varpi(\varpi,t)}{\Psi(\varpi,t)} - \beta \frac{\Psi_{\varpi\varpi}(\varpi,t)}{\Psi(\varpi,t)} < -\frac{1}{16}\Pi(\beta\Lambda_2).
\end{gathered}$$

Inequalities (1.4)-(1.5) imply that there exists a sufficiently large constant $X_2 > 0$ such that

$$\begin{gathered}
(B_3|\varrho| + B_4\varrho^2)\frac{|\Psi_{\varrho\varrho}(\varrho,t)|}{\Psi^\beta(\varrho,t)} < -\frac{1}{16}\Pi(\beta\Lambda_2), \\
(B_5 + B_6|\varrho|)\frac{|\Psi_\varrho(\varrho,t)|}{\Psi^\beta(\varrho,t)} < -\frac{1}{16}\Pi(\beta\Lambda_2)
\end{gathered}$$

for any $t \in [0,T]$ and $\varrho < -X_2$. In addition, we can choose $\alpha_1 \in (0,\beta)$ small enough such that

$$\alpha^2 B_1 + 3\alpha B_2 m_*\Lambda_2 + 3\alpha m_* M_*\Lambda_2 < -\frac{1}{16}\Pi(\beta\Lambda_2), \quad \forall \alpha \in (0,\alpha_1).$$

It follows from (2.6) that there exists a sufficiently large constant $X_3 > 0$ such that $-\varepsilon_1 < \Psi(\varrho,t) + \varepsilon\sigma(\mathbf{x}')\Psi^\beta(\varpi,t) < \varepsilon_1$ for any $0 < \varepsilon < \frac{\varepsilon_1}{2(c-c_*)}$. Therefore,

$$\begin{aligned}
f(\overline{\psi},t) - f(\Psi(\varrho,t),t) &= f_u(\Psi(\varrho,t) + \theta\varepsilon\sigma(\mathbf{x}')\Psi^\beta(\varpi,t),t)\varepsilon\sigma(\mathbf{x}')\Psi^\beta(\varpi,t) \\
&< -\frac{1}{16}\Pi(\beta\Lambda_2)
\end{aligned}$$

for $\varrho < -X_3$ and $t \in [0,T]$, where $\theta \in (0,1)$.

Let $X' = \max\{\frac{c}{c_*}, \frac{c}{c_*}X_1, X_2, \frac{c}{c_*}X_3\}$. Thus when $\varrho < -X'$, we have

$$\begin{aligned}
L(\overline{\psi}) \geq & -\varepsilon\sigma(\mathbf{x}')\Psi^\beta(\varpi,t)(B_3|\varrho| + B_4\varrho^2)\frac{|\Psi_{\varrho\varrho}(\varrho,t)|}{\Psi^\beta(\varrho,t)} \\
& - \varepsilon\sigma(\mathbf{x}')\Psi^\beta(\varpi,t)(B_5 + B_6|\varrho|)\frac{|\Psi_\varrho(\varrho,t)|}{\Psi^\beta(\varrho,t)} \\
& - \varepsilon\sigma(\mathbf{x}')\Psi^\beta(\varpi,t)\Big[\alpha^2 B_1 + 3\alpha B_2 m_*\Lambda_2 + 3\alpha m_* M_*\Lambda_2 \\
& + \beta\frac{c_*^2}{c^2}\Big| -\Big(\frac{\Psi_\varpi(\varpi,t)}{\Psi(\varpi,t)}\Big)^2 + \frac{\Psi_{\varpi\varpi}(\varpi,t)}{\Psi(\varpi,t)}\Big|\big(|\nabla\bar{\varphi}(\mathbf{z}')|^2 + 1\big) \\
& + \beta^2\frac{c_*^2}{c^2}\Big(\frac{\Psi_\varpi(\varpi,t)}{\Psi(\varpi,t)}\Big)^2\big(|\nabla\bar{\varphi}(\mathbf{z}')|^2 + 1\big) - \beta^2\Big(\frac{\Psi_\varpi(\varpi,t)}{\Psi(\varpi,t)}\Big)^2 \\
& + \beta^2\Big(\frac{\Psi_\varpi(\varpi,t)}{\Psi(\varpi,t)}\Big)^2 - c_*\beta\frac{\Psi_\varpi(\varpi,t)}{\Psi(\varpi,t)} \\
& + c_*\beta\frac{\Psi_\varpi(\varpi,t)}{\Psi(\varpi,t)} - \beta\frac{\Psi_{\varpi\varpi}(\varpi,t)}{\Psi(\varpi,t)} - \beta\frac{f(\Psi(\varpi,t),t)}{\Psi(\varpi,t)}\Big] \\
& - \varepsilon\sigma(\mathbf{x}')f_u(\Psi(\varrho,t) + \theta\varepsilon\sigma(\mathbf{x}')\Psi^\beta(\varpi,t),t)\Psi^\beta(\varpi,t)
\end{aligned}$$

$$\geq -\varepsilon\sigma(\mathbf{x}')\Psi^{\beta}(\varpi,t)\Big(\frac{1}{16}\Pi(\beta\Lambda_2)+\frac{1}{16}\Pi(\beta\Lambda_2)+\frac{1}{16}\Pi(\beta\Lambda_2)+\frac{1}{16}\Pi(\beta\Lambda_2) - \frac{1}{2}\Pi(\beta\Lambda_2)+\frac{1}{16}\Pi(\beta\Lambda_2)+\frac{1}{16}\Pi(\beta\Lambda_2)+\frac{1}{16}\Pi(\beta\Lambda_2)\Big) > 0.$$

Case 2: $\varrho > X'' > 0$, where X'' is a sufficiently large constant. Let $\varpi > 1$, (1.5) implies that there exists a sufficiently large constant $X_1' > 0$ such that for any $\varrho > X_1'$ and $t \in [0,T]$

$$(B_3|\varrho| + B_4\varrho^2)|\Psi_{\varrho\varrho}(\varrho,t)| < \frac{1}{8}P_1\Lambda_0,$$
$$(B_5 + B_6|\varrho|)|\Psi_{\varrho}(\varrho,t)| < \frac{1}{8}P_1\Lambda_0.$$

Since $\lim_{\varrho\to+\infty}\Psi(\varrho,t)=1$, there exists a sufficiently large constant $X_2' > 0$, such that for any $\varepsilon \in (0, \frac{\varepsilon_1}{P_2(c-c_*)})$, we have

$$1-\varepsilon_1 < \Psi(\varrho,t) + \theta\varepsilon\sigma(\mathbf{x}')\nu(t) < 1+\varepsilon_1, \quad \varrho > X_2',\ t\in[0,T].$$

Therefore, for each $\varrho > X_1'$, inequality (2.7) yields

$$(f_u(1,t) - f_u(\Psi(\varrho,t)+\theta\varepsilon\sigma(\mathbf{x}')\nu(t),t))\nu(t) > -\frac{1}{2}P_1\Lambda_0, \quad t\in[0,T].$$

In addition, from (3.7), one has

$$\varepsilon\sigma(\mathbf{x}')\Lambda_0\nu'(t) - \varepsilon\alpha^2\sum_{i=1}^{2}G_{z_iz_i}(\mathbf{z}')\nu(t) \geq \varepsilon\sigma(\mathbf{x}')(\Lambda_0 - \alpha^2B_1)P_1,$$

for $\mathbf{x}' \in \mathbb{R}^2$ and $t\in[0,T]$. Choosing $X'' = \max\Big\{X_1', X_2', \frac{c}{c_*}\Big\}$, then for any $\varrho > X_2''$, one has

$$\begin{aligned}
L(\overline{\psi}) &= \Big(1-\sum_{i=1}^{2}\varrho_{x_i}^2 - \frac{1}{1+|\nabla\bar{\varphi}(\mathbf{z}')|^2}\Big)\Psi_{\varrho\varrho}(\varrho,t) - \Psi_{\varrho}(\varrho,t)\sum_{i=1}^{2}\varrho_{x_ix_i} \\
&\quad - c_*\Psi_{\varrho}(\varrho,t) + f(\Psi(\varrho,t),t) - f(\overline{\psi},t) \\
&\quad + \varepsilon\sigma(\mathbf{x}')\nu'(t) - \varepsilon\alpha^2\sum_{i=1}^{2}G_{z_iz_i}(\mathbf{z}')\nu(t) \\
&= \Big(1-\sum_{i=1}^{2}\varrho_{x_i}^2 - \frac{1}{1+|\nabla\bar{\varphi}(\mathbf{z}')|^2}\Big)\Psi_{\varrho\varrho}(\varrho,t) - \Psi_{\varrho}(\varrho,t)\sum_{i=1}^{2}\varrho_{x_ix_i} - c_*\Psi_{\varrho}(\varrho,t) \\
&\quad - f_u(\Psi(\varrho,t)+\theta\varepsilon\sigma(\mathbf{x}')\nu(t),t)\varepsilon\sigma(\mathbf{x}')\nu(t) - \varepsilon\alpha^2\sum_{i=1}^{2}G_{z_iz_i}(\mathbf{z}')\nu(t) \\
&\quad + \varepsilon\sigma(\mathbf{x}')\nu(t)\big(f_u(1,t)+\Lambda_0\big) \\
&\geq -\varepsilon\sigma(\mathbf{x}')(B_3|\varrho| + B_4\varrho^2)|\Psi_{\varrho\varrho}(\varrho,t)| - \varepsilon\sigma(\mathbf{x}')(B_5+B_6|\varrho|)|\Psi_{\varrho}(\varrho,t)| \\
&\quad - \varepsilon\sigma(\mathbf{x}')\frac{1}{2}P_1\Lambda_0 + \varepsilon\sigma(\mathbf{x}')\left(\Lambda_0 - \alpha^2B_1\right)P_1 \\
&\geq \varepsilon\sigma(\mathbf{x}')\Big[-\frac{1}{8}P_1\Lambda_0 - \frac{1}{8}P_1\Lambda_0 - \frac{1}{2}P_1\Lambda_0 + (\Lambda_0-\alpha^2B_1)P_1\Big] > 0,
\end{aligned}$$

if $0 < \alpha < \sqrt{\frac{\Lambda_0}{4B_1}}$.

Case 3: $-X' < \varrho < X''$, where X' and X'' are defined in Cases 1 and 2. Let

$$\begin{gathered}
\psi_* = \min_{-X'\le\varrho\le X'', t\in[0,T]} \Psi_\varrho(\varrho,t), \quad Q_0 = \sup_{u\in[-\varepsilon_1,1+\varepsilon_1], t\in[0,T]} |f_u(u,t)|,\\
Q_1 = \sup_{\varrho\in\mathbb{R}, t\in[0,T]} |\Psi_\varrho(\varrho,t)|, \quad Q_2 = \sup_{\varrho\in\mathbb{R}, t\in[0,T]} |\varrho||\Psi_\varrho(\varrho,t)|,\\
Q_3 = \sup_{\varrho\in\mathbb{R}, t\in[0,T]} |\varrho||\Psi_{\varrho\varrho}(\varrho,t)|, \quad Q_4 = \sup_{\varrho\in\mathbb{R}, t\in[0,T]} \varrho^2|\Psi_{\varrho\varrho}(\varrho,t)|,\\
Q_5 = \sup_{\varrho\in\mathbb{R}, t\in[0,T]} \Big|\frac{\Psi_\varrho(\varrho,t)}{\Psi(\varrho,t)}\Big|, \quad Q_6 = \sup_{\varrho\in\mathbb{R}, t\in[0,T]} \Big|\frac{\Psi_{\varrho\varrho}(\varrho,t)}{\Psi(\varrho,t)}\Big|.
\end{gathered}$$

Since $\nu(t) = ke^{\Lambda_0 t + \int_0^t f_u(1,s)ds}$ and $\nu'(t) = ke^{\Lambda_0 t+\int_0^t f_u(1,s)ds}(\Lambda_0 + f_u(1,t))$, $\nu'(t)$ is bounded following from the boundedness of $\nu(t)$. By $\Psi_t = \Psi_{\varrho\varrho} - c_*\Psi_\varrho + f(\Psi(\varrho),t)$, we have Ψ_t is also bounded and

$$\max_{\varpi\in\mathbb{R}, t\in\mathbb{R}} |\omega(\varpi)\nu'(t) + (1-\omega(\varpi)\beta\Psi^{\beta-1}(\varpi,t)\Psi_t(\varpi,t)| \le C_0$$

for some constant $C_0 > 0$. Therefore we obtain

$$\begin{aligned}
L(\overline{\psi}) &= \Big(1 - \sum_{i=1}^2 \varrho_{x_i}^2 - \frac{1}{1+|\nabla\bar\varphi(\mathbf{z}')|^2}\Big)\Psi_{\varrho\varrho}(\varrho,t) - \Psi_\varrho(\varrho,t)\sum_{i=1}^2 \varrho_{x_ix_i}\\
&\quad - \varepsilon\sigma(\mathbf{x}')\Big\{\alpha^2\frac{\sum_{i=1}^2 G_{z_iz_i}(\mathbf{z}')}{\sigma(\mathbf{x}')}\omega(\varpi)\nu(t)\\
&\quad + (1-\omega(\varpi))\Big[\alpha^2\frac{\sum_{i=1}^2 G_{z_iz_i}(\mathbf{z}')}{\sigma(\mathbf{x}')}\Psi^\beta(\varpi,t)\\
&\quad + 2\alpha\beta\frac{c_*}{c}\frac{\sum_{i=1}^2 G_{z_i}(\mathbf{z}')\bar\varphi_{z_i}(\mathbf{z}')}{\sigma(\mathbf{x}')}\Psi^{\beta-1}(\varpi,t)\Psi_\varpi(\varpi,t)\\
&\quad + \alpha\beta\frac{c_*}{c}\sum_{i=1}^2 \bar\varphi_{z_iz_i}(\mathbf{z}')\Psi^{\beta-1}(\varpi,t)\Psi_\varpi(\varpi,t)\\
&\quad + \beta\Psi^{\beta-1}(\varpi,t)\Psi_{\varpi\varpi}(\varpi,t)\big(\frac{c_*}{c}\big)^2\big(|\nabla\bar\varphi(\mathbf{z}')|^2+1\big)\Big]\Big\}\\
&\quad - \varepsilon\sigma(\mathbf{x}')\Big[\big(\frac{c_*}{c}\big)^2\big(|\nabla\bar\varphi(\mathbf{z}')|^2+1\big)\omega''(\varpi)\big(\nu(t)-\Psi^\beta(\varpi,t)\big)\\
&\quad + 2\alpha\frac{c_*}{c}\frac{\sum_{i=1}^2 G_{z_i}(\mathbf{z}')\bar\varphi_{z_i(z_i)}}{\sigma(\mathbf{x}')}\omega'(\varpi)\big(\nu(t)-\Psi^\beta(\varpi,t)\big)\\
&\quad + \alpha\frac{c_*}{c}\sum_{i=1}^2 \bar\varphi_{z_iz_i}(\mathbf{z}')\omega'(\varpi)\big(\nu(t)-\Psi^\beta(\varpi,t)\big)\Big]\\
&\quad + f(\Psi(\varrho,t),t) - f(\overline{\psi},t) + \Big(\frac{c}{\sqrt{1+|\nabla\bar\varphi(\mathbf{z}')|^2}} - c_*\Big)\Psi_\varrho(\varrho,t)\\
&\quad + \varepsilon\sigma(\mathbf{x}')[\omega(\varpi)\nu'(t) + (1-\omega(\varpi))\beta\Psi^{\beta-1}(\varpi,t)\Psi_t(\varpi,t)]\\
&\ge -\varepsilon\sigma(\mathbf{x}')(B_3|\varrho| + B_4\varrho^2)|\Psi_{\varrho\varrho}(\varrho,t)|\\
&\quad - \varepsilon\sigma(\mathbf{x}')(B_5 + B_6|\varrho|)|\Psi_\varrho(\varrho,t)| + \sigma(\mathbf{x}')\Psi_\varrho(\varrho,t)\\
&\quad - \varepsilon\sigma(\mathbf{x}')\Big\{\alpha^2\frac{\sum_{i=1}^2 |G_{z_iz_i}(\mathbf{z}')|}{\sigma(\mathbf{x}')}[\omega(\varpi)\nu(t) + (1-\omega(\varpi))\Psi^\beta(\varpi,t)]
\end{aligned}$$

$$
\begin{aligned}
&+ 2\alpha \frac{\sum_{i=1}^{2} |G_{z_i}(\mathbf{z}')\bar{\varphi}_{z_i}(\mathbf{z}')|}{\sigma(\mathbf{x}')} \frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)} \\
&+ \alpha \sum_{i=1}^{2} |\bar{\varphi}_{z_i z_i}(\mathbf{z}')| \frac{\Psi_{\varpi}(\varpi,t)}{\Psi(\varpi,t)} + |\frac{\Psi_{\varpi\varpi}(\varpi,t)}{\Psi(\varpi,t)}| \Big\} \\
&- \varepsilon\sigma(\mathbf{x}')|\omega''(\varpi)|\big(\nu(t) - \Psi^{\beta}(\varpi,t)\big) \\
&- \varepsilon\sigma(\mathbf{x}') 2\alpha \frac{\sum_{i=1}^{2} |G_{z_i}(\mathbf{z}')\bar{\varphi}_{z_i(z_i)}|}{\sigma(\mathbf{x}')} \omega'(\varpi)\big(\nu(t) - \Psi^{\beta}(\varpi,t)\big) \\
&- \varepsilon\sigma(\mathbf{x}')\alpha \sum_{i=1}^{2} |\bar{\varphi}_{z_i z_i}(\mathbf{z}')| \omega'(\varpi)\big(\nu(t) - \Psi^{\beta}(\varpi,t)\big) \\
&- \varepsilon\sigma(\mathbf{x}')C_0 - \varepsilon\sigma(\mathbf{x}') f_u \Big[\Psi(\varrho,t) \\
&+ \theta\varepsilon\sigma(\mathbf{x}')(\omega(\varpi)\nu(t) + (1-\omega(\varpi))\Psi^{\beta}(\varpi,t)) \cdot (\omega(\varpi)\nu(t) \\
&+ (1-\omega(\varpi))\Psi^{\beta}(\varpi,t))\Big] \\
\geq\ & \sigma(\mathbf{x}')(-\alpha B_3 Q_3 - \alpha B_4 Q_4 - \alpha B_5 Q_1 - \alpha B_6 Q_2 - \alpha B_1 P_2 \\
&- 2\alpha B_1 m_* Q_5 - 2\alpha m_* M_* Q_5 - \varepsilon N_2 - \varepsilon \mathcal{A} + u_* - \varepsilon C_0 - Q_0 P_2) > 0,
\end{aligned}
$$

where

$$
\mathcal{A} = \Big(\sup_{x\in\mathbb{R}} |\omega''(x)| + 2B_2 m_* + 2M_* m_*\Big) P_2,
$$

$$
\alpha < \alpha_2 = \frac{u_*}{2\left(B_3 Q_3 + B_4 Q_4 + B_5 Q_1 + B_6 Q_2 + B_1 P_2 + 2B_1 m_* Q_5 + 2m_* M_* Q_5\right)},
$$

$$
\varepsilon < \varepsilon_2 = \frac{u_*}{2\left(N_2 + \mathcal{A} + C_0 + Q_0 P_2\right)}.
$$

To sum up, combining the above Cases 1– 3, $\overline{\psi}$ is the supersolution of (1.8)-(1.9) on $\mathbb{R}^3 \times (-\infty, +\infty)$.

Secondly, we prove (3.5). Let

$$
\vartheta(\mathbf{x}) = \frac{c_*}{c}(x_3 + h(\mathbf{x}')), \quad \eta(\mathbf{x}) = \frac{x_3 + h(\mathbf{x}')}{\sqrt{1 + |\nabla\bar{\varphi}(\alpha\mathbf{x}')|^2}}.
$$

Recall that

$$
\begin{gathered}
\varpi(\mathbf{x}) = \frac{c_*}{c}(x_3 + \bar{\varphi}(\mathbf{z}')/\alpha), \quad \varrho(\mathbf{x}) = \frac{x_3 + \bar{\varphi}(\mathbf{z}')/\alpha}{\sqrt{1 + |\nabla\bar{\varphi}(\alpha\mathbf{x}')|^2}}, \\
\overline{\psi}(\mathbf{x},t) - \underline{\psi}(\mathbf{x},t) = \Psi(\varrho(\mathbf{x}),t) - \Psi(\vartheta(\mathbf{x}),t) + \varepsilon\sigma(\mathbf{x}')(\omega(\varpi)\nu(t) \\
+ (1-\omega(\varpi))\Psi^{\beta}(\varpi,t)), \\
h(\mathbf{x}') \leq \bar{\varphi}(\alpha\mathbf{x}')/\alpha.
\end{gathered}
$$

We divide the proof into two cases.

Case 1: $\varrho(\mathbf{x}) \geq \vartheta(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^3$. Since the function $\Psi(\xi,t)$ is monotonically increasing in ξ, it is obvious that $\underline{\psi}(\mathbf{x},t) < \overline{\psi}(\mathbf{x},t)$ for any $(\mathbf{x},t) \in \mathbb{R}^3 \times (-\infty, +\infty)$.

Case 2: $\varrho(\mathbf{x}) < \vartheta(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^3$.

$$
\varrho(\mathbf{x}) - \vartheta(\mathbf{x}) = \frac{x_3 + \bar{\varphi}(\mathbf{z}')/\alpha}{\sqrt{1 + |\nabla\bar{\varphi}(\mathbf{z}')|^2}} - \frac{c_*}{c}(x_3 + h(\mathbf{x}'))
$$

$$= \Big(\frac{1}{\sqrt{1+|\nabla\bar{\varphi}(\mathbf{z}')|^2}} - \frac{c_*}{c}\Big)(x_3 + h(\mathbf{x}')) + \frac{\bar{\varphi}(\mathbf{z}') - h(\mathbf{z}')}{\alpha\sqrt{1+|\nabla\bar{\varphi}(\mathbf{z}')|^2}} < 0.$$

Since $\frac{1}{\sqrt{1+|\nabla\bar{\varphi}(\alpha\mathbf{x}')|^2}} > \frac{c^*}{c}$, we have

$$\begin{aligned} x_3 + h(\mathbf{x}') &< -\frac{\bar{\varphi}(\alpha\mathbf{x}') - h(\alpha\mathbf{x}')}{\alpha\sqrt{1+|\nabla\bar{\varphi}(\mathbf{z}')|^2}} \Big/ \Big(\frac{1}{\sqrt{1+|\nabla\bar{\varphi}(\mathbf{z}')|^2}} - \frac{c_*}{c}\Big) \\ &\le -\frac{a_1 c_*}{\alpha} < 0, \end{aligned}$$

where a_1 is defined in Lemma 2.2. Thus

$$\varrho(\mathbf{x}) < \vartheta(\mathbf{x}) \le -\frac{a_1 c_*^2}{c\alpha} < 0, \quad \frac{c}{c_*}\varpi(\mathbf{x}) < \varrho(\mathbf{x}) < \varpi(\mathbf{x}).$$

Furthermore,

$$\begin{aligned} &\overline{\psi}(\mathbf{x},t) - \underline{\psi}(\mathbf{x},t) \\ &= \Psi(\varrho(\mathbf{x}),t) - \Psi(\eta(\mathbf{x}),t) + \Psi(\eta(\mathbf{x}),t) - \Psi(\vartheta(\mathbf{x}),t) \\ &\quad + \varepsilon\sigma(\mathbf{x}')(\omega(\varpi)\nu(t) + (1-\omega(\varpi))\Psi^\beta(\varpi,t)) \\ &\ge \Psi(\eta(\mathbf{x}),t) - \Psi(\vartheta(\mathbf{x}),t) + \varepsilon\sigma(\mathbf{x}')(\omega(\varpi)\nu(t) + (1-\omega(\varpi))\Psi^\beta(\varpi,t)) \\ &\ge \Big(\frac{1}{\sqrt{1+|\nabla\bar{\varphi}(\alpha\mathbf{x}')|^2}} - \frac{c_*}{c}\Big)(x_3 + h(\mathbf{x}'))\Psi_\varpi(\theta\eta(\mathbf{x}) + (1-\theta)\vartheta(\mathbf{x}),t) \\ &\quad + \varepsilon\sigma(\mathbf{x}')\Psi^\beta(\varpi(\mathbf{x}),t) \\ &= \Big(\frac{1}{\sqrt{1+|\nabla\bar{\varphi}(\alpha\mathbf{x}')|^2}} - \frac{c_*}{c}\Big)\frac{c}{c_*}\vartheta(\mathbf{x})\Psi_\varpi(\theta\eta(\mathbf{x}) + (1-\theta)\vartheta(\mathbf{x}),t) \\ &\quad + \varepsilon\sigma(\mathbf{x}')\Psi^\beta(\varpi(\mathbf{x}),t), \end{aligned}$$

where $\theta \in (0,1)$. Since $x_3 + h(\mathbf{x}') < 0$, we have $\eta(\mathbf{x}) < \varrho(\mathbf{x}) < \vartheta(\mathbf{x}) < \varpi(\mathbf{x}) < 0$,

$$\Psi_\varpi(\theta\eta(\mathbf{x}) + (1-\theta)\vartheta(\mathbf{x}),t) \le L_2 e^{-\Lambda_2|\theta\eta(\mathbf{x}) + (1-\theta)\vartheta(\mathbf{x})|} \le L_2 e^{\Lambda_2\vartheta(\mathbf{x})}$$

and

$$\Psi^\beta(\varpi(\mathbf{x}),t) \ge L_1^\beta e^{\beta\Lambda_2\varpi(\mathbf{x})} > L_1 e^{\beta\Lambda_2\varpi(\mathbf{x})} > L_1 e^{\beta\Lambda_2\vartheta(\mathbf{x})}.$$

Thus we have

$$\begin{aligned} &\overline{\psi}(\mathbf{x},t) - \underline{\psi}(\mathbf{x},t) \\ &= \Big(\frac{1}{\sqrt{1+|\nabla\varphi(\alpha\mathbf{x}')|^2}} - \frac{c_*}{c}\Big)\frac{c}{c_*}\vartheta(\mathbf{x})\Psi_\varpi(\theta\eta(\mathbf{x}) + (1-\theta)\vartheta(\mathbf{x}),t) \\ &\quad + \varepsilon\sigma(\mathbf{x}')\Psi^\beta(\varpi(\mathbf{x}),t) \\ &\ge \sigma(\mathbf{x}')\Big(\frac{L_2}{c_*}\vartheta(\mathbf{x})e^{\Lambda_2\vartheta(\mathbf{x})} + \varepsilon L_1 e^{\Lambda_2\beta\vartheta(\mathbf{x})}\Big) \\ &\ge \sigma(\mathbf{x}')e^{\Lambda_2\beta\vartheta(\mathbf{x})}\Big(\frac{L_2}{c_*(1-\beta)^2\Lambda_2^2\vartheta(\mathbf{x})}\sup_{\omega>0}(\omega^2 e^{-\omega}) + \varepsilon L_1\Big) \\ &\ge \sigma(\mathbf{x}')e^{\Lambda_2\beta\vartheta(\mathbf{x})}\Big(-\frac{4L_2\alpha c}{c_*^3 e^2(1-\beta)^2 a_1\Lambda_2^2} + \varepsilon L_1\Big) > 0, \end{aligned}$$

if

$$\alpha < \alpha_3 = \frac{\varepsilon a_1 L_1 c_*^3 e^2(1-\beta)^2\Lambda_2^2}{4L_2 c}.$$

In conclusion, we can obtain $\overline{\psi}(\mathbf{x},t) > \underline{\psi}(\mathbf{x},t)$ for any $\mathbf{x} \in \mathbb{R}^3$ and $t \in [0,T]$.

Finally, we prove (3.4). We just need to prove

$$\lim_{\bar{\gamma}\to\infty}\sup_{\mathbf{x}\in\mathcal{D}(\bar{\gamma}),t\in[0,T]}\frac{|\Psi(\varrho(\mathbf{x}),t)-\Psi(\vartheta(\mathbf{x}),t)|}{\Psi^{\beta}(\varpi(\mathbf{x}),t)}=0. \tag{3.10}$$

We prove it by a contradiction argument. Assume that (3.10) is not true, then there exist a positive number ε^*, sequences $\{\bar{\gamma}_n\}_{n\in\mathbb{N}}\in\mathbb{R}$ and $\{\mathbf{x}_n\}_{n\in\mathbb{N}}\in\mathbb{R}^3$ such that

$$\lim_{n\to\infty}\bar{\gamma}_n=\infty,\quad \mathbf{x}_n\in\mathcal{D}(\bar{\gamma}_n), \tag{3.11}$$

$$\frac{|\Psi(\varrho(\mathbf{x}),t)-\Psi(\vartheta(\mathbf{x}),t)|}{\Psi^{\beta}(\varpi(\mathbf{x}),t)}\ge\varepsilon^*. \tag{3.12}$$

We denote $\mathbf{x}_n=(\mathbf{x}_n',x_{n,3})\in\mathbb{R}^3$, with $\mathbf{x}_n'=(x_{n,1},x_{n,2})\in\mathbb{R}^2$. Obviously,

$$\varrho(\mathbf{x}_n)=\frac{x_{n,3}+\frac{\bar{\varphi}(\alpha\mathbf{x}_n')}{\alpha}}{\sqrt{|\nabla\bar{\varphi}(\alpha\mathbf{x}_n')|^2+1}}=\frac{x_{n,3}+h(\mathbf{x}_n')+\frac{\bar{\varphi}(\alpha\mathbf{x}_n')-h(\alpha\mathbf{x}_n')}{\alpha}}{\sqrt{|\nabla\bar{\varphi}(\alpha\mathbf{x}_n')|^2+1}}.$$

Now we consider two cases and prove them separately.

Case 1: $\lim_{n\to+\infty}\operatorname{dist}(\mathbf{x}_n',E)=\infty$. In this case, we can obtain $\lim_{n\to+\infty}G(\mathbf{x}_n')=0$ and $\lim_{n\to+\infty}|\bar{\varphi}(\mathbf{x}_n')-h(\mathbf{x}_n')|=0$. Hence we can obtain $\lim_{n\to+\infty}|\varrho(\mathbf{x}_n)-\vartheta(\mathbf{x}_n)|=0$.

If $\vartheta(\mathbf{x}_n)=\frac{c_*}{c}(x_{n,3}+h(\mathbf{x}_n'))\to+\infty$ as $n\to+\infty$, then $\varrho(\mathbf{x}_n)\to+\infty$ and therefore

$$\lim_{n\to\infty}\frac{|\Psi(\varrho(\mathbf{x}_n),t)-\Psi(\vartheta(\mathbf{x}_n),t)|}{\Psi^{\beta}(\varpi(\mathbf{x}_n),t)}=0,$$

which contradicts with (3.12).

If $\vartheta(\mathbf{x}_n)=\frac{c_*}{c}(x_{n,3}+h(\mathbf{x}_n'))\to-\infty$ as $n\to+\infty$, then $\varrho(\mathbf{x}_n)\to-\infty$. Since

$$h(\mathbf{x}_n')<\frac{1}{\alpha}\bar{\varphi}(\alpha\mathbf{x}_n')\le h(\mathbf{x}_n')+\frac{2\pi m_*}{\alpha}\int_0^{\infty}r^2\tilde{\rho}(r)dr,$$

and

$$1<\sqrt{|\nabla\bar{\varphi}(\alpha\mathbf{x}_n')|^2+1}<\frac{c}{c_*},$$

it can be obtained that, when n is sufficiently large, there are

$$0>\varpi(\mathbf{x}_n)>\varrho(\mathbf{x}_n)>\frac{c}{c_*}\varpi(\mathbf{x}_n)$$

and

$$\vartheta(\mathbf{x}_n)<\varpi(\mathbf{x}_n)<\vartheta(\mathbf{x}_n)+\frac{2\pi m_*c_*}{\alpha c}\int_0^{\infty}r^2\tilde{\rho}(r)dr.$$

Letting $n\to+\infty$, we have

$$\begin{aligned}
&\frac{|\Psi(\varrho(\mathbf{x}_n),t)-\Psi(\vartheta(\mathbf{x}_n),t)|}{\Psi^{\beta}(\varpi(\mathbf{x}_n),t)}\\
&=\frac{|(\varrho(\mathbf{x}_n)-\vartheta(\mathbf{x}_n))\cdot\Psi'((1-\theta)\vartheta(\mathbf{x}_n)+\theta\varrho(\mathbf{x}_n),t)|}{\Psi^{\beta}(\varpi(\mathbf{x}_n),t)}\\
&\le\frac{L_2e^{\Lambda_2((1-\theta)\vartheta(\mathbf{x}_n)+\theta\varrho(\mathbf{x}_n))}}{L_1^{\beta}e^{\beta\Lambda_2\varpi(\mathbf{x}_n)}}|\varrho(\mathbf{x}_n)-\vartheta(\mathbf{x}_n)|\\
&\le\frac{L_2e^{\Lambda_2\varpi(\mathbf{x}_n)}}{L_1^{\beta}e^{\beta\Lambda_2\varpi(\mathbf{x}_n)}}\Big[(\frac{c}{c_*}+1)|\varpi(\mathbf{x}_n)|+\frac{2\pi m_*c_*}{\alpha c}\int_0^{\infty}r^2\tilde{\rho}(r)\,dr\Big]\to0,
\end{aligned}$$

which contradicts with (3.12).

If $\vartheta(\mathbf{x}_n) = \frac{c_*}{c}(x_{n,3} + h(\mathbf{x}'_n))$ is bounded for each $n \in \mathbb{N}$, then we have $\varpi(\mathbf{x}_n)$ is also bounded for each $n \in \mathbb{N}$. Since $\lim_{n\to+\infty} |\varrho(\mathbf{x}_n) - \vartheta(\mathbf{x}_n)| = 0$, it holds

$$\lim_{n\to\infty} \frac{|\Psi(\varrho(\mathbf{x}_n), t) - \Psi(\vartheta(\mathbf{x}_n), t)|}{\Psi^\beta(\varpi(\mathbf{x}_n), t)} = 0,$$

which also contradicts with (3.12).

Case 2: $\mathrm{dist}(\mathbf{x}'_n, E)$ is uniformly bounded in k. From (3.11), we can easily obtain $(x_{n,3} + h(\mathbf{x}'_n)) \to \pm\infty$ as $n \to +\infty$.

If $(x_{n,3} + h(\mathbf{x}'_n)) \to +\infty$ as $n \to +\infty$, then $\vartheta(\mathbf{x}_n) = \frac{c_*}{c}(x_{n,3} + h(\mathbf{x}'_n)) \to +\infty$ as $n \to +\infty$ and

$$\varrho(\mathbf{x}_n) = \frac{x_{n,3} + h(\mathbf{x}'_n) + \frac{\bar\varphi(\alpha\mathbf{x}'_n) - h(\alpha\mathbf{x}'_n)}{\alpha}}{\sqrt{|\nabla\bar\varphi(\alpha\mathbf{x}'_n)|^2 + 1}} \ge \vartheta(\mathbf{x}_n) \to +\infty \quad \text{as } n \to +\infty.$$

So we can obtain

$$\lim_{n\to\infty} \frac{|\Psi(\varrho(\mathbf{x}_n), t) - \Psi(\vartheta(\mathbf{x}_n), t)|}{\Psi^\beta(\varpi(\mathbf{x}_n), t)} = 0,$$

which contradicts with (3.12).

If $(x_{n,3} + h(\mathbf{x}'_n)) \to -\infty$ as $n \to +\infty$, then

$$\vartheta(\mathbf{x}_n) = \frac{c_*}{c}(x_{n,3} + h(\mathbf{x}'_n)) \to -\infty \quad \text{as } n \to +\infty$$

and

$$\begin{aligned}\varrho(\mathbf{x}_n) &= \frac{x_{n,3} + h(\mathbf{x}'_n) + \frac{\bar\varphi(\alpha\mathbf{x}'_n) - h(\alpha\mathbf{x}'_n)}{\alpha}}{\sqrt{|\nabla\bar\varphi(\alpha\mathbf{x}'_n)|^2 + 1}} \\ &\le \vartheta(\mathbf{x}_n) + \frac{1}{\sqrt{|\nabla\bar\varphi(\alpha\mathbf{x}'_n)|^2 + 1}} \frac{2\pi m_*}{\alpha} \int_0^\infty r^2 \tilde\rho(r) dr \to -\infty \quad \text{as } n \to +\infty.\end{aligned}$$

Similar to the argument in Case 1, we have

$$\frac{|\Psi(\varrho(\mathbf{x}_n), t) - \Psi(\vartheta(\mathbf{x}_n), t)|}{\Psi^\beta(\varpi(\mathbf{x}_n), t)} \le 0,$$

which contradicts (3.12). Summing up, (3.4) is true. In conclusion, letting

$$\varepsilon_0^+(\beta) = \{\frac{\varepsilon_1}{P_2}, \frac{\varepsilon_1}{c - c_*}, \varepsilon_2\}, \quad \alpha_0^+(\varepsilon, \beta) = \Big\{\varepsilon, \alpha_1, \alpha_2, \alpha_3, \sqrt{\frac{\Lambda_0}{4B_7}}\Big\},$$

we complete the proof. □

3.2. **Existence.** In this subsection, we give the proof of Theorem 1.1. That is, we prove the existence of three-dimensional periodic pyramidal traveling front.

Theorem 3.2. *Assume that* (H1) *and* (H2) *hold. For each $c > c_*$, equation* (1.1) *has a periodic nonplanar traveling front $V(\mathbf{x}, t)$ satisfying* (1.8)-(1.9) *and*

$$\underline{\psi}(\mathbf{x}, t) < V(\mathbf{x}, t) < \overline{\psi}(\mathbf{x}, t; \beta, \varepsilon, \alpha), \quad \mathbf{x} \in \mathbb{R}^3,\ t \in [0, T].$$

Moveover,

$$\lim_{\bar\gamma\to\infty} \sup_{\mathbf{x}\in\mathcal{D}(\bar\gamma), t\in[0,T]} \frac{|V(\mathbf{x}, t) - \underline{\psi}(\mathbf{x}, t)|}{(\underline{\psi}(\mathbf{x}, t))^\beta} = 0 \tag{3.13}$$

and $V_{x_3}(\mathbf{x}, t) > 0$ for all $(\mathbf{x}, t) \in \mathbb{R}^3 \times [0, T]$.

Proof. According to the parabolic estimation, there exists a constant $C > 0$ such that the solution $\psi(\mathbf{x}, t; \psi_0)$ of Eq. (1.7) with the initial value $\psi_0(\mathbf{x}, t) \in [0, 1]$ satisfies

$$\|\psi(\cdot, \cdot; \psi_0)\|_{C^{2+\theta,1+\frac{\theta}{2}}(\mathbb{R}^3\times[T,+\infty))} < C,$$

where $0 < \theta < 1$. Since $\underline{\psi}(\mathbf{x}, t)$ is the subsolution of (1.7) for any $\mathbf{x} \in \mathbb{R}^3$ and $t \in [0, T]$, and $\underline{\psi}(\mathbf{x}, t + T) = \underline{\psi}(\mathbf{x}, t)$, we have

$$0 < \psi(\mathbf{x}, t + kT; \underline{\psi}) \le \psi(\mathbf{x}, t + (k + 1)T; \underline{\psi}) < 1, \quad \mathbf{x} \in \mathbb{R}^3,\ t \in [0, T]$$

from the maximum principle. Thus $\psi(\mathbf{x}, t + kT; \underline{\psi})$ monotonically increasing converges to $V(\cdot, \cdot)$ under the norm $\|\cdot\|_{C^{2,1}_{loc}(\mathbb{R}^3\times[0,T])}$ as $k \to \infty$. That is

$$\lim_{k\to\infty} \big\|\psi(\mathbf{x}, t + kT; \underline{\psi}) - V(\mathbf{x}, t)\big\|_{C^{2,1}_{\rm loc}(\mathbb{R}^3\times[0,T])} = 0.$$

Meanwhile, since $\overline{\psi}(\mathbf{x}, t; \beta, \varepsilon, \alpha)$ is a supersolution, it can be obtained that $\underline{\psi}(\mathbf{x}, t) < V(\mathbf{x}, t) < \overline{\psi}(\mathbf{x}, t; \beta, \varepsilon, \alpha)$ by the comparison principle. Since $\underline{\psi}_{x_3}(\mathbf{x}, t) > 0$, it follows that $V_{x_3}(\mathbf{x}, t) \ge 0$. By the strong maximum principle, we can obtain $V_{x_3}(\mathbf{x}, t) > 0$. From (3.4), we have

$$\lim_{\bar\gamma\to\infty} \sup_{\mathbf{x}\in\mathcal{D}(\bar\gamma),t\in[0,T]} \frac{|V(\mathbf{x}, t) - \underline{\psi}(\mathbf{x}, t)|}{(\Psi(\mathbf{x}, t))^\beta} \le 2\varepsilon. \tag{3.14}$$

Then we fix $\beta \in (0, 1)$ and let $\overline{\beta} \in (\beta, 1)$. From (3.14), for any $0 < \varepsilon < \min\{\varepsilon_0^+(\beta), \varepsilon_0^+(\overline{\beta})\}$ and $0 < \alpha < \min\{\alpha_0^+(\beta, \varepsilon), \alpha_0^+(\overline{\beta}, \varepsilon)\}$, we can easily get

$$\lim_{\bar\gamma\to\infty} \sup_{\mathbf{x}\in\mathcal{D}(\bar\gamma),t\in[0,T]} \frac{|V(\mathbf{x}, t) - \underline{\psi}(\mathbf{x}, t)|}{\Psi^{\overline{\beta}}(\frac{c_*}{c}(x_3 + \frac{\varphi(\alpha\mathbf{x}')}{\alpha}), t)} \le 2\varepsilon. \tag{3.15}$$

Fix $\alpha \in (0, \min\{\alpha_0^+(\beta, \varepsilon), \alpha_0^+(\overline{\beta}, \varepsilon)\})$. Then there exists $\gamma' > 0$ such that for any $\bar\gamma > \gamma'$,

$$\sup_{\mathbf{x}\in\mathcal{D}(\bar\gamma),t\in[0,T]} \frac{|V(\mathbf{x}, t) - \underline{\psi}(\mathbf{x}, t)|}{\Psi^{\overline{\beta}}(\frac{c_*}{c}(x_3 + \frac{\bar\varphi(\alpha\mathbf{x}')}{\alpha}), t)} \le 4\varepsilon. \tag{3.16}$$

We divide the remaining part of the proof into two cases.

Case 1: $|x_3 + h(\boldsymbol{x}')| > K_1$, where $K_1 > 0$ is sufficiently large. Obviously if $\mathbf{x} \in \mathbb{R}^3$ satisfies $|x_3+h(\mathbf{x}')| > K_2$, we can obtain $\mathrm{dist}(\mathbf{x}, \Gamma) > \gamma'$, where $K_2 > 0$ is sufficiently large. Fix $K_3 > 0$ such that $\mathrm{dist}(\mathbf{x}, \Gamma) > \gamma'$ and $\Psi^\beta(\vartheta(\mathbf{x})) > \frac{4}{5}$ if $|x_3 + h(\mathbf{x}')| > K_3$. Because

$$\frac{|V(\mathbf{x}, t) - \underline{\psi}(\mathbf{x}, t)|}{\Psi^{\overline{\beta}}(\frac{c_*}{c}(x_3 + \frac{\bar\varphi(\alpha\mathbf{x}')}{\alpha}), t)} \ge \frac{|V(\mathbf{x}, t) - \underline{\psi}(\mathbf{x}, t)|}{(\underline{\psi}(\mathbf{x}), t)^\beta} \Psi^\beta\big(\frac{c_*}{c}(x_3 + h(\mathbf{x}')), t\big)$$

for any $t \in [0, T]$ and $\mathbf{x} \in \mathbb{R}^3$ with $|x_3 + h(\mathbf{x}')| > K_3$, we have

$$\frac{|V(\mathbf{x}, t) - \underline{\psi}(\mathbf{x}, t)|}{(\underline{\psi}(\mathbf{x}, t))^\beta} \le \frac{|V(\mathbf{x}, t) - \underline{\psi}(\mathbf{x}, t)|}{\Psi^{\overline{\beta}}(\frac{c_*}{c}(x_3 + \frac{\bar\varphi(\alpha\mathbf{x}')}{\alpha}), t)} \frac{1}{\Psi^\beta(\frac{c_*}{c}(x_3 + h(\mathbf{x}')), t)} \le 5\varepsilon.$$

Let $K_4 > \frac{2\pi m_*}{\alpha} \int_0^\infty r^2\tilde\rho(r)dr$ large enough satisfy

$$L_1^{-\beta} L_2^{\overline{\beta}} e^{\Lambda_2(\overline{\beta}-\beta)\vartheta} e^{\frac{2\pi m_*}{\alpha}\Lambda_2\overline{\beta}\int_0^\infty r^2\tilde\rho(r)dr} < \frac{5}{4}, \quad \forall\vartheta < -K_4,$$

where (1.4) gives the definitions of L_1 and L_2. Since

$$\begin{aligned}
&\frac{|V(\mathbf{x},t)-\underline{\psi}(\mathbf{x},t)|}{\Psi^{\overline{\beta}}(\frac{c_*}{c}(x_3+\frac{\bar{\varphi}(\alpha\mathbf{x}')}{\alpha}),t)}\\
&=\frac{(\underline{\psi}(\mathbf{x}),t)^{\beta}}{(\underline{\psi}(\mathbf{x}),t)^{\beta}}\frac{|V(\mathbf{x},t)-\underline{\psi}(\mathbf{x},t)|}{\Psi^{\overline{\beta}}(\frac{c_*}{c}(x_3+\frac{\bar{\varphi}(\alpha\mathbf{x}')}{\alpha}),t)}\\
&=\frac{\Psi^{\beta}(\frac{c_*}{c}(x_3+h(\mathbf{x}')),t)}{\Psi^{\overline{\beta}}(\frac{c_*}{c}(x_3+\frac{\bar{\varphi}(\alpha\mathbf{x}')}{\alpha}),t)}\frac{|V(\mathbf{x},t)-\underline{\psi}(\mathbf{x},t)|}{(\underline{\psi}(\mathbf{x}),t)^{\beta}}\\
&\geq\frac{\Psi^{\beta}(\frac{c_*}{c}(x_3+h(\mathbf{x}')),t)}{\Psi^{\overline{\beta}}(\frac{c_*}{c}(x_3+h(\mathbf{x}')+\frac{2\pi m_*}{\alpha}\int_0^\infty r^2\tilde{\rho}(r)dr),t)}\frac{|V(\mathbf{x},t)-\underline{\psi}(\mathbf{x},t)|}{(\underline{\psi}(\mathbf{x}),t)^{\beta}},
\end{aligned}$$

then from (1.4), for any $\mathbf{x}\in\mathbb{R}^3$ that satisfies $x_3+h(\mathbf{x}')<-K_4$, we can obtain

$$\frac{|V(\mathbf{x},t)-\underline{\psi}(\mathbf{x},t)|}{(\underline{\psi}(\mathbf{x},t))^{\beta}}\leq 5\varepsilon,\quad t\in[0,T].$$

Case 2: $|x_3+h(\mathbf{x}')|\leq K_1$ and $\overline{K}>0$ is sufficiently large such that $\operatorname{dist}(\mathbf{x}',E)>\overline{K}$, where $K_1=\max\{K_2,K_3,K_4\}$.

When $\overline{K}>0$ is sufficiently large for all $\mathbf{x}\in\mathbb{R}^3$ with $\operatorname{dist}(\mathbf{x}',E)>\overline{K}$ and $|x_3+h(\mathbf{x}')|\leq K_1$, one has

$$\frac{\Psi^{\beta}(\frac{c_*}{c}(x_3+h(\mathbf{x}')),t)}{\Psi^{\beta}(\frac{c_*}{c}(x_3+\frac{\bar{\varphi}(\alpha\mathbf{x}')}{\alpha}),t)}>\frac{4}{5}.$$

Thus we obtain

$$\frac{|V(\mathbf{x},t)-\underline{\psi}(\mathbf{x},t)|}{(\underline{\psi}(\mathbf{x},t))^{\beta}}\leq\frac{|V(\mathbf{x},t)-\underline{\psi}(\mathbf{x},t)|}{\Psi^{\beta}(\frac{c_*}{c}(x_3+h(\mathbf{x}')),t)}\frac{\Psi^{\beta}(\frac{c_*}{c}(x_3+\frac{\bar{\varphi}(\alpha\mathbf{x}')}{\alpha}),t)}{\Psi^{\overline{\beta}}(\frac{c_*}{c}(x_3+\frac{\bar{\varphi}(\alpha\mathbf{x}')}{\alpha}),t)}\leq 5\varepsilon.$$

According to the definition of $\mathcal{D}(\bar{\gamma})$, there exists $\gamma^*>0$ such that

$$\mathcal{D}(\gamma^*)\subset\{\mathbf{x}\in\mathbb{R}^3:|x_3+h(\mathbf{x}')|>K_1\text{ or }|x_3+h(\mathbf{x}')|\leq K_1\text{ and }\operatorname{dist}(\mathbf{x}',E)>\overline{K}\}.$$

Thus Cases 1 and 2 imply

$$\frac{|V(\mathbf{x},t)-\underline{\psi}(\mathbf{x},t)|}{(\underline{\psi}(\mathbf{x},t))^{\beta}}\leq 5\varepsilon,\quad \mathbf{x}\in\mathcal{D}(\gamma^*),\ t\in[0,T].$$

Therefore,

$$\sup_{\mathbf{x}\in\mathcal{D}(\bar{\gamma}),t\in[0,T]}\frac{|V(\mathbf{x},t)-\underline{\psi}(\mathbf{x},t)|}{(\underline{\psi}(\mathbf{x},t))^{\beta}}\leq 5\varepsilon,\quad\forall\gamma>\gamma^*.$$

Hence (3.13) holds by the arbitrariness of ε. The proof is complete. □

4. Periodic pyramidal traveling fronts in $\mathbb{R}^n$ with $n\geq 4$

In this section, we investigate the existence of periodic nonplanar traveling front to (1.1) in $\mathbb{R}^n$ ($n\geq 4$). We use the same notation as above. We denote $\mathbf{s}=(s_1,s_2,...s_n)\in\mathbb{R}^n$ and $\mathbf{s}'=(s_1,s_2,\dots,s_{n-1})\in\mathbb{R}^{n-1}$. Assume that the traveling fronts travel towards $-s_n$ direction at the speed of $c>c_*$. Let

$$u(\mathbf{s},t)=v(\mathbf{s}',s_n+ct,t)=v(\mathbf{s}',w,t).$$

We still express $v(\mathbf{s}', w, t)$ as $v(\mathbf{s}', s_n, t)$ for convenience. Substitute v into (1.1), then

$$\begin{gathered} v_t = \Delta v - c v_{s_n} + f(v,t), \quad \mathbf{s} \in \mathbb{R}^n,\ t > 0, \\ v(\mathbf{s}, 0) = v_0(\mathbf{s}), \quad \mathbf{s} \in \mathbb{R}^n. \end{gathered}$$

The purpose of this section is to find a functon $V(\mathbf{s}, t)$ satisfying the equations

$$V_t - \Delta V + cV_{s_n} - f(V,t) = 0, \quad \mathbf{s} \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{4.1}$$

$$V(\mathbf{s}, t) = V(\mathbf{s}, t+T), \quad \mathbf{s} \in \mathbb{R}^n,\ t \in \mathbb{R}. \tag{4.2}$$

Let $l \geq 3$ be a given integer and $\{\mathbf{A}_j\}_{j=1}^l \subset \mathbb{R}^n$ be a set of unit vectors such that $\{\mathbf{A}_i \neq \mathbf{A}_j\}$, if $i \neq j$. Then $\mathbf{A}_j = (A_{1,j}, A_{2,j}, \dots, A_{n-1,j})$ satisfies

$$|\mathbf{A}_j| = \sum_{i=1}^{n-1} A_{i,j}^2 = 1, \quad j = 1, 2, \dots, l.$$

Therefore $(m_* \mathbf{A}_j, 1) \in \mathbb{R}^n$ is a normal vector of $\{\mathbf{s} \in \mathbb{R}^n | -s_n = m_*(\mathbf{A}_j, \mathbf{s}')\}$, where $(\mathbf{A}_j, \mathbf{s}') = \sum_{i=1}^{n-1} A_{i,j} s_i$. Let

$$\begin{gathered} h_j(\mathbf{s}') = m_*(\mathbf{A}_j, \mathbf{s}'), \quad 1 \leq j \leq l, \\ h(\mathbf{s}') = \max_{1 \leq j \leq l} h_j(\mathbf{s}') = m_* \max_{1 \leq j \leq l} (\mathbf{A}_j, \mathbf{s}'), \end{gathered}$$

then $\{\mathbf{s} \in \mathbb{R}^n | -s_n = h(\mathbf{s}')\}$ is a pyramid in $\mathbb{R}^n$. Similar to the previous works, we define $\Omega_j, G_j, \Gamma_j, \mathcal{D}(\gamma), E$ as in Section 1 by replacing (x_1, x_2) and (x_1, x_2, x_3) with $\mathbf{s}'$ and $\mathbf{s}$, respectively. Let $\partial\Omega_j$ be the boundary of Ω_j. For any $1 \leq j \leq l$, it's obvious that $\Psi(\frac{c_*}{c}(s_n + h_j(\mathbf{s}')), t)$ is the solution of (4.1). Define

$$\underline{\psi}(\mathbf{s}, t) = \Psi\big(\frac{c_*}{c}(s_n + h(\mathbf{s}')), t\big) = \max_{1 \leq j \leq l} \Psi\big(\frac{c_*}{c}(s_n + h_j(\mathbf{s}')), t\big),$$

then $\underline{\psi}(\mathbf{s}, t)$ is the subsolution of (4.1).

Let function $\tilde{\rho}(r) \in C^\infty[0, \infty)$ satisfy the following properties:

(1) $\tilde{\rho}(r) > 0$, $\tilde{\rho}_r(r) \leq 0$, $r \geq 0$;
(2) If $r > 0$ is small enough, $\tilde{\rho}(r) = 1$;
(3) If $r > 0$ is large enough, say $r > R_0$, $\tilde{\rho}(r) = e^{-r}$, where $R_0 > 1$ is a constant;
(4) $\int_{\mathbb{R}^{n-1}} \tilde{\rho}(|\mathbf{s}'|)\, d\mathbf{s}' = 1$.

It is obvious that

$$\int_{\mathbb{R}^{n-1}} \tilde{\rho}(|\mathbf{s}'|) d\mathbf{s}' = \frac{(n-1)\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})} \int_0^\infty r^{n-2} \tilde{\rho}(r) dr.$$

Let $\rho(\mathbf{s}') = \tilde{\rho}(|\mathbf{s}'|)$, one has

$$\int_{\mathbb{R}^{n-1}} \rho(\mathbf{s}') d\mathbf{s}' = 1.$$

For all nonnegative integers $j_1, \dots, j_{n-1}$ satisfying $0 \leq \sum_{q=1}^{n-1} j_q \leq 3$, we have

$$|\mathcal{D}_1^{j_1} \dots \mathcal{D}_{n-1}^{j_{n-1}} \rho(\mathbf{s}')| \leq M_* \rho(\mathbf{s}'), \quad \mathbf{s}' \in \mathbb{R}^{n-1},$$

where M_* is a positive constant. Define $\bar{\varphi}(\mathbf{s}') = \rho * h$, then for each $\mathbf{s}' \in \mathbb{R}^{n-1}$,

$$\bar{\varphi}(\mathbf{s}') = \int_{\mathbb{R}^2} \rho(\mathbf{s}'') h(\mathbf{s}' - \mathbf{s}'') d\mathbf{s}'' = \int_{\mathbb{R}^2} \rho(\mathbf{s}' - \mathbf{s}'') h(\mathbf{s}'') d\mathbf{s}''. \tag{4.3}$$

The set $\{\mathbf{s} \in \mathbb{R}^n | -s_n = \bar{\varphi}(\mathbf{s}')\}$ is called the mollified pyramid of $\{\mathbf{s} \in \mathbb{R}^n | -s_n = h(\mathbf{s}')\}$. Let

$$G(\mathbf{s}') = \frac{c}{\sqrt{1 + |\nabla\bar{\varphi}(\mathbf{s}')|^2}} - c_*, \tag{4.4}$$

where $|\nabla\bar{\varphi}(\mathbf{s}')| = \sqrt{\sum_{i=1}^{n-1} \bar{\varphi}_{s_i}^2(\mathbf{s}')}$. The next two lemmas come from the [15, Lemma 2.2 and Prop. 2.3] and from [28, Remark 2.3].

Lemma 4.1. *Let $\bar{\varphi}(\mathbf{s}')$ and $G(\mathbf{s}')$ be as defined in (4.3) and (4.4) respectively. Then for any fixed $(j_1, \dots, j_{n-1}) \neq (0, \dots, 0)$ with $j_q \geq 0$ $(q = 1, \dots, n-1)$, one has*

$$\sup_{\mathbf{s}' \in \mathbb{R}^{n-1}} |\mathcal{D}_{s_1}^{j_1} \mathcal{D}_{s_2}^{j_2} \dots, \mathcal{D}_{s_{n-1}}^{j_{n-1}} \bar{\varphi}(\mathbf{s}')| < \infty,$$

$$h(s_1, s_2, \dots, s_{n-1}) < \bar{\varphi}(\mathbf{s}') \leq h(\mathbf{s}') + \frac{(n-1)\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})} m_* \int_0^\infty r^{n-1} \tilde{\rho}(r) dr,$$

$$|\nabla\bar{\varphi}(\mathbf{s}')| < m_*, \quad 0 < G(\mathbf{s}') \leq c - c_*, \quad \forall \mathbf{s}' \in \mathbb{R}^{n-1}.$$

Lemma 4.2. *There exist two constants b_1 and b_2 such that*

$$0 < b_1 = \inf_{\mathbf{s}' \in \mathbb{R}^{n-1}} \frac{\bar{\varphi}(\mathbf{s}') - h(\mathbf{s}')}{G(\mathbf{s}')} \leq \sup_{\mathbf{s}' \in \mathbb{R}^{n-1}} \frac{\bar{\varphi}(\mathbf{s}') - h(\mathbf{s}')}{G(\mathbf{s}')} = b_2 < \infty.$$

Moreover, for every integer $j_q \geq 0$ $(q = 1, \dots, n-1)$ with $2 \leq j_1 + j_2 + \dots + j_{n-1} \leq 3$, there exists a constant $\mathcal{K} > 0$ such that

$$\sup_{\mathbf{s}' \in \mathbb{R}^{n-1}} \Big| \frac{\mathcal{D}_{s_1}^{j_1} \mathcal{D}_{s_2}^{j_2} \dots \mathcal{D}_{s_{n-1}}^{j_{n-1}} \bar{\varphi}(\mathbf{s}')}{G(\mathbf{s}')} \Big| < \mathcal{K},$$

$$|\bar{\varphi}_{s_i s_i}(\mathbf{s}')| \leq m_* M_*, \quad i = 1, 2, \dots, n-1, \ \mathbf{s}' \in \mathbb{R}^{n-1}.$$

Proceeding as in the previous sections, we obtain the following lemma and theorem.

Lemma 4.3. *For each $\beta \in (0,1)$, there exist positive constants $\varepsilon_0^+(\beta)$ and $\alpha_0^+(\beta, \varepsilon)$ such that, for any $0 < \varepsilon < \varepsilon_0^+(\beta)$ and $0 < \alpha < \alpha_0^+(\beta, \varepsilon)$, the function*

$$\overline{\psi}(\mathbf{s}, t; \beta, \varepsilon, \alpha) = \Psi(\varrho(\mathbf{s}), t) + \varepsilon\sigma(\mathbf{s}')(\omega(\varpi(\mathbf{s}))\nu(t) + (1 - \omega(\varpi(\mathbf{s})))\Psi^\beta(\varpi(\mathbf{s}), t))$$

is a supersolution of (4.1)-(4.2) on $\mathbb{R}^n \times (-\infty, +\infty)$. In addition,

$$\lim_{\bar{\gamma} \to \infty} \sup_{\mathbf{s} \in \mathcal{D}(\bar{\gamma}), t \in [0,T]} \frac{|\overline{\psi}(\mathbf{s}, t; \beta, \varepsilon, \alpha) - \underline{\psi}(\mathbf{s}, t)|}{\underline{\psi}(\mathbf{s}, t)^\beta} \leq 2\varepsilon,$$

$$\underline{\psi}(\mathbf{s}, t) < \overline{\psi}(\mathbf{s}, t; \beta, \varepsilon, \alpha), \quad (\mathbf{s}, t) \in \mathbb{R}^n \times [0, T],$$

$$\overline{\psi}_{s_n}(\mathbf{s}, t; \beta, \varepsilon, \alpha) > 0, \quad (\mathbf{s}, t) \in \mathbb{R}^n \times [0, T].$$

Theorem 4.4. *Assume that (H1) and (H2) hold. Then for each $c > c_*$, equation (1.1) has a periodic nonplanar traveling front $V(\mathbf{s}, t)$ satisfying (4.1)-(4.2). Moreover,*

$$\lim_{\bar{\gamma} \to \infty} \sup_{\mathbf{s} \in \mathcal{D}(\bar{\gamma}), t \in [0,T]} \frac{|V(\mathbf{s}, t) - \underline{\psi}(\mathbf{s}, t)|}{(\underline{\psi}(\mathbf{s}, t))^\beta} = 0, \quad \forall \beta \in (0, 1),$$

$$V_{s_n}(\mathbf{s}, t) > 0, \quad (\mathbf{s}, t) \in \mathbb{R}^n \times \mathbb{R}.$$

5. Conclusion

Since the environment changes over time, it is of great practical significance to study the effect of time period on the dynamical behavior of reaction-diffusion equations. In this paper, we mainly consider the time periodic reaction-diffusion equation with degenerate monostable nonlinearity. We prove the existence of periodic pyramidal traveling fronts in $\mathbb{R}^n$ with $n \geq 3$. Due to the degeneration at the equilibrium point 0 and $f(u,t) = f(u,t+T) > 0$ on $(0,1) \times \mathbb{R}$, the dynamical properties of degenerate monostable periodic nonlinearity are essentially different from the bistable and combustion nonlinear terms. For the purpose of obtaining the existence of nonplanar traveling fronts, we use the super-sub solution method combined with comparison principle. It is worth noting that we adopt the method of adding small perturbation to the planar traveling front to overcome the difficulties in constructing the supersolution. Our results enrich the traveling front theory of degenerate monostable reaction-diffusion equation with time period.

Except for pyramidal traveling fronts, one expects that there exist conical-shaped traveling fronts and other kinds of nonplanar traveling fronts. These are interesting problems to study in the future.

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