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ASYMPTOTIC BEHAVIOR OF STOCHASTIC FUNCTIONAL DIFFERENTIAL EVOLUTION EQUATION

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Abstract. In this work we study the long time behavior of nonlinear stochastic functional-differential equations in Hilbert spaces. In particular, we start with establishing the existence and uniqueness of mild solutions. We proceed with deriving a priory uniform in time bounds for the solutions in the appropriate Hilbert spaces. These bounds enable us to establish the existence of invariant measure based on Krylov-Bogoliubov theorem on the tightness of the family of measures. Finally, under certain assumptions on nonlinearities, we establish the uniqueness of invariant measures.

1. Introduction

In this work we study the asymptotic behavior of the solutions of stochastic functional-differential equations. In a bounded domain, the equation reads as

$$\begin{gathered} du = [Au + f(u_t)]\,dt + \sigma(u_t)\,dW(t) \quad \text{in } D, t > 0; \\ u(t,x) = \phi(t,x), t \in [-h,0), u(0,x) = \varphi_0(x) \quad \text{in } D; \\ u(t,x) = 0, x \in \partial D, \quad t \geq 0. \end{gathered} \tag{1.1}$$

The corresponding problem in the entire space has the form

$$\begin{gathered} du = [Au + f(u_t)]\,dt + \sigma(u_t)\,dW(t) \quad \text{in } \mathbb{R}^d, t > 0; \\ u(t,x) = \phi(t,x), t \in [-h,0), u(0,x) = \varphi_0(x) \quad \text{in } \mathbb{R}^d. \end{gathered} \tag{1.2}$$

Here A is the elliptic operator

$$A = A(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} + c(x), \tag{1.3}$$

the interval $[-h, 0]$ is the interval of delay, and $u_t = u(t + \theta)$ with $\theta \in [-h, 0]$.

Functional differential equations of types (1.1) and (1.2) are mathematical models of processes, the evolution of which depends on the previous states. One of the natural examples of such behavior is heat conduction. In particular, the classic model of heat conduction $u_t = \Delta u$ has an essential shortcoming: it predicts infinite speed of propagation of thermal fluctuations in Fourier heat conductors. This

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observation suggests that the Fourier's law of heat conduction may be an approximation to a more general constitutive assumption relating the heat conduction to the material's thermal history. Gurtin and Pipkin [14] have proposed a memory theory of heat conduction, which has finite heat propagation speeds, and, in its linearized version reads as

$$\dot{u}(x,t) + \int_0^\infty \beta(s)\dot{u}(x,t-s)\,ds = C\Delta u(x,t),$$

which is a particular example of a functional-differential equation. Furthermore, [25] provides an example of temperature regularization through heat injection or extraction, controlled by a thermostat, which creates additional memory and delay effects. A closely related problem arises emerges in modeling partially diffused population dynamics with delay in the birth process [25]

$$\dot{u}(x,t) - Cu_{xx}(t,x) = u(t,x)\Big[1 - u(t,x) - \int_{-1}^0 u(t+r(s),x)\,ds\Big],$$

where $r(s)$ is a continuous delay function. In [27] we used a functional-differential equation to take into account the delay effects in modeling Performance-on-Demand Micro-electromechanical systems (POD MEMS). Similar memory effects emerge in Hodgkin-Huxley model, Dawson-Fleming model of population genetics [11], among others.

The classic results for deterministic functional-differential equations in finite dimensional spaces can be found in [13] and references therein. Stochastic functional differential equation in finite dimensions have be studies extensively as well. In particular, the existence of invariant measures for stochastic ordinary differential equations was established in [3, 12]. The work [15] addressed the stochastic stability, as well as various applications of stochastic delay equations in finite dimensions.

The results on functional differential equations in infinite dimensions are significantly more sparse. One example of analysis and applications of functional partial differential equations may be found in [1]. In this work, the authors study the nonlocal reaction-diffusion model of population dynamics. They establish the existence of time stationary solution and show that all other solutions converge to it.

The results on stochastic functional differential equations include [29, 8], which establish the existence of solutions and their stability. Stochastic differential equation of neutral type were studied in [26, 16, 31]. The work [28] established the comparison principle for such equations.

The main goal of the present work is to establish the existence and uniqueness of invariant measures for the equations (1.1) and (1.2) based on Krylov-Bogoliubov theorem on the tightness of the family of measures [17]. More precisely, we will use the compactness approach of Da Parto and Zabczyk [9], which involves the following key steps:

- (i) Establishing the existence of a Markovian solution of (1.1) or (1.2) in a certain functional space, in which the corresponding transition semigroup is Feller;
- (ii) Showing that the semigroup $S(t)$ generated by A is compact;
- (iii) Showing that the corresponding equation with a suitable initial condition has a solution, which is bounded in probability.

This approach was used in establishing the existence of invariant measure for a large class of stochastic nonlinear partial differential equations without delay, e.g.

[2, 5, 7, 10, 21, 22] and references therein. For functional differential equations in finite dimensions, the approach above was used in [6]. In this work, the author established the existence of an invariant measure in $\mathbb{R}^d \times L^2(-h,0;\mathbb{R}^d)$. In contrast, for stochastic partial differential equations, the natural phase space for the mild solutions of (1.2) is $L^2_\rho(\mathbb{R}^d) \times L^2(-h,0;L^2_\rho(\mathbb{R}^d))$, where $L^2_\rho(\mathbb{R}^d)$ is a weighted space. The equations of type (1.1) and (1.2) were studied in the space $C([-h,0];L^2_\rho(\mathbb{R}^d))$, which is a significantly easier problem [26, 28, 29]. In these spaces the authors studied the conditions for the existence and uniqueness of the solution, as well as their Markov's and Feller properties. However, in order to apply the compactness approach one needs to work in $L^2_\rho(\mathbb{R}^d) \times L^2(-h,0;L^2_\rho(\mathbb{R}^d))$, which is done in this work. We also establish the existence and uniqueness of the stationary solution, and the convergence of other solutions to it in square mean, which is the stochastic analog of the main result of [1].

This article is structured as follows. In Section 2 we introduce the notation and formulate the main results. Section 3 is devoted to the proof of the existence of invariant measure, as well as an example of application of this result to integral-differential equations. Section 4 establishes the uniqueness of invariant measure, and the convergence to the stationary solution.

2. Preliminaries and main results

Throughout this article, the domain D is either a bounded domain with ∂D satisfying the Lyapunov condition, or $D = \mathbb{R}^d$. Denote

$$\rho(x) := \frac{1}{1+|x|^r} \tag{2.1}$$

where $r > d$ if $D = \mathbb{R}^d$ and $r = 0$ (i.e. no weight) for bounded D. We introduce the following spaces:

$$B_0^\rho := L^2_\rho(D), \quad B_1^\rho := L^2(-h,0;L^2_\rho(D)), \quad B^\rho := B_0^\rho \times B_1^\rho, \quad H := L^2(D), \tag{2.2}$$

with the norms

$$\|u\|^2_{B^0_\rho} := \|u(\cdot)\|^2_\rho := \int_D u^2(x)\rho(x)\,dx,$$
$$\|u(\theta,\cdot)\|^2_{B^1_\rho} := \int_{-h}^0 \int_D u^2(\theta,x)\rho(x)\,dx\,d\theta,$$
$$\|(u(\cdot),u_1(\theta,\cdot))\|^2_{B^\rho} = \|u(x)\|^2_\rho + \|u_1(\theta,x)\|^2_{B^1_\rho},$$
$$\|u(\cdot)\|^2_H = \int_D u^2(x)\,dx.$$

The coefficients a_{ij} of the operator A defined in (1.3) are Holder continuous with the exponent $\beta \in (0,1)$, symmetric, bounded and satisfying the elipticity condition

$$\sum_{i,j=1}^d a_{i,j}\eta_i\eta_j \ge C_0|\eta|, \quad \forall \eta \in \mathbb{R}^d$$

for some $C_0 > 0$. The coefficients b_i and c are also bounded and Holder continuous with some positive Holder exponent.

If D is bounded, we impose homogeneous Dirichlet boundary conditions on ∂D. In this case,

$$D(A) = H^2(D) \cap H^1_0(D).$$

If $D=\mathbb{R}^d$, then $D(A)=H^2(\mathbb{R}^d)$. Denote $G(t,x,y)$ to be the fundamental solution (or the Green's function in the case of bounded D) for $\frac{\partial}{\partial t}-A$. It follows from, e.g., [18, p. 468], that there are positive constants $C_1(T),C_2(T)>0$ such that

$$0\le G(t,x,y)\le C_1(T)t^{-d/2}e^{-C_2(T)\frac{|x-y|^2}{t}} \tag{2.3}$$

for $t\in[0,T]$ and $x,y\in D$. Note that in (2.3), C_1 and C_2 depend not only on T, but on the constants C_0, d, T, maximum values of the coefficients of A, and the Holder constants. If the operator is in the divergence form $Au=\operatorname{div}(\mathrm{a}\nabla \mathrm{u})$, the estimates are of a different type, see e.g. [17], namely

$$g_1(t,x-y)\le G(t,x,y)\le g_2(t,x-y), \tag{2.4}$$

where

$$g_i(t,x)=K(C_0,d)t^{-d/2}e^{-K(C_0,d)\frac{|x|^2}{t}},\quad i=1,2,\ t\ge 0,\ x,y\in\mathbb{R}^d.$$

In this case, in contrast with (2.3), the constant $K(C_0,d)$ is independent of t.

Lemma 2.1. *For each $T>0$ there exists a positive $C(r,T)>0$ such that*

$$\int_D G(t,x,y)\rho(y)\,dy\le C(r,T)\rho(x),\quad t\in[0,T]. \tag{2.5}$$

Proof. Note that the weight (2.1) satisfies

$$\frac{\rho(x)}{\rho(y)}\le C(r)(1+|x-y|^r) \tag{2.6}$$

for some $C(r)>0$. Thus

$$\begin{aligned}\int_D G(t,x,y)\rho(y)\,dy &\le C(r)\int_D G(t,x,y)\rho^{-1}(x-y)\rho(x)\,dy\\ &\le C(r)C_1(T)\int_{\mathbb{R}^d} t^{-d/2}e^{-C_2(T)\frac{|y|^2}{t}}(1+|y|^r)\,dy\rho(x)\\ &\le C(r,T)\rho(x).\end{aligned}$$

□

We define

$$(S(t)\varphi)(x):=\int_D G(t,x,y)\varphi(y)\,dy,\quad t>0,x\in D,\varphi\in L^2(D), \tag{2.7}$$

and $S(0)=I$, where I is the identity map. This is a semigroup on $L^2(D)$ with generator A. Then for all $\varphi\in L^2(D)$ and for $t\in[0,T]$ by Lemma 2.1 we have

$$\begin{aligned}\|S(t)\varphi\|^2_{B^0_\rho} &= \int_D\Big(\int_D G(t,x,y)\varphi(y)\,dy\Big)^2\rho(x)\,dx\\ &\le \int_D\rho(x)\Big(\int_D G(t,x,y)\,dy\Big)\Big(\int_D G(t,x,y)\varphi^2(y)\,dy\Big)\,dx\\ &\le C\int_D\Big(\int_D G(t,x,y)\frac{\rho(x)}{\rho(y)}\,dx\Big)\rho(y)\varphi^2(y)\,dy\\ &\le C_\rho(T)\|\varphi\|^2_{B^0_\rho}.\end{aligned} \tag{2.8}$$

The above estimate allows the semigroup $S(t)$ to be extended to a linear map from B^ρ_0 to itself. Since $L^2(D)$ is dense in B^ρ_0, $S(t)$ is strongly continuous in B^ρ_0.

Let $a_i\ge 0$, $\sum_{i=1}^\infty a_i<\infty$, and e_n be orthonormal basis in H, such that $e_n\in L^\infty(D)$ and $\sup_n\|e_n\|_{L^\infty(D)}<\infty$. We introduce the operator $Q\in\mathcal{L}(H)$ such that Q is non-negative, $Tr(Q) < \infty$, $Qe_n = a_n e_n$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. We introduce

$$W(t) := \sum_{i=1}^{\infty} \sqrt{a_i}\beta_i(t)e_i(x),\ t \geq 0,$$

which is a Q-Wiener process on $t \geq 0$ with values in $L^2(Q)$. Here $\beta_i(t)$ are standard, one dimensional, mutually independent Wiener processes. Also let $\{F_t, t \geq 0\}$ be a normal filtration satisfying

- $W(t)$ is $\mathcal{F}_t$-measurable;
- $W(t+h) - W(t)$ is independent of $\mathcal{F}_t$ for all $h \geq 0, t \geq 0$.

Denote $U = Q^{\frac{1}{2}}(H)$. From [19, Lemma 2.2], $U \in L^\infty(D)$. Following [19] introduce the multiplication operator $\Phi : U \to B_0^\rho$ as follows: for a fixed $\varphi \in B_0^\rho$, let $\Phi(\psi) := \varphi\psi,\ \psi \in U$. Since $\varphi \in B_0^\rho$ and $\varphi \in L^\infty(D)$, the operator is well defined and hence $\Phi \circ Q^{1/2} : L^2(D) \to B_0^\rho$ defines a Hilbert-Schmidt operator. The operator Φ is also a Hilbert-Schmidt operator satisfying

$$\begin{aligned} \|\Phi \circ Q^{1/2}\|_{\mathcal{L}_2}^2 &:= \sum_{n=1}^{\infty} \|\Phi \circ Q^{1/2} e_n\|_{B_0^\rho}^2 \\ &= \sum_{n=1}^{\infty} a_n \int_D \varphi^2(x) e_n^2(x)\rho(x)\,dx \\ &\leq \operatorname{Tr}(Q) \sup_n \|e_n\|_\infty^2 \|\varphi\|_\rho^2, \end{aligned} \tag{2.9}$$

where $\operatorname{Tr}(Q) = \sum_{n=1}^{\infty} a_n = a$. Hence if $\Phi : \Omega \times [0,T] \to \mathcal{L}(U, B_0^\rho)$ is a predictable process satisfying

$$\mathbb{E}\int_0^T \|\Phi \circ Q^{1/2}\|_{\mathcal{L}_2}^2\,ds < \infty,$$

following [9] we can define

$$\int_0^t \Psi(s)\,dW(s) \in B_0^\rho$$

with the expansion

$$\int_0^t \Psi(s)\ dW(s) = \sum_{i=1}^{\infty} \sqrt{a_i} \int_0^t \Phi(s,\cdot)e_i(\cdot)\,d\beta_i(s).$$

Furthermore,

$$\mathbb{E}\Big\|\int_0^t \Psi(s)\,dW(s)\Big\|_{B_\rho^0}^2 \leq a \sup_n \|e_n\|_\infty^2 \int_0^t \mathbb{E}\|\Psi(s,\cdot)\|_{B_0^\rho}^2\,ds. \tag{2.10}$$

We assume f and σ satisfy the following conditions:

(i) The functionals f and σ map B_1^ρ to B_0^ρ,
(ii) There exists a constant $L > 0$ such that

$$\|f(\varphi_1) - f(\varphi_2)\|_{B_0^\rho} + \|\sigma(\varphi_1) - \sigma(\varphi_2)\|_{B_0^\rho} \leq L\|\varphi_1 - \varphi_2\|_{B_1^\rho}$$

for any $\varphi_1, \varphi_2 \in B_1^\rho$.

Definition 2.2. An $\mathcal{F}_t$ measurable random process $u(t,\cdot) \in B_0^\rho$ is a mild solution of (1.1) or (1.2), if

$$u(t,\cdot) = S(t)\varphi(0,\cdot) + \int_0^t S(t-s)f(u_s)\,ds + \int_0^t S(t-s)\sigma(u_s)\,dW(s) \tag{2.11}$$

where $u(0,\cdot) = \varphi(0,\cdot) \in B_0^\rho$, $u(t,\cdot) = \varphi(t,\cdot) \in B_1^\rho$, $t \in [-h,0]$.

Hence the phase space of the problem is the Hilbert space B^ρ. In this case $y(t) \in B^\rho$ if $y(t) = (u(t,\cdot), u_t) \in B_0^\rho \times B_1^\rho$, with $u_t = u(t+\theta,\cdot)$ and $\theta \in [-h,0]$.

Theorem 2.3 (Existence and uniqueness)**.** *Suppose f and σ satisfy the conditions (i) and (ii), and $\varphi(t,\cdot)$ is an $\mathcal{F}_0$ measurable random process for $t \in [-h,0]$, which is independent of W and such that*

$$\mathbb{E}\|\varphi(0,\cdot)\|_{B_0^\rho}^p < \infty, \quad \mathbb{E}\|\varphi(\cdot,\cdot)\|_{B_1^\rho}^p < \infty, \quad p \ge 2.$$

Then there exists a unique mild solution of (1.1) *(or 1.2) on* $[0,T]$*, and*

$$\mathbb{E}\|y(t)\|_{B^\rho}^p \le K(T)(1 + \mathbb{E}\|y(0)\|_{B^\rho}^p), \quad t \in [0,T]. \tag{2.12}$$

Theorem 2.4 (Continuous dependence on the initial data)**.** *Let $\phi \in B_1^\rho$, $\phi(0,\cdot) \in B_0^\rho$, $\phi_1 \in B_1^\rho$, $\phi_1(0,\cdot) \in B_0^\rho$ be two initial sets of data of two solutions*

$$y(t) = y(t,\phi) = \begin{pmatrix} u(t,\phi) \\ u_t(\phi) \end{pmatrix}, \quad y_1(t) = y(t,\phi_1) = \begin{pmatrix} u(t,\phi_1) \\ u_t(\phi_1) \end{pmatrix}$$

respectively. Then under the conditions of Theorem 2.3 there exists a constant $C(T)$ such that

$$\sup_{t\in[0,T]} \mathbb{E}\|y(t) - y_1(t)\|_{B^\rho}^2 \le C(T)\mathbb{E}\|\phi(t) - \phi_1(t)\|_{B^\rho}^2. \tag{2.13}$$

The following proposition shows the that the solution $u(t,\cdot)$ has continuous trajectories.

Proposition 2.5. *Let $u(t,\cdot)$ be a mild solution of* (1.1) *or* (1.2)*. Then, under the conditions of Theorem 2.3, u_t is continuous at $t = 0$ in probability with respect to the norm $\|\cdot\|_{B_1^\rho}$, i.e.*

$$\|u_t - u_0\|_{B_1^\rho}^2 = \int_{-h}^0 \mathbb{E}\|u(t+\theta) - \varphi(\theta)\|_{B_0^\rho}^2\,d\theta \to^P 0, \quad t \to 0.$$

Proof. Note that

$$\mathbb{E}\|u_t - u_0\|_{B_1^\rho}^2 \le \int_{-h}^{-t} \mathbb{E}\|\varphi(t+\theta) - \varphi(\theta)\|_{B_0^\rho}^2\,d\theta + \int_{-t}^0 \mathbb{E}\|u(t+\theta) - \varphi(\theta)\|_{B_0^\rho}^2\,d\theta.$$

The convergence of the first term to 0 follows from the density of $C([-h,0], B_0^\rho \times L^2(\Omega))$ in $L^2([-h,0], B_0^\rho \times B_0^\rho \times L^2(\Omega))$. The second term converges to zero as $t \to 0$ since the integrand is bounded. □

Let $B_b(B^\rho)$ be the Banach space of bounded real Borel functions from B^ρ to $\mathbb{R}$, and $C_b(B^\rho)$ be the space of bounded continuous functions. Since the choice of $T > 0$ in Theorem 2.3 is arbitrary, the solution exists for all $t \ge 0$, thus $y(t)$ also exists for all $t \ge 0$. Replacing the initial interval $[-h,0]$ with $[-h+s,s]$ for all $s \ge 0$, we can guarantee the existence and uniqueness of the solutions for $t \ge s \ge 0$

with the initial $\mathcal{F}_s$-measurable functions $\varphi(\theta,\cdot), \varphi(0,\cdot)$, which satisfy the conditions of Theorem 2.3 on $[s-h,s]$. This solution will be denoted with $u(t,s,\varphi)$. Similarly,

$$u_t(s,\varphi) = u(t+\theta,s,\varphi), \quad \theta \in [-h,0]$$

is a shift of the solution $u(t,\varphi)$, such that $u_s(s,\varphi) = u(s+\theta,s,\varphi) = \varphi(\theta)$ and for $\theta = 0$, $\varphi(0,\cdot) \in B_0^\rho$.

Following [4], we define the family of shift operators

$$U_s^t\varphi := u(t+\theta,s,\varphi) = u_t(s,\varphi). \tag{2.14}$$

Let $\mathcal{F}_s^t(dW)$ be the minimal σ-algebra containing $W(\tau) - W(s), \tau \in [s,t]$. Note that $u_t(s,\varphi)$ is independent of the σ-algebra G^t, which is the minimal sigma-algebra containing $W(\tau) - W(t)$ for $\tau \geq t$.

For any nonrandom $\varphi \in B^\rho$ with $s \geq 0$ and $t \geq s$, $U_s^t\varphi := u_t(s,\varphi)$ is an $\mathcal{F}_s^t(dW)$ measurable random function taking values in B_1^ρ, with $u(t,s,\varphi) \in B_0^\rho$ for $\theta = 0$. Defining $y(t,s,\varphi) = (u(s,t,\varphi), u_t(s,\varphi))$, we have that y maps B^ρ into itself. The next proposition follows from Theorem 2.3.

Proposition 2.6. *The family of the operators* (2.14) *satisfies*

$$U_\tau^t U_s^\tau \varphi = U_s^t\varphi \tag{2.15}$$

for all $t \geq \tau \geq s \geq 0$ and $\varphi \in B^\rho$.

Let D be a σ-algebra of Borel subset of B^ρ. Then $y(t,s,\varphi)$ naturally denotes the following probability measure μ_t on D,

$$\mu_t(A) = P\{y(t,s,\varphi) \in A\} = P\{U_s^t\varphi \in A\} = P(s,\varphi,t,A) \tag{2.16}$$

The measure μ is the transition function corresponding to the random process $y(t,s,\varphi)$. In a similar way as in the finite dimensional case [4], one can show that this function satisfies the properties of the transition probability. This way we have

Theorem 2.7 (Markov property)**.** *Under the assumptions of Theorem 2.3, the process $y(t,s,\varphi) \in B^\rho$ is the Markov process on B^ρ with the transition function $P(s,\varphi,t,A)$ given by* (2.16).

Proposition 2.8. *For any $t \geq s \geq 0$ we have*

$$P(s,\varphi,t,A) = P(0,\varphi,t-s,A)$$

Proof. Let $\tilde{u}(t) = u(s+t,s,\varphi)$. Then $\tilde{u}(0) = \varphi(0,\cdot)$ and $\tilde{u}_0 = u(s+\theta,s,\varphi) = \varphi(\theta,\cdot)$. On the other hand,

$$\begin{aligned} \tilde{u}(t) = u(s+t,s,\varphi) &= S(t)\varphi(0,\cdot) + \int_s^{s+t} S(s+t-\tau)f(u_\tau)\,d\tau \\ &\quad + \int_s^{s+t} S(s+t-\tau)\sigma(u_\tau)\,dW(\tau) = S(t)\varphi(0,\cdot) \\ &\quad + \int_0^t S(t-\tau)f(u_{\tau+s})\,d\tau + \int_0^t S(t-\tau)\sigma(u_{\tau+s})d\tilde{W}(\tau) \end{aligned}$$

where $\tilde{W}(\tau) := W(s+\tau) - W(s)$ is once again a Q-Wiener process. This way $\tilde{u}$ solves

$$\tilde{u}(t) = S(t)\varphi(0,\cdot) + \int_0^t S(t-\tau)f(\tilde{u}(\tau))\,d\tau + \int_0^t S(t-\tau)\sigma(\tilde{u}(\tau))d\tilde{W}(\tau) \tag{2.17}$$

The same equation is satisfied with $u(t,0,\varphi)$ such that $u(0,0,\varphi)=\varphi(0,\cdot)$ and $u_0=\varphi(\theta,\cdot)$. The only difference is that $u(t,0,\varphi)$ solves (2.17) with a different Wiener process W. However, since the distribution of W is the same as $\tilde W$, the distribution of $u(s+t,s,\varphi)$ is the same as the distribution of $u(t,0,\varphi)$, and hence independent of s. Thus the distribution of $u_t(s,\varphi)=u(t+\theta,s,\varphi)=u(t-s+\theta+s,s\varphi)$ coincides with the distribution of $u(t-s+\theta,0,\varphi)=u_{t-s}(0,\varphi)$. Hence

$$\begin{aligned} P(s,\varphi,t,A) &= P\{u_t(s,\varphi)\in A\} = P\{u(t+\theta,s,\varphi)\in A\} \\ &= P\{u(t-s+\theta,0,\varphi)\in A\} = P\{u_{t-s}(0,\varphi)\in A\} \end{aligned}$$

which yields the desired result. □

For $g\in B_b(B^\rho)$, for $\varphi\in B^\rho$ and $t\ge s\ge 0$, we define

$$P_{s,t}(\varphi) := \mathbb{E}g(y(t,s,\varphi)).$$

From proposition 2.8 we have $P_{0,t-s}(\varphi)$ and denote $P_t\varphi = P_{0,t}(\varphi)$. From Theorem 2.4 and Proposition 2.5 we have the following result.

Proposition 2.9. *Under the assumptions of Theorem 2.3 the transition semigroup $P_t, t\ge 0$ is stochastically continuous an satisfies the Feller property*

$$P_t : C_b(B^\rho)\to C_b(B^\rho),\quad \lim_{t\to 0} P_t\varphi(\theta)=\varphi(\theta).$$

We define $\bar\rho(x)=(1+|x|^{\bar r})^{-1}$. The main result of the paper is the following theorem.

Theorem 2.10. *Let the assumptions of Theorem 2.3 hold. Assume the equation (2.11) has a solution in $B^{\bar\rho}$ which is bounded in probability for $t\ge 0$ with*

$$r > d+\bar r. \tag{2.18}$$

Then there exists an invariant measure μ on B^ρ, i.e.

$$\int_{B^\rho} P_t\varphi(x)d\mu(x) = \int_{B^\rho}\varphi(x)d\mu,\quad \textit{for all } t\ge 0 \textit{ and } \varphi\in C_b(B^\rho).$$

Remark 2.11. Condition (2.18) is equivalent to

$$\int_{\mathbb{R}^d}\frac{\rho(x)}{\bar\rho(x)}\,dx<\infty.$$

The key condition in Theorem 2.10 is the existence of a globally bounded solution. The next theorem provides the sufficient conditions for the existence of such solution in terms of the coefficients, in the case when A is in the divergence form.

Theorem 2.12. *Assume*

- $D=\mathbb{R}^d, d\ge 3$;
- *the conditions of Theorem 2.3 hold;*
- *for some $\sigma_0>0$, we have $|\sigma(u)|\le\sigma_0$, for all $u\in B_1^\rho$;*
- *there exists $\Psi\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ such that $|f(u(\cdot))|\le\Psi(\cdot)$ for all $u\in B_1^\rho$;*
- $u(t,\cdot)=\varphi(t,\cdot), t\in[-h,0]$, $u(0,\cdot)=\varphi(0,x)$ *satisfy*

$$\mathbb{E}\int_{\mathbb{R}^d}|\varphi(0,x)|^2\,dx<\infty \textit{ and } \mathbb{E}\int_{\mathbb{R}^d}\int_{-h}^0|\varphi(\theta,x)|^2\,dx\,d\theta<\infty.$$

Then

$$\sup_{t\geq 0} \mathbb{E}\|y(t)\|^2_{B^\rho} < \infty,$$

which is a sufficient condition for the boundedness in probability.

Finally, for a bounded domain D we establish the uniqueness of the stationary solution as well as its stability. In this section, the weight $\rho \equiv 1$, thus

$$B_0 := L^2(D), \quad B_1 := L^2(-h, 0; B_0), \quad B := B_0 \times B_1.$$

The semigroup (2.7) now satisfies the exponential estimate

$$\|S(t)u_0\|^2_{B_0} \leq e^{-2\lambda_1 t}\|u_0\|^2_{B_0},$$

where $\lambda_1 > 0$ is the principle eigenvalue of $-A$. In a standard way, we can extend the Q-Weiner process $W(t)$ to $t \in \mathbb{R}$ as

$$W(t) = \begin{cases} W(t), t \geq 0; \\ V(-t), t \leq 0. \end{cases}$$

Here V is another Q-Weiner process, independent of W.

Definition 2.13. A B_0-valued process $u(t)$ is a mild solution of (1.1) for $t \in \mathbb{R}$ if

(1) for all $t \in \mathbb{R}$, $u(t)$ is $\mathcal{F}_t$ measurable;
(2) for all $t \in \mathbb{R}$
$$\mathbb{E}\|u(t)\|^2_{B_0} < \infty;$$
(3) for all $-\infty < t_0 < t < \infty$ with probability 1 we have
$$u(t) = S(t-t_0)u(t_0) + \int_{t_0}^t S(t-s)f(u_s)\,ds + \int_{t_0}^t S(t-s)\sigma(u_s)\,dW(s)$$

Theorem 2.14. *Assume the Lipschitz constant L is sufficiently small (see* (4.2) *for the exact condition), then equation* (1.1) *has a unique solution $u^*(t,x)$, defined for $t \in \mathbb{R}$, and*

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|u^*(t)\|^2_B < \infty.$$

Furthermore, this solution is exponentially attractive, that is exist $K, \gamma > 0$ such that for all $t_0 \in \mathbb{R}$ and $t > t_0 + h$, and for any other solution $\eta(t)$ with $\eta(t_0) \in B_0$ and $\eta_{t_0} \in B_1$ we have

$$\mathbb{E}\|u(\cdot,t) - \eta(\cdot,t)\|^2_B \leq Ke^{-\gamma(t-t_0)}\mathbb{E}\|u(\cdot,t_0) - \eta(\cdot,t_0)\|^2_B.$$

3. Proofs of main results

Proof of Theorem 2.3. Let $B_{p,T}$, $p \geq 2$ be the space of $\mathcal{F}_t$-measurable for $t \in [0,T]$ processes, equipped with the norm $\|\Phi\|^p_{B_{p,T}} := \mathbb{E}\int_{-h}^T \|\Phi(t,\cdot)\|^p_{B_0^\rho}\,dt$. We define

$$\begin{aligned} \Psi\Phi(t,\cdot) := S(t)\Phi(0,\cdot) + \int_0^t S(t-s)f(\Phi(s+\theta,\cdot))\,ds \\ + \int_0^t S(t-s)\sigma(\Phi(s+\theta,\cdot))\,dW(s) \end{aligned} \tag{3.1}$$

for $t \in [0,T]$, and

$$\Psi\Phi(t,\cdot) = \varphi(t,\cdot), t \in [-h,0], \quad \text{with } \Psi\Phi(0,\cdot) = \varphi(0,\cdot).$$

This way,

$$
\begin{aligned}
\Psi\Phi(t,\cdot)\|^p_{B_{p,T}} &\le \mathbb{E}\int_{-h}^0 \|\varphi(t,\cdot)\|^p_{B_0^\rho}\,dt + 3^{p-1}\mathbb{E}\int_0^T \|S(t)\varphi(0,\cdot)\|^p_{B_0^\rho}\,dt\\
&\quad + 3^{p-1}\mathbb{E}\int_0^T \Big\|\int_0^t S(t-s)f(\Phi(s+\theta,\cdot))\,ds\Big\|^p_{B_0^\rho}\,dt\\
&\quad + 3^{p-1}\mathbb{E}\int_0^T \Big\|\int_0^t S(t-s)\sigma(\Phi(s+\theta,\cdot))\,dW(s)\Big\|^p_{B_0^\rho}\,dt\\
&\le C_1(T) + 3^{p-1}(I_1+I_2+I_3).
\end{aligned}
$$

It follows from (2.8) that

$$
I_1 \le C^p_\rho(T)\int_0^T \mathbb{E}\|\varphi(0,\cdot)\|^p_{B_0^\rho}\,dt < \infty.
$$

Next, using the conditions (i) and (ii) for f, we have

$$
\begin{aligned}
I_2 &\le C^p_\rho(T)\int_0^T T^{p-1}\Big(\mathbb{E}\int_0^t \|f(\Phi_s,\cdot)\|^p_{B_0^\rho}\,ds\Big)\,dt\\
&\le C_2\int_0^T dt\int_0^t \Big(1+\mathbb{E}\|\Phi_s\|^p_{B_1^\rho}\Big)\,ds\\
&\le C_3 + C_2\int_0^T\int_0^t \mathbb{E}\Big(\int_{-h}^0 \|\Phi(s+\theta,\cdot)\|^2_{B_0^\rho}\,d\theta\Big)^{p/2}\,ds\,dt\\
&\le C_3 + C_4\mathbb{E}\int_{-h}^T \|\Phi(t,\cdot)\|^p_{B_0^\rho}\,dt < \infty.
\end{aligned}
\tag{3.2}
$$

To estimate I_3, we use [9, Lemma 7.2] and (2.10). Using the definition of Hilbert-Schmidt norm given in (2.9), we have

$$
\begin{aligned}
I_3 &\le C(p)\int_0^T \mathbb{E}\Big(\int_0^t \|S(t-s)\sigma(\Phi_s(\cdot))\|^2_{\mathcal{L}_2}\,ds\Big)^{p/2}dt\\
&\le C(p)a^p\sup_n\|e_n\|^p_\infty\int_0^T \mathbb{E}\Big(\int_0^t \|S(t-s)\sigma(\Phi_s(\cdot))\|^2_{B_0^\rho}\,ds\Big)^{p/2}\,dt\\
&\le C_4 + C_5\int_0^T\int_0^t \mathbb{E}\|\Phi_s(\cdot)\|^p_{B_0^\rho}\,ds\,dt < \infty
\end{aligned}
\tag{3.3}
$$

the same way as in (3.2). Combining these estimates, we have $\Psi : B_{p,T} \to B_{p,T}$.

We next show that Ψ is contractive. For any $\Phi, \tilde\Phi \in B_{p,t}$ we have

$$
\begin{aligned}
&\|\Psi\Phi(s,\cdot) - \Psi\tilde\Phi(s,\cdot)\|^p_{B_{p,T}}\\
&\le 2^{p-1}\int_0^t \mathbb{E}\Big\|\int_0^s S(s-\tau)(f(\Phi_\tau(\cdot) - f(\tilde\Phi_\tau(\cdot))\,d\tau\Big\|^p_{B_0^\rho}\,ds\\
&\quad + 2^{p-1}\int_0^t \mathbb{E}\Big\|\int_0^s S(s-\tau)(\sigma(\Phi_\tau(\cdot)) - \sigma(\tilde\Phi_\tau(\cdot))\,d\tau\Big\|^p_{B_0^\rho}\,ds\\
&:= 2^{p-1}(I_4+I_5).
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
I_4 &\le C_\rho^p(T)L^p\int_0^t \mathbb{E}\Big(\int_0^s \|\Phi_\tau(\cdot)-\tilde\Phi_\tau(\cdot)\|_{B_1^\rho}\,d\tau\Big)^p\,ds\\
&\le C(\rho,T,p)\int_0^t\int_0^s \mathbb{E}\Big(\int_{-h}^0 \|\Phi(\tau+\theta,\cdot)-\tilde\Phi(\tau+\theta,\cdot)\|_{B_0^\rho}^2\,d\theta\Big)^{p/2}\,d\tau\,ds\\
&\le C_5(\rho,T,p)t^2\|\Phi-\tilde\Phi\|_{B_{p,t}}^p
\end{aligned}
\tag{3.5}
$$

Now from estimate (3.3), we have

$$
\begin{aligned}
I_5 &\le C(p)a^p\sup_n\|e_n\|_\infty^p\int_0^t \mathbb{E}\Big(\int_0^s \|S(s-\tau)[\sigma(\Phi_\tau)-\sigma(\tilde\Phi_\tau)]\|_{B_0^\rho}^2\Big)^{p/2}\,dt\\
&\le C_6\int_0^t\int_0^s \mathbb{E}\Big(\int_{-h}^0 \|\Phi(\tau+\theta,\cdot)-\tilde\Phi(\tau+\theta,\cdot)\|_{B_0^\rho}^2\,d\theta\Big)^{p/2}\,d\tau\,ds\\
&\le C_6(\rho,T,p,h)t^2\|\Phi-\tilde\Phi\|_{B_{p,t}}^p.
\end{aligned}
\tag{3.6}
$$

Consequently, for $\tilde t$ small enough, (3.5) and (3.6) imply that the map Ψ has a unique fixed point in $B_{p,\tilde t}$, which is the solution of (2.11). If we consider the problem on $[0,\tilde t],[\tilde t,2\tilde t],\dots$ with $C_6\tilde t^2<1$. Since the solution is continuous with probability 1 in B_0^ρ norm, we obtain the existence and uniqueness of the solution on $[0,T]$.

It remains to prove estimate (2.12). It follows from (2.11) that for any $t\in[-h,T]$ we have

$$
\begin{aligned}
\mathbb{E}\|u(t,\cdot)\|_{B_0^\rho}^p &\le 3^{p-1}\mathbb{E}\|S(t)\varphi(0,\cdot)\|_{B_0^\rho}^p + 3^{p-1}\mathbb{E}\Big(\int_0^t \|S(t-s)f(u_s)\|_{B_0^\rho}\,ds\Big)^p\\
&\quad + 3^{p-1}\mathbb{E}\Big\|\int_0^t S(t-s)\sigma(u_s)\,dW(s)\Big\|_{B_0^\rho}^p\\
&\le 3^{p-1}C_\rho(T)\mathbb{E}\|\varphi(0,\cdot)\|_{B_0^\rho}^p + 3^{p-1}C_7\int_0^t(1+\mathbb{E}\|u_s\|_{B_1^\rho}^p)\,ds\\
&\quad + 3^{p-1}C_8\mathbb{E}\Big(\int_0^t \|S(t-s)\sigma(u_s)\|_{\mathcal{L}^2}^2\,ds\Big)^{p/2}\\
&\le C_9\Big(\mathbb{E}\|\varphi(0,\cdot)\|_{B_0^\rho}^p + \int_0^t(1+\mathbb{E}\|u_s\|_{B_1^\rho}^p)\,ds\Big).
\end{aligned}
\tag{3.7}
$$

We consider two separate cases $t\in[0,h]$ and $t\in[h,T]$: If $t\in[0,h]$, then

$$
\begin{aligned}
\mathbb{E}\|u_t\|_{B_1^\rho}^p &= \mathbb{E}\Big(\int_{-h}^0 \|u(t+\theta,\cdot)\|_{B_0^\rho}^2\,d\theta\Big)^{p/2}\\
&\le 2^{\frac p2-1}\Big(\mathbb{E}\Big(\int_{-h}^{-t}\|u(s,\cdot)\|_{B_0^\rho}^2\,ds\Big)^{p/2} + \mathbb{E}\Big(\int_{-t}^0\|u(s,\cdot)\|_{B_0^\rho}^2\,ds\Big)^{p/2}\Big)\\
&\le 2^{\frac p2-1}\Big(\mathbb{E}\|\varphi(t,\cdot)\|_{B_1^\rho}^p + h^{\frac{p-2}{p}}\int_0^t \mathbb{E}\|u(s,\cdot)\|_{B_0^\rho}^p\,ds\Big)\\
&\le 2^{\frac p2-1}\Big(\mathbb{E}\|\varphi(t,\cdot)\|_{B_1^\rho}^p + C_{10}\sup_{s\in[0,t]}\mathbb{E}\|u(s,\cdot)\|_{B_0^\rho}^p\Big).
\end{aligned}
\tag{3.8}
$$

If $t\in[h,T]$, then

$$
\mathbb{E}\|u_t\|_{B_1^\rho}^p = \mathbb{E}\Big(\int_{-h}^0 \|u(t+\theta,\cdot)\|_{B_0^\rho}^2\,d\theta\Big)^{p/2} \le C_{11}(T)\sup_{s\in[0,t]}\mathbb{E}\|u(s)\|_{B_0^\rho}^p. \tag{3.9}
$$

From (3.7)–(3.9) we have

$$
\begin{aligned}
&\sup_{s\in[0,t]} \mathbb{E}\|u(s,\cdot)\|^p_{B^\rho_0} \\
&\le C_{12}(T)\Big(\mathbb{E}\|\varphi(0,\cdot)\|^p_{B^\rho_0} + \mathbb{E}\|\varphi(t,\cdot)\|^p_{B^\rho_1} + \int_0^t \sup_{\tau\in[0,s]} \mathbb{E}\|u(\tau,\cdot)\|^p_{B^\rho_0}\,ds\Big).
\end{aligned}
$$

Estimating the last term separately, we have

$$
\sup_{s\in[0,t]} \mathbb{E}\|u(s,\cdot)\|^p_{B^\rho_0} \le C_{13}(T)[1+\mathbb{E}\|\varphi(0,\cdot)\|^p_{B^\rho_0} + \mathbb{E}\|\varphi(t,\cdot)\|^p_{B^\rho_1}].
$$

Combining the estimates above, we obtain

$$
\mathbb{E}\|u_t\|^p_{B^\rho_1} \le C_{14}(T)(1+\mathbb{E}\|y(0)\|^p_{B^\rho},
$$

which completes the proof. □

Proof of Theorem 2.4. By definition of y_1 and y_2,

$$
\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\|y(t)-y_1(t)\|^2_{B^\rho} \\
&\le \sup_{t\in[0,T]} \mathbb{E}\|u(t,\phi)-u(t,\phi_1)\|^2_{B^\rho_0} + \sup_{t\in[0,T]} \mathbb{E}\|u_t(\phi)-u_t(\phi_1)\|^2_{B^\rho_1}.
\end{aligned} \tag{3.10}
$$

The first term in (3.10) can be estimated as follows

$$
\sup_{t\in[0,T]} \mathbb{E}\|u(t,\phi)-u(t,\phi_1)\|^2_{B^\rho_0} \le C_{15} \sup_{t\in[0,T]} \mathbb{E}\|\phi(t)-\phi_1(t)\|^2_{B^\rho_0}. \tag{3.11}
$$

As for the second term in (3.10), once again we consider separately the cases $t\in[0,h]$ and $t\in[h,T]$. Taking into account the estimate (3.11), we obtain

$$
\sup_{t\in[0,T]} \mathbb{E}\int_{-h}^0 \|u(t+\theta,\phi)-u(t+\theta,\phi_1)\|^2_{B^\rho_0}\,d\theta \le C_{16} \sup_{t\in[0,T]} \mathbb{E}\|\phi(t)-\phi_1(t)\|^2_{B^\rho},
$$

which completes the proof. □

For the proof of Theorem 2.10, we need the following auxiliary lemmas.

Lemma 3.1. *For any fixed $T_0 > 2h$, the operator*

$$
A\varphi_0 := S(T_0+\theta) : B^{\bar\rho}_0 \to B^\rho_1
$$

is a Hilbert-Schmidt operator.

Proof. By [23], there exists an orthonormal basis $\{h_n, n\ge 1\}$ in B^ρ_0 such that $\sup_n \|h_n\|_{L^\infty(D)} < \infty$. It is straightforward to verify that if $\{e_n, n\ge 1\}$ is an orthonormal basis in $H = L^2(D)$, then $\{\frac{e_n}{\bar\rho^{1/2}}, n\ge 1\}$ is an orthonormal basis in $B^{\bar\rho}_0$. Therefore

$$
\begin{aligned}
&\|A\|^2_{\mathcal{L}^2} \\
&= \sum_{i=1}^\infty \|A\frac{e_i}{\sqrt{\bar\rho}}\|^2_{B^\rho_1} \\
&= \sum_{i=1}^\infty \|S(T_0+\theta)\frac{e_i}{\sqrt{\bar\rho}}\|^2_{B^\rho_1} \\
&= \sum_{i=1}^\infty \int_{-h}^0 d\theta \int_D \Big|S(T_0+\theta)\frac{e_i}{\sqrt{\bar\rho}}\Big|^2 \rho(x)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \int_{-h}^{0} d\theta \int_D \Big| \int_D G(T_0+\theta,x,y) \frac{e_i(y)}{\sqrt{\bar\rho(y)}}\,dy \Big|^2 \rho(x)\,dx \\
&= \int_{-h}^{0} d\theta \int_D \rho(x) \int_D \frac{G^2(T_0+\theta,x,y)}{\bar\rho(y)}\,dy\,dx \\
&\le \int_{-h}^{0} d\theta \int_D \int_D \frac{\rho(x)}{\bar\rho(y)} C_1(T_0)(T_0+\theta)^{-d} \exp\{-2C_2(T_0)\frac{|x-y|^2}{T_0+\theta}\}\,dy\,dx \\
&\le C_{17} \int_{-h}^{0} \frac{d\theta}{(T_0+\theta)^{d/2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{(T_0+\theta)^{d/2}} \exp\{-2C_2(T_0)\frac{|x-y|^2}{T_0+\theta}\} \frac{\rho(x)}{\bar\rho(y)}\,dx\,dy.
\end{aligned}
$$

But

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{(T_0+\theta)^{d/2}} \exp\{-2C_2(T_0)\frac{|x-y|^2}{T_0+\theta}\} \frac{\rho(x)}{\rho(y)}\,dx\rho(y) \\
&\le C(r) \int_{\mathbb{R}^d} \frac{1}{(T_0+\theta)^{d/2}} \exp\Big\{-2C_2(T_0)\frac{|x-y|^2}{T_0+\theta}\Big\}(1+|x-y|^r)\,dx\rho(y) \\
&\le C_{18}(T,r)\rho(y).
\end{aligned}
$$

Thus

$$
\|A\|^2_{\mathcal{L}^2} \le C_{19}(T,r) \int_{-h}^{0} \frac{d\theta}{(T_0+\theta)^{d/2}} \int_{\mathbb{R}^d} \frac{1+|y|^{\bar r}}{1+|y|^r}\,dy < \infty,
$$

which completes the proof. □

Corollary 3.2. *Following the lines of the proof of Lemma 3.1 we can show that $S(t)$ is a compact operator from $B_0^{\bar\rho}$ to B_0^{ρ} for $t>0$.*

We now return to the proof of Theorem 2.10. Following the approach in [9, Theorem 11.29], we have

$$
\begin{aligned}
&u(T_0) \\
&= S(T_0)\varphi(0,\cdot) + \int_0^{T_0} S(T_0-s)f(u_s)\,ds + \int_0^{T_0} S(T_0-s)\sigma(u_s)\,dW(s),
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
u_{T_0} &= u(T_0+\theta) \\
&= S(T_0+\theta)\varphi(0,\cdot) + \int_0^{T_0+\theta} S(T_0+\theta-s)f(u_s)\,ds \\
&\quad + \int_0^{T_0+\theta} S(T_0+\theta-s)\sigma(u_s)\,dW(s).
\end{aligned} \tag{3.13}
$$

The arguments in [9, Theorem 11.29] can be applied to (3.12) directly.

Lemma 3.3. *For $p>2$ and $\alpha \ge \frac{1}{p}$, the operator*

$$
(G_\alpha\varphi)(\theta) = \int_0^{T_0+\theta} (T_0+\theta-s)^{\alpha-1} S(T_0+\theta-s)\varphi(s)\,ds
$$

is compact from $L^p(0,T_0;B_0^{\bar\rho})$ to $C([-h,0],B_0^{\rho})$.

Remark 3.4. *Compactness in $C([-h,0],B_0^{\rho})$ implies compactness in B_1^{ρ}.*

Proof of Lemma 3.3. We denote

$$
\|\varphi\|^p_{L_p} := \int_0^{T_0} \|\varphi\|^p_{B_0^{\bar\rho}}\,dt
$$

We will use the infinite dimensional version of Arzela-Ascoli Theorem. To this, we need to show

(i) For any fixed $\theta \in [-h, 0]$ the set $\{G_\alpha(\varphi)(\theta), \|\varphi\|_{L^p} \le 1\}$ is compact in B_0^ρ;
(ii) for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\varphi\|_{L^p} \le 1$ such that if $\|\varphi\|_{L^p} \le 1$ and for all θ_1, θ_2 with $|\theta_1 - \theta_2| \le \delta$ we have
$$\|G_\alpha(\varphi)(\theta_1) - G_\alpha(\varphi)(\theta_2)\|_{B_0^\rho} < \varepsilon.$$

To check (i), for fixed $\theta \in [-h, 0]$ and $0 < \varepsilon < T_0 + \theta$, introduce
$$\begin{aligned} G_\alpha^\varepsilon \varphi &:= \int_0^{T_0+\theta-\varepsilon} (T_0 + \theta - s)^{\alpha-1} S(T_0 + \theta - s)\varphi(s)\, ds \\ &= S(\varepsilon) \int_0^{T_0+\theta-\varepsilon} (T_0 + \theta - s)^{\alpha-1} S(T_0 + \theta - s - \varepsilon)\varphi(s)\, ds \end{aligned}$$
Clearly $\int_0^{T_0+\theta-\varepsilon} (T_0 + \theta - s)^{\alpha-1} S(T_0 + \theta - s - \varepsilon)\varphi(s)\, ds$ is in $B_0^{\bar\rho}$. Using Corollary 3.2, $S(\varepsilon)$ is a compact operator from $B_0^{\bar\rho}$ to B_0^ρ. Then, following [9, p.227], G_α^ε converges to G_α strongly as $\varepsilon \to 0$, hence G_α is compact and (i) follows.

To prove (ii), fix θ and r such that $-h \le \theta \le \theta + r \le 0$, and $\|\varphi\|_{L^p} \le 1$. Then
$$\begin{aligned} &\|(G_\alpha\varphi)(\theta + r) - (G_\alpha\varphi)(\theta)\|_{B_0^\rho} \\ &= \Big\| \int_0^{T_0+\theta+r} (T_0 + \theta + r - s)^{\alpha-1} S(T_0 + \theta + r - s)\varphi(s)\, ds \\ &\quad - \int_0^{T_0+\theta} (T_0 + \theta - s)^{\alpha-1} S(T_0 + \theta - s)\varphi(s)\, ds \Big\|_{B_0^\rho} \\ &\le \int_0^{T_0+\theta} \Big\|(T_0 + \theta + r - s)^{(\alpha-1)} S(T_0 + \theta + r - s) \\ &\quad - (T_0 + \theta - s)^{(\alpha-1)} S(T_0 + \theta - s)\Big\| \Big\|\varphi(s)\Big\|\, ds \\ &\quad + \int_{T_0+\theta}^{T_0+\theta+r} \|(T_0 + \theta + r - s)^{(\alpha-1)} S(T_0 + \theta + r - s)\varphi(s)\|\, ds \\ &\le \Big(\int_0^{T_0} \|(r + s)^{\alpha-1} S(s + r) - s^{\alpha-1} S(s)\|^q\, ds \Big)^{1/q} \|\varphi\|_{L^p} \\ &\quad + C_{20} \Big(\int_0^{T_0} s^{(\alpha-1)q}\, ds \Big)^{1/q} \|\varphi\|_{L^p} := J_1 + J_2. \end{aligned}$$
Direct calculations yield
$$J_2 = C_{20} \frac{r^{\alpha-\frac{1}{p}}}{((\alpha - 1)q + 1)^{1/q}} \|\varphi\|_{L^p} \to 0 \quad \text{as } r \to 0.$$
We now proceed with estimating J_1. Since $S(t)$ is compact, then $S(t)$ is strongly continuous for $t > 0$ (see [24, Theorem 3.27]), hence $\|S(s + r) - S(s)\| \to 0$ as $r \to 0$, for any $s > 0$. Furthermore, the integrand in J_1 is bounded by $2C_{20}s^{(\alpha-1)q}$. Hence, by Dominated Convergence Theorem, $J_1 \to 0$ as $r \to 0$, which concludes the proof of the Lemma. □

We now complete the proof of Theorem 2.10. For any $r > 0$ introduce
$$K(r) := \{(\mu, \nu), \mu \in B_0^\rho, \nu \in B_1^\rho\}$$

such that

$$\mu := S(T_0)v + (G_1\varphi)(0) + (G_\alpha h)(0),$$
$$\nu := S(T_0+\theta)v + (G_1\varphi)(0) + (G_\alpha h)(0)$$

with $\|v\|_{B_0^\rho} \le r$, $\|\varphi\|_{L^p(0,T_0,B_0^{\bar\rho})} \le r$ and $\|h\|_{L^p(0,T_0,B_0^{\bar\rho})} \le r$. It follows from Lemma 3.1, Corollary 3.2 and Lemma 3.3 that $K(r)$ is compact in B^ρ.

Lemma 3.5. *Under the conditions of Theorem 2.3, there is $C>0$ such that for arbitrary $r>0$ and $y=(x,z)\in B^{\bar\rho}$ such that $\|y\|_{B^{\bar\rho}} \le r$ we have*

$$P\{(u(T_0,x,z),u_{T_0}(x,z))\in K(r)\} \ge 1 - cr^{-p}(1+\|y\|^p_{B^{\bar\rho}}), \tag{3.14}$$

where $u(0,x,z)=x\in B_0^{\bar\rho}$ and $u_0(x,z)=z\in B_1^{\bar\rho}$.

Proof. From the factorization formula [10, Thm. 5.2.5], we have

$$u(T_0,y) = S(T_0)x + (G_1 f(u_s))(0) + \frac{\sin(\alpha\pi)}{\pi}(G_\alpha Y(s))(0), \tag{3.15}$$

$$u_{T_0}(y) = S(T_0+\theta)x + (G_1 f(u_s))(\theta) + \frac{\sin(\alpha\pi)}{\pi}(G_\alpha Y(s))(\theta), \tag{3.16}$$

$$Y(s) = \int_0^s (s-\tau)^{-\alpha} S(s-\tau)\sigma(u_\tau)\,dW(\tau). \tag{3.17}$$

Using Lemma 7.2 [9], we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^{T_0} \|Y(s)\|^p_{B_0^{\bar\rho}}\,ds \\
&= \mathbb{E}\int_0^{T_0} \|\int_0^s (s-\tau)^{-\alpha} S(s-\tau)\sigma(u_\tau)\,dW(\tau)\|^p_{B_0^{\bar\rho}}\,ds \\
&\le C_{p,T_0}\mathbb{E}\int_0^{T_0}\Big(\int_0^s (s-\tau)^{-2\alpha}\|S(s-\tau)\sigma(u_\tau)\circ Q^{1/2}\|_{\mathcal{L}_2(H,B_0^{\bar\rho})}\Big)^{p/2}\,ds \\
&\le C_{21}\mathbb{E}\int_0^{T_0}\Big(\int_0^s (s-\tau)^{-2\alpha}\|\sigma(u_\tau)\|^2_{B_0^{\bar\rho}}\,d\tau\Big)^{p/2}\,ds.
\end{aligned} \tag{3.18}$$

Using Hausdorff-Young's inequality and (2.11), we have

$$\begin{aligned}
\mathbb{E}\int_0^{T_0} \|Y(s)\|^p_{B_0^{\bar\rho}} &\le C_{21}\Big(\int_0^{T_0} t^{-2\alpha}\,dt\Big)^{p/2}\int_0^{T_0}\mathbb{E}\|\sigma(u_t)\|^p_{B_0^{\bar\rho}}\,dt \\
&\le C_{22}\int_0^{T_0}(1+\mathbb{E}\|u_t\|^p_{B_1^{\bar\rho}})\,dt \\
&\le C_{23}(1+\|y\|^p_{B^{\bar\rho}}).
\end{aligned} \tag{3.19}$$

In a similar way,

$$\mathbb{E}\int_0^{T_0}\|f(u_s)\|^p_{B_0^{\bar\rho}}\,ds \le C_{23}(1+\|y\|^p_{B^{\bar\rho}}). \tag{3.20}$$

Hence, if $\|y\|_{B^{\bar\rho}} \le r$, $\|f(u_s)\|_{L^p(0,T_0,B_0^{\bar\rho})} \le r$, and

$$\|\sigma(u_s)\|_{L^p(0,T_0,B_0^{\bar\rho})} \le \frac{\pi r}{\sin(\alpha\pi)},$$

then from the definition of $K(r)$ we have $(u(T_0,y),u_{T_0}(y)) \in K(r)$. Assume $\|y\|_{B^{\bar\rho}} \le r$. Then

$$P\{(u(T_0,y),u_{T_0}(y))\notin K(r)\} \le P\{\|f(u_s)\|_{L^p(0,T_0;B_0^{\bar\rho})} > r\} + P\{\|Y(s)\|_{L^p(0,T_0,B_0^{\bar\rho})}\}$$

$$\leq 2r^p C_{23}(1 + \|y\|^p_{B^{\bar\rho}})$$

where we used (3.19) and (3.20). The proof is complete. □

The rest of the proof of Theorem 2.10 follows the lines of the proof of [9, Theorem 11.29].

Proof of Theorem 2.12. This proof a lot in common with the proof of [20, Theorem 1]. Let us point out the differences caused by the presence of the delay. We have

$$\mathbb{E}\|y(t)\|^2_{B^\rho} = \mathbb{E}\int_{\mathbb{R}^d} |u(t,x)|^2\rho(x)\,dx + \mathbb{E}\int_{-h}^0 d\theta \int_{\mathbb{R}^d} |u(t+\theta,x)|^2\rho(x)\,dx. \tag{3.21}$$

By definition of a mild solution (2.11), we have

$$\|u(t,x)\|^2_{B_0^\rho} \leq 3(I_1(t) + I_2(t) + I_3(t))$$

where

$$\begin{gathered} I_1(t) = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} G(t,x,y)\varphi(0,y)\,dy\Big)^2 \rho(x)\,dx, \\ I_2(t) = \int_{\mathbb{R}^d} \Big(\int_0^t \int_{\mathbb{R}^d} G(t-s,x,y) f(u_s(y))\,dy\,ds\Big)^2 \rho(x)\,dx, \\ I_3(t) = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} G(t-s,x,y)\sigma(u_s(y))\,dW(s)\,dy\Big)^2 \rho(x)\,dx. \end{gathered}$$

It follows from (2.4) that for all $t \geq 0$,

$$\begin{aligned} \mathbb{E} I_1 &\leq \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} G(t,x,y)\,dy \int_{\mathbb{R}^d} G(t,x,y)\varphi^2(0,y)\,dy\Big)\rho(x)\,dx \\ &\leq C_{24}\mathbb{E}\int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} K(t,x-y)\varphi^2(0,y)\,dy\Big)\rho(x)\,dx \\ &\leq C_{24}\|\rho\|_\infty \mathbb{E}\|\varphi(0,\cdot)\|^2_{B_0^\rho} < \infty, \end{aligned}$$

where K is the heat kernel in $\mathbb{R}^d$. The estimates for I_2 and I_3 can be estimated along the lines of [20, Theorem 1] using the Nash-Aranson type estimates for the kernel (2.3).

To estimate the second term in (3.21), once again we consider two cases: $t \in [0,h]$ and $t \geq h$. If $t \in [0,h]$, then

$$\begin{aligned} \mathbb{E}\|u_t\|^2_{B_1^\rho} &= \mathbb{E}\int_{-h}^0 \|u(t+\theta)\|^2_{B_0^\rho}\,d\theta \\ &\leq \mathbb{E}\int_{-h}^0 \|u(s)\|^2_{B_0^\rho}\,ds + \mathbb{E}\int_0^h \|u(s)\|^2_{B_0^\rho}\,ds \\ &\leq \mathbb{E}\|\varphi(t,\cdot)\|^2_{B_1^\rho} + h\sup_{t\geq 0}\mathbb{E}\|u(t)\|^2_{B_0^\rho} < \infty. \end{aligned}$$

Finally, if $t \geq h$, then

$$\mathbb{E}\|u_t\|^2_{B_1^\rho} = \mathbb{E}\int_{-h}^0 \|u(t+\theta)\|^2_{B_0^\rho}\,d\theta \leq \sup_{t\geq 0}\mathbb{E}\|u(t)\|^2_{B_0^\rho} < \infty,$$

which completes the proof. □

Example 3.6. Let $\bar{f} : \mathbb{R} \to \mathbb{R}$ and $\bar{\sigma} : \mathbb{R} \to \mathbb{R}$ be Lipschitz functions with Lipschitz constants L. We define

$$f[\varphi] := \bar{f}\Big(\int_{-h}^{0} \varphi(\theta)\, d\theta\Big), \quad \sigma[\varphi] := \bar{\sigma}\Big(\int_{-h}^{0} \varphi(\theta)\, d\theta\Big)$$

Then for all $\varphi_1, \varphi_2 \in B_1^\rho$ we have

$$|f[\varphi_1] - f[\varphi_2]| \le L \int_{-h}^{0} |\varphi_1(\theta) - \varphi_2(\theta)|\, d\theta.$$

Hence

$$\|f[\varphi_1] - f[\varphi_2]\|_{B_0^\rho}^2 \le L^2 \int_{\mathbb{R}^d} \Big(\int_{-h}^{0} |\varphi_1(\theta) - \varphi_2(\theta)|\, d\theta\Big)^2 \rho\, dx \le L^2 h \|\varphi_1 - \varphi_2\|_{B_1^\rho}^2.$$

Similarly,

$$\|\sigma[\varphi_1] - \sigma[\varphi_2]\|_{B_0^\rho}^2 \le L^2 h \|\varphi_1 - \varphi_2\|_{B_1^\rho}^2.$$

Thus f and σ are examples of Lipschits maps from B_1^ρ to B_0^ρ, for which the theorems above apply.

4. Uniqueness of the invariant measure

Proof of Theorem 2.14. Let $\mathcal{B}$ be the class of $\mathcal{F}_t$ measurable B_0-valued processes $\xi(t)$, such that

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|\xi(t)\|_{B_0}^2 < \infty.$$

Since

$$\sup_{t\in\mathbb{R}} \|\xi(t)\|_{B}^2 \le (1+h) \sup_{t\in\mathbb{R}} \mathbb{E}\|\xi(t)\|_{B_0}^2,$$

we follow the procedure in [20] and define the successive approximations $u^{(0)} \equiv 0$ and

$$du^{(n+1)} = (Au^{(n+1)} + f(u_t^{(n)}))\, dt + \sigma(u_t^{(n)})\, dW(t). \tag{4.1}$$

Then

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|f(u_t^{(n)})\|_{B_0}^2 \le 2\|f(0)\|_{B_0}^2 + 2L^2h^2 \sup_{t\in\mathbb{R}} \mathbb{E}\|u^{(n)}(t)\|_{B_0}^2 < \infty.$$

Similarly,

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|\sigma(u_t^{(n)})\|_{B_0}^2 < \infty.$$

Thus by Theorem 5 [20], equation (4.1) has the unique solution $u^{(n+1)}(t)$ such that

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|u^{(n+1)}(t)\|_{B_0}^2 < \infty,$$

and therefore,

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|u^{(n+1)}(t)\|_{B}^2 < \infty.$$

But

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|u^{(n)}\|_{B_0}^2 \le (1+h) \sup_{t\in\mathbb{R}} \mathbb{E}\|u^{(n)}(t)\|_{B_0}^2 \le C + hL^2\Big(\frac{4}{\lambda_1^2} + \frac{2a}{\lambda_1}\Big) \sup_{t\in\mathbb{R}} \mathbb{E}\|u^{(n-1)}\|_{B_0}^2.$$

Hence for

$$hL^2\Big(\frac{4}{\lambda_1^2} + \frac{2a}{\lambda_1}\Big) < 1 \tag{4.2}$$

in a similar way to [20] we can argue that the sequence is in fact Cauchy, and there exists a unique $u^*(t)$ such that

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|u^*(t)\|_B < \infty$$

and

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|u^n(t) - u^*(t)\|_B^2 \to 0, \quad n \to \infty.$$

Furthermore, we can argue that u^* satisfies

$$u^*(t) = S(t-t_0)u^*(t_0) + \int_{t_0}^t S(t-t_0)f(u_s^*)\,ds + \int_{t_0}^t S(t-s)\sigma(u_s^*)\,dW(s). \tag{4.3}$$

Consider any other solution (4.3) such that $\eta(t_0)$ is $\mathcal{F}_{t_0}$-measurable, and $\mathbb{E}\|\eta(t_0)\|_B^2 < \infty$. Here $\eta_{t_0} = \varphi(\theta, x)$ is defined on $[-h, 0]$. Let us show that the solution η converges to u^* exponentially. Since we are interested in the behavior of the solutions for large t, suppose $t > t_0 + h$. Then $t + \theta > t_0$ and $\eta(t)$ is defined via the formula (4.3). Hence

$$\begin{aligned}
&\mathbb{E}\|u^*(t) - \eta(t)\|_{B_0}^2 \\
&\le 3e^{-\lambda_1(t-t_0)}\mathbb{E}\|u^*(t_0) - \eta(t_0)\|_{B_0}^2 + 3\frac{L^2}{\lambda_1}\int_{t_0}^t e^{-\lambda_1(t-s)}\mathbb{E}\|u_s^* - \eta_s\|_{B_1}^2\,ds \\
&\quad + 3L^2 a\int_{t_0}^t e^{-\lambda_1(t-s)}\mathbb{E}\|u_s^* - \eta_s\|_{B_1}^2\,ds \\
&= 3e^{-\lambda_1(t-t_0)}\mathbb{E}\|u^*(t_0) - \eta(t_0)\|_{B_0}^2 \\
&\quad + 3\Big(\frac{L^2}{\lambda_1} + L^2 a\Big)\int_{t_0}^t e^{-\lambda_1(t-s)}\mathbb{E}\|u_s^* - \eta_s\|_{B_1}^2\,ds.
\end{aligned}$$

In addition,

$$\begin{aligned}
\mathbb{E}\|u_t^* - \eta_t\|_{B_1}^2 &= \int_{-h}^0 \mathbb{E}\|u^*(t+\theta) - \eta(t+\theta)\|_{B_0}^2\,d\theta \\
&\quad + 3\int_{-h}^0 e^{-\lambda_1(t+\theta-t_0)}\mathbb{E}\|u^*(t_0) - \eta(t_0)\|_{B_0}^2\,d\theta \\
&\quad + 3\int_{-h}^0 \Big(\frac{L^2}{\lambda_1}\int_{t_0}^{t+\theta} e^{-\lambda_1(t+\theta-s)}\mathbb{E}\|u_s^* - \eta_s\|_{B_1}^2\,ds\Big)\,d\theta \\
&\quad + 3\int_{-h}^0 \Big(L^2 a\int_{t_0}^{t+\theta} e^{-\lambda_1(t+\theta-s)}\mathbb{E}\|u_s^* - \eta_s\|_{B_1}^2\,ds\Big)\,d\theta.
\end{aligned}$$

However,

$$e^{-\lambda_1(t+\theta-s)} \le e^{-\lambda_1(t-s)} \cdot e^{\lambda_1 h},$$

thus

$$\begin{aligned}
\mathbb{E}\|u_t^* - \eta_t\|_{B_1}^2 &\le 3he^{\lambda_1 h}e^{-\lambda_1(t-t_0)}E\|u^*(t_0) - \eta(t_0)\|_{B_0}^2 \\
&\quad + 3e^{\lambda_1 h}h\Big(\frac{L^2}{\lambda_1} + L^2 a\Big)\int_{t_0}^t e^{-\lambda_1(t-s)}\mathbb{E}\|u_s^* - \eta_s\|_{B_1}^2\,ds.
\end{aligned}$$

Altogether,

$$\mathbb{E}\|u^*(t) - \eta(t)\|_B^2 \le (3e^{\lambda_1 h}h + 3)e^{-\lambda_1(t-t_0)}\mathbb{E}\|u^*(t_0) - \eta(t_0)\|_B^2$$

$$+ (3+3he^{\lambda_1 h})\Big(\frac{L^2}{\lambda_1}+L^2 a\Big)\int_{t_0}^t e^{-\lambda_1(t-s)}\mathbb{E}\|u^*(s)-\eta(s)\|_B^2\,ds.$$

Therefore, if

$$(3+3he^{\lambda_1 h})\Big(\frac{L^2}{\lambda_1}+L^2 a\Big) := \gamma_0 L^2 < \lambda_1 \tag{4.4}$$

we have

$$\mathbb{E}\|u^*(t)-\eta(t)\|_B^2 \le (3e^{\lambda_1 h}h+3)e^{(\gamma_0 L^2-\lambda_1)(t-t_0)}\mathbb{E}\|u^*(t_0)-\eta(t_0)\|_B^2.$$

Then the existence and uniqueness of invariant measure can be established in the same manner as in [20]. □

5. Conclusions

In summary, we completed the analysis of the long time behavior of nonlinear stochastic functional-differential equations in Hilbert spaces is several steps. In Theorem 2.3 and Theorem 2.4 we establish the existence and uniqueness of mild solutions, as well as their continuous dependence on the initial data. Next, in Theorem 2.12 we obtain a priory, uniform in time bounds for these solutions in the appropriate Hilbert spaces, which were further used to deduce the main result, namely, the existence of invariant measure, in Theorem 2.10. Furthermore, in Theorem 2.14 we exploit the further properties of the problem, which enable us to deduce the exponential stability and thus the uniqueness of invariant measures.

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