

## VISCOSITY SOLUTIONS TO THE INFINITY LAPLACIAN EQUATION WITH LOWER TERMS

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ABSTRACT. We establish the existence and uniqueness of viscosity solutions to the Dirichlet problem

$$\begin{aligned} \Delta_{\infty}^h u &= f(x, u), & \text{in } \Omega, \\ u &= q, & \text{on } \partial\Omega, \end{aligned}$$

where  $q \in C(\partial\Omega)$ ,  $h > 1$ ,  $\Delta_{\infty}^h u = |Du|^{h-3} \Delta_{\infty} u$ . The operator  $\Delta_{\infty} u = \langle D^2 u Du, Du \rangle$  is the infinity Laplacian which is strongly degenerate, quasilinear and it is associated with the absolutely minimizing Lipschitz extension. When the nonhomogeneous term  $f(x, t)$  is non-decreasing in  $t$ , we prove the existence of the viscosity solution via Perron's method. We also establish a uniqueness result based on the perturbation analysis of the viscosity solutions. If the function  $f(x, t)$  is nonpositive (nonnegative) and non-increasing in  $t$ , we also give the existence of viscosity solutions by an iteration technique under the condition that the domain has small diameter. Furthermore, we investigate the existence and uniqueness of viscosity solutions to the boundary-value problem with singularity

$$\begin{aligned} \Delta_{\infty}^h u &= -b(x)g(u), & \text{in } \Omega, \\ u &> 0, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{aligned}$$

when the domain satisfies some regular condition. We analyze asymptotic estimates for the viscosity solution near the boundary.

### 1. INTRODUCTION

In this manuscript, we investigate the following inhomogeneous problem for  $q \in C(\partial\Omega)$ ,

$$\begin{aligned} \Delta_{\infty}^h u &= f(x, u), & \text{in } \Omega, \\ u &= q, & \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where  $\Delta_{\infty}^h$  is strongly degenerate and is defined as

$$\Delta_{\infty}^h u := |Du|^{h-3} \langle D^2 u Du, Du \rangle = |Du|^{h-3} \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i x_j}, \quad h > 1.$$

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Throughout this article,  $\Omega$  is assumed to be a bounded domain of  $\mathbb{R}^n$ ,  $n \geq 2$ . Note that the operator  $\Delta_\infty^h$  is not in divergence form. Hence, the solution is usually understood in the viscosity frame introduced by Crandall and Lions [23], and Crandall, Evans and Lions [20].

For the special case  $h = 3$ , the operator  $\Delta_\infty^h$  is the infinity Laplacian which is often denoted by  $\Delta_\infty u = \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i x_j}$ . The operator  $\Delta_\infty$  was motivated in studying the absolutely minimizing Lipschitz extension (AMLE) by Aronsson [2, 3, 4, 5] in 1960's. Jensen [25] showed the uniqueness of AMLE and the equivalence of the AMLE and infinity harmonic functions (viscosity solutions to  $\Delta_\infty u = 0$ ). Crandall, Gunnarsson and Wang [21] studied the uniqueness of infinity harmonic functions in an unbounded domain. Crandall, Evans and Gariepy [19] showed that infinity harmonic functions enjoy comparison property with linear cones. Aronsson, Crandall and Juutinen [6] gave a systematic treatment of the theory of AMLEs. For more results of AMLEs, see for example Armstrong and Smart [1], Barron, Jensen and Wang [1, 9], Barles and Busca [7], Barron, Evans and Jensen [8], Evans [24], Yu [44] and the references therein.

For  $h = 1$ ,  $\Delta_\infty^h u$  is the 1-homogeneous normalized  $\infty$ -Laplacian operator,

$$\Delta_\infty^N u := |Du|^{-2} \langle D^2 u Du, Du \rangle.$$

There is a ‘‘tug-of-war’’ game when approaching the normalized infinity Laplacian Dirichlet problem which was first introduced by Peres et al. [40] based on a probability view,

$$\begin{aligned} \Delta_\infty^N u(x) &= H(x), \quad \text{in } \Omega, \\ u(x) &= q(x), \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

The continuum value function of the game is proven to satisfy (1.2) and  $\Delta_\infty^N$  is also called game infinity Laplacian (denoted also by  $\Delta_\infty^G$ ). Lu and Wang [34] studied the well-posedness of (1.2) from the partial differential equation perspective. Note that the uniqueness is valid if the nonlinear source term  $H(x) > 0 (< 0)$ . A counterexample was shown in [37, 40] that the uniqueness does not hold if  $H(x)$  changes its sign. One can see [36] for more uniqueness results of infinity Laplacian equations. We direct the reader to [27, 28, 29, 30, 31, 33, 36, 37, 39, 42, 44] and the references therein for the  $\infty$ -Laplacian operator.

Lu and Wang [35] proved that the inhomogeneous Dirichlet problem

$$\begin{aligned} \Delta_\infty u &= H(x), \quad \text{in } \Omega, \\ u &= q, \quad \text{on } \partial\Omega \end{aligned}$$

has a unique viscosity solution  $u \in C(\bar{\Omega})$  under the assumptions that the continuous function  $H$  has one sign. They also proved the comparison property with special functions for the viscosity solutions which extended the result of Crandall, Evans and Gariepy [19]. Bhattacharya and Mohammed [10] studied the existence and nonexistence of viscosity solutions to the Dirichlet problem

$$\begin{aligned} \Delta_\infty u &= f(x, u), \quad \text{in } \Omega, \\ u &= g, \quad \text{on } \partial\Omega \end{aligned} \tag{1.3}$$

for  $f$  with the sign and the monotonicity restrictions and  $g \in C(\partial\Omega)$ . In [11], they further removed the sign and the monotonicity restrictions and gave the existence result from the general structure condition on  $f$ . Bhattacharya and Mohammed [10] also investigated the bounds and boundary behavior of viscosity solutions to the

problem (1.3). For the general boundary behavior of the viscosity solution to (1.3), one can see [38]. In [32], the existence of the viscosity solutions of the following inhomogeneous problem was obtained,

$$\begin{aligned}\Delta_{\infty}^h u &= f(x), & \text{in } \Omega, \\ u &= g, & \text{on } \partial\Omega.\end{aligned}$$

And for  $1 \leq h \leq 3$ , under suitable conditions on  $\alpha$  and  $f$ , Biswas and Vo [14] studied the existence, nonexistence and uniqueness of positive viscosity solutions to the Dirichlet problem of the equation

$$\Delta_{\infty}^h u + \alpha(x) \cdot \nabla u |\nabla u|^{h-1} + f(x, u) = 0.$$

Bhattacharya and Mohammed [10] obtained bounds and boundary behavior of viscosity solutions to the problem

$$\begin{aligned}-\Delta_{\infty} u &= f(u), & \text{in } \Omega, \\ u &> 0, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega,\end{aligned}$$

when  $f \in C^1((0, \infty), (0, \infty))$ ,  $\lim_{t \rightarrow 0^+} f(t) = \infty$  and  $f$  is decreasing on  $(0, \infty)$ . By Karamata regular variation theory, Mi [38] gave the boundary asymptotic estimate of solutions to the problem

$$\begin{aligned}-\Delta_{\infty} u &= b(x)f(u), & \text{in } \Omega, \\ u &> 0, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega\end{aligned}$$

for a wide range of the functions  $b(x)$ . Biset and Mohammed [13] established the existence of ground state solutions to the problem

$$\begin{aligned}-\Delta_{\infty} u &= \lambda f(x, u), & \text{in } \Omega, \\ u &> 0, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega,\end{aligned}$$

in a bounded domain and in the whole Euclidean space. The study is based on the subsolution/supersolution method and the existence of the principal Dirichlet eigenfunctions.

Inspired by the previous work, we study the Dirichlet problem (1.1). The  $h$ -degree operators  $\Delta_{\infty}^h$ , besides their wide applications, are not only degenerate, singular for  $1 < h < 3$ , but also not in divergence form and have no variational structure. They constitute a class of operators with particular properties. Our main results are summarized as follows.

**Theorem 1.1.** *Let  $q \in C(\partial\Omega)$ . Suppose that  $f(x, t) \in C(\Omega \times \mathbb{R})$  is non-negative and non-decreasing in  $t$  and  $\sup_{x \in \Omega} f(x, t) < \infty$  for each  $t \in \mathbb{R}$ . Then (1.1) admits a viscosity solution. Furthermore, if  $f$  is positive, the solution is unique.*

**Remark 1.2.** Similar results are still valid if the conditions of  $f$  are replaced by  $f$  non-positive and  $\inf_{x \in \Omega} f(x, t) > -\infty$  for each  $t \in \mathbb{R}$ .

The existence of viscosity solutions is proved via Perron's method. The key is to construct a suitable viscosity subsolution. Thanks to the "good" structure of the operator  $\Delta_{\infty}^h$ , we can use the cone functions to construct the subsolution. The uniqueness can be derived from the comparison principle. We remark that the

uniqueness is still open when the function  $f(x, t)$  is nonpositive or nonnegative. In fact, when  $f(x, t)$  does not depend on the second variable  $t$  and  $h = 1$ , Peres, Schramm, Sheffield and Wilson [40] construct a counterexample to show that the uniqueness does not hold if  $f$  changes its sign. For  $h = 3$ , Lu and Wang [35] gave a counterexample to show the uniqueness is invalid if the function  $f(x)$  changes its sign.

With Theorem 1.1 in hand, when  $f(x, t)$  is non-increasing in the variable  $t$ , we can also prove the existence of the viscosity solution to the Dirichlet problem (1.1) if the domain is of small diameter. Note that the small diameter condition (1.4) guarantees the existence of a viscosity subsolution and then we can use an iteration technique to establish the following existence result.

**Theorem 1.3.** *Let  $q \in C(\partial\Omega)$  and  $\ell_1 := \inf_{\partial\Omega} q$ . Suppose that  $f(x, t) \in C(\Omega \times \mathbb{R})$  is positive, non-increasing in  $t$  and  $\sup_{x \in \Omega} f(x, t) < \infty$  for each  $t \in \mathbb{R}$ . If  $\Omega$  satisfies the condition*

$$\text{diam}(\Omega) \leq \left( \frac{\ell_1 - \lambda_0}{\gamma C} \right)^{h/(h+1)}, \quad (1.4)$$

where  $\gamma = \frac{1}{h+1} h^{(h+1)/h}$ ,  $C \geq (\sup_{\Omega} f(x, \lambda_0))^{1/h} > 0$ , and  $\lambda_0 < \ell_1$ , then (1.1) has a viscosity solution  $u \in C(\bar{\Omega})$ .

**Remark 1.4.** Similarly, let  $\ell_2 = \sup_{\partial\Omega} q$  and suppose that  $f(x, t) \in C(\Omega \times \mathbb{R})$  is negative, non-increasing in  $t$  and  $\inf_{x \in \Omega} f(x, t) > -\infty$  for each  $t \in \mathbb{R}$ . If  $\Omega$  satisfies

$$\text{diam}(\Omega) \leq \left( -\frac{\ell_2 + \lambda_0}{\gamma C} \right)^{h/(h+1)},$$

where  $\gamma = \frac{1}{h+1} h^{(h+1)/h}$ ,  $C > 0$  and  $\lambda_0 < -\ell_2$ , then (1.1) has a viscosity solution  $u \in C(\bar{\Omega})$ .

For  $h = 3$ , Bhattacharya and Mohammed [10] constructed a counter-example in the appendix to show that the uniqueness of the viscosity solution does not generally hold when  $f(x, t)$  is non-increasing in  $t$ .

To establish the existence of viscosity solutions to (1.1), a difficulty with respect to the degenerate operators is the lack of existence of barriers. Thanks to the particular structure of  $\Delta_{\infty}^h$ , we can construct ‘good’ barriers and use the standard Perron’s method to get the existence of the approximate solutions. Due to the strong degeneracy of the operator  $\Delta_{\infty}^h$ , we combine the iteration method, Theorem 1.1 and stability method to establish Theorem 1.3. The key idea is that the existence of an appropriate viscosity subsolution leads to the existence of a viscosity solution.

Furthermore, we investigate the singular Dirichlet problem

$$\begin{aligned} \Delta_{\infty}^h u &= -b(x)g(u), & \text{in } \Omega, \\ u &> 0, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega. \end{aligned} \quad (1.5)$$

We first construct a viscosity supersolution of (1.5) and then prove the existence of viscosity solutions to (1.5) using the comparison principle and the stability of viscosity solutions.

**Theorem 1.5.** *Let  $b \in C(\Omega)$  be a positive function such that  $\sup_{x \in \Omega} b(x) < \infty$ . If  $g \in C^1((0, \infty), (0, \infty))$  is non-increasing with  $\lim_{t \rightarrow 0^+} g(t) = \infty$ , then the problem (1.5) has a unique viscosity solution.*

When the bounded domain  $\Omega$  has smooth boundary, we also investigate the boundary behavior of viscosity solutions to (1.5). The functions  $b(\cdot)$  and  $g(\cdot)$  satisfy the following conditions:

- (H1)  $b \in C(\bar{\Omega})$  is positive in  $\Omega$ .  
 (H2) For some  $\delta_0 > 0$ , there exist a function  $k \in \Lambda$  and a positive constant  $\ell \in \mathbb{R}$  such that

$$\lim_{d(x) \rightarrow 0} \frac{b(x)}{k^{h+1}(d(x))} = \ell, \quad (1.6)$$

where  $\Lambda$  is the set of all positive, non-decreasing functions  $k \in C^1(0, \delta_0)$  satisfying

$$\lim_{t \rightarrow 0^+} \left( \frac{K(t)}{k(t)} \right)' = \tau, \quad \text{where } K(t) = \int_0^t k(s) ds, \quad (1.7)$$

and  $\tau$  is a positive constant.

- (H3)  $g \in C^1((0, \infty), (0, \infty))$ ,  $\lim_{t \rightarrow 0^+} g(t) = \infty$  and  $g$  is decreasing on  $(0, \infty)$ .  
 (H4) There exists  $\gamma > 1$  such that

$$\lim_{t \rightarrow 0^+} \frac{g'(t)t}{g(t)} = -\gamma. \quad (1.8)$$

**Theorem 1.6.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary. Suppose (H1)–(H4) are satisfied, and  $\phi$  is the solution to the problem*

$$\int_0^{\phi(t)} \frac{ds}{(g(s))^{1/h}} = t, \quad \forall t > 0.$$

*If  $\tau(\gamma + h) > h + 1$ , then for the unique viscosity solution  $u$  to (1.5), then it holds*

$$\lim_{d(x) \rightarrow 0} \frac{u(x)}{\phi(K^{(h+1)/h}(d(x)))} = \xi_0,$$

*with*

$$\xi_0 = \left( \frac{h^h \ell (h + \gamma)}{(h + 1)^h (h + \gamma) \tau - h - 1} \right)^{1/(h+\gamma)}.$$

To obtain the existence of the viscosity solutions of the singular problem (1.5), we adopt the truncation method to deal with the singularity of the equation and then use the stability and compactness methods. Based on the comparison principle, the uniqueness result of the viscosity solution follows immediately.

One should notice that the distance function is a solution of  $\Delta_\infty^h v = 0$  near the boundary. Therefore, we can perturb the distance function to analyze the asymptotic behavior near the boundary of viscosity solutions to the singular boundary value problem (1.5). The idea is based on Karamata regular variation theory which was first introduced by Cîrstea and Rădulescu in stochastic process to study the boundary behavior and uniqueness of solutions to boundary blow-up elliptic problems. And a series of rich and significant information about the boundary behavior of solutions was obtained based on such theory [15, 16, 17]. Note that, unlike the case  $h = 1$ , the operator  $\Delta_\infty^h$  is quasi-linear even in one-dimension for  $h > 1$ . Therefore, we must make subtle analysis.

This article is organized in the following way. In Section 2, we prove the comparison principle to  $\Delta_\infty^h u = f(x, u)$ , based on the double variables method. In Section 3, by Perron's method and the comparison principle, we prove the existence and

uniqueness of viscosity solutions to (1.1) when  $f(x, t)$  is non-decreasing in  $t$ . In Section 4, when  $f(x, t)$  is non-increasing in  $t$ , we use an iteration technique to obtain the existence of the viscosity solution to (1.1) in domains with small diameter. In Section 5, we establish the existence and boundary behavior of viscosity solutions to (1.5).

## 2. COMPARISON PRINCIPLES

In this section, we give the comparison principles via the perturbation method for the equation

$$\Delta_\infty^h u = f(x, u), \quad \text{in } \Omega. \quad (2.1)$$

Since  $\Delta_\infty^h$  has no divergence structure, we define the viscosity solution by the semi-continuous extension. See for example [22, 30, 32, 34]. For  $F_h : \mathbb{S} \times (\mathbb{R}^n \setminus \{0\}) \rightarrow \mathbb{R}$  and  $F_h(M, p) := |p|^{h-3}(Mp) \cdot p$ , where  $\mathbb{S}$  denotes the set of  $n \times n$  real symmetric matrices, (2.1) can be rewritten as

$$F_h(D^2u, Du) = f(x, u), \quad x \in \Omega.$$

Since  $h > 1$ , we have  $\lim_{p \rightarrow 0} F_h(M, p) = 0$  for all  $M \in \mathbb{S}$ . Therefore, we define the following continuous extension of  $F_h$ ,

$$\bar{F}_h(M, p) := \begin{cases} F_h(M, p), & \text{if } p \neq 0, \\ 0, & \text{if } p = 0. \end{cases}$$

Now we state the definition of viscosity solutions to the equation (2.1).

**Definition 2.1.** *Suppose that  $u : \Omega \rightarrow \mathbb{R}$  is an upper semi-continuous function. If, for any  $x_0 \in \Omega$  and  $\varphi \in C^2(\Omega)$  such that  $u(x_0) = \varphi(x_0)$  and  $u(x) \leq \varphi(x)$  for all  $x \in \Omega$  near  $x_0$ , it holds*

$$\bar{F}_h(D^2\varphi(x_0), D\varphi(x_0)) \geq f(x_0, \varphi(x_0)),$$

*then we say that  $u$  is a viscosity subsolution to (2.1) in  $\Omega$ .*

*Similarly, suppose that  $u : \Omega \rightarrow \mathbb{R}$  is a lower semi-continuous function. If, for any  $x_0 \in \Omega$  and  $\varphi \in C^2(\Omega)$  such that  $u(x_0) = \varphi(x_0)$  and  $u(x) \geq \varphi(x)$  for all  $x \in \Omega$  near  $x_0$ , it holds*

$$\bar{F}_h(D^2\varphi(x_0), D\varphi(x_0)) \leq f(x_0, \varphi(x_0)),$$

*then we say that  $u$  is a viscosity supersolution to (2.1) in  $\Omega$ .*

*A function  $u \in C(\Omega)$  is called a viscosity solution to (2.1) in  $\Omega$  if it is both a viscosity subsolution and viscosity supersolution of (2.1).*

Next, we state the strong maximum principle for infinity subharmonic functions (see for example [6, 18]).

**Lemma 2.2.** *Assume  $u \in C(\bar{\Omega})$  is infinity subharmonic ( $\Delta_\infty u \geq 0$ ). Then  $u$  attains its maximum only on the boundary  $\partial\Omega$  unless  $u$  is a constant.*

We also need the comparison principle which was established by Li and Liu [26].

**Lemma 2.3.** *Suppose that  $f(x, t) \in C(\Omega \times \mathbb{R})$  is positive (negative) and non-decreasing in  $t$ . Assume that  $u \in C(\bar{\Omega})$  and  $v \in C(\bar{\Omega})$  satisfy*

$$\Delta_\infty^h u \geq f(x, u), \quad x \in \Omega,$$

$$\Delta_\infty^h v \leq f(x, v), \quad x \in \Omega$$

*respectively, in the viscosity sense. If  $u \leq v$  on  $\partial\Omega$ , then  $u \leq v$  in  $\Omega$ .*

Now we present a comparison result applicable to singular problems and prove it by a truncating function.

**Theorem 2.4.** *Suppose that  $f(x, t) \in C(\Omega \times (0, \infty))$  is negative and non-decreasing in  $t$  with  $\lim_{t \rightarrow 0^+} f(x, t) = -\infty$ . If  $u, v \in C(\overline{\Omega})$  are positive functions satisfying  $\Delta_\infty^h u \geq f(x, u)$  and  $\Delta_\infty^h v \leq f(x, v)$  in  $\Omega$ , respectively, then  $u \leq v$  on  $\partial\Omega$  implies  $u \leq v$  in  $\Omega$ .*

*Proof.* We set

$$v_\varepsilon := v + \varepsilon, \quad \varepsilon > 0.$$

We claim that  $u \leq v_\varepsilon$  in  $\Omega$  for each  $\varepsilon > 0$ . We assume to the contrary. Set

$$\Omega_0 := \{x \in \Omega : u(x) > v_\varepsilon(x)\}.$$

Since  $u \leq v_\varepsilon$  on  $\partial\Omega$ , we see that  $\Omega_0$  is compactly contained in  $\Omega$  and  $u = v_\varepsilon$  on  $\partial\Omega_0$ . Moreover,

$$\Delta_\infty^h v_\varepsilon = \Delta_\infty^h v \leq f(x, v) \leq f(x, v_\varepsilon) \quad \text{in } \Omega_0,$$

where we have used that  $f(x, t)$  is non-decreasing in  $t$ .

We define

$$\vartheta(x, t) = \begin{cases} f(x, t), & t \geq \varepsilon, \\ f(x, \varepsilon), & t < \varepsilon. \end{cases}$$

Since  $u > v_\varepsilon \geq \varepsilon$  in  $\Omega_0$ , we have

$$\Delta_\infty^h u \geq \vartheta(x, u) \quad \text{and} \quad \Delta_\infty^h v_\varepsilon \leq \vartheta(x, v_\varepsilon), \quad \text{in } \Omega_0.$$

Since  $u = v_\varepsilon$  on  $\partial\Omega_0$ , by Lemma 2.3, we have  $u \leq v_\varepsilon$  in  $\Omega_0$ , which is a contradiction.  $\square$

### 3. EXISTENCE WHEN $f(x, t)$ NON-DECREASING IN $t$

We first construct the viscosity subsolution to the problem (1.1), and then establish the existence result using Perron’s method and the Lipschitz continuity of infinity harmonic functions.

**Lemma 3.1.** *Let  $q \in C(\partial\Omega)$ . Suppose that  $f(x, t) \in C(\Omega \times \mathbb{R}, \mathbb{R})$  is non-decreasing in  $t$ . If  $f$  satisfies the condition*

$$\sup_{x \in \Omega} f(x, t) < \infty, \quad \text{for each } t \in \mathbb{R}, \tag{3.1}$$

*then (1.1) has a viscosity subsolution  $\underline{u} \in C(\overline{\Omega})$ .*

*Proof.* Let

$$\ell_1 := \inf_{x \in \partial\Omega} q(x). \tag{3.2}$$

Choose a positive constant  $M_1$  such that  $M_1^h \geq \sup_{x \in \Omega} f(x, \ell_1)$  and then a constant  $d_1$  such that

$$d_1 \leq \frac{\ell_1}{M_1} - \gamma(\text{diam}(\Omega))^{(h+1)/h}, \quad \text{where } \gamma = \frac{1}{h+1} h^{(h+1)/h}.$$

We define

$$\underline{u}(x) := M_1(\gamma|x - z|^{(h+1)/h} + d_1), \quad z \in \partial\Omega.$$

Obviously,  $\underline{u} \in C(\overline{\Omega})$ . One can verify that  $\Delta_\infty^h \underline{u} = M_1^h \geq f(x, \ell_1)$  and  $\underline{u} \leq \ell_1$  in  $\Omega$  due to the choice of  $M_1$  and  $d_1$ . Since  $f(x, t)$  is non-decreasing in  $t$ , we obtain

$$\Delta_\infty^h \underline{u} \geq f(x, \underline{u}), \quad \text{in } \Omega,$$

$$\underline{u} \leq q, \quad \text{on } \partial\Omega.$$

That is,  $\underline{u}$  is a desired viscosity subsolution to (1.1).  $\square$

We denote

$$\aleph_+ := \{\tilde{u} \in C(\bar{\Omega}) : \Delta_\infty^h \tilde{u} \geq f(x, \tilde{u}) \text{ in } \Omega, \text{ and } \tilde{u} \leq q \text{ on } \partial\Omega\}.$$

Lemma 3.1 shows that the set  $\aleph_+$  is non-empty. We define the function

$$u(x) := \sup_{\alpha \in \aleph_+} \tilde{u}(x), \quad x \in \bar{\Omega}. \quad (3.3)$$

**Remark 3.2.** If  $f$  is non-negative and  $\ell_2 := \sup_{x \in \partial\Omega} q(x)$ , the comparison principle, Lemma 2.2, implies  $\tilde{u} \leq \ell_2$  for all  $\tilde{u} \in \aleph_+$ . Then the function defined in (3.3) satisfies  $\tilde{u} \leq u \leq \ell_2$  in  $\bar{\Omega}$ .

**Remark 3.3.** If  $f$  is non-negative and  $\tilde{u}$  is a viscosity subsolution of (2.1), then  $\tilde{u}$  is locally Lipschitz continuous in  $\Omega$  (see for example [18, Lemma 4.1]). Hence, we have the function  $u$  defined in (3.3) is locally Lipschitz continuous.

*Proof of Theorem 1.1.* The existence is an application of standard Perron's method. Since  $f$  is non-negative and  $\sup_{x \in \Omega} f(x, t) < \infty$ , Lemma 3.1 implies that problem (1.1) has a viscosity subsolution  $\underline{u} \in C(\bar{\Omega})$ .

Indeed, the function  $u$  defined in (3.3) is a viscosity solution of (1.1).

**Step 1.** We first claim that  $u$  is a viscosity subsolution of (1.1). Indeed, we have  $u$  is locally Lipschitz continuous in  $\Omega$  by Remark 3.3. For every  $x_0 \in \Omega$  and  $\varphi \in C^2(\Omega)$ , if  $u - \varphi$  has a local maximum at  $x_0$ , i.e. for some small  $1 > \rho > 0$ ,

$$u(x) - \varphi(x) \leq u(x_0) - \varphi(x_0), \quad x \in B_\rho(x_0) \subseteq \Omega,$$

we want to show that

$$\Delta_\infty^h \varphi(x_0) \geq f(x_0, u(x_0)).$$

Since  $u(x_0) = \sup_{\tilde{u} \in \aleph_+} \tilde{u}(x_0)$ , we take a sequence  $\{\tilde{u}_k\}$  in  $\aleph_+$  such that

$$u(x_0) - \tilde{u}_k(x_0) < \delta/k,$$

for each positive integer  $k$  and  $0 < \delta < \rho^{2(h+1)}$ . Then

$$\tilde{u}_k(x) - \varphi(x) \leq u(x) - \varphi(x) \leq u(x_0) - \varphi(x_0) \leq \tilde{u}_k(x_0) - \varphi(x_0) + \delta/k, \quad (3.4)$$

for  $x \in B_\rho(x_0)$ . Therefore,

$$\tilde{u}_k(x) - \varphi(x) - \delta/k \leq \tilde{u}_k(x_0) - \varphi(x_0), \quad x \in B_\rho(x_0),$$

which yields

$$\tilde{u}_k(x) - [\varphi(x) + |x - x_0|^{2(h+1)}] < \tilde{u}_k(x) - \varphi(x) - \delta/k \leq \tilde{u}_k(x_0) - \varphi(x_0), \quad (3.5)$$

for  $x \in B_\rho(x_0) \setminus \bar{B}_{(\rho/k)^{1/[2(h+1)]}}(x_0)$ . Let

$$\varphi_0(x) := \varphi(x) + |x - x_0|^{2(h+1)}.$$

Then inequality (3.5) implies that the maximum of the function  $\tilde{u}_k(x) - \varphi_0(x)$  in  $\bar{B}_\rho(x_0)$ , occurs at some  $x_k \in \bar{B}_{(\rho/k)^{1/[2(h+1)]}}(x_0)$ . In particular,

$$\tilde{u}_k(x_0) - \varphi(x_0) = \tilde{u}_k(x_0) - \varphi_0(x_0) \leq \tilde{u}_k(x_k) - \varphi_0(x_k). \quad (3.6)$$

Since  $x_k \neq x_0$ , a direct calculation yields

$$D\varphi_0(x_k) = D\varphi(x_k) + 2(h+1)|x_k - x_0|^{2h+1} \frac{x_k - x_0}{|x_k - x_0|}$$



and

$$D^2\varphi_0(x_k) = D^2\varphi(x_k) + 2(h+1)(2h+1)|x_k - x_0|^{2h} \frac{x_k - x_0}{|x_k - x_0|} \otimes \frac{x_k - x_0}{|x_k - x_0|} + 2(h+1)|x_k - x_0|^{2h+1} \left( \frac{I}{|x_k - x_0|} - \frac{(x_k - x_0) \otimes (x_k - x_0)}{|x_k - x_0|^3} \right),$$

and one can check that

$$\Delta_\infty^h \varphi_0(x_k) = \Delta_\infty^h \varphi(x_k) + O((\delta/k)^{1/(2h)}) \geq f(x_k, \tilde{u}(x_k)). \tag{3.7}$$

Combining (3.4) and (3.6), we have

$$\begin{aligned} \tilde{u}_k(x_0) - \varphi(x_0) &\leq \tilde{u}_k(x_k) - [\varphi(x_k) + |x_k - x_0|^{2(h+1)}] \\ &\leq u(x_0) - \varphi(x_0) - |x_k - x_0|^{2(h+1)}. \end{aligned} \tag{3.8}$$

Inequality (3.8) shows that  $\lim_{k \rightarrow \infty} \tilde{u}_k(x_k) = u(x_0)$ . Letting  $k \rightarrow \infty$  in (3.7), we have

$$\Delta_\infty^h \varphi(x_0) \geq f(x_0, u(x_0)).$$

**Step 2.** Next we show that  $u \in C(\bar{\Omega})$  and  $u = q$  on  $\partial\Omega$ . We first prove that  $u = q$  on  $\partial\Omega$ . By the definition of  $u$ , we have  $u \leq q$  on  $\partial\Omega$ . Then we just have to prove  $u \geq q$  on  $\partial\Omega$ . Let  $z \in \partial\Omega$  and  $\varepsilon > 0$  be arbitrary. Since  $q \in C(\partial\Omega)$ , there exists some  $r > 0$  such that

$$|q(x) - q(z)| < \varepsilon, \quad x \in \partial\Omega \cap B_r(z).$$

Set

$$U(x) := q(z) - \varepsilon - \frac{C}{h+1} [M^{(h+1)/h} - (M - h|x - z|)^{(h+1)/h}],$$

where  $M > h(\text{diam}(\Omega))$ ,  $\ell := \sup_{\partial\Omega} |q|$ , and  $C > 0$  is chosen large enough such that

$$\frac{C}{h+1} [M^{(h+1)/h} - (M - hr)^{(h+1)/h}] \geq 2\ell \quad \text{and} \quad C^h \geq \sup_{x \in \Omega} f(x, \ell).$$

Direct computations show that

$$\begin{aligned} \Delta_\infty^h U(x) &= C^h \geq f(x, \ell), \quad \text{in } \Omega, \\ U(x) &\leq q(z) - \varepsilon, \quad \text{in } B_{\text{diam}(\Omega)}(z). \end{aligned}$$

Hence

$$U(x) \leq q(z) - \varepsilon \leq q(x), \quad \text{in } \partial\Omega \cap B_r(z).$$

On  $\partial\Omega \setminus B_r(z)$ , we have

$$U(x) \leq q(z) - \varepsilon - \frac{C}{h+1} [M^{(h+1)/h} - (M - hr)^{(h+1)/h}] \leq q(z) - 2\ell \leq -\ell \leq q(x).$$

Then,  $U \leq q$  on  $\partial\Omega$ . In particular,  $U \leq \ell$  in  $\Omega$ . Since  $f$  is non-decreasing,

$$\Delta_\infty^h U(x) \geq f(x, \ell) \geq f(x, U), \quad \text{in } \Omega.$$

Therefore,  $U \in \mathbb{N}_+$ . Consequently, we have  $U \leq u$  in  $\bar{\Omega}$ . In particular,  $U(z) = q(z) - \varepsilon \leq u(z)$ . Since  $\varepsilon > 0$  is arbitrary, we obtain  $q(z) \leq u(z)$ . Hence, we have  $u = q$  on  $\partial\Omega$ .

Now we show that  $u \in C(\bar{\Omega})$ . Let  $z \in \partial\Omega$  and  $B_r(z)$  be as above, and  $\{x_k\}$  a sequence in  $\Omega$  such that  $\lim_{k \rightarrow \infty} x_k = z$ . Since the lower semi-continuity of  $u$ , we have

$$\liminf_{k \rightarrow \infty} u(x_k) \geq u(z) = q(z).$$

Next, we show that  $\limsup_{k \rightarrow \infty} u(x_k) \leq q(z)$ . Let

$$w_\varepsilon(x) = q(z) + \varepsilon + [\ell_1 - q(z)] \frac{|x - z|}{r}, \quad \text{in } B_r(z) \cap \Omega,$$

where  $\ell_1$  is as in (3.2). Clearly,  $w_\varepsilon \in C(B_r(z))$  and  $w_\varepsilon(x) = \ell_1 + \varepsilon$  on  $\partial B_r(z) \cap \Omega$ . By the selection of  $B_r(z)$ , we have

$$w_\varepsilon(x) \geq q(z) + \varepsilon > q(x), \quad \forall x \in \partial\Omega \cap B_r(z).$$

Direct computations show that

$$\Delta_\infty^h w_\varepsilon(x) = 0, \quad \text{in } \Omega \cap B_r(z).$$

For any  $\tilde{u} \in \mathbb{N}_+$ , we obtain  $\tilde{u} \leq w_\varepsilon$  on  $\partial(\Omega \cap B_r(z))$ . Since  $\Delta_\infty^h \tilde{u} \geq 0$  in the viscosity sense, by the comparison principle of infinity harmonic functions [7, 21], we have  $\tilde{u} \leq w_\varepsilon$  in  $\Omega \cap B_r(z)$ . Then  $u \leq w_\varepsilon$  in  $\Omega \cap B_r(z)$ . If  $x_k \in \Omega \cap B_r(z)$  and  $x_k \rightarrow z$ , then

$$u(z) \leq \liminf_{k \rightarrow \infty} u(x_k) \leq \limsup_{k \rightarrow \infty} u(x_k) \leq \lim_{k \rightarrow \infty} w_\varepsilon(x_k) = q(z) + \varepsilon = u(z) + \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we have  $u \in C(\overline{\Omega})$ . Moreover, we obtain  $u \in \mathbb{N}_+$ .

**Step 3.** Next, we will prove that  $u$  is a viscosity supersolution. We assume to the contrary. Then there exist  $x_0 \in \Omega$  and  $\varphi \in C^2(\Omega)$  such that

$$\varphi(x_0) = u(x_0), \quad u(x) - \varphi(x) \geq 0, \quad x \in B_\rho(x_0) \subseteq \Omega,$$

for some  $\rho > 0$ , but

$$\Delta_\infty^h \varphi(x_0) > f(x_0, u(x_0)).$$

Let  $d(x) := \text{dist}(x, \partial\Omega)$ . Since the continuity of  $f(x, t)$ , we can choose  $0 < \varepsilon_0 < \min\{1, \rho, (d(x_0)/2)^{4(h+1)}\}$  such that

$$\Delta_\infty^h \varphi(x_0) > f(x_0, \varphi(x_0) + \varepsilon), \quad \forall 0 < \varepsilon \leq \varepsilon_0. \quad (3.9)$$

For  $0 < \varepsilon \leq \varepsilon_0$ , we define

$$\varphi_\varepsilon(x) := \varphi(x) - \sqrt{\varepsilon}|x - x_0|^{2(h+1)} + \varepsilon.$$

For  $x \neq x_0$ , a direct calculation yields

$$D\varphi_\varepsilon(x) = D\varphi(x) - 2(h+1)\sqrt{\varepsilon}|x - x_0|^{2h+1} \frac{x - x_0}{|x - x_0|}$$

and

$$\begin{aligned} D^2\varphi_\varepsilon(x) = & D^2\varphi(x) - 2(h+1)(2h+1)\sqrt{\varepsilon}|x - x_0|^{2h} \frac{x - x_0}{|x - x_0|} \otimes \frac{x - x_0}{|x - x_0|} \\ & - 2(h+1)\sqrt{\varepsilon}|x - x_0|^{2h+1} \left( \frac{I}{|x - x_0|} - \frac{(x - x_0) \otimes (x - x_0)}{|x - x_0|^3} \right). \end{aligned}$$

Hence, we obtain

$$\Delta_\infty^h \varphi_\varepsilon(x) = \Delta_\infty^h \varphi(x) + O(\sqrt{\varepsilon}|x - x_0|^{2h}), \quad \text{as } x \rightarrow x_0.$$

Then, by (3.9), we have

$$\Delta_\infty^h \varphi_\varepsilon(x_0) = \Delta_\infty^h \varphi(x_0) > f(x_0, \varphi(x_0) + \varepsilon) = f(x_0, \varphi_\varepsilon(x_0)).$$

We claim that there exists an  $\varepsilon_1$ , with  $0 < \varepsilon_1 \leq \varepsilon_0$ , such that

$$\Delta_\infty^h \varphi_{\varepsilon_1}(x) > f(x, \varphi_{\varepsilon_1}(x)), \quad \forall x \in B_{\varepsilon_1^{1/[4(h+1)]}}(x_0).$$

We argue by contradiction. Then, for each  $\varepsilon > 0$  with  $\varepsilon$  small enough, there exists an  $x_\varepsilon \in B_{\varepsilon_1^{1/[4(h+1)]}}(x_0)$  such that  $\Delta_\infty^h \varphi_\varepsilon(x_\varepsilon) \leq f(x_\varepsilon, \varphi_\varepsilon(x_\varepsilon))$ . Since  $x_\varepsilon \rightarrow x_0$ , we observe that

$$\lim_{\varepsilon \rightarrow 0} \Delta_\infty^h \varphi_\varepsilon(x_\varepsilon) = \Delta_\infty^h \varphi(x_0) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} f(x_\varepsilon, \varphi_\varepsilon(x_\varepsilon)) = f(x_0, \varphi(x_0)).$$

We conclude that  $\Delta_\infty^h \varphi(x_0) \leq f(x_0, \varphi(x_0))$ .

Since  $\varphi_\varepsilon(x_0) > u(x_0)$ , we can take  $0 < s_1 < \varepsilon_1^{1/[4(h+1)]}$  such that  $u(x) < \varphi_{\varepsilon_1}(x)$  for all  $x \in B_{s_1}(x_0)$ . Thus, we have

$$\begin{aligned} u(x) &< \varphi_{\varepsilon_1}(x), \quad \forall x \in B_{s_1}(x_0), \\ \Delta_\infty^h \varphi_{\varepsilon_1}(x) &> f(x, \varphi_{\varepsilon_1}(x)), \quad \forall x \in B_{\varepsilon_1^{1/[4(h+1)]}}(x_0), \\ u(x) &> \varphi_{\varepsilon_1}(x), \quad \forall x \in B_\rho(x_0) \setminus \overline{B_{\varepsilon_1^{1/[4(h+1)]}}}(x_0). \end{aligned} \tag{3.10}$$

We define

$$u_*(x) = \begin{cases} u(x), & x \in \Omega \setminus \overline{B_{\varepsilon_1^{1/[4(h+1)]}}}(x_0), \\ \sup\{\varphi_{\varepsilon_1}(x), u(x)\}, & x \in B_{\varepsilon_1^{1/[4(h+1)]}}(x_0). \end{cases} \tag{3.11}$$

It is obvious that  $u_* \in C(\overline{\Omega})$  is in  $\mathbb{N}_+$ . However, by (3.10), we see that

$$u_*(x) = \varphi_{\varepsilon_1}(x) > u(x), \quad x \in B_{s_1}(x_0),$$

which is impossible due to the definition of  $u$ . Thus,  $u$  is a viscosity supersolution, and this completes the proof that  $u$  is a viscosity solution to (1.1) in  $\Omega$ .

The uniqueness follows by the comparison principle, Theorem 2.4. □

**Remark 3.4.** If  $f$  is non-positive and  $\inf_{x \in \Omega} f(x, t) > -\infty$  for each  $t \in \mathbb{R}$  as in Remark 1.2, we consider  $\hat{f}(x, t) := -f(x, -t)$ . Then  $\hat{f} \in C(\Omega \times \mathbb{R}, [0, \infty))$  is non-decreasing and  $\sup_{x \in \Omega} \hat{f}(x, t) < \infty$  for each  $t$ . Therefore, the Dirichlet problem (1.1) has a viscosity solution  $u$  with the right-hand side  $\hat{f}$  and boundary data  $-q$  by Theorem (1.1).

#### 4. EXISTENCE WHEN $f(x, t)$ NON-INCREASING IN $t$

In this section, we investigate the existence of viscosity solutions to the problem (1.1), when  $f(x, t)$  is non-increasing in  $t$ . We first prove a stability result of the viscosity solutions. Then under the assumption that the problem (1.1) has a viscosity subsolution with  $f$  replaced by  $f + \varepsilon$ ,  $\varepsilon > 0$ , we prove the existence of the viscosity solution to (1.1). Finally, we use the iteration method and Theorem 1.1 to establish the existence of the viscosity solution to (1.1).

**Lemma 4.1.** *Let  $\{\xi_k\}_{k=1}^\infty$  be a sequence of non-negative functions in  $C(\Omega)$  such that  $\xi_k \rightarrow \xi$  locally uniformly in  $\Omega$  for some  $\xi \in C(\Omega)$ . For each positive integer  $k$ , let  $u_k \in C(\overline{\Omega})$  be a viscosity solution to the problem*

$$\begin{aligned} \Delta_\infty^h u_k &= \xi_k, \quad \text{in } \Omega, \\ u_k &= q, \quad \text{on } \partial\Omega \end{aligned}$$

*such that  $u_0 \leq u_k \leq u_\infty$  in  $\overline{\Omega}$ , for some functions  $u_0$  and  $u_\infty$  in  $C(\overline{\Omega})$ , with  $u_0 = u_\infty = q$  on  $\partial\Omega$ . Then  $\{u_k\}$  has a subsequence that converges locally uniformly in  $\Omega$  to a viscosity solution  $u \in C(\overline{\Omega})$  to the problem*

$$\begin{aligned} \Delta_\infty^h u &= \xi, \quad \text{in } \Omega, \\ u &= q, \quad \text{on } \partial\Omega. \end{aligned}$$

*Proof.* Set  $M := \sup_{\Omega} u_{\infty} - \inf_{\Omega} u_0$ . Clearly, we have  $\sup_{\Omega} u_k - \inf_{\Omega} u_k \leq M$ , for every  $k = 1, 2, \dots$ . Let  $K$  be any compact subset of  $\Omega$  and  $d := \text{dist}(K, \partial\Omega)$ . We take  $R > 0$  such that  $4R < d$ . Since  $\Delta_{\infty}^h u_k \geq 0$  in  $\Omega$ , by [6, Lemma 2.9], we obtain

$$|u_k(x) - u_k(y)| \leq M \frac{|x - y|}{R}, \quad \forall z \in K, x, y \in B_{R/2}(z).$$

By compactness, we obtain  $\{u_k\}$  are equicontinuous in  $K$ . On taking an exhaustion of  $\Omega$  by subdomains compactly contained in  $\Omega$ , we apply the standard method of Cantor diagonalization to extract a subsequence of  $\{u_k\}$  that converge uniformly on compact subsets of  $\Omega$ . For simplicity we will continue to denote such subsequence by  $\{u_k\}$ . Set

$$u(x) := \lim_{k \rightarrow \infty} u_k(x), \quad x \in \Omega.$$

We extend this definition to the closure  $\bar{\Omega}$  by defining  $u = q$  on  $\partial\Omega$ . By the assumption, we have  $u_0 \leq u \leq u_{\infty}$  in  $\bar{\Omega}$ . This means that  $u \in C(\bar{\Omega})$ .

Next, we show that  $\Delta_{\infty}^h u = \xi$  in the viscosity sense. Suppose that  $\varphi \in C^2(\Omega)$  and  $u - \varphi$  has a local maximum at some  $x_0 \in \Omega$ , i.e.

$$u(x) - \varphi(x) \leq u(x_0) - \varphi(x_0), \quad x \in B_r(x_0) \subseteq \Omega$$

for some  $r > 0$ . Suppose that  $x_k$  is a point of maximum of

$$u_k(x) - (\varphi(x) + \frac{\varepsilon}{2}|x - x_0|^2), \quad \varepsilon > 0, x \in \bar{B}_r(x_0).$$

Particularly,

$$u_k(x_k) - (\varphi(x_k) + \frac{\varepsilon}{2}|x_k - x_0|^2) \geq u_k(x_0) - \varphi(x_0). \quad (4.1)$$

Since  $x_k \in \bar{B}_r(x_0)$ , by passing to a subsequence,  $x_k \rightarrow \hat{x}$ , for some  $\hat{x} \in \bar{B}_r(x_0)$ , letting  $k \rightarrow \infty$  in (4.1), we have

$$u(\hat{x}) - (\varphi(\hat{x}) + \frac{\varepsilon}{2}|\hat{x} - x_0|^2) \geq u(x_0) - \varphi(x_0),$$

i.e.

$$\frac{\varepsilon}{2}|\hat{x} - x_0|^2 \leq u(\hat{x}) - \varphi(\hat{x}) - (u(x_0) - \varphi(x_0)) \leq 0.$$

Then we have  $\hat{x} = x_0$ . Thus,  $x_k \in B_{r/2}(x_0)$  for sufficiently large  $k$ . Since  $u_k$  is a viscosity subsolution and  $x_k$  is a point of local maximum of  $u_k(x) - (\varphi(x) + \frac{\varepsilon}{2}|x - x_0|^2)$  in  $B_r(x_0)$ , we have

$$\Delta_{\infty}^h \varphi(x_k) + O(\varepsilon) \geq \xi_k(x_k). \quad (4.2)$$

Taking the limit in (4.2) and recalling that  $\xi_k \rightarrow \xi$  locally uniformly in  $\Omega$ , we obtain

$$\Delta_{\infty}^h \varphi(x_0) + O(\varepsilon) \geq \xi(x_0).$$

Letting  $\varepsilon \rightarrow 0$ , we have  $\Delta_{\infty}^h u \geq \xi$  in the viscosity sense. Similarly, we can prove that  $u$  is a viscosity supersolution.  $\square$

Now we first give an existence result under the condition that problem (1.1) has a viscosity subsolution with  $f$  replaced by  $f + \varepsilon$ . Then we combine the iteration method and Theorem 1.1 to establish the existence result. The idea is that the existence of an appropriate viscosity subsolution leads to the existence of a viscosity solution.

**Theorem 4.2.** *Let  $f(x, t) \in C(\Omega \times \mathbb{R})$  be positive, non-increasing in  $t$ , and satisfy the condition*

$$\sup_{x \in \Omega} f(x, t) < \infty, \quad \text{for each } t \in \mathbb{R}.$$

*If the problem (1.1) has a viscosity subsolution with  $f$  replaced by  $f + \varepsilon$ , for some  $\varepsilon > 0$ , then there exists a viscosity solution  $u \in C(\bar{\Omega})$  to (1.1).*

*Proof.* By the assumption, let  $\eta_0 \in C(\bar{\Omega})$  satisfy

$$\begin{aligned} \Delta_\infty^h \eta_0 &\geq f(x, \eta_0) + \varepsilon, & \text{in } \Omega, \\ \eta_0 &\leq q, & \text{on } \partial\Omega. \end{aligned}$$

Then we define a sequence  $\{\eta_k\}_{k=1}^\infty$  satisfying

$$\begin{aligned} \Delta_\infty^h \eta_k &= f(x, \eta_{k-1}) + \varepsilon/k, & \text{in } \Omega, \\ \eta_k &= q, & \text{on } \partial\Omega. \end{aligned}$$

The existence of  $\eta_k$  is ensured by Theorem 1.1. Since  $\Delta_\infty^h \eta_1 = f(x, \eta_0) + \varepsilon$  and  $\Delta_\infty^h \eta_0 \geq f(x, \eta_0) + \varepsilon$  with  $\eta_0 \leq \eta_1$  on  $\partial\Omega$ , by Lemma 2.3, we have  $\eta_0 \leq \eta_1$  in  $\bar{\Omega}$ . Suppose  $\eta_{k-1} \leq \eta_k$  in  $\Omega$  for  $k \geq 2$ . Then

$$\Delta_\infty^h \eta_k = f(x, \eta_{k-1}) + \varepsilon/k > f(x, \eta_k) + \varepsilon/(k+1) = \Delta_\infty^h \eta_{k+1},$$

and hence Lemma 2.3 implies  $\eta_k \leq \eta_{k+1}$  in  $\bar{\Omega}$ . Then we have  $\eta_k \leq \eta_{k+1}$  in  $\bar{\Omega}$  for all  $k$ . By Theorem 1.1, we let  $V \in C(\bar{\Omega})$  satisfy

$$\begin{aligned} \Delta_\infty^h V &= 0, & \text{in } \Omega, \\ V &= q, & \text{on } \partial\Omega. \end{aligned}$$

By the comparison principle of infinity harmonic functions [7, 21], we have that  $\eta_k \leq V$  in  $\bar{\Omega}$  for all  $k = 0, 1, 2, \dots$ . Therefore, we have

$$\eta_0 \leq \eta_1 \leq \dots \leq \eta_k \leq \eta_{k+1} \leq \dots \leq V, \quad \text{in } \bar{\Omega}.$$

Hence, the sequence  $\eta_k$  converges uniformly in  $\Omega$ . Let

$$\eta(x) := \lim_{k \rightarrow \infty} \eta_k(x), \quad x \in \bar{\Omega}.$$

It is clear that  $\eta \in C(\Omega)$ . We take

$$\xi_k := f(x, \eta_{k-1}) + \varepsilon/k, \quad \xi := f(x, \eta).$$

Lemma 4.1 implies that  $\eta \in C(\bar{\Omega})$  is a viscosity solution to (1.1). □

Theorem 4.2 provides us with an approach to the existence problem, but it suffers from the shortcoming that we need a viscosity subsolution for the function  $f + \varepsilon$ , for some  $\varepsilon > 0$ . Next we impose the condition (4.3) on the domain to remove the assumption on the existence of the viscosity subsolution.

**Lemma 4.3.** *Let  $q \in C(\partial\Omega)$ ,  $f$  be a non-increasing function that satisfies condition*

$$\sup_{x \in \Omega} f(x, t) < \infty, \quad \text{for each } t \in \mathbb{R}.$$

*If a bounded domain  $\Omega$  satisfies*

$$\text{diam}(\Omega) \leq \left( \frac{\ell_1 - \lambda_0}{\gamma C} \right)^{h/(h+1)}, \tag{4.3}$$

*where  $\gamma = \frac{1}{h+1} h^{(h+1)/h}$ ,  $\lambda_0 < \ell_1 := \inf_{\partial\Omega} q$ , and  $C \geq (\sup_{\Omega} f(x, \lambda_0))^{1/h}$ , then (1.1) has a viscosity subsolution in  $C(\bar{\Omega})$ .*

*Proof.* If  $f(x, t) \equiv g(x)$ , the existence follows immediately by Theorem 1.1. Thus we consider the inhomogeneous term depending on the variable  $t$ . We choose  $d$  satisfying

$$\frac{\lambda_0}{C} \leq d \leq \frac{\ell_1}{C} - \gamma(\text{diam}(\Omega))^{(h+1)/h}. \quad (4.4)$$

We define

$$W(x) := C(\gamma|x - z|^{(h+1)/h} + d), \quad z \in \partial\Omega. \quad (4.5)$$

Clearly  $W \in C^\infty(\Omega)$  and one can verify that

$$\Delta_\infty^h W = C^h \geq \sup_\Omega f(x, \lambda_0) \geq f(x, \lambda_0), \quad \text{in } \Omega.$$

With the choice of  $d$  as in (4.4), we have  $\lambda_0 \leq W \leq \ell_1$  in  $\bar{\Omega}$ . Since  $f(x, t)$  is non-increasing in  $t$ , we obtain that  $W$  satisfies

$$\Delta_\infty^h W \geq f(x, W), \quad \text{in } \Omega.$$

Recalling that  $W \leq q$  on  $\partial\Omega$ , we conclude that  $W$  is a viscosity subsolution of (1.1) in  $\Omega$ .  $\square$

Note that the existence of viscosity subsolution depends on the size of the domain when  $f$  is non-increasing. Based on this point, we are ready to prove the existence result with the iteration method and Theorem 1.1.

*Proof of Theorem 1.3.* Since  $C \geq (\sup_\Omega f(x, \lambda_0))^{1/h} > 0$  and  $\Omega$  satisfies the condition (4.3), Theorem 4.2 implies the existence of a viscosity subsolution  $w$  defined in (4.5) satisfying

$$\begin{aligned} \Delta_\infty^h W &= C^h \geq \sup_\Omega f(x, \lambda_0) > 0, \quad \text{in } \Omega, \\ W &\leq q, \quad \text{on } \partial\Omega, \end{aligned}$$

and  $\lambda_0 \leq W \leq \ell_1$ . Then we have  $f(x, W) \leq \sup_\Omega f(x, \lambda_0) \leq C^h$ . Denote  $u_0 := W$ , and we recursively define a sequence  $\{u_k\}$  in  $C(\bar{\Omega})$  as follows for  $k \geq 1$ . By Theorem 1.1, we let  $u_k$  satisfy

$$\begin{aligned} \Delta_\infty^h u_k &= f(x, u_{k-1}), \quad \text{in } \Omega, \\ u_k &= q, \quad \text{on } \partial\Omega. \end{aligned} \quad (4.6)$$

By induction, we show that  $W \leq u_k$  in  $\bar{\Omega}$  for all  $k \geq 1$ . Note that

$$\Delta_\infty^h u_1 = f(x, u_0) = f(x, W) \leq f(x, \lambda_0) \leq C^h.$$

Thus,

$$\begin{aligned} \Delta_\infty^h u_1 &\leq C^h, \quad \text{in } \Omega, \\ u_1 &= q, \quad \text{on } \partial\Omega, \end{aligned}$$

while

$$\begin{aligned} \Delta_\infty^h W &= C^h, \quad \text{in } \Omega, \\ W &\leq q, \quad \text{on } \partial\Omega. \end{aligned}$$

Therefore, by Lemma 2.3, we have  $W \leq u_1$  in  $\bar{\Omega}$ . Suppose  $W \leq u_k$  in  $\bar{\Omega}$  for some  $k \geq 1$ . Then

$$\begin{aligned} \Delta_\infty^h u_{k+1} &= f(x, u_k) \leq f(x, W) \leq f(x, \lambda_0) \leq C^h, \quad \text{in } \Omega, \\ u_{k+1} &= q, \quad \text{on } \partial\Omega. \end{aligned}$$

Lemma 2.3 again implies  $W \leq u_{k+1}$  in  $\bar{\Omega}$ . This proves the claim.

Since  $\lambda_0 \leq W \leq u_k$  in  $\Omega$  for all  $k$ , we have

$$\begin{aligned} \Delta_{\infty}^h u_k &= f(x, u_{k-1}) \leq f(x, \lambda_0) \leq C^h, \quad \text{in } \Omega, \\ u_k &= q, \quad \text{on } \partial\Omega. \end{aligned}$$

Let  $v_1 \in C(\bar{\Omega})$  be the viscosity solution to the problem

$$\begin{aligned} \Delta_{\infty}^h v_1 &= C^h, \quad \text{in } \Omega, \\ v_1 &= q, \quad \text{on } \partial\Omega. \end{aligned}$$

By Lemma 2.3, we obtain  $u_k \geq v_1$  in  $\bar{\Omega}$ . Finally, let  $v_2 \in C(\bar{\Omega})$  satisfy

$$\begin{aligned} \Delta_{\infty}^h v_2 &= 0, \quad \text{in } \Omega, \\ v_2 &= q, \quad \text{on } \partial\Omega. \end{aligned}$$

Since  $\Delta_{\infty}^h u_k \geq 0$  in  $\Omega$ , by the comparison principle in [7, 21], we see that  $u_k \leq v_2$  in  $\bar{\Omega}$  for all  $k$ . In summary, we have constructed a sequence  $\{u_k\}$  of viscosity solutions to (4.6) such that

$$\begin{aligned} v_1 &\leq u_k \leq v_2, \quad \text{in } \bar{\Omega}, \\ v_1 &= u_k = v_2 = q, \quad \text{on } \partial\Omega. \end{aligned}$$

Therefore, by Lemma 4.1, we can get the existence of the viscosity solution to (1.1).  $\square$

## 5. SINGULAR BOUNDARY VALUE PROBLEM

In this section, we show the existence and the uniqueness of the viscosity solution to the singular problem (1.5). Moreover, when the domain satisfies some regular condition, we analyze the asymptotic behavior near the boundary of the viscosity solution. We now prove that the singular problem (1.5) has a viscosity supersolution.

**Lemma 5.1.** *Let  $b \in C(\Omega)$  be positive and  $\sup_{x \in \Omega} b(x) < \infty$ . If  $g$  belongs to  $C^1((0, \infty), (0, \infty))$  is non-increasing with  $\lim_{t \rightarrow 0^+} g(t) = \infty$ , then problem (1.5) has a viscosity supersolution.*

*Proof.* By Theorem 1.1, the following problem has a viscosity solution  $w \in C(\bar{\Omega})$ ,

$$\begin{aligned} \Delta_{\infty}^h w &= -b(x), \quad \text{in } \Omega, \\ w &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

We define  $v = \eta^{-1}(w) \in C(\bar{\Omega})$ , where

$$\eta(t) = \int_0^t \frac{1}{\sqrt[h]{g(s)}} ds, \quad t > 0. \tag{5.2}$$

Because  $w \in C(\bar{\Omega})$ , we have  $v \in C(\bar{\Omega})$ . Suppose that  $\varphi \in C^2(\Omega)$  and  $v - \varphi$  has a local minimum at  $z \in \Omega$ , i.e. for some small  $\delta > 0$ ,

$$v(z) = \varphi(z), \quad v(x) \geq \varphi(x), \quad x \in B_{\delta}(z) \subseteq \Omega.$$

By Lemma 2.2, we have  $w > 0$  in  $\Omega$ , and then  $v > 0$  in  $\Omega$ . In particular,  $v(z) > 0$  and we can assume that  $\delta$  is small enough such that  $\varphi > 0$  in  $B_{\delta}(z)$ . Since  $\eta$  is increasing, we have

$$w(z) = \eta(\varphi(z)), \quad w(x) \geq \eta(\varphi(x)), \quad x \in B_{\delta}(z).$$

Let  $\psi(x) := \eta(\varphi(x)) \in C^2(\Omega)$  such that  $w - \psi$  has a local minimum at  $z$ . Since  $w$  is a viscosity solution to (5.1), we have

$$\Delta_\infty^h \psi(z) \leq -b(z), \quad (5.3)$$

in the viscosity sense. In  $B_\delta(z)$ , by a simple calculation,

$$\begin{aligned} \Delta_\infty^h \psi &= |\eta'(\varphi)D\varphi|^{h-1} \eta''(\varphi)|D\varphi|^2 + |\eta'(\varphi)|^h \Delta_\infty^h \varphi \\ &= -\frac{g'(\varphi)}{hg(\varphi)^2} |D\varphi|^{h+1} + \frac{1}{g(\varphi)} \Delta_\infty^h \varphi, \end{aligned}$$

where we have used

$$\eta'(t) = \frac{1}{\sqrt[h]{g(t)}} \quad \text{and} \quad \eta''(t) = -\frac{g'(t)}{hg(t)^{1+1/h}}.$$

Since  $g$  is non-increasing, we obtain

$$\Delta_\infty^h \psi \geq \frac{1}{g(\varphi)} \Delta_\infty^h \varphi, \quad x \in B_\delta(z).$$

From (5.3), we see that

$$-b(z) \geq \Delta_\infty^h \psi(z) \geq \frac{1}{g(\varphi(z))} \Delta_\infty^h \varphi(z),$$

and then

$$\Delta_\infty^h \varphi(z) \leq -b(z)g(\varphi(z)) = -b(z)g(v(z)).$$

Thus, problem (1.5) has a viscosity supersolution  $v$ .  $\square$

Now, we prove the existence of a solution to (1.5) though the truncation method and the stability theory.

*Proof of Theorem 1.5.* For some  $\mu > 0$ , let

$$\hat{g}(t) = \begin{cases} g(\mu + t), & t \geq 0, \\ g(\mu), & t < 0. \end{cases}$$

By Theorem 1.1, the problem

$$\begin{aligned} \Delta_\infty^h u &= -b(x)\hat{g}(u), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

has a viscosity solution  $u$ . By Lemma 2.2, we have  $u \geq 0$ . And then we actually have

$$\begin{aligned} \Delta_\infty^h u &= -b(x)g(u + \mu), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

Then for each positive integer  $k$ , the perturbed Dirichlet problem

$$\begin{aligned} \Delta_\infty^h \lambda_k &= -b(x)g(\lambda_k + k^{-1}), \quad \text{in } \Omega, \\ \lambda_k &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

has a viscosity solution  $\lambda_k$ . And by Lemma 2.2, we see that  $\lambda_k > 0$  for all  $k$  in  $\Omega$ .

Let  $w$  be a viscosity solution of (5.1) and  $v := \eta^{-1}(w)$ , where  $\eta$  is as in (5.2). By Lemma 5.1, we have

$$\Delta_\infty^h v \leq -b(x)g(v + k^{-1}),$$



for every  $k$ . Theorem 2.4 shows that  $\lambda_k \leq v$  in  $\bar{\Omega}$ . Note that

$$\begin{aligned} \Delta_\infty^h \lambda_k &= -b(x)g(\lambda_k + k^{-1}), \\ \Delta_\infty^h \lambda_{k+1} &= -b(x)g(\lambda_{k+1} + (k+1)^{-1}) \leq -b(x)g(\lambda_k + k^{-1}), \end{aligned}$$

where we have used that  $g$  is non-increasing. Theorem 2.4 implies  $\lambda_k \leq \lambda_{k+1}$  for all  $k$ . Then one has

$$0 < \lambda_1 \leq \dots \leq \lambda_k \leq \lambda_{k+1} \leq \dots \leq v, \quad \text{in } \bar{\Omega}.$$

Then  $\{\lambda_k\}$  are locally uniformly Lipschitz continuous and locally uniformly bounded. And the limit  $\lim_{k \rightarrow \infty} \lambda_k = \lambda$  is also locally Lipschitz continuous in  $\Omega$ . In Lemma 4.1, let

$$\xi_k := b(x)g(\lambda_k + k^{-1}), \quad \xi := b(x)g(\lambda), \quad \text{and} \quad u_k := -\lambda_k.$$

Since  $\lambda \in C(\Omega)$ , we also have

$$\lim_{k \rightarrow \infty} \xi_k = b(x)g(\lambda),$$

locally uniformly, and the limit function is continuous in  $\Omega$ . We set

$$\begin{aligned} u_0 &= -v, \quad u_\infty = 0, \quad \text{in } \Omega, \\ u_0 &= u_\infty = 0, \quad \text{on } \partial\Omega. \end{aligned}$$

Obviously,  $u_0, u_\infty \in C(\bar{\Omega})$ . From Lemma 4.1, we see that the problem (1.5) has a viscosity solution  $\lambda$ . Finally, the uniqueness can be obtained by Theorem 2.4.  $\square$

Next, we give some definitions and properties of Karamata’s regular variation theory which was first introduced by Karamata in 1930’s (see [15, 16, 17] and the references therein), and then based on Karamata’s regular variation theory, we proceed to discuss the boundary behavior of viscosity solutions to the singular boundary value problem (1.5).

Now we recall some definitions and properties of regularly varying functions (see [12, 41, 43]).

**Definition 5.2.** A positive measurable function  $f$  defined on  $(0, a)$ , for some  $a > 0$ , is called regularly varying at zero with index  $\rho \in \mathbb{R}$ , written  $f \in RVZ_\rho$ , if for each  $\xi > 0$ ,

$$\lim_{s \rightarrow 0^+} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{5.4}$$

In particular, when  $\rho = 0$ ,  $f$  is called slowly varying at zero.

Clearly, if  $f \in RVZ_\rho$ , then  $L(s) := f(s)/s^\rho$  is slowly varying at zero.

**Proposition 5.3** (Representation theorem). *A function  $L$  is slowly varying at zero if and only if it may be written in the form*

$$L(s) = c(s) \exp\left(\int_s^{a_1} \frac{y(t)}{t} dt\right), \quad s \in (0, a_1),$$

for some  $a_1 \in (0, a)$ , where the functions  $c$  and  $y$  are measurable and for  $s \rightarrow 0^+$ ,  $y(s) \rightarrow 0$  and  $c(s) \rightarrow c_0$ , with  $c_0 > 0$ .

We say that

$$\hat{L}(s) = c_0 \exp\left(\int_s^{a_1} \frac{y(t)}{t} dt\right), \quad s \in (0, a_1),$$

is normalized slowly varying at zero and

$$f(s) = s^\rho \hat{L}(s), \quad s \in (0, a_1),$$

is normalized regularly varying at zero with index  $\rho$  and written  $f \in NRVZ_\rho$ .

Recall that a function  $f \in RVZ_\rho$  belongs to  $NRVZ_\rho$  if and only if  $f \in C^1(0, a_1)$  for some  $a_1 > 0$  and  $\lim_{s \rightarrow 0^+} \frac{sf'(s)}{f(s)} = \rho$ .

**Proposition 5.4.** *If functions  $L_1, L_2$  are slowly varying at zero, then*

- (i)  $L_1^\rho$  (for every  $\rho \in \mathbb{R}$ ),  $a_1 L_1 + a_2 L_2$  ( $a_1 \geq 0, a_2 \geq 0$  with  $a_1 + a_2 > 0$ ),  $L_1 \circ L_2$  (if  $L_2(s) \rightarrow \infty$  as  $s \rightarrow 0^+$ ), are also slowly varying at zero.
- (ii) For every  $\rho > 0$  and  $s \rightarrow 0^+$ ,  $s^\rho L_1(s) \rightarrow 0$ ,  $s^{-\rho} L_1(s) \rightarrow \infty$ .
- (iii) For  $\rho \in \mathbb{R}$  and  $s \rightarrow 0^+$ ,  $\log(L_1(s))/\log s \rightarrow 0$ , and  $\log(s^\rho L_1(s))/\log(s) \rightarrow \rho$ .

**Proposition 5.5** (Asymptotic behavior). *If a function  $L$  is slowly varying at zero, then for  $a > 0$  and  $s \rightarrow 0^+$ ,*

$$\int_0^s t^\rho L(t) dt \cong (\rho + 1)^{-1} s^{1+\rho} L(s), \quad \text{for } \rho > -1,$$

$$\int_s^a t^\rho L(t) dt \cong (-\rho - 1)^{-1} s^{1+\rho} L(s), \quad \text{for } \rho < -1.$$

**Proposition 5.6.** (i) *If  $f_1 \in RVZ_{\rho_1}$ ,  $f_2 \in RVZ_{\rho_2}$  with  $\lim_{t \rightarrow 0^+} f_2(t) = 0$ , then  $f_1 \circ f_2 \in RVZ_{\rho_1 \rho_2}$ .*

(ii) *If  $f \in RVZ_\rho$ , then  $f^\alpha \in RVZ_{\rho\alpha}$  for every  $\alpha \in \mathbb{R}$ .*

Now we state some important results that we can use to prove Theorem 1.6.

**Lemma 5.7.** *Let  $g$  satisfy (H3), (H4) and  $\phi$  be the solution to the problem*

$$\int_0^{\phi(t)} \frac{ds}{(g(s))^{1/h}} = t, \quad \forall t > 0.$$

Then

- (i)  $\phi \in NRVZ_{h/(h+\gamma)}$  and  $\phi' \in NRVZ_{-\gamma/(h+\gamma)}$ ;
- (ii)  $\lim_{t \rightarrow 0^+} \frac{t}{\phi(K^{(h+1)/h}(t)})} = 0$ , if  $k \in \Lambda$  and  $\tau(\gamma + h) > h + 1$ .

*Proof.* By the definition of  $\phi$ , we have

$$\phi'(t) = (g \circ \phi(t))^{1/h}, \quad \phi(t) > 0, t > 0, \quad \phi(0) = 0, \tag{5.5}$$

$$\phi''(t) = \frac{1}{h} (g \circ \phi(t))^{(2-h)/h} (g' \circ \phi(t)), \quad t > 0. \tag{5.6}$$

(i) By (H4), we have  $g \in RVZ_{-\gamma}$ . Proposition 5.6 implies  $g^{-1/h} \in RVZ_{\gamma/h}$ . We define

$$L_1(t) := \frac{g^{-1/h}(t)}{t^{\gamma/h}}.$$

Then  $L_1$  is slowly varying at zero. From  $\gamma > 1$  and Proposition 5.5, we see that

$$\lim_{t \rightarrow 0^+} \frac{tg^{-1/h}(t)}{\int_0^t g^{-1/h}(s) ds} = \lim_{t \rightarrow 0^+} \frac{tL_1(t)t^{\gamma/h}}{\int_0^t L_1(s)s^{\gamma/h} ds} = \frac{h + \gamma}{h}. \tag{5.7}$$

Then

$$\lim_{t \rightarrow 0^+} \frac{t\phi'(t)}{\phi(t)} = \lim_{t \rightarrow 0^+} \frac{t(g(\phi(t)))^{1/h}}{\phi(t)} = \lim_{s \rightarrow 0^+} \frac{(g(s))^{1/h} \int_0^s \frac{dv}{(g(v))^{1/h}}}{s} = \frac{h}{h + \gamma},$$

that is,  $\phi \in NRVZ_{\frac{h}{h+\gamma}}$ .

From (5.5), (5.6), and (5.7), it follows that

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{t\phi''(t)}{\phi'(t)} &= \frac{1}{h} \lim_{t \rightarrow 0^+} \frac{(g' \circ \phi(t)) \int_0^{\phi(t)} (g(u))^{-1/h} du}{(g \circ \phi(t))^{(h-1)/h}} \\ &= \frac{1}{h} \lim_{s \rightarrow 0^+} \frac{g'(s) \int_0^s (g(\nu))^{-1/h} d\nu}{(g(s))^{(h-1)/h}} \\ &= \frac{1}{h} \lim_{s \rightarrow 0^+} \frac{sg'(s) \int_0^s (g(\nu))^{-1/h} d\nu}{g(s) s(g(s))^{-1/h}} = -\frac{\gamma}{h + \gamma}. \end{aligned}$$

(ii) Since  $k \in \Lambda$ , we have

$$\lim_{t \rightarrow 0^+} \frac{K(t)}{tk(t)} = \lim_{t \rightarrow 0^+} \frac{d}{dt} \left( \frac{K(t)}{k(t)} \right) = \tau.$$

And then

$$\lim_{t \rightarrow 0^+} \frac{tk(t)}{K(t)} = \frac{1}{\tau},$$

i.e.,  $K \in NRVZ_{1/\tau}$ . By Proposition 5.6 (i), we have

$$\phi \circ K^{(h+1)/h} \in NRVZ_{\frac{h+1}{\tau(\gamma+h)}} \quad \text{and} \quad \frac{t}{\phi(K^{(h+1)/h}(t))} \in NRVZ_{\frac{\tau(\gamma+h)-h-1}{\tau(\gamma+h)}}.$$

Since  $\tau(\gamma + h) > h + 1$ , by Proposition 5.4 (ii),

$$\lim_{t \rightarrow 0^+} \frac{t}{\phi(K^{(h+1)/h}(t))} = 0. \quad \square$$

*Proof of Theorem 1.6.* We define

$$d(x) := \text{dist}(x, \partial\Omega), \quad \Omega_\delta := \{x \in \Omega : d(x) < \delta\}.$$

Since  $\Omega$  is a bounded domain with smooth boundary, it follows that  $d(x) \in C^1(\Omega_\delta)$  for some  $\delta > 0$ . Moreover,  $|Dd(x)| = 1$  and  $\Delta_\infty^h d(x) = 0$  in  $\Omega_\delta$ , in the viscosity sense.

We set

$$\begin{aligned} \eta(t) &:= (\xi_0 + \varepsilon)\phi(K^{(h+1)/h}(t)), \quad t \in (0, \delta), \\ u^*(x) &:= \eta(d(x)), \quad x \in \Omega_\delta. \end{aligned}$$

Note that  $K$  and  $\phi$  are both increasing in their respective definition domains. Therefore, when  $\delta$  is small enough,  $\eta$  is increasing in  $(0, \delta)$ . Let  $\zeta$  be the inverse of  $\eta$ . It is easy to check that

$$\zeta'(t) = \frac{1}{\eta'(\zeta(t))} = \left( \frac{h+1}{h} (\xi_0 + \varepsilon) \phi'(K^{(h+1)/h}(\zeta(t))) K^{1/h}(\zeta(t)) k(\zeta(t)) \right)^{-1} \quad (5.8)$$

and

$$\begin{aligned} \zeta''(t) &= -\frac{\eta''(\zeta(t))}{[\eta'(\zeta(t))]^3} \\ &= -\left( \frac{h+1}{h} (\xi_0 + \varepsilon) \phi'(K^{(h+1)/h}(\zeta(t))) K^{1/h}(\zeta(t)) k(\zeta(t)) \right)^{-3} \zeta_0, \end{aligned} \quad (5.9)$$

where

$$\zeta_0 = \frac{(h+1)^2}{h^2} (\xi_0 + \varepsilon) \phi''(K^{(h+1)/h}(\zeta(t))) K^{2/h}(\zeta(t)) k^2(\zeta(t))$$

$$\begin{aligned} & + \frac{h+1}{h^2}(\xi_0 + \varepsilon)\phi'(K^{(h+1)/h}(\zeta(t)))K^{(1-h)/h}(\zeta(t))k^2(\zeta(t)) \\ & + \frac{h+1}{h}(\xi_0 + \varepsilon)\phi'(K^{(h+1)/h}(\zeta(t)))K^{1/h}(\zeta(t))k'(\zeta(t)). \end{aligned}$$

Let  $\psi \in C^2(\Omega_\delta)$  and suppose  $u^* - \psi$  has a local minimum at  $x_0 \in \Omega_\delta$ , i.e.,  $u^* \geq \psi$  in a neighborhood  $\mathcal{N}$  of  $x_0$  and  $u^*(x_0) = \psi(x_0)$ . Then  $\varphi = \zeta(\psi) \in C^2(\Omega_\delta)$  and

$$d(x_0) = \varphi(x_0), \quad d(x) \geq \varphi(x) \quad \text{in } \mathcal{N}.$$

Since  $\Delta_\infty^h d = 0$ , we have  $\Delta_\infty^h \varphi(x_0) \leq 0$ . Direct computations yield

$$\begin{aligned} D\varphi &= \zeta'(\psi)D\psi, \\ D^2\varphi &= \zeta''(\psi)D\psi \otimes D\psi + \zeta'(\psi)D^2\psi, \\ \Delta_\infty^h \varphi &= |D\varphi|^{h-3} \langle D^2\varphi D\varphi, D\varphi \rangle \\ &= |\zeta'(\psi)|^{h-1} |D\psi|^{h+1} \zeta''(\psi) + |\zeta'(\psi)|^{h-1} \zeta'(\psi) \Delta_\infty^h \psi. \end{aligned}$$

Since  $\Delta_\infty^h \varphi(x_0) \leq 0$  and  $\zeta' > 0$ , we have

$$\Delta_\infty^h \psi(x_0) \leq -(\zeta'(\psi(x_0)))^{-1} \zeta''(\psi(x_0)) |D\psi(x_0)|^{h+1}.$$

Noting that  $|Dd(x)| = 1$  for all  $x \in \Omega_\delta$  and  $d - \varphi$  attains a local maximum at  $x_0$ , we have

$$1 = |Dd(x_0)| = |\zeta'(\psi(x_0))D\psi(x_0)|.$$

Then

$$\Delta_\infty^h \psi(x_0) \leq -|\zeta'(\psi(x_0))|^{-h-2} \zeta''(\psi(x_0)).$$

Combing this with (5.8) and (5.9), we obtain

$$\begin{aligned} & \Delta_\infty^h \psi(x_0) \\ & \leq \left(\frac{h+1}{h}(\xi_0 + \varepsilon)\right)^h \left(\phi'(K^{(h+1)/h}(\varphi(x_0)))\right)^h k^{h+1}(\varphi(x_0)) \\ & \quad \times \left[ \frac{h+1}{h} \frac{\phi''(K^{(h+1)/h}(\varphi(x_0)))K^{(h+1)/h}(\varphi(x_0))}{\phi'(K^{(h+1)/h}(\varphi(x_0)))} + \frac{1}{h} + \frac{K(\varphi(x_0))k'(\varphi(x_0))}{k^2(\varphi(x_0))} \right]. \end{aligned}$$

Hence,

$$\begin{aligned} & \Delta_\infty^h \psi(x_0) + b(x_0)g(u^*(x_0)) \\ & \leq \left(\frac{h+1}{h}(\xi_0 + \varepsilon)\right)^h \left(\phi'(K^{(h+1)/h}(d(x_0)))\right)^h k^{h+1}(d(x_0))A(x_0), \end{aligned}$$

where

$$\begin{aligned} A(x_0) &:= \frac{h+1}{h} \frac{\phi''(K^{(h+1)/h}(d(x_0)))K^{(h+1)/h}(d(x_0))}{\phi'(K^{(h+1)/h}(d(x_0)))} + \frac{1}{h} + \frac{K(d(x_0))k'(d(x_0))}{k^2(d(x_0))} \\ & + \left(\frac{h+1}{h}(\xi_0 + \varepsilon)\right)^{-h} \frac{b(x_0)}{k^{h+1}(d(x_0))} \frac{g(u^*(x_0))}{(\phi'(K^{(h+1)/h}(d(x_0))))^h}. \end{aligned}$$

Note that  $K^{\frac{h+1}{h}}(d(x_0)) \rightarrow 0$  as  $\delta \rightarrow 0$ . Then, by Lemma 5.7 and  $\lim_{t \rightarrow 0^+} \frac{k'(t)K(t)}{k^2(t)} = 1 - \tau$ , we have that

$$A(x_0) \rightarrow \frac{h+1 - (h+\gamma)\tau}{h+\gamma} + \ell \left(\frac{h}{h+1}\right)^h (\xi_0 + \varepsilon)^{-h-\gamma}, \quad \delta \rightarrow 0.$$

Since

$$\xi_0 = \left( \frac{h^h \ell(h + \gamma)}{(h + 1)^h [(h + \gamma)\tau - h - 1]} \right)^{1/(h+\gamma)},$$

we have  $A(x_0) < 0$  when  $\delta_0 \in (0, \frac{\delta}{2})$  small enough. Thus

$$\Delta_\infty^h \psi(x_0) \leq -b(x_0)g(u^*(x_0)),$$

that is,  $u^*$  is a viscosity supersolution of (1.5) in  $\Omega_{\delta_0}$ . Similarly, we can prove that

$$u_*(x) = (\xi_0 - \varepsilon)\phi(K^{(h+1)/h}(d(x)))$$

is a viscosity subsolution of (1.5) in  $\Omega_{\delta_0}$ .

Let  $v \in C(\bar{\Omega})$  be the unique viscosity solution of the problem

$$\begin{aligned} -\Delta_\infty^h v &= 1, & \text{in } \Omega, \\ v &> 0, & \text{in } \Omega, \\ v &= 0, & \text{on } \partial\Omega. \end{aligned}$$

According to [10, Theorem 7.7], there are two positive constants  $a$  and  $c$ , with  $0 < a < c$  such that

$$ad(x) \leq v(x) \leq cd(x), \quad d(x) \rightarrow 0. \tag{5.10}$$

Let  $u \in C(\Omega)$  be the unique viscosity solution to (1.5) and  $M$  large enough such that

$$u(x) \leq u^*(x) + Mv(x) \text{ and } u_*(x) \leq u(x) + Mv(x) \quad \text{on } \{x \in \Omega : d(x) = \delta_0\}.$$

By (H3), we see that  $u^*(x) + Mv(x)$  and  $u(x) + Mv(x)$  are also viscosity supersolutions of (1.5) in  $\Omega_{\delta_\varepsilon}$ . Since

$$u(x) = u^*(x) + Mv(x) = u(x) + Mv(x) = u_*(x) = 0, \quad \text{on } \partial\Omega,$$

by (H3) and Lemma 2.3, we have

$$u(x) \leq u^*(x) + Mv(x), \quad u_*(x) \leq u(x) + Mv(x), \quad x \in \Omega_{\delta_0}.$$

Hence, for  $x \in \Omega_{\delta_0}$ , one has

$$\xi_0 - \varepsilon - \frac{Mv(x)}{\phi(K^{(h+1)/h}(d(x)))} \leq \frac{u(x)}{\phi(K^{(h+1)/h}(d(x)))}$$

and

$$\frac{u(x)}{\phi(K^{(h+1)/h}(d(x)))} \leq \xi_0 + \varepsilon + \frac{Mv(x)}{\phi(K^{(h+1)/h}(d(x)))}.$$

By Lemma 5.7 (ii) and (5.10), we have

$$\xi_0 - \varepsilon \leq \liminf_{d(x) \rightarrow 0} \frac{u(x)}{\phi(K^{(h+1)/h}(d(x)))} \quad \text{and} \quad \limsup_{d(x) \rightarrow 0} \frac{u(x)}{\phi(K^{(h+1)/h}(d(x)))} \leq \xi_0 + \varepsilon.$$

Thus, letting  $\varepsilon \rightarrow 0$ , we obtain

$$\lim_{d(x) \rightarrow 0} \frac{u(x)}{\phi(K^{(h+1)/h}(d(x)))} = \xi_0.$$

□

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## REFERENCES

- [1] S. Armstrong, C. Smart; An easy proof Jensen's theorem on the uniqueness of infinity harmonic functions, *Calc. Var. Partial Differ. Equ.*, **37** (2010), 381-384.
- [2] G. Aronsson; Minimization problems for the functional  $\sup_x F(x, f(x), f'(x))$ , *Ark. Mat.*, **6** (1965), 33-53.
- [3] G. Aronsson; Minimization problems for the functional  $\sup_x F(x, f(x), f'(x))$  II, *Ark. Mat.*, **6** (1966), 409-431.
- [4] G. Aronsson; Extension of functions satisfying Lipschitz conditions, *Ark. Mat.*, **6** (1967), 551-561.
- [5] G. Aronsson; Minimization problems for the functional  $\sup_x F(x, f(x), f'(x))$  III, *Ark. Mat.*, **7** (1969), 509-512.
- [6] G. Aronsson, M. G. Crandall, P. Juutinen; A tour of the theory of absolutely minimizing functions, *Bull. Amer. Math. Soc.*, **41** (2004), 439-505.
- [7] G. Barles, J. Busca; Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth-order term, *Comm. Partial Diff. Equations*, **26** (2001), 2323-2337.
- [8] E.N. Barron, L.C. Evans, R. Jensen; The infinity Laplacian, Aronsson's equation and their generalizations, *Trans. Amer. Math. Soc.*, **360** (2008), 77-101.
- [9] E. Barron, R. Jensen, C. Wang; The Euler equation and absolute minimizers of  $L^\infty$  functionals, *Arch. Ration. Mech. Anal.*, **157** (2001), 255-283.
- [10] T. Bhattacharya, A. Mohammed; On solutions to Dirichlet problems involving the infinity-Laplacian, *Adv. Calc. Var.*, **4** (2011), 445-487.
- [11] T. Bhattacharya, A. Mohammed; Inhomogeneous Dirichlet problems involving the infinity-Laplacian, *Adv. Differential Equations*, **17** (2012), no.3-4, 225-266.
- [12] N. H. Bingham, C. M. Goldie, J. L. Teugels; *Regular variation*, Cambridge: Cambridge University Press, 1987.
- [13] T. Biset, A. Mohammed; A singular boundary value problem for a degenerate elliptic PDE, *Nonlinear Analysis: Theory, Methods & Applications*, **119** (2015), 222-234.
- [14] A. Biswas, H. H. Vo; Liouville theorems for infinity Laplacian with gradient and KPP type equation, preprint.
- [15] F. Cirstea, V. Rădulescu; Uniqueness of the blow-up boundary solution of logistic equations with absorption, *C. R. Acad. Sci. Paris, Sér. I*, **335** (2002), 447-452.
- [16] F. Cirstea, V. Rădulescu; Asymptotics for the blow-up boundary solution of the logistic equation with absorption, *C. R. Acad. Sci. Paris, Sér. I*, **336** (2003), 231-236.
- [17] F. Cirstea, V. Rădulescu; Nonlinear problems with boundary blow-up: aKaramata regular variation theory approach, *Asymptot. Anal.*, **46** (2006), 275-298.
- [18] M. G. Crandall; A visit with the  $\infty$ -Laplace equation, Calculus of variations and nonlinear partial differential equations, 75-122, *Lecture Notes in Math.*, **1927**, Springer, Berlin, 2008.
- [19] M. G. Crandall, L. C. Evans, R. F. Gariepy; Optimal Lipschitz extensions and the infinity-Laplacian, *Calc. Var. Partial Differ. Equ.*, **13** (2001), 123-139.
- [20] M. G. Crandall, L. C. Evans, P. L. Lions; Some properties of viscosity solutions of Hamilton-Jacobi equations, *Trans. Am. Math. Soc.*, **282** (1984), 487-502.
- [21] M. G. Crandall, G. Gunnarsson, P. Wang; Uniqueness of  $\infty$ -harmonic functions and the eikonal equation, *Comm. Partial Differential Equations*, **32** (2007), no.10-12, 1587-1615.
- [22] M. G. Crandall, H. Ishii, P. L. Lions; User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N. S.)*, **27** (1992), 1-67.
- [23] M. G. Crandall, P. L. Lions; Viscosity solutions and Hamilton-Jacobi equations, *Trans. Am. Math. Soc.*, **277** (1983), 1-42.
- [24] L. C. Evans; The 1-Laplacian, the infinity Laplacian and differential games, *Contemp. Math.*, **446** (2007), 245-254.
- [25] R. Jensen; Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient, *Arch. Rational Mech. Anal.*, **123** (1993), 51-74.
- [26] C. Li, F. Liu; Large solutions of a class of degenerate equations associated with infinity Laplacian, *Adv. Nonlinear Stud.*, **22** (2022), 67-87.
- [27] C. Li, F. Liu, Peibiao Zhao; Boundary blow-up solutions to the equation involved in infinity Laplacian, *J. Aust. Math. Soc.*, **114** (2023), 337-358.
- [28] T. Lin, F. Liu; Viscosity solutions to the infinity Laplacian equation with strong absorptions, *Comm. Pure and Applied Anal.*, **21** (2022), no. 12, 4251-4267.

- [29] F. Liu; The eigenvalue problem for a class of degenerate operators related to the normalized  $p$ -Laplacian, *Disc. & Conti. Dynamical Systems(B)*, **27** (2022), no. 5, 2701-2720.
- [30] F. Liu, L. Tian, P. Zhao; A weighted eigenvalue problem of the degenerate operator associated with infinity Laplacian, *Nonlinear Analysis:TMA*, **200** (2020), 112001, 15 pp.
- [31] F. Liu, X. Yang; A weighted eigenvalue problem of the biased infinity Laplacian, *Nonlinearity*, **34** (2021), 1197-1237.
- [32] F. Liu, X. Yang; Solutions to an inhomogeneous equation involving infinity-Laplacian, *Non-linear Analysis: Theory, Methods & Applications*, **75** (2012), 5693-5701.
- [33] R. López-Soriano, J. C. Navarro-Climent, J. D. Rossi; The infinity Laplacian with a transport term, *J. Math. Anal. Appl.*, **398** (2013), 752-765.
- [34] G. Lu, P. Wang; A PDE perspective of the normalized infinity Laplacian, *Comm. Part. Diff. Eqns.*, **33** (2008), no.10, 1788-1817.
- [35] G. Lu, P. Wang; Inhomogeneous infinity Laplace equation. *Adv. Math.*, **217** (2008), 1838-1868.
- [36] G. Lu, P. Wang; A uniqueness theorem for degenerate elliptic equations, *Seminario Interdisciplinare di Matematica*, **7** (2008), 207-222.
- [37] G. Lu, P. Wang; Infinity Laplace equation with non-trivial right-hand side, *Electronic Journal of Differential Equations*, **77** (2010), 517-532.
- [38] L. Mi, Boundary behavior for the solutions to Dirichlet problems involving the infinity-Laplacian, *J. Math. Anal. Appl.*, **425** (2015), 1061-1070.
- [39] Y. Peres, G. Pete, S. Somersille; Biased tug-of-war, the biased infinity Laplacian, and comparison with exponential cones, *Calc. Var. PDE*. **38** (2010), no.3-4, 541-564.
- [40] Y. Peres, O. Schramm, S. Sheffield, D. Wilson; Tug-of-war and the infinity Laplacian, *J. Amer. Math. Soc.*, **22** (2009), no.1, 167-210.
- [41] S. I. Resnick; *Extreme Values, Regular Variation, and Point Processes*, Applied Probability. A Series of the Applied Probability Trust, 4, Springer-Verlag, New York, Berlin, 1987. xii+320 pp.
- [42] J. D. Rossi; Tug-of-war games and PDEs, *Proc. Roy. Soc. Edinburgh Sect. A*, **141** (2011), no.2, 319-369.
- [43] E. Seneta; *Regularly Varying Functions*, Lecture Notes in Math., **508**, Springer-Verlag, Berlin-New York, 1976, v+112 pp.
- [44] Y. Yu; Uniqueness of values of Aronsson operators and running costs in “tug-of-war” games, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **26** (2009), no.4, 1299-1308.

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