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# SOLUTIONS OF COMPLEX NONLINEAR FUNCTIONAL EQUATIONS INCLUDING SECOND ORDER PARTIAL DIFFERENTIAL AND DIFFERENCE IN $\mathbb{C}^2$

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Abstract. This article is devoted to exploring the existence and the form of finite order transcendental entire solutions of Fermat-type second order partial differential-difference equations

$$\Big(\frac{\partial^2 f}{\partial z_1^2}+\delta\frac{\partial^2 f}{\partial z_2^2}+\eta\frac{\partial^2 f}{\partial z_1 \partial z_2}\Big)^2+f(z_1+c_1,z_2+c_2)^2=e^{g(z_1,z_2)}$$

and

$$\Big(\frac{\partial^2 f}{\partial z_1^2}+\delta\frac{\partial^2 f}{\partial z_2^2}+\eta\frac{\partial^2 f}{\partial z_1 \partial z_2}\Big)^2+(f(z_1+c_1,z_2+c_2)-f(z_1,z_2))^2=e^{g(z)},$$

where $\delta,\eta\in\mathbb{C}$ and $g(z_1,z_2)$ is a polynomial in $\mathbb{C}^2$. Our results improve the results of Liu and Dong [23], Liu et al. [24], and Liu and Yang [25]. Several examples confirm that the form of transcendental entire solutions of finite order in our results are precise.

## 1. Introduction

It is well known that for a positive integer $m$, the equation

$$f^m+g^m=1 \tag{1.1}$$

is regarded as Fermat type equation over function fields. With the help of Nevanlinna theory [11, 16], Montel [27], Iyer [15], and Gross [5] studied the existence and form of the solutions of the functional equation (1.1) and pointed out that for $m=2$, the entire solutions of (1.1) are $f(z)=\cos(\xi(z))$ and $g(z)=\sin(\xi(z))$, where $\xi$ is an entire function, and for $m>2$, there are no non-constant entire solutions of (1.1). In 2004, Yang and Li [42] investigated (1.1) by replacing $g$ with $f'$ when $m=2$, and proved that the transcendental entire solution of $f(z)^2+f'(z)^2=1$ has the form $f(z)=Ae^{\alpha z}/2+e^{-\alpha z}/2A$, where $A,\alpha$ are non-zero complex constants.

After the development of difference Nevanlinna theory (see [4, 6]), many researcher began to study the existence and form of entire or meromorphic solutions of Fermat-type difference and differential-difference equations (see [7, 21, 22, 23, 24, 25]). In 2012, Liu et al. [24] proved that the transcendental entire solutions with finite order of the Fermat-type difference equation $f(z)^2+f(z+c)^2=1$ must

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satisfy $f(z) = \sin(Az + B)$, where $B$ is a constant and $A = (4k+1)\pi/2c$, where $k$ is an integer. In 2019, Han and Lü [10] investigated the more general complex difference equation $f(z)^2 + f(z+c)^2 = e^{\alpha z+\beta}$, $\alpha, \beta \in \mathbb{C}$, and proved that the nontrivial meromorphic solutions of this equation are of the form $de^{(\alpha z+\beta)/2}$, where $d \in \mathbb{C}$ such that $d^2(1 + e^{\alpha c}) = 1$.

Hereinafter, we denote by $z + w = (z_1 + w_1, z_2 + w_2)$ for any $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{C}^2$. The study of several characteristics of the solutions to partial differential equations in several complex variables is an important topic; see [1, 2, 3, 8, 9, 12, 14, 18, 26, 34, 35, 36, 37, 38]). It was Saleebly, who in 1999, first studied the existence and form of entire and meromorphic solutions of Fermat-type partial differential equations (see [30, 31, 32]). Most noticeably, Khavinson [14] proved that any entire solution of the partial differential equation $f_{z_1}^2 + f_{z_2}^2 = 1$ must be linear, i.e., $f(z_1, z_2) = az_1 + bz_2 + c$, where $a, b, c \in \mathbb{C}$, and $a^2 + b^2 = 1$. Here $f_{z_1}$ and $f_{z_2}$ are the partial derivatives of $f$ with respect to $z_1$ and $z_2$, respectively. Later, Li [19, 20] investigated on the partial differential equations with more general forms such as $f_{z_1}^2 + f_{z_2}^2 = p$, $f_{z_1}^2 + f_{z_2}^2 = e^q$, etc, where $p, q$ are polynomials in $\mathbb{C}^2$. Recently, Xu and Cao [40] extended several results from one complex variable to several complex variables. We recall some of them here.

**Theorem 1.1** ([40])**.** *Let $c = (c_1, c_2, \dots, c_n) \in \mathbb{C}^n \setminus \{(0, 0, \dots, 0)\}$. Then, any non-constant entire solution $f : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ with finite order of the Fermat-type difference equation*

$$f(z)^2 + f(z+c)^2 = 1 \tag{1.2}$$

*has the form of $f(z) = \cos(L(z) + B)$, where $L$ is a linear function of the form $L(z) = a_1z_1 + \dots + a_nz_n$ on $\mathbb{C}^n$ such that $L(c) = -\pi/2 - 2k\pi$ ($k \in \mathbb{Z}$), and $B$ is a constant on $\mathbb{C}$.*

**Theorem 1.2** ([40])**.** *Let $c = (c_1, c_2)$ be a constant in $\mathbb{C}^2$. Then any transcendental entire solution with finite order of the Fermat-type partial differential-difference equation*

$$\Big(\frac{\partial f(z_1, z_2)}{\partial z_1}\Big)^2 + f^2(z_1 + c_1, z_2 + c_2) = 1 \tag{1.3}$$

*has the form $f(z_1, z_2) = \sin(Az_1 + Bz_2 + H(z_2))$, Where $A, B$ are constant on $\mathbb{C}$ satisfying $A^2 = 1$ and $Ae^{i(Ac_1+Bc_2)} = 1$, and $H(z_2)$ is a polynomial in one variable $z_2$ such that $H(z_2) \equiv H(z_2 + c_2)$. In the special case whenever $c_2 \neq 0$, we have $f(z_1, z_2) = \sin(Az_1 + Bz_2 + Constant)$.*

In 2021, Zheng and Xu [43] obtained the following result.

**Theorem 1.3** ([43])**.** *Let $c = (c_1, c_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Then there are no finite order transcendental entire solutions of*

$$f(z)^2 + [f(z+c) - f(z)]^2 = 1. \tag{1.4}$$

In 2022, Xu et al. [41] extended Theorems 1.1 and 1.2 by replacing 1 with $e^{g(z_1,z_2)}$ in the right-hand side of equations (1.2) and (1.3), and $\frac{\partial f(z_1,z_2)}{\partial z_1}$ with $\alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2}$ in equation (1.3). We list some of the results here.

**Theorem 1.4** ([41])**.** *Let $c = (c_1, c_2) \in \mathbb{C}^2$, and let $\alpha, \beta$ be constants in $\mathbb{C}$ that are not zero at the same time. If the partial differential-difference equation*

$$\Big(\alpha\frac{\partial f(z_1, z_1)}{\partial z_1} + \beta\frac{\partial f(z_1, z_1)}{\partial z_2}\Big)^2 + f(z_1 + c_1, z_2 + c_2)^2 = e^{g(z_1,z_1)} \tag{1.5}$$

*admits a transcendental entire solution of finite order, then $f$ and $g$ must satisfy one of the following cases:*

(i) $f(z_1,z_2)=\pm e^{\frac{1}{2}g(z-c)}$, *where* $g(z)=\phi(\beta z_1-\alpha z_2)$ *and* $\phi$ *is a polynomial in* $\mathbb{C}$*;*

(ii) $g(z)$ *must be of the form* $g(z)=L(z)+H(s_1)+B$, *where* $L(z)$ *is a linear function of the form* $L(z)=A_1z_1+A_2z_2$, $H(s_1)$ *is a polynomial in* $s_1:=c_2z_1-c_1z_2$, $A_1,A_2,B\in\mathbb{C}$ *and*

$$f(z_1,z_2)=\frac{\xi^2+1}{\xi(\alpha A_1+\beta A_2)}e^{\frac{1}{2}(L(z)+H(s_1)+B)},$$

*where* $\xi(\neq 0)$, $A_1,A_2,B\in\mathbb{C}$ *satisfying*

$$(\alpha c_2-\beta c_1)H'\equiv 0,\quad \frac{1}{2i}\frac{\xi^2-1}{\xi^2+1}(\alpha A_1+\beta A_2)=e^{\frac{1}{2}(A_1c_1+A_2c_2)};$$

(iii)

$$f(z_1,z_2)=\frac{e^{L_1(z)+H_1(s_1)+B_1}}{2(\alpha A_{11}+\beta A_{12})}+\frac{e^{L_2(z)+H_2(s_1)+B_2}}{2(\alpha A_{21}+\beta A_{22})},$$

*where* $L_1(z)=A_{11}z_1+A_{12}z_2+B_1$ *and* $L_2(z)=A_{21}z_1+A_{22}z_2+B_2$, *with* $A_{j1},A_{j2},B_j\in\mathbb{C}$*( j =1, 2), satisfy*

$$\begin{gathered}g(z)=L_1(z)+L_2(z)+H_1(s_1)+H_2(s_1)+B_1+B_2,\\ L_1(z)+H_1(s_1)\neq L_2(z)+H_2(s_1),\quad (\alpha c_2-\beta c_1)H_j'\equiv 0\\ -i(\alpha A_{11}+\beta A_{12})e^{-L_1(c)}=i(\alpha A_{21}+\beta A_{22})e^{-L_2(c)}=1,\end{gathered}$$

*where* $H_j(s_1)$ *for* $j=1,2$ *are polynomials in* $s_1=c_2z_1-c_1z_2$*.*

In the same paper [41], they also explored the existence and the forms of entire and meromorphic solutions of the partial differential difference equation

$$\Big(\alpha\frac{\partial f}{\partial z_1}+\beta\frac{\partial f}{\partial z_2}\Big)^2+(f(z+c)-f(z))^2=e^{g(z)},\tag{1.6}$$

where $g(z)$ is a polynomial in $\mathbb{C}^2$ and $\alpha,\beta$ are constants in $\mathbb{C}$ and obtained the following result.

**Theorem 1.5.** [41] *Let* $c=(c_1,c_2)\in\mathbb{C}^2$, $\alpha(\neq 0),\beta$ *constants in* $\mathbb{C}$*, and* $\alpha c_2-\beta c_1\neq 0$*. Let* $f$ *be a finite order transcendental entire solution of the partial differential-difference equation* (1.6)*, then* $f$ *must satisfy one of the following cases:*

(i) $f(z_1,z_2)=\phi_1(\beta z_1-\alpha z_2)$, *where* $\phi_1$ *is a finite order transcendental entire function such that*

$$\pm e^{\frac{1}{2}g(z)}=\phi_1(\beta z_1-\alpha z_2+\beta c_1-\alpha c_2)-\phi_1(\beta z_1-\alpha z_2),$$

(ii) $g(z)=A_1z_1+A_2z_2+H(s_1)+B$ *and*

$$f(z)=\pm\frac{1}{\alpha}\int_0^{z_1/\alpha}e^{\frac{1}{2}(A_1z_1+A_2z_2+H(s_1)+B)}dz_1+G\big(\frac{\alpha z_2-\beta z_1}{\alpha}\big),$$

*where* $e^{\frac{1}{2}(A_1c_1+A_2c_2)}=1$, $H(s_1)$ *is a polynomial in* $s_1=c_2z_1-c_1z_2$, $G$ *is a finite order period entire function with period* $(\alpha c_2-\beta c_1)/\alpha$*, and* $A_1,A_2\in\mathbb{C}$*;*

(iii) $g(z)=A_1z_1+A_2z_2+B$ *and*

$$f(z)=\big(\xi+\frac{1}{\xi}\big)\frac{e^{\frac{1}{2}(A_1z_1+A_2z_2+B)}}{\alpha A_1+\beta A_2}+G\big(\frac{\alpha z_2-\beta z_1}{\alpha}\big),$$

*where* $A_1,A_2,B\in\mathbb{C}$, $G$ *is a finite order entire period function with period* $(\alpha c_2-\beta c_1)/\alpha$ *and* $\xi(\neq 0),A_1,A_2,B\in\mathbb{C}$ *satisfying*

$$\frac{1}{2i}\frac{\xi^2-1}{\xi^2+1}(\alpha A_1+\beta A_2)+1=e^{\frac{1}{2}(A_1c_1+A_2c_2)};$$

(iv) $g(z)=A_1z_1+A_2z_2$ *and*

$$f(z_1,z_2)=\frac{e^{L_1(z)+B_1}}{2(\alpha A_{11}+\beta A_{12})}+\frac{e^{L_2(z)+B_2}}{2(\alpha A_{21}+\beta A_{22})}+G\big(\frac{\alpha z_2-\beta z_1}{\alpha}\big),$$

*where* $A_1,A_2,B\in\mathbb{C}$, $G$ *is a finite order entire period function with period* $(\alpha c_2-\beta c_1)/\alpha$ *and* $L_1(z)=A_{11}z_1+A_{12}z_2+B_1$ *and* $L_2(z)=A_{21}z_1+A_{22}z_2+B_2$, *with* $A_{j1},A_{j2},B_j\in\mathbb{C}$*( j =1, 2), satisfy*

$$L_1(z)\neq L_2(z),\quad g(z)=L_1(z)+L_2(z)+B_1+B_2,$$
$$[1-i(\alpha A_{11}+\beta A_{12})]e^{-L_1(c)}=[1+i(\alpha A_{21}+\beta A_{22})]e^{-L_2(c)}=1.$$

r For the second-order partial differential-difference equations of Fermat type in $\mathbb{C}^2$, Xu et al. [39] obtained the following important results.

**Theorem 1.6.** [39] *Let* $c=(c_1,c_2)\in\mathbb{C}^2$ *and* $c_2\neq 0$. *If the difference equation*

$$\Big(\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}\Big)^2+f(z_1+c_1,z_2+c_2)^2=e^{g(z_1,z_2)} \tag{1.7}$$

*admits a finite order transcendental entire solution, then* $g(z)$ *must be of the form* $g(z)=L(z)+H(s_1)+B$, *where* $L(z)=A_1z_1+A_2z_2$, $H(s_1)$ *is a polynomial in* $s_1:=c_2z_1-c_1z_2$, *and* $A_1,A_2\in\mathbb{C}$. *Further,* $f(z)$ *must satisfy one of the following cases:*

(i)
$$f(z_1,z_2)=\frac{4(\xi^2+1)}{A_1^2\xi}e^{\frac{1}{2}[A_1z_1+A_2z_2+B]},$$

*where* $\xi$ *is a non-zero complex number in* $\mathbb{C}$ *and* $e^{(A_1c_1+A_2c_2)/2}=A_1^2(\xi^2-1)/4i(\xi^2+1)$.

(ii)
$$f(z_1,z_2)=\frac{A_{21}^2e^{L_1(z)+B_1}+A_{11}^2e^{L_2(z)+B_2}}{2},$$

*where* $L_1(z)=A_{11}z_1+A_{12}z_2+B_1$ *and* $L_2(z)=A_{21}z_1+A_{22}z_2+B_2$, *with* $A_{j1},A_{j2},B_j\in\mathbb{C}$*( j =1, 2), satisfy*

$$g(z)=L_1(z)+L_2(z)+B_1+B_2,$$
$$L_1(z)\neq L_2(z),\quad -iA_{21}^2e^{L_1(c)}=iA_{21}^2e^{L_2(c)}=1.$$

Theorems 1.3–1.6 suggest the following questions as open problems.

(1) What can be said about the existence and forms of solutions of the equation (1.4) when the constant 1 is replaced by a function $e^{g(z_1,z_2)}$ in Theorem 1.3?

(2) What can be said about the existence and forms of solutions of the equation (1.7) when $\frac{\partial^2 f(z_1,z_2)}{\partial z_1^2}$ is replaced by more general operator $\frac{\partial^2 f}{\partial z_1^2} + \delta\frac{\partial^2 f}{\partial z_2^2} + \eta\frac{\partial^2 f}{\partial z_1 \partial z_2}$ in Theorem 1.6?
(3) What can be said about the existence and forms of solutions of (1.5) and (1.6) when $\alpha\frac{\partial f}{\partial z_1} + \beta\frac{\partial f}{\partial z_2}$ is replace by second order homogeneous linear partial differential operator $\frac{\partial^2 f}{\partial z_1^2} + \delta\frac{\partial^2 f}{\partial z_2^2} + \eta\frac{\partial^2 f}{\partial z_1 \partial z_2}$ in Theorems 1.5 and 1.6?

## 2. Results

Motivated by the above questions and utilizing difference analogues of Nevanlinna theory of several complex variables [1, 2, 3], we obtain Theorems 2.1, 2.6, and 2.10. Theorem 2.1 is an extension and generalization of Theorems 1.4 and 1.6. Theorem 2.6 is an extension of Theorem 1.5. And Theorem 2.10 is the extension of Theorem 1.3. Now we consider the second-order partial differential difference equations

$$\Big(\frac{\partial^2 f}{\partial z_1^2} + \delta\frac{\partial^2 f}{\partial z_2^2} + \eta\frac{\partial^2 f}{\partial z_1 \partial z_2}\Big)^2 + f(z_1+c_1, z_2+c_2)^2 = e^{g(z_1,z_2)}, \tag{2.1}$$

$$\Big(\frac{\partial^2 f}{\partial z_1^2} + \delta\frac{\partial^2 f}{\partial z_2^2} + \eta\frac{\partial^2 f}{\partial z_1 \partial z_2}\Big)^2 + Big(f(z_1+c_1, z_2+c_2) - f(z_1,z_2)\Big)^2 = e^{g(z)}, \tag{2.2}$$

and the difference equation

$$f(z)^2 + [f(z+c) - f(z)]^2 = e^{g(z_1,z_2)}, \tag{2.3}$$

where $\delta, \eta \in \mathbb{C}$, $c = (c_1, c_2) \in \mathbb{C}^2$ and $g(z_1, z_2)$ is a polynomial in $\mathbb{C}^2$.

Before we state our main results, let us first set the following.

$$\begin{gathered} A_1 = a_1 + \frac{1}{2}\eta a_2, \quad A_2 = \delta a_2 + \frac{1}{2}\eta a_1, \quad A_3 = c_2^2 + \delta c_1^2 - \eta c_1 c_2, \\ A_4 = \frac{1}{2}(a_1^2 + \delta a_2^2 + \eta a_1 a_2), \quad A_{j5} = 2a_{j1} + \eta a_{j2}, \quad A_{j6} = 2\delta a_{j2} + \eta a_{j1}, \\ A_{j7} = a_{j1}^2 + \delta a_{j2}^2 + \eta a_{j1} a_{j2}, \quad j = 1, 2. \end{gathered} \tag{2.4}$$

Now we state our results as follows.

**Theorem 2.1.** *Let $c = (c_1, c_2) \in \mathbb{C}^2$ and $g(z_1, z_2)$ be a polynomial in $\mathbb{C}^2$. If $f(z)$ be a finite order transcendental entire solution of* (2.1)*, then one of the following cases occurs.*

(i) *$f(z_1, z_2) = \phi_1(z_2 - \alpha z_1) + \phi_2(z_2 - \beta z_1)$, where $\phi_1, \phi_2$ are finite order transcendental entire functions in $\mathbb{C}^2$ such that*

$$\phi_1(z_2 - \alpha z_1 + c_2 - \alpha c_1) + \phi_2(z_2 - \beta z_1 + c_2 - \beta c_1) = \pm e^{\frac{1}{2}g(z_1,z_2)},$$

*$\alpha, \beta \in \mathbb{C}$ with $\alpha + \beta = \eta$, $\alpha\beta = \delta$.*
(ii) *$g(z_1, z_2)$ is of the form $g(z_1, z_2) = L(z) + H(s_1) + B$, where $L(z) = a_1 z_1 + a_2 z_2$, $H(s_1)$ is a polynomial in $s_1 := c_2 z_1 - c_1 z_2$, $a_1, a_2, B \in \mathbb{C}$, and the form of the solution is*

$$f(z_1, z_2) = \frac{\xi^2 - 1}{2i\xi} e^{\frac{1}{2}[L(z) + H(s_1) - L(c) + B]},$$

*where $\xi \neq 0, \pm 1, \pm i$ and $L(z)$ satisfies the relation*

$$e^{\frac{1}{2}[a_1c_1+a_2c_2]} = \frac{\xi^2-1}{2i(\xi^2+1)}[A_4 + (A_1c_2 - A_2c_1)a_0 + \frac{1}{2}A_3a_0^2],$$

*where $a_0$ is the coefficient of linear term of the polynomial $H(s_1)$ and $A_j$'s are defined in (2.4). In particular, if $A_1c_2 - A_2c_1 \neq 0$ or $A_3 \neq 0$, then $H(s_1)$ becomes linear in $s_1$.*

(iii) *$g(z_1, z_2)$ is of the form $g(z_1, z_2) = L(z) + H(s_1) + B$, where $L(z) = L_1(z) + L_2(z)$, $H(s_1) = H_1(s_1) + H_2(s_1)$ with $L_1(z) + H_1(s_1) \neq L_2(z) + H_2(s_1)$, $L_j(z) = a_{j1}z_1 + a_{j2}z_2$, $B = B_1 + B_2$, $H_j(s_1)$ is a polynomial in $s_1 = c_2z_1 - c_1z_2$ for $j = 1, 2$, $B_1, B_2, a_{ji}$ are constants in $\mathbb{C}$, and the form of the solution is*

$$f(z_1, z_2) = \frac{1}{2i}[A_2e^{(L_1(z)+H_1(s_1)-L_1(c)+B_1)} + A_1e^{(L_2(z)+H_2(s_1)-L_2(c)+B_2)}],$$

*where $L_1(z)$ and $L_2(z)$, respectively satisfy the relations*

$$e^{L_1(c)} = -i[A_{17}+(A_{15}c_2-A_{16}c_1)a_0+A_3a_0^2]e^{L_2(c)} = i[A_{27}+(A_{25}c_2-A_{26}c_1)a_{00}+A_3a_{00}^2],$$

*$a_0$ and $a_{00}$, respectively the coefficients of the linear term of the polynomials $H_1(s_1)$ and $H_2(s_1)$, and $A_{ij}$'s are defined in (2.4) In particular, if $A_{15}c_2 - A_{16}c_1 \neq 0$ or $A_3 \neq 0$, then $H_1$ becomes linear in $s_1$. Similarly, if $A_{25}c_2 - A_{26}c_1 \neq 0$ or $A_3 \neq 0$, then $H_2$ becomes linear in $s_1$.*

Next, we exhibit some examples in support of the Theorem 2.1.

**Example 2.2.** Let $\alpha = \beta = 1$, $c_1 = 2, c_2 = 3$ and $g(z) = 4(z_2 - z_1 + 1)^2$. Then, in view of Theorem 2.1(i), it can be easily seen that $f(z_1, z_2) = e^{(z_2-z_1)^2}$ is a solution of (2.1).

**Example 2.3.** Let $c_1 = c_2 = 1, \xi = 3$, $\delta = 1, \eta = 2$ and $g(z_1, z_2) = z_1 + z_2 + (z_1 - z_2)^n + 10$, $n \in \mathbb{N}$. Then in view of of Theorem 2.1(ii), one can easily verify that $f(z) = \frac{5}{3}e^{[z_1+z_2+(z_1-z_2)^n+10]/2}$ is a solution of (2.1).

**Example 2.4.** Let $\delta = \eta = 4$, $\xi = 5$, $c_1 = 2, c_2 = 3$, $a_0 = 1$, $L(z) = z_1 - z_2$ and $g(z_1, z_2) = 4z_1 - 3z_2$. Then in view of Theorem 2.1(ii), we can easily verify that $f(z_1, z_2) = -\frac{12i}{25}e^{\frac{1}{2}(4z_1-3z_2)}$ is a solution of (2.1).

**Example 2.5.** Let $c = (c_1, c_2) \in \mathbb{C}$ such that $c_1 \neq c_2$, $\delta = 1$, $\eta = 2$, $L_1(z) = z_1 + z_2$, $L_2(z) = z_1 + 2z_2$, $H_1(s_1) = H_2(s_2) = 0$ and $B_1 = B_2 = 1$, and $g(z_1, z_2) = 2z_1 + 3z_2 + 2$. Then in view of Theorem 2.1(iii), it can be easily verified that $f(z_1, z_2) = \frac{1}{2}[\frac{1}{4}e^{z_1+z_2+1} + \frac{1}{9}e^{z_1+2z_2+1}]$ is a solution of (2.1).

**Theorem 2.6.** *Let $c = (c_1, c_2) \in \mathbb{C}^2$, $\delta, \eta \in \mathbb{C}$ and $g(z)$ is a polynomial in $\mathbb{C}^2$. Let $f(z)$ be a finite order transcendental entire solution of (2.2). Then, one of the following cases must occur.*

(i) *$f(z_1, z_2) = \phi_1(z_2 - \alpha z_1) + \phi_2(z_2 - \beta z_1)$, where $\phi_1, \phi_2$ are finite order transcendental entire functions in $\mathbb{C}^2$ satisfying*

$$\begin{aligned}&\phi_1(z_2 - \alpha z_1 + c_2 - \alpha c_1) + \phi_2(z_2 - \beta z_1 + c_2 - \beta c_1)\\&- \phi_1(z_2 - \alpha z_1) - \phi_2(z_2 - \beta z_1) = \pm e^{g(z)/2},\end{aligned}$$

*with $\alpha, \beta \in \mathbb{C}$ such that $\alpha + \beta = \eta$ and $\alpha\beta = \delta$.*

(ii) $g(z_1,z_2)=a_1z_1+a_2z_2+H(c_2z_1-c_1z_2)+B$, *where* $H$ *is a polynomial in* $c_2z_1-c_1z_2$ *and* $a_1c_1+a_2c_2=4k\pi i$, $k\in\mathbb{Z}$,

$$\begin{aligned} f(z_1,z_2)=&\pm\int_0^{z_1}\int_0^{z_1}e^{\frac{1}{2}[a_1z_1+a_2z_2+H(c_2z_1-c_1z_2)+B]}dz_1dz_1\\ &+\int_0^{z_1/\alpha_2}G_0(z_2-\beta z_1)dz_1+G_1(z_2-\alpha z_1), \end{aligned}$$

*where* $G_0,G_1$ *are finite order transcendental entire functions in* $\mathbb{C}^2$ *satisfying*

$$\begin{aligned} &\int_0^{z_1}[G_0(z_2-\beta z_1+c_2-\beta c_1)-G_0(z_2-\beta z_1)]dz_1\\ &+G_1(z_2-\alpha z_1+c_2-\alpha c_1)-G_1(z_2-\alpha z_1)=0. \end{aligned}$$

(iii) *If* $\gamma c_2^2+\delta c_1^2\neq\eta c_1c_2$, *then* $g(z)$ *must be of the form* $g(z)=a_1z_1+a_2z_2+B$, $a_1,a_2,B\in\mathbb{C}$, *and the solution has the form*

$$\begin{aligned} f(z_1,z_2)=&\phi_1(z_2-\alpha z_1)+\phi_2(z_2-\beta z_1)\\ &+\frac{2(\xi+\xi^{-1})}{a_1^2+\delta a_2^2+\eta a_1a_2}e^{\frac{1}{2}[a_1z_1+a_2z_2+B]}, \end{aligned}$$

*where* $\xi(\neq 0)\in\mathbb{C}$, $a_1^2+\delta a_2^2+\eta a_1a_2\neq 0$, $\alpha,\beta$ *are same as in (i)*, $\phi_1,\phi_2$ *are finite order transcendental entire functions in* $\mathbb{C}^2$ *such that*

$$\begin{aligned} &\phi_1(z_2-\alpha z_1+c_2-\alpha c_1)+\phi_2(z_2-\beta z_1+c_2-\beta c_1)\\ &=\phi_1(z_2-\alpha z_1)+\phi_2(z_2-\beta z_1) \end{aligned}$$

*and*

$$e^{\frac{1}{2}[a_1c_1+a_2c_2]}=\frac{(\xi-\xi^{-1})(a_1^2+\delta a_2^2+\eta a_1a_2)}{4i(\xi+\xi^{-1})}+1.$$

(iv) *If* $\gamma c_2^2+\delta c_1^2\neq\eta c_1c_2$, *then* $g(z)$ *must be of the form* $g(z)=L_1(z)+L_2(z)+B_1+B_2$, *where* $L_j(z)=a_{j1}z_1+a_{j2}z_2$ *with* $L_1(z)\neq L_2(z)$, $a_{ij},B_j\in\mathbb{C}$ *and the form of the solution is*

$$\begin{aligned} f(z_1,z_2)=&\phi_1(z_2-\alpha z_1)+\phi_2(z_2-\beta z_1)+\frac{e^{L_1(z)+B_1}}{2(a_{11}^2+\delta a_{12}^2+\eta a_{11}a_{12})}\\ &+\frac{e^{L_2(z)+B_2}}{2(a_{21}^2+\delta a_{22}^2+\eta a_{21}a_{22})}, \end{aligned}$$

*where* $a_{21}^2+\delta a_{22}^2+\eta a_{21}a_{22}\neq 0$, $a_{11}^2+\delta a_{12}^2+\eta a_{11}a_{12}\neq 0$, $\alpha,\beta$ *are same as in (i)*, $\phi_1,\phi_2$ *are finite order transcendental entire functions in* $\mathbb{C}^2$ *such that*

$$\begin{aligned} &\phi_1(z_2-\alpha z_1+c_2-\alpha c_1)+\phi_2(z_2-\beta z_1+c_2-\beta c_1)\\ &=\phi_1(z_2-\alpha z_1)+\phi_2(z_2-\beta z_1) \end{aligned}$$

*and* $L_1(z),L_2(z)$ *satisfy the relations*

$$\begin{aligned} e^{L_1(c)}&=-i(a_{11}^2+\delta a_{12}^2+\eta a_{11}a_{12})+1,\\ e^{L_2(c)}&=i(a_{21}^2+\delta a_{22}^2+\eta a_{21}a_{22})+1. \end{aligned}$$

The following examples show that the forms of solutions are precise.

**Example 2.7.** Let $\alpha=\beta=-1$. Choose $c=(c_1,c_2)\in\mathbb{C}^2$ such that $c_1+c_2=2k\pi i$, $k\in\mathbb{C}$. Then in view of Theorem 2.6(i), we can easily deduce that $f(z_1,z_2)=e^{z_1+z_2}$ is a solution of (2.2) with $g(z_1,z_2)=2(z_1+z_2)$.

**Example 2.8.** Let $\alpha=\beta=1$ and $\xi=2$. Choose $c=(c_1,c_2)\in\mathbb{C}^2$ such that $c_1\neq c_2$ and $c_2-c_1=2k\pi i$, $k\in\mathbb{Z}$. Let $\psi(z_2-z_1)=\phi_1(z_2-z_1)+\phi_2(z_2-z_1)=e^{z_2-z_1}$ and $g(z)=L(z)+1=z_1+2z_2+1$. Then, in view of Theorem 2.6(iii), we can easily verify that $f(z_1,z_2)=e^{z_1-z_2}+\frac{5}{9}e^{(z_1+z_2+1)/2}$ is a solution of (2.2).

**Example 2.9.** Let $\delta=1,\eta=2$, $c_1=\log(10-8i)/4$, and $c_2=[\log(1-9i)-\log(1+i)]/2$. Let $L_1(z)=z_1+2z_2$, $L_2(z)=-z_1+2z_2$. Then in view of Theorem 2.6(iv), we can easily deduce that $f(z_1,z_2)=e^{L_1(z)+1}/18+e^{L_2(z)+2}/2$ is a solution of (2.2) with $g(z_1,z_2)=L_1(z)+L_2(z)+2$.

**Theorem 2.10.** Let $c=(c_1,c_2)\in\mathbb{C}^2$ and $g(z_1,z_2)$ be a polynomial in $\mathbb{C}^2$. If $f$ be a finite order transcendental entire solution of (2.3), then one of the following cases must occur.

(i) $g(z_1,z_2)$ must be of the form $g(z_1,z_2)=L(z)+H(s)+B$, where $L(z)=a_1z_1+a_2z_2$, $H(s)$ is a polynomial in $s:=c_2z_1-c_1z_2$, $a_1,a_2,B\in\mathbb{C}$ and
$$f(z_1,z_2)=\pm e^{\frac{1}{2}[L(z)+H(s)+B]},\quad \text{where } e^{\frac{1}{2}L(c)}=1.$$

(ii) $g(z_1,z_2)$ must be of the form $g(z_1,z_2)=L(z)+H(s)+B$, $L(z),H(s)$ and $B$ are same as (i) and
$$f(z_1,z_2)=\frac{\xi^2+1}{2\xi}e^{\frac{1}{2}[L(z)+H(s)+B]},$$
where $\xi\neq 0,\pm i,\pm 1$ and $L(z)$ satisfies the relation
$$e^{\frac{1}{2}L(c)}=\frac{(1-i)\xi^2+1+i}{\xi^2+1}.$$

(iii)
$$f(z_1,z_2)=\frac{e^{L_1(z)+H_1(s_1)+B_1}+e^{L_2(z)+H_2(s_1)+B_2}}{2},$$
where $L_1(z)=a_{11}z_1+a_{12}z_2+B_1$ and $L_2(z)=a_{21}z_1+a_{22}z_2+B_2$, with $a_{j1},a_{j2},B_j\in\mathbb{C}$( j =1, 2), satisfy
$$g(z_1,z_2)=L_1(z)+L_2(z)+H_1(s_1)+H_2(s_1)+B_1+B_2,$$
$$L_1(z)+H_1(s_1)\neq L_2(z)+H_2(s_1),\quad e^{L_1(c)}=1-i,\quad e^{L_2(c)}=1+i.$$

**Example 2.11.** Let $L(z)=z_1+2z_2$, $H(s)=-\pi^2(z_1-2z_2)^2$, $B=1$, $c_1=2\pi i$ and $c_2=\pi i$. Then in view of Theorem 2.10(i), it can be shown that $f(z_1,z_2)=e^{\frac{1}{2}[z_1+2z_2-\pi^2(z_1-2z_2)^2+1]}$ is a solution of (2.3), where $g(z)=z_1+2z_2-\pi^2(z_1-2z_2)^2+1$.

**Example 2.12.** Let $L(z)=z_1-z_2$ and $c=(c_1,c_2)\in\mathbb{C}$ such that $c_1-c_2=(5-3i)/5$. Let $H(s)=(c_2z_1-c_1z_2)^n$, $n\in\mathbb{N}$. Then in view of Theorem 2.10(ii), it can be shown that $f(z_1,z_2)=e^{\frac{5}{4}[z_1-z_2+(c_2z_1-c_1z_2)^n+2]}$ is a solution of (2.3), where $g(z)=z_1-z_2+(c_2z_1-c_1z_2)^n+2$.

## 3. Proofs of main results

Before we starting, we present some necessary lemmas which will play key role to prove the main results.

**Lemma 3.1** ([13]). *Let $f_j \not\equiv 0$ $(j = 1, 2, 3)$ be meromorphic functions on $\mathbb{C}^n$ such that $f_1$ are not constant, $f_1 + f_2 + f_3 = 1$, and such that*

$$\sum_{j=1}^{3} \left\{ N_2\big(r, \frac{1}{f_j}\big) + 2\overline{N}(r, f_j) \right\} < \lambda T(r, f_j) + O(\log^+ T(r, f_j))$$

*holds for all $r$ outside possibly a set with finite logarithmic measure, where $\lambda < 1$ is a positive number. Then, either $f_2 = 1$ or $f_3 = 1$.*

**Lemma 3.2** ([17, 29, 33]). *For an entire function $F$ on $\mathbb{C}^n$, $F(0) \not\equiv 0$ and put $\rho(n_F) = \rho < \infty$. Then there exist a canonical function $f_F$ and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_F(z)e^{g_F(z)}$. For the special case $n = 1$, $f_F$ is the canonical product of Weierstrass.*

**Lemma 3.3** ([28]). *If $g$ and $h$ are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible cases: either*

(i) *the internal function $h$ is a polynomial and the external function $g$ is of finite order; or else*
(ii) *the internal function $h$ is not a polynomial but a function of finite order, and the external function $g$ is of zero order.*

**Lemma 3.4** ([13]). *Let $a_0(z), a_1(z), \dots, a_n(z)$ $(n \ge 1)$ be meromorphic functions on $\mathbb{C}^m$ and $g_0(z), g_1(z), \dots, g_n(z)$ are entire functions on $\mathbb{C}^m$ such that $g_j(z) - g_k(z)$ are not constants for $0 \le j < k \le n$. If $\sum_{j=0}^{n} a_j(z)e^{g_j(z)} \equiv 0$, and $||T(r, a_j) = o(T(r))$, where $T(r) = \min_{0 \le j < k \le n} T(r, e^{g_j - g_k})$ for $j = 0, 1, \dots, n$, then $a_j(z) \equiv 0$ for each $j = 0, 1, \dots, n$.*

*Proof of Theorem 2.1.* Let $f(z)$ be a transcendental entire solution of (2.1). First rewrite (2.1) as

$$\Big(\frac{P(f)}{e^{g(z)/2}} + i\frac{f(z_1 + c_1, z_2 + c_2)}{e^{g(z)/2}}\Big)\Big(\frac{P(f)}{e^{g(z)/2}} - i\frac{f(z_1 + c_1, z_2 + c_2)}{e^{g(z)/2}}\Big) = 1, \tag{3.1}$$

where $P(f) = \frac{\partial^2 f}{\partial z_1^2} + \delta\frac{\partial^2 f}{\partial z_2^2} + \eta\frac{\partial^2 f}{\partial z_1 \partial z_2}$.

Since $f$ is a transcendental entire function of finite order, in view of (3.1), we conclude that $(P(f) + if(z_1 + c_1, z_2 + c_2))/e^{g(z)/2}$ and $(P(f) - if(z_1 + c_1, z_2 + c_2))/e^{g(z)/2}$ have no zeros and poles. Thus, by Lemmas 3.2 and 3.3, there exists a non-constant polynomial $h(z)$ in $\mathbb{C}^2$ such that

$$\begin{gathered}
\frac{\frac{\partial^2 f}{\partial z_1^2} + \delta\frac{\partial^2 f}{\partial z_2^2} + \eta\frac{\partial^2 f}{\partial z_1 \partial z_2}}{e^{g(z)/2}} + i\frac{f(z_1 + c_1, z_2 + c_2)}{e^{g(z)/2}} = e^{h(z)}, \\
\frac{\frac{\partial^2 f}{\partial z_1^2} + \delta\frac{\partial^2 f}{\partial z_2^2} + \eta\frac{\partial^2 f}{\partial z_1 \partial z_2}}{e^{g(z)/2}} - i\frac{f(z_1 + c_1, z_2 + c_2)}{e^{g(z)/2}} = e^{-h(z)}.
\end{gathered} \tag{3.2}$$

We set

$$\gamma_1(z) = \frac{g(z)}{2} + h(z), \quad \gamma_2(z) = \frac{g(z)}{2} - h(z). \tag{3.3}$$

Therefore, in view of (3.2) and (3.3), we obtain that

$$\begin{gathered}\frac{\partial^2 f}{\partial z_1^2}+\delta\frac{\partial^2 f}{\partial z_2^2}+\eta\frac{\partial^2 f}{\partial z_1\partial z_2}=\frac{e^{\gamma_1(z)}+e^{\gamma_2(z)}}{2},\\ f(z_1+c_1,z_2+c_2)=\frac{e^{\gamma_1(z)}-e^{\gamma_2(z)}}{2i}.\end{gathered}\tag{3.4}$$

After simple computations, it follows from the two equations of (3.4) that

$$-iQ_1(z)e^{\gamma_1(z)-\gamma_1(z+c))}+iQ_2(z)e^{\gamma_2(z)-\gamma_1(z+c)}-e^{\gamma_2(z+c)-\gamma_1(z+c)}=1,\tag{3.5}$$

where

$$\begin{aligned}Q_j(z)=\Big(\frac{\partial\gamma_j}{\partial z_1}\Big)^2+\frac{\partial^2\gamma_j}{\partial z_1^2}+\delta\Big(\Big(\frac{\partial\gamma_j}{\partial z_2}\Big)^2+\frac{\partial^2\gamma_j}{\partial z_2^2}\Big)\\ +\eta\Big(\frac{\partial\gamma_j}{\partial z_1}\frac{\partial\gamma_j}{\partial z_2}+\frac{\partial^2\gamma_j}{\partial z_1\partial z_2}\Big),\quad j=1,2.\end{aligned}\tag{3.6}$$

Now, we discuss two possible cases.

**Case 1.** Let $\gamma_2(z+c)-\gamma_1(z+c)$ be a constant, say $k\in\mathbb{C}$. In view of (3.3), we conclude that $h(z)$ is constant. Let $\xi=e^{h(z)}\in\mathbb{C}$. Then, (3.4) yields that

$$\frac{\partial^2 f}{\partial z_1^2}+\delta\frac{\partial^2 f}{\partial z_2^2}+\eta\frac{\partial^2 f}{\partial z_1\partial z_2}=D_1e^{g(z)/2},\quad f(z+c)=D_2e^{g(z)/2},\tag{3.7}$$

where $D_1=\left(\xi+\xi^{-1}\right)/2$, $D_2=-i\left(\xi-\xi^{-1}\right)/2$. Note that $D_2\neq 0$ and $D_1^2+D_2^2=1$.

If $D_1=0$, in view of (3.7), it follows that

$$\begin{gathered}\frac{\partial^2 f(z)}{\partial z_1^2}+\delta\frac{\partial^2 f(z)}{\partial z_2^2}+\eta\frac{\partial^2 f(z)}{\partial z_1\partial z_2}=0,\\ f(z_1+c_1,z_2+c_2)=\pm e^{g(z)/2}.\end{gathered}\tag{3.8}$$

From the first equation of (3.8), we obtain $f(z_1,z_2)=\phi_1(z_2-\alpha z_1)+\phi_2(z_2-\beta z_1)$, where $\phi_1,\phi_2$ are finite order transcendental entire functions in $\mathbb{C}^2$, and $\alpha,\beta\in\mathbb{C}$ such that $\alpha+\beta=\eta$, $\alpha\beta=\delta$. Hence, from the second equation of (3.8), it follows that

$$\phi_1(z_2-\alpha z_1+c_2-\alpha c_1)+\phi_2(z_2-\beta z_1+c_2-\beta c_1)=\pm e^{\frac12 g(z_1,z_2)}.$$

This is the conclusion (i).

If $D_1\neq 0$, then from (3.7), we obtain that

$$\begin{aligned}&\frac12\Big(\frac{\partial g}{\partial z_1}\Big)^2+\frac{\partial^2 g}{\partial z_1^2}+\delta\Big(\frac12\Big(\frac{\partial g}{\partial z_2}\Big)^2+\frac{\partial^2 g}{\partial z_2^2}\Big)+\eta\Big(\frac12\frac{\partial g}{\partial z_1}\frac{\partial g}{\partial z_2}+\frac{\partial^2 g}{\partial z_1\partial z_2}\Big)\\ &=\frac{2D_1}{D_2}e^{\frac12[g(z+c)-g(z)]}.\end{aligned}\tag{3.9}$$

Since $g(z)$ is a polynomial in $\mathbb{C}^2$, it follows from (3.9) that $g(z+c)-g(z)$ must be constant. Then, $g(z)$ can be written as $g(z)=L(z)+H(s_1)+B$, where $L(z)=a_1z_1+a_2z_2$, $H(s_1)$ is a polynomial in $s_1=c_2z_1-c_1z_2$, $a_1,a_2,B$ are constants in $\mathbb{C}$. Hence, it follows from (3.9) that

$$(A_1c_2-A_2c_1)H'+A_3\Big(\frac12 H'^2+H''\Big)=\frac{2(\xi+\xi^{-1})}{\xi-\xi^{-1}}e^{\frac12 L(c)}-A_4,\tag{3.10}$$

where $A_j$'s are defined in (2.4).

If $A_1c_2 - A_2c_1 = 0 = A_3$, then from (3.10), we obtain that

$$e^{\frac{1}{2}L(c)} = \frac{\xi - \xi^{-1}}{2(\xi + \xi^{-1})} A_4.$$

If $A_1c_2 - A_2c_1 \neq 0$ or $A_3 \neq 0$, then it follows from (3.10) that $H'$ must be constant, say $a_0$, which is the coefficient of $s_1$ in the polynomial $H(s_1)$.

Therefore, from (3.10), we obtain that

$$e^{\frac{1}{2}L(c)} = \frac{\xi^2 - 1}{2i(\xi^2 + 1)}[A_4 + (A_1c_2 - A_2c_1)a_0 + \frac{1}{2}A_3a_0^2]. \tag{3.11}$$

Hence, in either case $L(z)$ satisfies the relation (3.11).

Therefore, in view of the second equation of (3.7), we obtain the form of the solution as

$$f(z) = \frac{\xi^2 - 1}{2i\xi} e^{\frac{1}{2}[L(z) + H(s_1) - L(c) + B]}.$$

This is conclusion **(ii)**.

**Case 2** Let $\gamma_2(z + c) - \gamma_1(z + c)$ be non-constant. Then in view of (3.5), it follows that $Q_1(z)$ and $Q_2(z)$ both can not be zero at the same time.

If $Q_1(z) \equiv 0$ and $Q_2(z) \not\equiv 0$, then (3.5) yields that

$$iQ_2(z)e^{\gamma_2(z)-\gamma_1(z+c)} - e^{\gamma_2(z+c)-\gamma_1(z+c)} = 1.$$

In view of the above equation, it follows that

$$N\Big(r, \frac{1}{e^{\gamma_2(z+c)-\gamma_1(z+c)} + 1}\Big) = N\Big(r, \frac{1}{Q_2(z)e^{\gamma_2(z)-\gamma_1(z+c)}}\Big) = S\Big(r, e^{\gamma_2(z)-\gamma_1(z+c)}\Big).$$

Also, notice that

$$\begin{gathered}
N\Big(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}\Big) = S\Big(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}\Big), \\
N\Big(r, \frac{1}{e^{\gamma_2(z+c)-\gamma_1(z+c)}}\Big) = S\Big(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}\Big).
\end{gathered}$$

By the second main theorem of Nevanlinna for several complex variables, we obtain

$$\begin{aligned}
T\Big(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}\Big) &\leq \overline{N}\Big(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}\Big) + \overline{N}\Big(r, \frac{1}{e^{\gamma_2(z+c)-\gamma_1(z+c)}}\Big) \\
&\quad + \overline{N}\Big(r, \frac{1}{e^{\gamma_2(z+c)-\gamma_1(z+c)} + 1}\Big) + S\Big(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}\Big) \\
&\leq S\Big(r, e^{\gamma_2(z+c)-\gamma_1(z+c)}\Big) + S\Big(r, e^{\gamma_2(z)-\gamma_1(z+c)}\Big).
\end{aligned}$$

This implies that $\gamma_2(z + c) - \gamma_1(z + c)$ is constant, which is a contradiction.

Similarly, we can get a contradiction for the case $Q_1(z) \not\equiv 0$ and $Q_1(z) \equiv 0$. Hence, $Q_1(z) \not\equiv 0$ and $Q_2(z) \not\equiv 0$.

Since $\gamma_1(z)$ and $\gamma_2(z)$ are polynomials in $\mathbb{C}^2$ and $\gamma_2(z + c) - \gamma_1(z + c)$ is non-constant, applying Lemma 3.1 to the equation (3.5), we obtain that either $-iQ_1(z)e^{\gamma_1(z)-\gamma_1(z+c)} = 1$, or $iQ_2(z)e^{\gamma_2(z)-\gamma_1(z+c)} = 1$.

If

$$-iQ_1(z)e^{\gamma_1(z)-\gamma_1(z+c)} = 1, \tag{3.12}$$

then from (3.5), it follows that

$$iQ_2(z)e^{\gamma_2(z)-\gamma_2(z+c)} = 1. \tag{3.13}$$

As $\gamma_1(z)$ and $\gamma_2(z)$ are polynomials, in view of (3.12) and (3.13), we conclude that $\gamma_1(z) - \gamma_1(z+c)$ and $\gamma_2(z) - \gamma_2(z+c)$ both are constants in $\mathbb{C}$, and hence we obtain that $\gamma_1(z) = L_1(z) + H_1(s_1) + B_1$ and $\gamma_2(z) = L_2(z) + H_2(s_1) + B_2$, where $L_j(z) = a_{j1}z_1 + a_{j2}z_2$, $H_j(s_1)$ is a polynomial in $s_1 = c_2z_1 - c_1z_2$, $a_{j1}, a_{j2}, B_1, B_2$ are constants in $\mathbb{C}$ for $j = 1, 2$. Note that $L_1(z) + H_1(s_1) \neq L_2(z) + H_2(s_1)$. Otherwise, $\gamma_2(z+c) - \gamma_1(z+c)$ would become constant, a contradiction to our assumption. Hence, the form of the polynomial $g(z)$ is $g(z) = L(z) + H(s_1) + B$, where $L(z) = L_1(z) + L_2(z)$, $H(s_1) = H_1(s_1) + H_2(s_1)$ and $B = B_1 + B_2$.

Therefore, in view of (3.12) and (3.13), we obtain that

$$\begin{gathered}(A_{15}c_2 - A_{16}c_1)H_1' + A_3(H_1'^2 + H_1'') = ie^{L_1(c)} - A_{17},\\ (A_{25}c_2 - A_{26}c_1)H_2' + A_3(H_2'^2 + H_2'') = -ie^{L_2(c)} - A_{27},\end{gathered} \tag{3.14}$$

where $A_{ij}$'s are defined in (2.4).

Then, by similar arguments as in Case 1, we obtain from (3.14) that

$$\begin{aligned}e^{L_1(c)} &= -i\left[A_{17} + (A_{15}c_2 - A_{16}c_1)a_0 + A_3a_0^2\right],\\ e^{L_2(c)} &= i[A_{27} + (A_{25}c_2 - A_{26}c_1)a_{00} + A_3a_{00}^2],\end{aligned} \tag{3.15}$$

where $a_0$ and $a_{00}$, respectively the coefficients of the linear term of the polynomials $H_1(s_1)$ and $H_2(s_1)$.

Therefore, in view of the second equation of (3.4), we obtain

$$f(z) = \frac{1}{2i}\Big(e^{L_1(z)+H_1(s_1)-L_1(c)+B_1} - e^{L_2(z)+H_2(s_1)-L_2(c)+B_2}\Big),$$

where $L_1(c)$ and $L_2(c)$ can be found from (3.15). This is conclusion (iii).

If $iQ_2(z)e^{\gamma_2(z)-\gamma_1(z+c)} = 1$, then it follows from equation (3.5) that

$$-iQ_1(z)e^{\gamma_1(z)-\gamma_2(z+c)} = 1.$$

Since $\gamma_1(z)$ and $\gamma_2(z)$ are both polynomials in $\mathbb{C}^2$, it follows that $\gamma_2(z)-\gamma_1(z+c) = \eta_1$ and $\gamma_1(z)-\gamma_2(z+c) = \eta_2$, where $\eta_1, \eta_2 \in \mathbb{C}$. This implies that $\gamma_1(z)-\gamma_1(z+c) = \gamma_2(z) - \gamma_2(z+c) = \eta_1 + \eta_2$. Therefore, we can write $\gamma_1(z) = L(z) + H(s_1) + \zeta_1$ and $\gamma_2(z) = L(z) + H(s_1) + \zeta_2$. But, then we obtain $\gamma_2(z+c) - \gamma_1(z+c) = \zeta_2 - \zeta_1$, a constants, which is a contradiction. $\square$

*Proof of Theorem 2.6.* Let $f(z)$ be a transcendental entire solution of the equation (2.2). First rewrite (2.2) as

$$\Big(\frac{P(f)}{e^{g(z)/2}} + i\frac{f(z+c)-f(z)}{e^{g(z)/2}}\Big)\Big(\frac{P(f)}{e^{g(z)/2}} - i\frac{f(z+c)-f(z)}{e^{g(z)/2}}\Big) = 1, \tag{3.16}$$

where

$$P(f) = \frac{\partial^2 f}{\partial z_1^2} + \delta\frac{\partial^2 f}{\partial z_2^2} + \eta\frac{\partial^2 f}{\partial z_1\partial z_2}.$$

Since $f$ is a transcendental entire function of finite order, in view of (3.16), we conclude that $(P(f)+i(f(z+c)-f(z)))/e^{g(z)/2}$ and $(P(f)-i(f(z+c)-f(z)))/e^{g(z)/2}$ have no zeros and poles. Thus, by Lemmas 3.2 and 3.3, there exists a non-constant polynomial $h(z)$ in $\mathbb{C}^2$ such that

$$\begin{aligned}
&\frac{\frac{\partial^2 f}{\partial z_1^2}+\delta\frac{\partial^2 f}{\partial z_2^2}+\eta\frac{\partial^2 f}{\partial z_1\partial z_2}}{e^{g(z)/2}}+i\frac{f(z+c)-f(z)}{e^{g(z)/2}}=e^{h(z)},\\
&\frac{\frac{\partial^2 f}{\partial z_1^2}+\delta\frac{\partial^2 f}{\partial z_2^2}+\eta\frac{\partial^2 f}{\partial z_1\partial z_2}}{e^{g(z)/2}}-i\frac{f(z+c)-f(z)}{e^{g(z)/2}}=e^{-h(z)}.
\end{aligned}\tag{3.17}$$

We set

$$\gamma_1(z)=\frac{g(z)}{2}+h(z),\quad \gamma_2(z)=\frac{g(z)}{2}-h(z).\tag{3.18}$$

Then, in view of (3.17) and (3.18), we obtain that

$$\begin{gathered}
\frac{\partial^2 f}{\partial z_1^2}+\delta\frac{\partial^2 f}{\partial z_2^2}+\eta\frac{\partial^2 f}{\partial z_1\partial z_2}=\frac{1}{2}[e^{\gamma_1(z)}+e^{\gamma_2(z)}]\\
f(z+c)-f(z)=\frac{1}{2i}[e^{\gamma_1(z)}-e^{\gamma_2(z)}]
\end{gathered}.\tag{3.19}$$

After simple computations, it follows from the two equations of (3.19) that

$$[1-iQ_1(z)]e^{\gamma_1(z)-\gamma_1(z+c))}+[1+iQ_2(z)]e^{\gamma_2(z)-\gamma_1(z+c)}-e^{\gamma_2(z+c)-\gamma_1(z+c)}=1,\tag{3.20}$$

where $Q_1(z)$ and $Q_2(z)$ are defined in (3.6). Now we consider two possible cases.

**Case 1.** Let $\gamma_2(z+c)-\gamma_1(z+c)=k\in\mathbb{C}$. In view of (3.18), we conclude that $h(z)$ is constant. Set $e^h=\xi\in\mathbb{C}$. Then (3.19) yields that

$$\frac{\partial^2 f}{\partial z_1^2}+\delta\frac{\partial^2 f}{\partial z_2^2}+\eta\frac{\partial^2 f}{\partial z_1\partial z_2}=D_1e^{g(z)/2},\quad f(z+c)-f(z)=D_2e^{g(z)/2},\tag{3.21}$$

where $D_1=\frac{1}{2}(\xi+\xi^{-1})$, $D_2=\frac{1}{2i}(\xi-\xi^{-1}t)$. Note that $D_1^2+D_2^2=1$.

**Subcase 1.1.** Let $D_1=0$. Therefore, it follows from (3.21) that

$$\begin{gathered}
\frac{\partial^2 f(z)}{\partial z_1^2}+\delta\frac{\partial^2 f(z)}{\partial z_2^2}+\eta\frac{\partial^2 f(z)}{\partial z_1\partial z_2}=0,\\
f(z+c)-f(z)=\pm e^{g(z)/2}.
\end{gathered}\tag{3.22}$$

Now, in view of the first equation of (3.22), we obtain that

$$f(z_1,z_2)=\phi_1(z_2-\alpha z_1)+\phi_2(z_2-\beta z_1),$$

where $\phi_1,\phi_2$ are finite order transcendental entire functions in $\mathbb{C}^2$, and $\alpha,\beta$ are constants in $\mathbb{C}$ such that $\alpha+\beta=\eta$ and $\alpha\beta=\delta$.

In view of the second equation of (3.22), we obtain that

$$\begin{aligned}
&\phi_1(z_2-\alpha z_1+c_2-\alpha c_1)+\phi_2(z_2-\beta z_1+c_2-\beta c_1)-\phi_1(z_2-\alpha z_1)-\phi_2(z_2-\beta z_1)\\
&=\pm e^{\frac{1}{2}g(z_1,z_2)}.
\end{aligned}$$

This is conclusion **(i)**.

**Subcase 1.2.** Let $D_2=0$. Therefore, it follows from (3.21) that

$$\begin{gathered}
\frac{\partial^2 f(z)}{\partial z_1^2}+\delta\frac{\partial^2 f(z)}{\partial z_2^2}+\eta\frac{\partial^2 f(z)}{\partial z_1\partial z_2}=\pm e^{\frac{1}{2}g(z)},\\
f(z+c)-f(z)=0.
\end{gathered}\tag{3.23}$$

Clearly, the second equation of (3.23) shows that $f$ is a periodic function of period $c$. In view of the two equations in (3.23), it follows that $e^{\frac{1}{2}(g(z+c)-g(z))}=1$. This implies that $g(z_1, z_2) = a_1z_1 + a_2z_2 + H(c_2z_1 - c_1z_2) + B$, where $H$ is a polynomial in $c_2z_1 - c_1z_2$ and $a_1c_1 + a_2c_2 = 4k\pi i$, $k \in \mathbb{Z}$.

Now, in view of the results in [41, page 2178, Line 21], the first equation of (3.23) can be written as

$$(D + \alpha D')(D + \beta D')f(z) = \pm e^{\frac{1}{2}g(z)}, \tag{3.24}$$

where $D \equiv \frac{\partial}{\partial z_1}$, $D' \equiv \frac{\partial}{\partial z_2}$, $\alpha, \beta \in \mathbb{C}$ such that $\alpha + \beta = \eta$ and $\alpha\beta = \delta$.

Let $(D + \beta D')f(z) = u(z)$. Then (3.24) yields that

$$\frac{\partial u}{\partial z_1} + \beta \frac{\partial u}{\partial z_2} = \pm e^{\frac{1}{2}g(z_1,z_2)}. \tag{3.25}$$

The characteristic equations of (3.25) are

$$\frac{dz_1}{dt} = 1, \quad \frac{dz_2}{dt} = \beta, \quad \frac{du}{dt} = e^{\frac{1}{2}g(z_1,z_2)}.$$

Using the initial conditions: $t = 0$, $z_1 = 0$, $z_2 = s$, and $u = u(0, s) := G_0(s)$, with a parameter $s$, we obtain the following parametric representation for the solutions of the characteristic equations: $z_1 = t$, $z_2 = \beta t + s$,

$$u(z_1, z_2) = \pm \int_0^{z_1} e^{\frac{1}{2}g(z)} dz_1 + G_0(z_2 - \beta z_1),$$

where $G_0$ is a finite order transcendental entire function in $\mathbb{C}^2$.

Since, we have assumed that $(D + \beta D')f(z) = u(z)$, in view of (3.24), it follows that

$$\frac{\partial f(z)}{\partial z_1} + \alpha \frac{\partial f(z)}{\partial z_2} = \pm \int_0^{z_1} e^{\frac{1}{2}g(z)} dz_1 + G_0(z_2 - \beta z_1). \tag{3.26}$$

By similar arguments as above, we obtain from (3.26) that

$$\begin{aligned} f(z_1, z_2) = &\pm \int_0^{z_1} \int_0^{z_1} e^{\frac{1}{2}[a_1z_1 + a_2z_2 + H(c_2z_1 - c_1z_2) + B]} dz_1 dz_1 \\ &+ \int_0^{z_1} G_0(z_2 - \beta z_1) dz_1 + G_1(z_2 - \alpha z_1), \end{aligned}$$

where $G_1$ is a finite order transcendental entire function in $\mathbb{C}^2$.

In view of the fact that $a_1c_1 + a_2c_2 = 4k\pi i$, $k \in \mathbb{Z}$, it follows from the second equation of (3.23) that

$$\begin{aligned} &\int_0^{z_1} [G_0(z_2 - \beta z_1 + c_2 - \beta c_1) - G_0(z_2 - \beta z_1)] dz_1 \\ &+ G_1(z_2 - \alpha z_1 + c_2 - \alpha c_1) - G_1(z_2 - \alpha z_1) = 0. \end{aligned}$$

This is the conclusion **(ii)**.

**Subcase 1.3.** Let $D_1 \neq 0$ and $D_2 \neq 0$. Then after simple calculations, (3.21) yields that

$$\begin{aligned} &\Big(\frac{\partial^2 g}{\partial z_1^2} + \frac{1}{2}\big(\frac{\partial g}{\partial z_1}\big)^2\Big) + \delta\Big(\frac{\partial^2 g}{\partial z_2^2} + \frac{1}{2}\big(\frac{\partial g}{\partial z_2}\big)^2\Big) + \eta\Big(\frac{\partial^2 g}{\partial z_1 \partial z_2} + \frac{1}{2}\frac{\partial g}{\partial z_1}\frac{\partial g}{\partial z_2}\Big) \\ &= \frac{2D_1}{D_2}[e^{\frac{1}{2}[g(z+c) - g(z)]} - 1. \end{aligned} \tag{3.27}$$

Since $g(z)$ is a polynomial in $\mathbb{C}^2$, in view of (3.27) we conclude that $g(z+c) - g(z) = \xi$, $\xi \in \mathbb{C}$. This implies that $g(z) = L_1(z) + H(s_1) + B_1$, where $L_1(z) = a_{11}z_1 + a_{12}z_2$,

$H(s_1)$ is a polynomial in $s_1 := c_2z_1 - c_1z_2$, $a_{11}, a_{12}, B_1 \in \mathbb{C}$. Hence, we obtain from (3.27) that

$$\begin{aligned}&\big[\big(a_{11} + \frac{1}{2}\eta a_{12}\big)c_2 - \big(\delta a_{12} + \frac{1}{2}\eta a_{11}\big)c_1\big]H' \\ &+ \big(c_2^2 + \delta c_1^2 - \eta c_1c_2\big)\big(\frac{1}{2}{H'}^2 + H''\big) = \frac{2D_1}{D_2}[e^{\frac{1}{2}L_1(c)} - 1].\end{aligned} \tag{3.28}$$

Since $c_2^2 + \delta c_1^2 \neq \eta c_1c_2$, in view of (3.28), we conclude that $H'$ is constant. This implies that $H(s_1) = a_0s_1 + b_0$. Hence, $g(z)$ reduces to the form

$$g(z) = L(z) + B = a_1z_1 + a_2z_2 + B, \tag{3.29}$$

where $a_1 = a_{11} + a_0c_2$, $a_2 = a_{12} - a_0c_1$ and $B = B_1 + b_0$.

Therefore, in view of (3.27) and (3.29) we obtain that

$$e^{\frac{1}{2}[a_1c_1+a_2c_2]} = \frac{D_2}{4D_1}\left(a_1^2 + \delta a_2^2 + \eta a_1a_2\right) + 1. \tag{3.30}$$

Now, in view of the results in [41, page 2178, Line 21], the first equation of (3.21) can be written as

$$(D^2 + \delta D'^2 + \eta DD')f(z) = D_1e^{\frac{1}{2}[a_1z_1+a_2z_2+B]}, \tag{3.31}$$

where $D \equiv \frac{\partial}{\partial z_1}$ and $D' \equiv \frac{\partial}{\partial z_2}$. Therefore, complementary function of (3.31) is C.F. $= \phi_1(z_2 - \alpha z_1) + \phi_2(z_1 - \beta z_1)$, where $\phi_1$, $\phi_2$ are finite order transcendental entire functions in $\mathbb{C}^2$, $\alpha, \beta \in \mathbb{C}$ such that $\alpha + \beta = \eta$ and $\alpha\beta = \delta$. Particular integral of (3.31) is

$$\text{P.I.} = \frac{4D_1e^{B/2}}{a_1^2 + \delta a_2^2 + \eta a_1a_2}\int\int e^v dvdv = \frac{4D_1}{a_1^2 + \delta a_2^2 + \eta a_1a_2}e^{\frac{1}{2}[a_1z_1+a_2z_2+B]},$$

where $v = a_1z_1 + a_2z_2$. Hence, from (3.21), we obtain

$$\begin{aligned}f(z_1, z_2) &= \phi_1(z_2 - \alpha z_1) + \phi_2(z_2 - \beta z_1) \\ &\quad + \frac{2(\xi + \xi^{-1})}{a_1^2 + \delta a_2^2 + \eta a_1a_2}e^{\frac{1}{2}[a_1z_1+a_2z_2+B]}.\end{aligned}$$

Substituting $f(z_1, z_2)$ into the second equation of (3.21) and combining with (3.30), we obtain that

$$\phi_1(z_2 - \alpha z_1 + c_2 - \alpha c_1) + \phi_2(z_2 - \beta z_1 + c_2 - \beta c_1) = \phi_1(z_2 - \alpha z_1) + \phi_2(z_2 - \beta z_1).$$

This is the conclusion **(iii)**.

**Case 2.** Let $\gamma_2(z+c) - \gamma_1(z+c)$ be non-constant. Then, obviously $1 - iQ_1(z)$ and $1+iQ_2(z)$ can not be identically zero at the same time. Otherwise, in view of (3.20), it follows that $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is a constant, which implies that $\gamma_2(z+c) - \gamma_1(z+c)$ is a constant. This is a contradiction to our assumption.

If $1 - iQ_1(z) \equiv 0$ and $1 + iQ_2(z) \not\equiv 0$, the (3.20) it yields that

$$(1 + iQ_2(z))e^{\gamma_2(z)} - e^{\gamma_2(z+c)} - e^{\gamma_1(z+c)} \equiv 0. \tag{3.32}$$

Note that $\gamma_2(z) - \gamma_2(z + c)$ is non-constant. Otherwise, if $\gamma_2(z) - \gamma_2(z + c) = \zeta \in \mathbb{C}$, then (3.32) yields that $[(1 + iQ_2(z))e^{\zeta} - 1]e^{\gamma_1(z+c)-\gamma_2(+c)} = 1$. But, then $\gamma_1(z + c) - \gamma_2(+c)$ becomes a constant, which is a contradiction. Also, note that $\gamma_2(z) - \gamma_1(z + c)$ is non-constant. Otherwise, in view of (3.32), we obtain that $\gamma_1(z + c) - \gamma_2(+c)$ is constant, which is a contradiction. Hence, in view of (3.32) and the Lemma 3.4, we can easily get a contradiction. Similarly, we can get a contradiction for the case $1 - iQ_1(z) \not\equiv 0$ and $1 + iQ_2(z) \equiv 0$. Therefore, we must have $1 - iQ_1(z) \not\equiv 0$ and $1 + iQ_2(z) \not\equiv 0$.

Now, in view of Lemma 3.1, we obtain from (3.20) that either

$$[1 - iQ_1(z)]e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1, \text{ or } [1 + iQ_2(z)]e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1.$$

If $[1 + iQ_2(z)]e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1$, then in view of (3.20), it follows that $[1 - iQ_1(z)]e^{\gamma_1(z)-\gamma_2(z+c)} \equiv 1$. Therefore, we must obtain that $\gamma_2(z) - \gamma_1(z + c) = \xi_1$ and $\gamma_1(z) - \gamma_2(z + c) = \xi_2$, $\xi_1, \xi_2 \in \mathbb{C}$. Thus, it follows that $\gamma_1(z) - \gamma_1(z + 2c) = \gamma_2(z) - \gamma_2(z + 2c) = \xi_1 + \xi_2$. This implies that $\gamma_1(z) = L(z) + H(s_1) + B_1$ and $\gamma_2(z) = L(z) + H(s_1) + B_2$, where $L(z) = a_1z_1 + a_2z_2$ and $H(s_1)$ is a polynomial in $s_1 := c_2z_1 - c_1z_2$, $a_1, a_2, B_1, B_2 \in \mathbb{C}$. Hence, we must have that $\gamma_2(z + c) - \gamma_1(z + c) = B_2 - B_1$, a constant in $\mathbb{C}$, which is a contradiction to the assumption. Therefore, we must have

$$[1 - iQ_1(z)]e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1. \tag{3.33}$$

In view of (3.20) and (3.33), we obtain that

$$[1 + iQ_2(z)]e^{\gamma_2(z)-\gamma_2(z+c)} \equiv 1. \tag{3.34}$$

Since $\gamma_1(z)$ and $\gamma_2(z)$ are polynomials in $\mathbb{C}^2$, from (3.33) and (3.34), we can conclude that $\gamma_1(z) - \gamma_1(z + c) = \eta_1$ and $\gamma_2(z) - \gamma_2(z + c) = \eta_2$, $\eta_1, \eta_2 \in \mathbb{C}$. Thus, we have $\gamma_1(z) = L_1(z) + H_1(s_1) + B_1$ and $\gamma_2(z) = L_2(z) + H_2(s_1) + B_2$, where $L_j(z) = a_{j1}z_1 + a_{j2}z_2$ and $H_j(s_1)$ is a polynomial in $s_1 := c_2z_1 - c_1z_2$, $a_{j1}, a_{j2}, B_j \in \mathbb{C}$ for $j = 1, 2$. Therefore, in view of (3.5), (3.33), we obtain that

$$\begin{aligned}
&[(2a_{11} + \eta a_{12})c_2 - (2\delta a_{12} + \eta a_{11})c_1]H_1' + (c_2^2 + \delta c_1^2 - \eta c_1c_2)(H_1'^2 + H_1'') \\
&= i[e^{L_1(c)} - 1] - (a_{11}^2 + \delta a_{12}^2 + \eta a_{11}a_{12}).
\end{aligned}$$

Since $c_2^2 + \delta c_1^2 - \eta c_1c_2 \neq 0$, in view of the above equation, we conclude that $H_1(s_1)$ is a linear polynomial in $s_1$, and thus $L_1(z) + H_1(s_1)$ becomes linear in $\mathbb{C}$. For the sake of convenience, we still denote that $\gamma_1(z) = L_1(z) + B_1$. In a similar manner, from (3.5) and (3.34), we can conclude that $\gamma_2(z) = L_2(z) + B_2$. Therefore, in view of (3.5), it follows from (3.33) and (3.34) that

$$\begin{aligned}
e^{L_1(c)} &= -i(a_{11}^2 + \delta a_{12}^2 + \eta a_{11}a_{12}) + 1, \\
e^{L_2(c)} &= i(a_{21}^2 + \delta a_{22}^2 + \eta a_{21}a_{22}) + 1.
\end{aligned} \tag{3.35}$$

Now, in view of the results in [41, page 2178, Line 21], and the form of $\gamma_1(z)$ and $\gamma_2(z)$, the first equation of (3.19) can be written as

$$(D^2 + \delta D'^2 + \eta DD')f(z) = \frac{1}{2}[e^{L_1(z)+B_1} + e^{L_2(z)+B_2}], \tag{3.36}$$

where $D \equiv \frac{\partial}{\partial z_1}$ and $D' \equiv \frac{\partial}{\partial z_2}$.

The complementary function of (3.36) is $\phi_1(z_2 - \alpha z_1) + \phi_2(z_1 - \beta z_1)$, where $\phi_1$, $\phi_2$ are finite order transcendental entire functions in $\mathbb{C}^2$, $\alpha, \beta \in \mathbb{C}$ such that $\alpha + \beta = \eta$ and $\alpha\beta = \delta$, and the particular integral is

$$\text{P.I.} = \frac{e^{L_1(z)+B_1}}{2(a_{11}^2 + \delta a_{12}^2 + \eta a_{11}a_{12})} + \frac{e^{L_2(z)+B_2}}{2(a_{21}^2 + \delta a_{22}^2 + \eta a_{21}a_{22})}.$$

Hence, the form of the solution of (3.36) is

$$\begin{aligned} f(z_1,z_2) = \phi_1(z_2-\alpha z_1)+\phi_2(z_2-\beta z_1)+\frac{e^{L_1(z)+B_1}}{2(a_{11}^2+\delta a_{12}^2+\eta a_{11}a_{12})} \\ +\frac{e^{L_2(z)+B_2}}{2(a_{21}^2+\delta a_{22}^2+\eta a_{21}a_{22})}. \end{aligned} \tag{3.37}$$

Substituting (3.37) into the second equation of (3.19) and combining with (3.35), we obtain that

$$\phi_1(z_2-\alpha z_1+c_2-\alpha c_1)+\phi_2(z_2-\beta z_1+c_2-\beta c_1)=\phi_1(z_2-\alpha z_1)+\phi_2(z_2-\beta z_1).$$

This is conclusion **(iv)**. □

Theorem 2.10 can be proved by similar arguments as in Theorem 2.1, Therefore, we omit its proof.

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