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OSCILLATION CRITERIA FOR NON-CANONICAL SECOND-ORDER NONLINEAR DELAY DIFFERENCE EQUATIONS WITH A SUPERLINEAR NEUTRAL TERM

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Abstract. We obtain oscillation conditions for non-canonical second-order nonlinear delay difference equations with a superlinear neutral term. To cope with non-canonical types of equations, we propose new oscillation criteria for the main equation when the neutral coefficient does not satisfy any of the conditions that call it to either converge to 0 or ∞. Our approach differs from others in that we first turn into the non-canonical equation to a canonical form and as a result, we only require one condition to weed out non-oscillatory solutions in order to induce oscillation. The conclusions made here are new and have been condensed significantly from those found in the literature. For the sake of confirmation, we provide examples that cannot be included in earlier works.

1. Introduction

The article concerns the oscillation of the nonlinear delay difference equation with a superlinear neutral term,

$$\Delta(\delta(\iota)\Delta\phi(\iota)) + \theta(\iota)\eta^{\beta}(\iota-\sigma) = 0; \quad \iota \ge \iota_0, \tag{1.1}$$

where $\phi(\iota) = \eta(\iota) + \rho(\iota)\eta^{\alpha}(\iota-\tau)$ and ι_0 is a positive integer.

We use the following assumptions:

- (A1) α (≥ 1) and β are the ratios of odd positive integers;
- (A2) $\{\delta(\iota)\}, \{\rho(\iota)\}$ and $\{\theta(\iota)\}$ are positive real-valued sequences with $0 < \rho(\iota) < \rho < 1$ for all $\iota \ge \iota_0$;
- (A3) $\tau, \sigma \in \mathbb{Z}^+$;
- (A4) equation (1.1) is in non-canonical form, that is,

$$\Psi(\iota) = \sum_{s=\iota}^{\infty} \frac{1}{\delta(s)} \quad \text{with } \Psi(\iota_0) < \infty;$$

- (A5) $\liminf_{\iota\to\infty} \Psi(\iota+1)\theta(\iota) > 0$.

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Let $\nu = \max\{\tau, \sigma\}$. A solution $\{\eta(\iota)\}$ of (1.1) is a nontrivial real-valued sequence defined for all $\iota \geq \iota_0 - \nu$ satisfying (1.1) for all $\iota \geq \iota_0$. Identically vanishing solutions in a neighborhood of infinity will not be considered in the paper. A solution of (1.1) is called *oscillatory* if it has arbitrarily large generalized zeros; otherwise it is called *nonoscillatory*. If all solutions of (1.1) are (non)oscillatory, then equation (1.1) is said to be *(non)oscillatory*.

Oscillation theory has expanded and developed greatly since this phenomena take part in different models from real world applications, see, e.g., the papers [9, 23] dealing with biological mechanisms (for models from mathematical biology where oscillation and or delay actions may be formulated by means of cross-diffusion terms). Moreover, the study of neutral functional differential equations received significant attention because it arise in many fields such as control theory, communication, mechanical engineering, biodynamics, physics, economics and so on, see, [15, 28, 30] and the references therein. In view of the above observations, the researchers paid attention to the oscillation area for various classes of second-order difference, differential and dynamic equations, see [2, 6, 7, 8, 10, 12, 13, 14, 15, 17, 18, 21, 24, 25, 26, 29, 31] and the references cited therein. As far as second-order difference equations with positive superlinear neutral terms are considered, not many results are known about the oscillation, see [3, 4, 11, 16, 19, 27, 32, 33]. A close look at these papers reveals that the neutral coefficient $\{\rho(\iota)\}$ must rectify explicitly or implicitly either $\rho(\iota) \to 0$ or $\rho(\iota) \to \infty$ as $\iota \to \infty$. Further, they dealt with the non-canonical type of equations without changing its form and therefore required two conditions to eliminate all nonoscillatory solutions of these equations to get oscillatory solutions.

The purpose of the article is to study the oscillation of equation (1.1) when $\{\rho(\iota)\}$ fails to satisfy any of the above mentioned conditions. Our approach is different in the sense that; first we require one condition to eliminate nonoscillatory solutions of (1.1) to achieve oscillation via transforming the non-canonical equation (1.1) into canonical form. Next, we obtain oscillation of (1.1) by using comparison technique with first-order delay difference equations and Riccati transformation. Finally, we emphasize the practicality of the main results obtained via some particular examples, which cannot be discussed using any of the previously known results.

2. Main results

In this section, several oscillation criteria for (1.1) are presented. Without loss of generality, we study the nonoscillatory solutions of (1.1) by restricting our attention to eventually positive solutions. Let equation (1.1) have a positive solution $\{\eta(\iota)\}$. Then it is well known that the corresponding sequence $\{\phi(\iota)\}$ has the following structure:

(I) $\phi(\iota) > 0$, $\Delta\phi(\iota) > 0$, and $\Delta(\delta(\iota)\Delta\phi(\iota)) < 0$;
(II) $\phi(\iota) > 0$, $\Delta\phi(\iota) < 0$, and $\Delta(\delta(\iota)\Delta\phi(\iota)) < 0$.

We transform (1.1) into the canonical form that essentially simplifies the examination of (1.1). For $\iota \geq \iota_*$ and some $\iota_* \geq \iota_0$, we set

$$w(\iota) = \delta(\iota)\Psi(\iota)\Psi(\iota+1), \quad y(\iota) = \frac{\phi(\iota)}{\Psi(\iota)}, \quad q(\iota) = \Psi(\iota+1)\theta(\iota);$$

$$m(\iota) = \Big(1 - p(\iota)\frac{\Psi(\iota-\tau)}{\Psi(\iota)}\Big) > 0, \quad \Gamma(\iota) = \sum_{s=\iota_0}^{\iota-1} \frac{1}{w(s)},$$

$$\Omega(\iota) = q(\iota)\Psi^{\beta}(\iota-\sigma)m^{B}(\iota-\sigma),$$
$$Q(\iota) = q(\iota)\Psi^{B}(\iota-\sigma)m^{\beta}(\iota-\sigma)\Gamma(\iota-\sigma)\Gamma(\iota)w(\iota).$$

The following lemma is crucial to prove the main results.

Lemma 2.1. *Let* (A1)–(A4) *be satisfied. Then*

$$\Delta(\delta(\iota)\Delta\phi(\iota)) = \frac{1}{\Psi(\iota+1)}\Big(\delta(\iota)\Psi(\iota)\Psi(\iota+1)\Delta\Big(\frac{\phi(\iota)}{\Psi(\iota)}\Big)\Big). \tag{2.1}$$

Proof. By a direct computation, we can easily show that (2.1) holds for any sequence $\phi(\iota)$. Indeed,

$$\begin{aligned}
&\frac{1}{\Psi(\iota+1)}\Delta\Big(\delta(\iota)\Psi(\iota)\Psi(\iota+1)\Delta\Big(\frac{\phi(\iota)}{\Psi(\iota)}\Big)\Big)\\
&= \frac{1}{\Psi(\iota+1)}\Delta\Big(\delta(\iota)\Psi(\iota)\Psi(\iota+1)\frac{\Psi(\iota)\delta(\iota)\Delta\phi(\iota)+\phi(\iota)}{\delta(\iota)\Psi(\iota)\Psi(\iota+1)}\Big)\\
&= \frac{1}{\Psi(\iota+1)}\Delta(\Psi(\iota)\delta(\iota)\Delta\phi(\iota)+\phi(\iota))\\
&= \frac{1}{\Psi(\iota+1)}[\Psi(\iota)\Delta(\delta(\iota)\Delta\phi(\iota)) - \Delta\phi(\iota) + \Delta\phi(\iota)]\\
&= \Delta(\delta(\iota)\Delta\phi(\iota)).
\end{aligned}$$

Further,

$$\sum_{\iota=\iota_0}^{\infty}\frac{1}{\delta(\iota)\Psi(\iota)\Psi(\iota+1)} = \lim_{\iota\to\infty}\frac{1}{\Psi(\iota)} - \frac{1}{\Psi(\iota_0)} = \infty,$$

that is, the operator on the right-hand side of (2.1) is canonical. This proves the lemma. □

As a result of Lemma 2.1, we see that the non-canonical equation (1.1) can be equivalently written as

$$\Delta(w(\iota)\Delta y(\iota)) + q(\iota)\eta^{\beta}(\iota-\sigma) = 0 \tag{2.2}$$

which is in canonical form. The next result directly follows from the above discussion.

Theorem 2.2. $\{\eta(\iota)\}$ *is a solution of the non-canonical difference equation* (1.1) *if and only if it is a solution of the canonical equation* (2.2) *with the companion sequence* $y(\iota) = \eta(\iota)/\Psi(\iota)$.

Corollary 2.3. *Both the non-canonical difference equation* (1.1) *and the canonical equation* (2.2) *have an eventually positive solution.*

Corollary 2.3 makes easy to study (1.1) significantly since using (2.2), the companion sequence $\{y(\iota)\}$ satisfies only one class, namely,

$$y(\iota) > 0, \quad \Delta y(\iota) > 0 \quad \text{and} \quad \Delta(w(\iota)\Delta y(\iota)) < 0. \tag{2.3}$$

This follows similarly from [22, Lemma 2.1].

Lemma 2.4. *Assume that* $y(\iota)$ *satisfies* (2.3) *for* $\iota \in [\iota_0,\infty)$. *Then there exists an integer* $\iota_0^* \geq \iota_0$ *such that*

$$y(\iota) \geq \Gamma(\iota)w(\iota)\Delta y(\iota), \tag{2.4}$$

and

$$\frac{y(\iota)}{\Gamma(\iota)} \text{ is decreasing.} \tag{2.5}$$

for $\iota \geq \iota_0^*$.

Proof. From the monotonicity of $y(\iota)$, we have

$$y(\iota) = y(\iota_0^*) + \sum_{s=\iota_0^*}^{\iota-1} \frac{w(s)\Delta y(s)}{w(s)} \geq \Gamma(\iota)w(\iota)\Delta y(\iota)$$

which proves (2.4). Moreover

$$\Delta\Big(\frac{y(\iota)}{\Gamma(\iota)}\Big) = \frac{\Gamma(\iota)w(\iota)\Delta y(\iota) - y(\iota)}{w(\iota)\Gamma(\iota+1)\Gamma(\iota)} \leq 0$$

by (2.4). This implies that $y(\iota)/\Gamma(\iota)$ is decreasing which completes the proof of the lemma. □

Lemma 2.5. *Let* $y(\iota)$ *be defined for* $\iota \geq \iota_0$ *and satisfy* (2.3) *for all* $\iota \geq \iota_0$. *Then* $y^{\beta-1}(\iota) \geq \mathcal{D}(\iota)$, *where* $\mathcal{D}(\iota)$ *is given by*

$$\mathcal{D}(\iota) = \begin{cases} 1 & \text{if } \beta = 1 \\ d_1 & \text{if } \beta > 1 \\ d_2\Gamma^{\beta-1}(\iota) & \text{if } \beta < 1 \end{cases}$$

for all large $\iota \geq \iota_0^* \geq \iota_0$, *where* d_1 *and* d_2 *are positive constants.*

Proof. The proof is similar to those of [2, Lemma 2.2] and [22, Lemma 2.2], and hence is omitted. □

Theorem 2.6. *Assume that* (A1)–(A5) *hold. If the difference equation*

$$\Delta\mu(\iota) + \Omega(\iota)\mathcal{D}(\iota-\sigma)\Gamma(\iota-\sigma)\mu(\iota-\sigma) = 0 \tag{2.6}$$

is oscillatory for all large $\iota \geq \iota_0^*$, *then* (1.1) *is oscillatory.*

Proof. Without loss of generality we may assume that $\{\eta(\iota)\}$ is an eventually positive solution of (1.1), i.e., $\eta(\iota-\nu) > 0$ for all $\iota \geq \iota_1$ for some $\iota_1 \geq \iota_0$. Then by Corollary 2.3, equation (2.2) has a positive solution $\eta(\iota)$ with the corresponding fraction $y(\iota)$ satisfying (2.3). From the definition of $y(\iota)$, we have

$$\Psi(\iota)y(\iota) = \eta(\iota) + \rho(\iota)\eta^{\alpha}(\iota-\tau) \quad \text{and} \quad \Psi(\iota)y(\iota) \geq \eta(\iota) \tag{2.7}$$

for all $\iota \geq \iota_1$.

Now, we claim that $\lim_{\iota\to\infty}\eta(\iota) = 0$. Summing (2.2) from ι_1 to ∞, we obtain

$$\sum_{\iota=\iota_1}^{\infty} q(\iota)\eta^{\beta}(\iota-\tau) < w(\iota_1)\Delta y(\iota_1) - M,$$

where

$$0 \leq M = \lim_{\iota\to\infty} w(\iota)\Delta y(\iota) < \infty.$$

Since

$$\sum_{\iota=\iota_1}^{\infty} q(\iota)\eta^{\beta}(\iota-\tau) < \infty,$$

we see that

$$\lim_{\iota\to\infty} q(\iota)\eta^{\beta}(\iota-\tau) = 0.$$

But, in view of (A5) we obtain that $\lim_{\iota\to\infty}\eta(\iota) = 0$. Therefore, there exists a $\iota_2 \geq \iota_1$ such that $0 \leq \eta^{\alpha}(\iota) \leq \eta(\iota)$ for $\iota \leq \iota_2$, or

$$0 \leq \eta^{\alpha-1}(\iota) \leq 1, \quad \iota \geq \iota_2. \tag{2.8}$$

Since $y(\iota)$ is increasing, taking into account (2.8), (2.7) turns out to be

$$\begin{aligned}\Psi(\iota)y(\iota) &= \eta(\iota) + \rho(\iota)\eta^{\beta}(\iota-\tau) - \rho(\iota)\eta(\iota-\tau) + \rho(\iota)\eta(\iota-\tau)\\ &\leq \eta(\iota) + \rho(\iota)\eta(\iota-\tau)(\eta^{\alpha-1}(\iota-\tau)-1) + \rho(\iota)\eta(\iota-\tau)\\ &\leq \eta(\iota) + \rho(\iota)\Psi(\iota-\tau)y(\iota-\tau)\\ &\leq \eta(\iota) + \rho(\iota)\Psi(\iota-\tau)y(\iota),\end{aligned}$$

or

$$\Big(1-\rho(\iota)\frac{\Psi(\iota-\tau)}{\Psi(\iota)}\Big)\Psi(\iota)y(\iota) \leq \eta(\iota). \tag{2.9}$$

Using (2.9) in (2.2), we obtain

$$\Delta(w(\iota)\Delta y(\iota)) + q(\iota)\Psi^{\beta}(\iota-\sigma)m^{\beta}(\iota-\sigma)y^{\beta}(\iota-\sigma) \leq 0 \tag{2.10}$$

for $\iota \leq \iota_2$. Now, using Lemma 2.5, inequality (2.10) turns that

$$\Delta(w(\iota)\Delta y(\iota)) + \Omega(\iota)\mathcal{D}(\iota-\sigma)y(\iota-\sigma) \leq 0. \tag{2.11}$$

Now using (2.4) and (2.11) and letting $\mu(\iota) = w(\iota)\Delta y(\iota)$, it is shown that $\mu(\iota) > 0$ satisfies the inequality

$$\Delta\mu(\iota) + \Omega(\iota)\mathcal{D}(\iota-\sigma)\Gamma(\iota-\sigma)\mu(\iota-\sigma) \leq 0 \tag{2.12}$$

for $\iota \leq \iota_2$. Summing (2.12) from ι ($\geq \iota_2$) to j and letting $j \to \infty$, we obtain

$$\mu(\iota) \geq \sum_{s=\iota}^{\infty} \Omega(s)\mathcal{D}(s-\sigma)\Gamma(s-\sigma), \quad \iota \leq \iota_2.$$

The function $\mu(\iota)$, is strictly decreasing for $\iota \leq \iota_2$. On the other hand [2, Lemma 2.1] implies that the corresponding difference equation (2.6) has also a positive solution which contradicts the assumption of the theorem. □

By summing equation (2.6) from $\iota-\sigma$ to $\iota-1$ we obtain the following results.

Corollary 2.7. *Assume* (A1)–(A5) *hold. If*

$$\liminf_{\iota\to\infty} \sum_{s=\iota-\sigma}^{\iota-1} \Omega(s)\Gamma^{\beta}(s-\sigma) = \infty, \quad \iota \geq \iota_0^* \quad (0 < \beta < 1) \tag{2.13}$$

then equation (1.1) *is oscillatory.*

Corollary 2.8. *Assume* (A1)–(A5) *hold. If* $\beta = 1$ *and*

$$\liminf_{\iota\to\infty} \sum_{s=\iota-\sigma}^{\iota-1} \Omega(s)\Gamma(s-\sigma) > \Big(\frac{\sigma}{\sigma+1}\Big)^{\sigma+1}, \tag{2.14}$$

then equation (1.1) *is oscillatory.*

Proof. In view of (2.14) and [1, Theorem 7.6.1], clearly (2.6) is oscillatory, and hence equation (1.1) is oscillatory too by Theorem 2.6. □

Theorem 2.9. *Assume* (A1)–(A5) *hold. If there exists a positive nondecreasing real sequence* $\{\rho(\iota)\}$ *such that for any* $\iota \geq \iota_0$

$$\limsup_{\iota\to\infty} \sum_{s=\iota_0}^{\iota} \Big[\rho(s)\Omega(s)\mathcal{D}(s-\sigma) - \frac{w(s-\sigma)(\Delta\rho(s))^2}{4\rho(s)}\Big] = \infty, \tag{2.15}$$

then (1.1) *is oscillatory.*

Proof. Assume (1.1) is a nonoscillatory equation having an eventually positive solution $\eta(\iota)$, i.e., $\eta(\iota-\nu) > 0$ for all $\iota \geq \iota_1$ for some $\iota_1 \geq \iota_0$. Then following the similar steps as in the proof of Theorem 2.6, we obtain (2.11);

$$\Delta(w(\iota)\Delta y(\iota)) + \Omega(\iota)\mathcal{D}(\iota-\sigma)y(\iota-\sigma) \leq 0, \quad \iota \geq \iota_1. \tag{2.16}$$

We define

$$F(\iota) := \rho(\iota)\frac{w(\iota)\Delta y(\iota)}{y(\iota-\sigma)}, \quad \iota \geq \iota_1. \tag{2.17}$$

Then $F(\iota) > 0$ for $\iota \geq \iota_1$. Now using (2.16) and (2.17), we obtain

$$\begin{aligned} &\Delta F(\iota) \\ &= \rho(\iota)\frac{\Delta(w(\iota)\Delta y(\iota))}{y(\iota-\sigma)} + \frac{\Delta\rho(\iota)}{\rho(\iota+1)}F(\iota+1) - \frac{\rho(\iota)}{\rho(\iota+1)}F(\iota+1)\frac{\Delta y(\iota-\sigma)}{y(\iota-\sigma)} \\ &\leq -\rho(\iota)\Omega(\iota)\mathcal{D}(\iota-\sigma) + \frac{\Delta\rho(\iota)}{\rho(\iota+1)}F(\iota+1) - \frac{\rho(\iota)}{\rho(\iota+1)}\frac{F^2(\iota+1)}{w(\iota-\sigma)}, \end{aligned} \tag{2.18}$$

where we have used that $w(\iota-\sigma)\Delta y(\iota-\sigma)$ is positive and decreasing. Completing the square the latter inequality in (2.18), we obtain

$$\Delta F(\iota) \leq -\rho(\iota)\Omega(\iota)\mathcal{D}(\iota-\sigma) + \frac{(\Delta\rho(\iota))^2 w(\iota-\sigma)}{4\rho(\iota)}, \quad \iota \geq \iota_1. \tag{2.19}$$

Summing the both sides of inequality (2.19) from ι_1 to ι, we obtain

$$\sum_{s=\iota_1}^{\iota} \Big[\rho(s)\Omega(s)\mathcal{D}(s-\sigma) - \frac{w(s-\sigma)(\Delta\rho(s))^2}{4\rho(s)}\Big] < \infty.$$

Taking limsup as $\iota \to \infty$, we obtain

$$\limsup_{\iota\to\infty} \sum_{s=\iota_1}^{\iota} \big[\rho(s)\Omega(s)\mathcal{D}(s-\sigma) - \frac{w(s-\sigma)(\Delta\rho(s))^2}{4\rho(s)}\big] < \infty$$

which contradicts (2.15). $\square$

Theorem 2.10. *Assume* (A1)–(A5) *hold and* $\beta = 1$. *If*

$$\begin{aligned} &\limsup_{\iota\to\infty} \Big\{\frac{1}{\Gamma(\iota-\sigma)} \sum_{s=\iota_1}^{\iota-\sigma-1} \Gamma(s)\Gamma(s-\sigma)\Omega(s) + \sum_{s=\iota-\sigma}^{\iota-1} \Omega(s)\Gamma(s-\sigma) \\ &+ \Gamma(\iota-\sigma)\sum_{s=\iota}^{\infty}\Omega(s)\Big\} > 1, \end{aligned} \tag{2.20}$$

then (1.1) *is oscillatory.*

Proof. Assume (1.1) is a nonoscillatory equation having an eventually positive solution $\eta(\iota)$, i.e., $\eta(\iota-\nu)>0$ for all $\iota\ge\iota_1$ for some $\iota_1\ge\iota_0$. Then following the same steps as in the proof of Theorem 2.6, we obtain (2.10);

$$\Delta(w(\iota)\Delta y(\iota))+\Omega(\iota)y(\iota-\sigma)\le 0,\quad \iota\ge\iota_1. \tag{2.21}$$

Summing both sides of (2.21) from ι_1 to $\iota-1$ and solving $y(\iota)$, we obtain

$$\begin{aligned}
y(\iota) &\ge \sum_{s=\iota_1}^{\iota-1}\frac{1}{w(s)}\sum_{t=s}^{\infty}\Omega(t)y(t-\sigma)\\
&= \sum_{s=\iota_1}^{\iota-1}\frac{1}{w(s)}\sum_{t=s}^{\iota-1}\Omega(t)y(t-\sigma)+\sum_{s=\iota_1}^{\iota-1}\frac{1}{w(s)}\sum_{t=\iota}^{\infty}\Omega(t)y(t-\sigma)\\
&= \sum_{s=\iota_1}^{\iota-1}\Gamma(s+1)\Omega(s)y(s-\sigma)+\Gamma(\iota)\sum_{t=\iota}^{\infty}\Omega(t)y(t-\sigma).
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(\iota-\sigma) &\ge \sum_{s=\iota_1}^{\iota-\sigma-1}\Gamma(s+1)\Omega(s)y(s-\sigma)+\Gamma(\iota-\sigma)\sum_{t=\iota-\sigma}^{\infty}\Omega(t)y(t-\sigma)\\
&= \sum_{s=\iota_1}^{\iota-\sigma-1}\Gamma(s+1)\Omega(s)y(s-\sigma)+\Gamma(\iota-\sigma)\sum_{t=\iota-\sigma}^{\iota-1}\Omega(t)y(t-\sigma)\\
&\quad+\Gamma(\iota-\sigma)\sum_{t=\iota}^{\infty}\Omega(t)y(t-\sigma).
\end{aligned} \tag{2.22}$$

Since $y(\iota)$ is increasing and $y(\iota)/\Gamma(\iota)$ is decreasing, we have

$$\begin{aligned}
y(\iota-\sigma) &\ge \frac{y(\iota-\sigma)}{\Gamma(\iota-\sigma)}\sum_{s=\iota_1}^{\iota-\sigma-1}\Gamma(s+1)\Omega(s)\Gamma(s-\sigma)+y(\iota-\sigma)\sum_{s=\iota-\sigma}^{\iota-1}\Omega(s)\Gamma(s-\sigma)\\
&\quad+\Gamma(\iota-\sigma)y(\iota-\sigma)\sum_{s=\iota}^{\infty}\Omega(s).
\end{aligned}$$

That is,

$$1\ge\frac{1}{\Gamma(\iota-\sigma)}\sum_{s=\iota_1}^{\iota-\sigma-1}\Gamma(s+1)\Gamma(s-\sigma)\Omega(s)+\sum_{s=\iota-\sigma}^{\iota-1}\Gamma(s)\Gamma(s-\sigma)+\Gamma(\iota-\sigma)\sum_{s=\iota}^{\infty}\Omega(s)$$

which contradicts (2.20). $\square$

For the final result of this section, the following lemma from [22] is needed.

Lemma 2.11. *Assume that*

$$\sum_{\iota=\iota_0}^{\infty}\Omega(\iota)\Gamma^{\beta}(\iota-\sigma)=\infty, \tag{2.23}$$

and there exists a constant $\gamma\in(0,1)$ *such that*

$$Q(\iota)\ge\gamma,\quad \iota\ge\iota_0. \tag{2.24}$$

If $\{y(\iota)\}$ *is a positive solution of* (2.10)*, then*

$$y(\iota) \geq \frac{\Gamma(\iota-\sigma)w(\iota)\Delta y(\iota)}{(1-\gamma)}, \quad \iota \geq \iota_1, \tag{2.25}$$

$$\frac{y(\iota)}{\Gamma^{\gamma}(\iota)} \text{ is increasing.} \tag{2.26}$$

Theorem 2.12. *Assume* (A1)–(A5)*,* (2.23) *and* (2.24) *hold. If*

$$\liminf_{\iota\to\infty} \sum_{s=\iota-\sigma}^{\iota-1} \Omega(s)\mathcal{D}(s-\sigma)\Gamma(s-\sigma) > (1-\gamma)\Big(\frac{\sigma}{\sigma+1}\Big)^{\sigma+1}, \tag{2.27}$$

then (1.1) *is oscillatory.*

Proof. Assume that $\eta(\iota)$ is an eventually positive solution of (1.1), i.e., $\eta(\iota-\nu) > 0$ for all $\iota \geq \iota_1$ for some $\iota_1 \geq \iota_0$. Then following the same steps as in the proof of Theorem 2.6, we obtain (2.11); $y(\iota) > 0$ is an increasing solution of

$$\Delta(w(\iota)\Delta y(\iota)) + \Omega(\iota)\mathcal{D}(\iota-\sigma)y(\iota-\sigma) \leq 0, \quad \iota \geq \iota_1. \tag{2.28}$$

Let $G(\iota) = w(\iota)\Delta y(\iota)$. Using (2.25) in (2.28), we obtain

$$\Delta G(\iota) + \frac{\Omega(\iota)\mathcal{D}(\iota-\sigma)\Gamma(\iota-\sigma)}{(1-\gamma)} G(\iota-\sigma) \leq 0. \tag{2.29}$$

This shows that $G(\iota)$ is a positive solution of (2.29) by [1, Theorem 7.6.1] which contradicts (2.27). □

Theorem 2.13. *Assume* (A1)–(A5)*,* (2.23) *and* (2.24) *hold. If*

$$\begin{aligned} \limsup_{\iota\to\infty} \Big\{ &\frac{1}{\Gamma(\iota-\sigma)} \sum_{s=\iota_1}^{\iota-\sigma-1} \Omega(s)\mathcal{D}(\iota-\sigma)\Gamma(s)\Gamma(s-\sigma) + \sum_{s=\iota-\sigma}^{\iota-1} \Omega(s)\mathcal{D}(s-\sigma)\Gamma(s-\sigma) \\ &+ \Gamma^{1-\gamma}(\iota-\sigma) \sum_{s=\iota}^{\infty} \Omega(s)\mathcal{D}(s-\sigma)\Gamma^{\gamma}(s-\sigma) \Big\} > 1, \end{aligned} \tag{2.30}$$

then (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.6, we arrive at (2.11), that is,

$$\Delta(w(\iota)\Delta y(\iota)) + \Omega(\iota)\mathcal{D}(\iota-\sigma)y(\iota-\sigma) \leq 0, \quad \iota \geq \iota_1. \tag{2.31}$$

Now arguing as in the proof of Theorem 2.10, we obtain

$$\begin{aligned} y(\iota-\sigma) \geq &\sum_{s=\iota_1}^{\iota-\sigma-1} \Gamma(s+1)\Omega(s)\mathcal{D}(s-\sigma)y(s-\sigma) \\ &+ \Gamma(\iota-\sigma) \sum_{s=\iota-\sigma}^{\iota-1} \Omega(s)\mathcal{D}(s-\sigma)y(s-\sigma) \\ &+ \Gamma(\iota-\sigma) \sum_{s=\iota}^{\infty} \Omega(s)\mathcal{D}(s-\sigma)y(s-\sigma). \end{aligned}$$

Using the fact that $y(\iota)/\Gamma(\iota)$ is decreasing and $y(\iota)/\Gamma^{\gamma}(\iota)$ is increasing, the latter inequality gives

$$\begin{aligned} y(\iota-\sigma) \geq \frac{y(\iota-\sigma)}{\Gamma(\iota-\sigma)} \sum_{s=\iota_1}^{\iota-\sigma-1} \Omega(s)\mathcal{D}(s-\sigma)\Gamma(s+1)\Gamma(s-\sigma) \\ + y(\iota-\sigma) \sum_{s=\iota-\sigma}^{\iota-1} \Omega(s)\mathcal{D}(s-\sigma)\Gamma(s-\sigma) \\ + \frac{\Gamma(\iota-\sigma)y(\iota-\sigma)}{\Gamma^{\gamma}(\iota-\sigma)} \sum_{s=\iota}^{\infty} \Omega(s)\mathcal{D}(s-\sigma)\Gamma^{\gamma}(s-\sigma). \end{aligned}$$

After simplification, we obtain

$$\begin{aligned} &\frac{1}{\Gamma(\iota-\sigma)} \sum_{s=\iota_1}^{\iota-\sigma-1} \Omega(s)\mathcal{D}(s-\sigma)\Gamma(s+1)\Gamma(s-\iota) \\ &+ \sum_{s=\iota-\sigma}^{\iota-1} \Omega(s)\mathcal{D}(s-\sigma)\Gamma(s-\sigma) \\ &+ \Gamma^{1-\gamma}(\iota-\sigma) \sum_{s=\iota}^{\infty} \Omega(s)\mathcal{D}(s-\sigma)\Gamma^{\gamma}(s-\sigma) \leq 1 \end{aligned}$$

which contradicts (2.30). □

3. Applications

In this section, we present three examples to illustrate the emphasize of the main results.

Example 3.1. If we take $\phi(\iota) = \eta(\iota) + \eta^3(\iota-1)/2$ in equation (1.1) together with that $\delta(\iota) = \iota(\iota+1)$, $\rho(\iota) = 1/2, \theta(\iota) = (\iota+1)^3, \tau = 1, \sigma = 2, \alpha = 3$ and $\beta = 3$, then it takes the second-order nonlinear neutral difference equation of the form

$$\Delta(\iota(\iota+1)\Delta\phi(\iota)) + (\iota+1)^3\eta^3(\iota-2) = 0, \quad \iota \geq 4. \tag{3.1}$$

Elementary calculations give $\Psi(\iota) = 1/\iota$, $w(\iota) = 1$, $q(\iota) = (\iota+1)^2$, $\Gamma(\iota) \approx \iota$, $\mathcal{D}(\iota) = d_1 > 0$, and that

$$m(\iota) = \Big(1 - \frac{\iota}{2(\iota-1)}\Big) \geq \frac{1}{6} \quad \text{and} \quad \Omega(\iota) \simeq \frac{(\iota+1)^2}{216(\iota-2)^3}.$$

Clearly conditions (A1)–(A5) are satisfied. Choosing $\rho(\iota) = 1$, then condition (2.15) becomes

$$\limsup_{\iota\to\infty} \sum_{s=4}^{\iota} \frac{d_1(s+1)^2}{216(s-2)^3} = \infty,$$

that is, condition (2.15) holds. Therefore by Theorem 2.9, equation (3.1) is oscillatory.

Example 3.2. Consider the second-order neutral difference equation

$$\Delta(2^{\iota}\Delta\phi(\iota)) + 2^{\iota}\eta(\iota-2) = 0, \quad \iota \geq 1, \tag{3.2}$$

for which the case $\phi(\iota) = \eta(\iota) + \eta^3(\iota-1)/3$ in equation (1.1) together with that $\delta(\iota) = 2^{\iota}$, $\rho(\iota) = 1/3$, $\theta(\iota) = 2^{\iota}$,, $\tau = 1$, $\sigma = 2$, $\alpha = 3$ and $\beta = 1$. Some simple computations yield that $\Psi(\iota) = w(\iota) = 2^{1-\iota}$, $m(\iota) = 1/3$, $q(\iota) = a > 0$, $\Gamma(\iota) \approx 2^{\iota-1} - 1$ and $\Omega(\iota) = 2^{3-5\iota} a$.

Clearly the conditions (A1)–(A5) are satisfied. Condition (2.14) becomes

$$\limsup_{\iota\to\infty} \sum_{s=\iota-2}^{\iota-1} 2^{3-5\iota}\left(2^{\iota-1} - 1\right) a = \frac{8a}{3} > \frac{8}{27},$$

that is, (2.14) holds if $a > 1/9$. Thus by Corollary 2.7, equation (3.2) is oscillatory for $a > 1/9$.

Example 3.3. Equation (1.1) turns out to be the second-order neutral difference equation

$$\Delta(\iota(\iota+1)\Delta\phi(\iota)) + (\iota+1)\eta(\iota-2) = 0, \quad \iota \geq 4, \tag{3.3}$$

if $\phi(\iota) = \eta(\iota) + \eta^3(\iota-1)/2$ with that $\delta(\iota) = \iota(\iota+1)$, $\rho(\iota) = 1/2$, $\theta(\iota) = \iota+1$, $\tau = 1$, $\sigma = 2$, $\alpha = 3$ and $\beta = 1$. A simple calculation shows that $\Psi(\iota) = 1/\iota$, $w(\iota) = 1$, $q(\iota) = 1$, $\Gamma(\iota) \approx \iota$, $Q(\iota) = \iota/6 \geq 2/3 = \gamma$, and that

$$m(\iota) = \Big(1 - \frac{\iota}{2(\iota-1)}\Big) \geq \frac{1}{6} \quad \text{and} \quad \Omega(\iota) \simeq \frac{1}{6(\iota-2)}.$$

Clearly (A1)–(A5) hold. The condition (2.23) becomes

$$\sum_{\iota=4}^{\infty} \frac{\iota-2}{6(\iota-2)} = \sum_{\iota=4}^{\infty} \frac{1}{6} = \infty,$$

that is, (2.23) holds. The condition (2.24) holds with $\gamma = 2/3$.

Condition (2.27) becomes

$$\liminf_{\iota\to\infty} \sum_{s=\iota-2}^{\iota-1} \frac{s-2}{6(s-2)} = \frac{2}{3} > \big(\frac{1}{3}\big)\big(\frac{8}{27}\big),$$

that is, condition (2.27) holds. Therefore equation (3.3) is oscillatory by Theorem 2.12.

We remark that Corollary 2.8 does not yield this conclusion since condition (2.14) is not satisfied. Therefore, Theorem 2.12 improves Corollary 2.8.

4. Conclusions

By putting the equation in canonical form, we offer oscillation conditions for (1.1) in this work, which makes it easier to examine (1.1). Furthermore, the oscillation criteria developed here are novel and add to the findings previously reported in the literature. The neutral coefficient $\rho(t) \in (0,1)$ prevents the results presented in [3, 4, 11, 16, 19, 27, 32, 33] from being applicable to our equations (3.1)–(3.3). As a result, our findings constitute a highly valuable addition to the oscillation theory of second-order neutral difference equations with superlinear neutral terms. When $-1 < \rho(\iota) < 0$ or $\{\rho(\iota)\}$ is oscillatory, it is also intriguing to extend the findings of this paper.

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References

[1] R. P. Agarwal, M. Bohner, S.R. Grace, D. O'Regan; *Discrete Oscillation Theory*, Hindawi Publ. Corp., New York, 2005.

[2] R. P. Agarwal, M. Bohner, T. Li, C. Zhang; *Oscillation of second-order differential equations with a sublinear neutral term*, Carpathian J. Math., **30**, 1–6, 2014.

[3] G. Ayyappan, G. Nithyakala; *Oscillation of second-order nonlinear difference equation with superlinear neutral term*, Malaya J. Matematik, **7**, 366–377, 2019.

[4] G. Ayyappan, G. Nithyakala; *Some oscillation results for even-order delay difference equations with a sublinear neutral term*, Abstr. Appl. Anal., 6 pp., 2018. (Article ID 2590158).

[5] M. Bohner, H. A. El-Marshedy, S. R. Grace, I. Sager; *Oscillation of second-order nonlinear difference equations with sublinear neutral term*, Math. Murav. **23**, 1–10, 2019.

[6] M. Bohner, T. Li; *Oscillation of second orderp Laplace dynamic equations with a nonpositive neutral coefficient* , Appl. Math. Lett., 37(2014), 72-76.

[7] E. Chandrasekaran, G. E. Chatzarakis, G. Palani, E. Thandapani; *Oscillation criteria for advanced difference equations of second order*, Appl. Math. Comput., **372** (124963), 6 pp., 2020.

[8] G. E. Chatzarakis, R. KanagaSabapathi, S. Selvarangam, E. Thandapani; *Oscillation is second-order damped difference equations with a superlinear neutral term*, Adv. Math.: Sci. J., **9**, 10969–10981, 2020.

[9] A. Columbu, S. Frassu, G. Viglialoro; *Refined criteria toward boundedness in an attraction-repulsion chemotaxis system with nonlinear productions*, Appl. Anal., (2023), 1-17, DOI:10.1080/00036811.223.2187789.

[10] C. Dharuman, J. R. Graef, E. Thandapani, K.S. Vidhyaa; *Oscillation of second-order difference equation with a sublinear neutral term*, J. Math. Appl. **40**, 59–67, 2017.

[11] C. Dharuman, E. Thandapani; *Oscillation of solutions of nonlinear difference equation with a super-linear neutral term,* Nonauton. Dyn. Syst., **5**, 52–58, 2018.

[12] J. Dzurina, S. R. Grace, I. Jadlovska, T. Li; *Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term*, Math.Naghr.,293(5) (2020), 910-922.

[13] S. R. Grace, J. R. Graef; *Oscillatory behaviour of second order nonlinear differential equations with a sublinear neutral term*, Math. Model. Anal., **23**, 217–226, 2018.

[14] S. Grace, J. Alzabut, *Oscillation results for nonlinear second order difference equations with mixed neutral terms*, Adv. Difference Equ., (8), 12 pp., 2020.

[15] S. R. Grace, I. Jadlovska,d A. Zafer; *On oscillation of second order delay differential equations with a sublinear neutral term*, Mediterr, J. Math. **17** (116), 11 pp., 2020.

[16] S. R. Grace, J. R. Graef; *Oscillation theorems for second-order delay difference equations with superlinear neutral terms,* Georgian Math. J. **28**, 725–731, 2021.

[17] S. R. Grace; *New oscillation criteria of nonlinear second order delay difference equations*, Mediterr. J. Math., **19** (166), 11 pp., 2022.

[18] J. R. Graef, S. R. Grace, E. Tunc; *Oscillatory behavior of even-order nonlinear differential equations with a sublinear neutral term*, Opuscula Math., **39**, 39–47, 2019.

[19] I. Gyori, G. Ladas; *Oscillation theory of delay differential equations with applications*, Clareuden Press, Oxford, 1991.

[20] J. K. Hale; *Functional Differential Equations*, Springer, Berlin,1977.

[21] B. Kamaraj, R. Vasuki; *Oscillation of second order difference equations with a superlinear neutral term*, J. Adv. Math. Comput. Sci. **23**, 1–10 , 2017.

[22] R. Kanagasabapathi, S. Selvarangam, J. R. Graef, E. Thandapani; *Oscillation results using linearization of quasilinear second-order delay difference equations*, Mediterr. J. Math., **18** (248), 14 pp., 2021.

[23] T. Li, S.Frassu, G. Viglisloro; *Combining effects ensuring boundedness in an attraction-repulsion chemotaxis model with production and consumption*, Z. Angew. Math. Phys., 74(3) (2023), Art.109.

[24] W. T. Li, S.H. Sager; *Oscillation of second-order sublinear neutral delay difference equations*, Appl. Math. Comput. **146**, 543–551, 2003.

[25] T. Li, Yuriy V. Rogovchenko; *Oscillation of second-order neutral differential equations* , Math.Nachr., 288(10) (2015), 1150-1162.

[26] S. Meharbanu, S. Nalini; *Oscillation of second order difference equations with several superlinear neutral terms*, Adv. Differ. Equ. **345**, 10 p., 2018.

[27] G. Nithyakala, G. Ayyappan; *Oscillation theorems for second-order nonlinear difference equations with advanced superlinear neutral term*, J. Anal. **30** (4), 1475–1484, 2022.

[28] F. A. Rihan; *Delay Differential Equations and Applications to Biology*, Springer Nature, Singapore,2021.

[29] S. H. Saker, A. K. Sethi, Osman Tunc, Jehad Alzabut; *Riccati technique for oscillation of second order nonlinear neutral delay dynamic equations*, J. Math.Comput. Sci., **29** (4), 387–398, 2023.

[30] W.Soedel; *Vibrations of Shells and Plates*, Marcel Dekker, New York, 1991.

[31] E. Thandapani, Z. Hiu, R. Arul, P.S. Raja; *Oscillation and asymmetric behavior of second order difference equations with nonlinear neutral term*, Appl. Math. E. Notes **4**, 59–67, 2004.

[32] M. K. Yildiz, H. Ogunmez; *Oscillation results of higher order nonlinear neutral delay differences equations with a nonlinear neutral term*, Hacet. J. Math. Stat. **43**, 809–844, 2014.

[33] Z. Zhang, J. Chen, C. Zhang; *Oscillation of solutions for second order nonlinear difference equations with nonlinear neutral term*, Comput. Math. Appl. **41**, 1487–1494, 2001.

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