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# MULTIPLICITY OF SOLUTIONS FOR A GENERALIZED KADOMTSEV-PETVIASHVILI EQUATION WITH POTENTIAL IN $\mathbb{R}^2$

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ABSTRACT. In this article, we study the generalized Kadomtsev-Petviashvili equation with a potential

$$(-u_{xx} + D_x^{-2}u_{yy} + V(\varepsilon x, \varepsilon y)u - f(u))_x = 0 \quad \text{in } \mathbb{R}^2,$$

where $D_x^{-2}h(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{t} h(s,y)\,ds\,dt$, $f$ is a nonlinearity, $\varepsilon$ is a small positive parameter, and the potential $V$ satisfies a local condition. We prove the existence of nontrivial solitary waves for the modified problem by applying penalization techniques. The relationship between the number of positive solutions and the topology of the set where $V$ attains its minimum is obtained by using Ljusternik-Schnirelmann theory. With the help of Moser's iteration method, we verify that the solutions of the modified problem are indeed solutions of the original problem for $\varepsilon > 0$ small enough.

## 1. Introduction

This article is devoted to studying solitary waves for the generalized Kadomtsev-Petviashvili equation with a potential

$$v_t + \varepsilon^2 v_{xxx} + (f(v) - \widetilde{V}(x,y)v)_x = D_x^{-1}v_{yy}, \tag{1.1}$$

where $v = v(t,x,y)$ with $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$, $\widetilde{V}$ is a potential function, parameter $\varepsilon > 0$ and $D_x^{-1}h(x,y) = \int_{-\infty}^{x} h(s,y)ds$.

A solitary wave is a solution of the form $u(x-ct,y)$, with $c > 0$. Hence, inserting this into (1.1), we obtain

$$-cu_x + \varepsilon^2 u_{xxx} + (f(u) - \widetilde{V}(x,y)u)_x = D_x^{-1}u_{yy} \quad \text{in } \mathbb{R}^2,$$

or equivalently,

$$\left(-\varepsilon^2 u_{xx} + D_x^{-2}u_{yy} + V(x,y)u - f(u)\right)_x = 0, \quad \text{in } \mathbb{R}^2,$$

where $V = \widetilde{V} + c$, and by a simple scaling calculus, it is easy to see that the above equation becomes

$$\left(-u_{xx} + D_x^{-2}u_{yy} + V(\varepsilon x, \varepsilon y)u - f(u)\right)_x = 0 \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

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Let $\tau > 0$ be a constant, if $V(\varepsilon x, \varepsilon x) = \tau$, then problem (1.2) becomes

$$\big(-u_{xx} + D_x^{-2} u_{yy} + \tau u - f(u)\big)_x = 0 \quad \text{in } \mathbb{R}^2. \tag{1.3}$$

If we choose $f(t) = t^2$ in (1.3), then above equation becomes a well-known two-dimensional Korteweg-de Vries type equation, which models long dispersive waves, essentially unidimensional, but having small transverse effects, see [14]. In the pioneering work, De Bouard and Saut [7, 8] used the concentration compactness principle from [15, 16] to show the existence of solitary waves for (1.3) in the plane, with $f(t) = |t|^p t$ and $p = \frac{m}{n}$, where $m$ and $n$ are relatively prime numbers and $n$ is odd. Willem [25] extended the above results to the case $N = 2$ and $f \in C(\mathbb{R}, \mathbb{R})$ via the mountain pass theorem. Thereafter, Wang and Willem [26] proved multiple solutions for problem (1.3) by the Lyusternik-Schnirelman category theory. Liang and Su [17] obtained the existence of solutions for equation (1.3) with $f(x, y, u) = Q(x, y)|u|^{p-2}u$ and $N \geq 2$, where $Q \in C(\mathbb{R} \times \mathbb{R}^{N-1}, \mathbb{R})$, satisfying some structural conditions and $2 < p < \bar{p} = \frac{2(2N-1)}{2N-3}$. Combining the variational method and linking theorems were appeared in [24], He and Zou [12] studied the existence of nontrivial solutions for the above equation in multi-dimensional spaces. Similar techniques were used in [30]. For further works about nontrivial solitary waves for the generalized Kadomtsev-Petviashvili equation, we refer to [2, 22, 27, 28, 29, 31] and the references therein.

Recently, Alves and Miyagaki [3] obtained the existence, regularity and concentration phenomenon of nontrivial solitary waves for a class of generalized variable coefficient Kadomtsev-Petviashvili equation in $\mathbb{R}^2$. Lately, inspired by the similar method of researching concentration in [3], Li, Wei and Yang [19] also studied concentration of solitary waves for problem (1.2) in $\mathbb{R}^2$ under the assumption of $f(u) = |u|^{p-2}u$, where $2 < p < 6$ and potential $V$ satisfies a global condition. With the help of variational methods, the authors obtained that the existence of the least energy solution for all $\varepsilon > 0$ small enough and these solutions concentrate to the minimum point of the potential $V$ as $\varepsilon \to 0$.

We notice that potentials $V$ in [3, 19] satisfy global conditions. However, the problem (1.2) becomes more complicated when potential $V$ satisfies the local condition. Alves and Ji [1] proved the existence and concentration of nontrivial solitary waves for (1.2) in $\mathbb{R}^2$ by using penalization method in [21]. Looking at their conditions, potential $V$ and nonlinearity $f$ required that $V \in C^2(\mathbb{R}^2, \mathbb{R}) \cap L^\infty(\mathbb{R}^2, \mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ is $C^2$ with $|f''(t)| \leq C|t|^{p-1}$ for all $t \in \mathbb{R}$, which are so strong to obtain more regularity. Furthermore, by analyzing the regularity of the solutions, the authors obtained the concentration and estimation of the solutions. To the best of our knowledge, these analyses are crucial to prove that the solutions of the modified problem are indeed solutions of the original problem for $\varepsilon > 0$ small enough.

As we know, there are only a few works about the multiple solutions for problem (1.2), Figueiredo and Montenegro [10] investigated the multiple solitary waves for problem (1.2). In their work, they assumed that both global potential $V$ and nonlinearity $f$ belong to $C^1$ and the following conditions hold:

- (A1) There is a constant $V_0 > 0$ such that $V_0 := \inf_{x \in \mathbb{R}^2} V(x)$.
- (A2) $V_\infty = \lim_{|(x,y)| \to \infty} V(x, y) > V_0$.
- (A3) $V^{-1}(\{V_0\}) = \{a_1, a_2, \dots, a_m\} \subset \mathbb{R}^2$ with $a_1 = 0$ and $a_j \neq a_s$ if $j \neq s$.
- (A4) $f \in C^1(\mathbb{R}, \mathbb{R})$ with $f(0) = f'(0) = 0$.

(A5) There exist constants $q, \sigma \in (2,6), C_0 > 0$ such that
$$f(t) \geq C_0 t^{q-1}, \quad \text{for all } t \geq 0, \quad \text{and} \quad \lim_{t\to\infty} \frac{f(t)}{t^{\sigma-1}} = 0.$$
(A6) There is $\theta \in (2,6)$ such that
$$0 < \theta F(t) = \theta \int_0^t f(r)dr \leq tf(t) \quad \text{for all } t > 0.$$
(A7) $f(t) > 0$ for all $t > 0$, and $f(t) = 0$ for all $t < 0$.
(A8) The function $t \mapsto f(t)/t$ is increasing for $t > 0$.

By using the variational method and the concentration-compactness principle, the authors obtained that the number of solitary waves corresponds to the number of global minimum points of the potential $V$ when positive parameter $\varepsilon$ is small enough. Notice that different from the works in [3, 19], the use of special condition (A3) further ensures that the exact number of solutions can be secured in [10].

Motivated by the ideas developed in [1, 13, 18], we study problem (1.2) by considering a local assumption on $V$, the penalization scheme, and the Ljusternik-Schnirelmann theory. We aim to investigate the existence of multiple solutions for problem (1.2) without assuming (A2), (A3), (A6), and (A8). Furthermore, we assume additionally that $V \in C(\mathbb{R}^2, \mathbb{R})$ $f \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfy the following conditions:

(A9) There exists an open and bounded set $\Omega \subset \mathbb{R}^2$ satisfying $V_0 < \min_{\partial\Omega} V$ and $M = \{x \in \Omega : V(x) = V_0\} \neq \emptyset$.
(A10) There exists a positive number $\alpha \in (1, +\infty)$ such that
$$t \mapsto \frac{f(t)}{t^\alpha} \text{ is nondecreasing on } (0, \infty).$$

Throughout this article, without loss of generality, we assume that $V(0,0) = V_0 = \min_{x\in\mathbb{R}^2} V(x)$. A typical example of function $f$ which satisfying assumptions (A4), (A5), (A7), (A10) is
$$f(t) = \begin{cases} \lambda t^{q-1} \text{ with } \lambda > 0, 2 < q < 6, & \text{for } t > 0, \\ 0 & \text{for } t \leq 0 \end{cases}$$
Comparing this article with [10], we not only improve the above conditions of $V$ and $f$, but also adopt different methods from [10]. In the proof of Theorem 1.1, we adopt the penalization method to restore the modified functional compactness. But in our equation (1.2), we notice that the local condition of potential $V$ make the modified problem more complicated, thus we use the truncation trick from [21] to overcome this difficulties. It consists in making a suitable modification on the nonlinearity $f$, solving a modified problem and then check that for $\varepsilon$ small enough, the solutions of the modified problem are indeed solutions of the original one. Moreover, we apply the method introduced by Benci and Cerami [4] to describe the multiplicity result. By considering the relationship between the category of some sublevel sets of the modified functional and the category of set $M$, we prove the existence of multiple solutions for the modified problem. Specifically, the main ingredient is to make precisely comparisons between the category of some sublevel sets of the modified functional and the category of the set $M$ given in (A9). Remarkably, unlike the method in [1], we don't analyze the regularity and concentration of solutions, and we obtain the existence of estimates involving the $L^\infty$-norm of the

modified problem by using Moser's iteration method [11]. To this end, we believe that the idea of combining penalization scheme with topological arguments to get the multiple solutions can be widely applied in different equations or systems to cope with local conditions on the potential $V$.

Finally, by Chang's definition of category in [9], we remark that the category $\operatorname{cat}_X(A)$ of a subset $A$ of a topological space $X$ is defined as the minimal $k \in \mathbb{N}$ such that $A$ is covered by $k$ closed subsets of $X$ which are contractible in $X$, namely

$$\begin{aligned}\operatorname{cat}_X(A) = \inf \big\{ & k \in \mathbb{N} \cup \{+\infty\} : \exists k \text{ contractible closed subsets of} \\ & X : F_1, F_2, \dots, F_k \text{ such that } A \subset \cup_{i=1}^{k} F_i \big\}.\end{aligned}$$

where set $F$ is called contractible (in $X$) if there exists $\kappa : [0,1] \times X \mapsto X$ such that

$$\kappa(0, F) = id_X \quad \text{and} \quad \kappa(1, F) \text{ is a one-point set.}$$

Our main results can be stated as follows.

**Theorem 1.1.** *Suppose that the conditions* (A1), (A4), (A5), (A7), (A9), (A10) *hold. Then, for any $\delta > 0$ such that*

$$M_\delta = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, M) \le \delta\} \subset \Omega,$$

*there exists $\hat{\varepsilon} > 0$ such that, for any $\varepsilon \in (0, \hat{\varepsilon})$, problem* (1.2) *has at least* $\operatorname{cat}_{M_\delta}(M)$ *solutions.*

The article is organized as follows. In Section 2, we give the variational setting and we modify the original problem. In Section 3, we study the autonomous problem associated with the modified problem. From this study, we obtain that the modified problem has multiple solutions by means of the Ljusternik-Schnirelmann theory. In Section 4, for $\varepsilon > 0$ small enough, we prove that the solutions of the modified problem are indeed solutions of the original problem by using Moser iteration method.

Throughout the article, we use the following notation:

- $\|\cdot\|_p$ denotes the norm of the Lebesgue space $L^p(\mathbb{R}^2)$.
- for $x \in \mathbb{R}^2$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^2 : |y - x| < r\}$.
- $C_1, C_2, C_3, \dots$ denote positive constants possibly different in different places.
- $u^\pm = \max\{\pm u, 0\}$.
- The symbols "$\to$" and "$\rightharpoonup$" denote strong and weak convergence, respectively.

## 2. Variational framework

Arguing as in [25], in the set $Y = \{g_x : g \in C_0^\infty(\mathbb{R}^2)\}$, we define the inner product

$$(u, v)_\varepsilon = \int_{\mathbb{R}^2} [u_x v_x + D_x^{-1} u_y D_x^{-1} v_y + V(\varepsilon x, \varepsilon y) uv] \, dx \, dy, \tag{2.1}$$

and the corresponding norm

$$\|u\|_\varepsilon = \Big( \int_{\mathbb{R}^2} [|u_x|^2 + |D_x^{-1} u_y|^2 + V(\varepsilon x, \varepsilon y) u^2] \, dx \, dy \Big)^{1/2}. \tag{2.2}$$

Now we define a working space of functions. A function $u : \mathbb{R}^2 \to \mathbb{R}$ belongs to $X_\varepsilon$ if there exists $\{u_n\} \subset Y$ such that

(i) $u_n \to u$ a.e. in $\mathbb{R}^2$;
(ii) $\|u_j - u_k\|_\varepsilon \to 0$ as $j, k \to \infty$.

The space $X_\varepsilon$ with inner product (2.1) and norm (2.2) is a Hilbert space. By a solution of (1.2) we mean a function $u \in X_\varepsilon$ such that

$$(u,\phi)_\varepsilon - \int_{\mathbb{R}^2} f(u)\phi\,dx\,dy = 0 \quad \text{for all } \phi \in X_\varepsilon,$$

we define the Euler-Lagrange functional associated with (1.2) by

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^2} [|u_x|^2 + |D_x^{-1}u_y|^2 + V(\varepsilon x,\varepsilon y)u^2]\,dx\,dy - \int_{\mathbb{R}^2} F(u)\,dx\,dy \\ &= \frac{1}{2}\|u\|_\varepsilon^2 - \int_{\mathbb{R}^2} F(u)\,dx\,dy, \end{aligned} \tag{2.3}$$

which is $C^1$ with Gateaux derivative

$$\begin{aligned} &\langle I_\varepsilon'(u), v\rangle \\ &= \int_{\mathbb{R}^2} [u_x v_x + D_x^{-1}u_y D_x^{-1}v_y + V(\varepsilon x,\varepsilon y)uv]\,dx\,dy - \int_{\mathbb{R}^2} f(u)v\,dx\,dy \\ &= (u,v)_\varepsilon - \int_{\mathbb{R}^2} f(u)v\,dx\,dy, \quad \forall u,v \in X_\varepsilon. \end{aligned} \tag{2.4}$$

The embedding $X_\varepsilon \hookrightarrow L^p(\mathbb{R}^2)$ is continuous for $2 \le p \le 6$ and $X_\varepsilon \hookrightarrow L^p_{\rm loc}(\mathbb{R}^2)$ is compact for $2 \le p < 6$ (see [8, 5]).

As in [21], to study problem (1.2) by variational methods, we modify suitably the nonlinearity $f$ so that, for parameter $\varepsilon > 0$ small enough, the solutions of the modified problem are also solutions of the original problem (1.2). To establish the multiplicity of solutions of (1.2), we will adapt for our case an argument explored by the penalization method introduced by Del and Felmer [21]. To this end, we need to fix some notation.

Let $k > 2$ and $a > 0$ such that $f(a) = V_0 a/k$ with $V_0$ given by (A1). We set

$$\widehat{f}(t) = \begin{cases} f(t), & \text{i ft} \le a, \\ \frac{V_0}{k}t, & \text{if } t \le a. \end{cases}$$

Moreover, we fix $t_0, t_1 \in (0,+\infty)$ such that $t_0 < a < t_1$ and $\omega \in C^1([t_0,t_1])$, satisfying

$$\omega(t) \le \widehat{f}(t) \quad \text{for all } t \in [t_0,t_1], \tag{2.5}$$

$$\omega(t_0) = \widehat{f}(t_0), \quad \omega(t_1) = \widehat{f}(t_1), \tag{2.6}$$

$$\omega'(t_0) = \widehat{f}'(t_0), \quad \omega'(t_1) = \widehat{f}'(t_1), \tag{2.7}$$

$$t \mapsto \frac{\omega(t)}{t} \text{ is nondecreasing for } t \in [t_0,t_1]. \tag{2.8}$$

Using the functions $\omega$ and $\widehat{f}$, let us consider two new functions

$$\widetilde{f}(t) = \begin{cases} \widehat{f}(t), & \text{if } t \notin [t_0,t_1], \\ \omega(t), & \text{if } t \in [t_0,t_1]. \end{cases}$$

and

$$g(x,y,t) = \chi(x,y)f(t) + (1-\chi(x,y))\widetilde{f}(t),$$

where $\chi(x,y) \in C_0^\infty(\mathbb{R}^2)$ is the characteristic function of set $\Omega$, and $g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is the penalized nonlinearity.

From the definition of $\widetilde{f}$, it follows that $\widetilde{f} \in C^1(\mathbb{R},\mathbb{R})$, $\widetilde{f}(t) \le \frac{V_0}{k} t$ for all $t \ge 0$. Hereafter, for the above $\delta > 0$, we have

$$V_0 + \delta < \min_{\bar{\Omega}_\delta} V(x,y)$$

where

$$\bar{\Omega}_\delta = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \bar{\Omega}) \le \delta\}.$$

Now, we illustrate the properties of $g$. First, it follows from (A4) and (A10) that

$$0 < 2F(t) < (\alpha+1)F(t) \le f(t)t \quad \text{for all } t > 0. \tag{2.9}$$

In view of (A4), (A5), (A7), and (A10), it is easy to deduce that $g$ is a Carathéodory function and satisfying the following properties:

(A11) $g(x,y,t) \le \delta t + f(t)$ for any $t \ge 0$ and $\delta \ge 0$;

(A12) $\lim_{t\to 0} \frac{g(x,y,t)}{t} = 0$ uniformly in $(x,y) \in \mathbb{R}^2$;

(A13) $0 < 2G(x,y,t) = 2\int_0^t g(x,y,r)dr < g(x,y,t)t \le V_0 t^2/k$ for all $(x,y) \in \mathbb{R}^2 \setminus \Omega$ and all $t > 0$;

(A14) For each $(x,y) \in \Omega$, $\alpha > 1$, the function $t \mapsto \frac{g(x,y,t)}{t^\alpha}$ is increasing on $(0,\infty)$, and for each $(x,y) \in \mathbb{R}^2 \setminus \Omega$, the function $t \mapsto \frac{g(x,y,t)}{t^\alpha}$ is increasing on $(0,a)$.

Now we study the modified problem

$$\big(-u_{xx} + D_x^{-2} u_{yy} + V(\varepsilon x, \varepsilon x)u - g(\varepsilon x, \varepsilon x, u)\big)_x = 0 \quad \text{in } \mathbb{R}^2. \tag{2.10}$$

Notice that solutions of (2.10) with $|u(x,y)| \le a$ for each $(x,y) \in \mathbb{R}^2 \setminus \Omega_\varepsilon$ are also the solutions of (1.2), where $\Omega_\varepsilon = \{(x,y) \in \mathbb{R}^2 : \varepsilon(x,y) \in \Omega\}$.

The energy functional associated with problem (2.10) is

$$\varphi_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 - \int_{\mathbb{R}^2} G(\varepsilon x, \varepsilon y, u)\,dx\,dy. \tag{2.11}$$

It is standard to prove that $\varphi_\varepsilon \in C^1(X_\varepsilon, \mathbb{R})$ and its critical points are the weak solutions of the modified problem (2.10). Next, we define the Nehari manifold associated with $\varphi_\varepsilon$ given by

$$\mathcal{N}_\varepsilon = \{u \in X_\varepsilon \setminus \{0\} : \langle \varphi_\varepsilon'(u), u\rangle = 0\}. \tag{2.12}$$

The first lemma is related to the fact that $\varphi_\varepsilon$ satisfies the mountain pass geometry(see [25]).

**Lemma 2.1.** *The functional $\varphi_\varepsilon$ satisfies the following properties.*

(i) *There exist $r, \rho > 0$ such that $\varphi_\varepsilon(u) \ge \rho$ with $\|u\|_\varepsilon = r$.*

(ii) *There exists $\|e\|_\varepsilon > r$ satisfying $\varphi_\varepsilon(e) < 0$.*

*Proof.* (i) For any $u \in X_\varepsilon \setminus \{0\}$ and $\delta > 0$ small, it follows from (A5), (A11), and (A12) that there exists $C_\delta >$ such that

$$|g(\varepsilon x, \varepsilon y, t)| \le \delta|t| + C_\delta |t|^{\sigma-1}, \quad \text{for all } (x,y) \in \mathbb{R}^2,\ t \in \mathbb{R},$$

$$|G(\varepsilon x, \varepsilon y, t)| \le \frac{\delta}{2} t^2 + \frac{C_\delta}{\sigma}|t|^\sigma, \quad \text{for all } (x,y) \in \mathbb{R}^2,\ t \in \mathbb{R}.$$

Now by the Sobolev continuous embedding $X_\varepsilon \hookrightarrow L^p(\mathbb{R}^2)$ for $p \in [2,6]$, we have

$$\begin{aligned} \varphi_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^2} \big(|u_x|^2 + |D_x^{-1}u_y|^2 + V(\varepsilon x, \varepsilon y)u^2\big)\,dx\,dy - \int_{\mathbb{R}^2} G(\varepsilon x, \varepsilon y, u)\,dx\,dy \\ &\ge \frac{1}{2}\|u\|_\varepsilon^2 - \frac{\delta}{2}\int_{\mathbb{R}^2} u^2\,dx\,dy - \frac{C_\delta}{\sigma}\int_{\mathbb{R}^2} u^\sigma\,dx\,dy \end{aligned}$$

$$\geq \frac{1+C_1}{4}\|u\|_\varepsilon^2.$$

Hence, we can choose some $r, \rho > 0$ such that $\varphi_\varepsilon(u) \geq \rho$ with $\|u\|_\varepsilon = r$ small enough.

(ii) For each $u \in X_\varepsilon \setminus \{0\}$ with $\operatorname{supp}(u) \subset \Omega_\varepsilon$, and $t > 0$, we obtain that

$$\begin{aligned}
\varphi_\varepsilon(tu) &= \frac{t^2}{2}\|u\|_\varepsilon^2 - \int_{\mathbb{R}^2} G(\varepsilon x, \varepsilon y, tu)\,dx\,dy \\
&= \frac{t^2}{2}\|u\|_\varepsilon^2 - \int_{\bar{\Omega}_\varepsilon} F(tu)\,dx\,dy \\
&\leq \frac{t^2}{2}\|u\|_\varepsilon^2 - t^{(\alpha+1)}\int_{\bar{\Omega}_\varepsilon} u^{(\alpha+1)}\,dx\,dy.
\end{aligned}$$

Here we have used property (A10), which implies that $\varphi_\varepsilon(tu) \to -\infty$ as $t \to +\infty$, and conclusion (ii) follows. $\square$

Now, let $c_\varepsilon$ denote the mountain pass level associated with $\varphi_\varepsilon$, that is

$$c_\varepsilon = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} \varphi_\varepsilon(\eta(t)),$$

where $\Gamma = \{\eta \in C([0,1], X_\varepsilon) : \eta(0) = 0 \text{ and } \varphi_\varepsilon(\eta(1)) < 0\}$.

**Lemma 2.2.** *If the conditions* (A1), (A9), (A11)-(A14) *hold, then, for each $u \in X_\varepsilon$ with $u \neq 0$, there exists a unique $t_\varepsilon = t_\varepsilon(u) > 0$ such that $t_\varepsilon u \in \mathcal{N}_\varepsilon$ and $\varphi_\varepsilon(t_\varepsilon u) = \max_{t\geq 0}\varphi_\varepsilon(tu)$. Moreover,*

$$c_\varepsilon = \inf_{u\in X_\varepsilon\setminus\{0\}} \max_{t\geq 0} \varphi_\varepsilon(tu) = \inf_{\mathcal{N}_\varepsilon} \varphi_\varepsilon. \tag{2.13}$$

*Proof.* For a fixed $w \in X_\varepsilon \setminus \{0\}$, let $l(t) = \varphi_\varepsilon(tw)$ for all $t > 0$. By the proof of Lemma 2.1, we observe that $l(0) = 0, l(t) > 0$ for $t$ small enough and $l(t) > 0$ for $t$ sufficiently large. Thus, there exists $t_\varepsilon > 0$ such that

$$l(t_\varepsilon) = \varphi_\varepsilon(t_\varepsilon w) = \max_{t\geq 0}\varphi_\varepsilon(tw) \quad \text{and} \quad l'(t_\varepsilon) = 0.$$

Therefore, $t_\varepsilon w \in \mathcal{N}_\varepsilon$. Then we will prove the uniqueness of $t_\varepsilon$, and we have

$$\|w\|_\varepsilon^2 = t_\varepsilon^{(\alpha-1)} \int_{\mathbb{R}^2} \frac{g(\varepsilon x, \varepsilon y, t_\varepsilon w)}{(t_\varepsilon w)^\alpha} w^{(\alpha+1)}\,dx\,dy.$$

Hence, we define a function $q : \mathbb{R}^+ \mapsto [0, +\infty]$ by

$$q(t) := t^{(\alpha-1)} \int_{\mathbb{R}^2} \frac{g(\varepsilon x, \varepsilon y, tw)}{(tw)^\alpha} w^{(\alpha+1)}\,dx\,dy.$$

By calculation and condition (A14), we prove that the function $q$ is increasing on $(0, +\infty)$. Therefore, $t_\varepsilon$ is unique. The conclusion

$$c_\varepsilon = \inf_{w\in X_\varepsilon\setminus\{0\}} \max_{t\geq 0} \varphi_\varepsilon(tw)$$

is obvious and the proof is complete. $\square$

**Lemma 2.3** ([1])**.** *For any fixed $\varepsilon > 0$, the functional $\varphi_\varepsilon$ satisfies the $(PS)_c$ condition.*

**Lemma 2.4.** *The functional $\varphi_\varepsilon$ has a nontrivial critical point $u_\varepsilon \in X_\varepsilon$ such that*

$$\varphi_\varepsilon(u_\varepsilon) = c_\varepsilon \quad \textit{and} \quad \varphi_\varepsilon'(u_\varepsilon) = 0.$$

Lemmas 2.1 and 2.3 permit us to apply the Mountain Pass Lemma due to Ambrosetti and Rabinowitz [25] to conclude that $c_\varepsilon$ is a critical value for $\varphi_\varepsilon$.

**Lemma 2.5.** *Suppose that* (A1), (A4), (A5), (A7), (A9) *hold. Then, for $\varepsilon > 0$ small, the functional $\varphi_\varepsilon$ restricted to $\mathcal{N}_\varepsilon$ satisfies the (PS) condition. And the critical points of the functional $\varphi_\varepsilon$ on $\mathcal{N}_\varepsilon$ are critical points of $\varphi_\varepsilon$ in $X_\varepsilon$.*

*Proof.* Let $\{w_n\} \subset \mathcal{N}_\varepsilon$ be such that $\varphi_\varepsilon(w_n) \to c$ and $\varphi'_\varepsilon(w_n)|_{\mathcal{N}_\varepsilon} \to 0$. Then there exists $\{\mu_n\} \subset \mathbb{R}$ satisfying

$$\varphi'_\varepsilon(w_n) = \mu_n \Phi'_\varepsilon(w_n) + o_n(1), \tag{2.14}$$

where $\Phi_\varepsilon : X_\varepsilon \to \mathbb{R}$ is

$$\Phi_\varepsilon(w) = \langle \varphi'_\varepsilon(w), w\rangle = \|w\|_\varepsilon^2 - \int_{\mathbb{R}^2} g(\varepsilon x, \varepsilon y, w) w \, dx\, dy.$$

From $\langle \varphi'_\varepsilon(w_n), w_n\rangle = 0$, we have

$$\|w_n\|_\varepsilon^2 = \int_{\mathbb{R}^2} g(\varepsilon x, \varepsilon y, w_n) w_n \, dx\, dy. \tag{2.15}$$

By $f \in C^1(\mathbb{R},\mathbb{R})$ and the definition of $g$, we see that $g$ is of $C^1$. Then, by calculations, we have

$$\begin{aligned}
\langle \Phi'_\varepsilon(w_n), w_n\rangle &= 2\|w_n\|_\varepsilon^2 - \int_{\mathbb{R}^2} [g'(\varepsilon x, \varepsilon y, w_n) w_n^2 + g(\varepsilon x, \varepsilon y, w_n) w_n] \, dx\, dy \\
&= 2\int_{\mathbb{R}^2} g(\varepsilon x, \varepsilon y, w_n) w_n \, dx\, dy \\
&\quad - \int_{\mathbb{R}^2} [g'(\varepsilon x, \varepsilon y, w_n) w_n^2 + g(\varepsilon x, \varepsilon y, w_n) w_n] \, dx\, dy \\
&= \int_{\mathbb{R}^2} [g(\varepsilon x, \varepsilon y, w_n) w_n - g'(\varepsilon x, \varepsilon y, w_n) w_n^2] \, dx\, dy.
\end{aligned} \tag{2.16}$$

Let $\mathbb{R}^2 = \Omega_1 \cup \Omega_2$, where

$$\begin{gathered}
\Omega_1 = [\Omega_\varepsilon \cup \{w_n < t_0\}] \cup [(\mathbb{R}^2 \setminus \Omega_\varepsilon) \cap \{t_0 \le w_n \le t_1\}], \\
\Omega_2 = [(\mathbb{R}^2 \setminus \Omega_\varepsilon) \cap \{w_n \ge t_1\}].
\end{gathered}$$

On the one hand, for $(x, y) \in \Omega_1$, from (A14) we have

$$\begin{gathered}
g(\varepsilon x, \varepsilon y, w_n) w_n + g'(\varepsilon x, \varepsilon y, w_n) w_n^2 \ge 0, \\
g(\varepsilon x, \varepsilon y, w_n) w_n - g'(\varepsilon x, \varepsilon y, w_n) w_n^2 \le 0,
\end{gathered}$$

One the other hand, for $(x, y) \in \Omega_2$, we have $g(\varepsilon x, \varepsilon y, w_n) = \frac{V_0}{k} w_n$, and then

$$g(\varepsilon x, \varepsilon y, w_n) w_n + g'(\varepsilon x, \varepsilon y, w_n) w_n^2 = \frac{2V_0}{k} w_n^2, \quad \forall (x, y) \in \Omega_2.$$

Since $\{w_n\}$ is bounded, up to a subsequence, we may suppose that

$$\langle \Phi'_\varepsilon(w_n), w_n\rangle \to a \le 0.$$

If $a = 0$, by the Sobolev continuous embedding $X_\varepsilon \hookrightarrow L^p(\mathbb{R}^2)$ for $p \in [2, 6]$ and choose some $C_2 < \frac{k}{V_0}$, we obtain

$$|\langle \Phi'_\varepsilon(w_n), w_n\rangle| \ge 2\|w_n\|_\varepsilon^2 - \frac{2V_0}{k}\|w_n\|_2^2$$

$$
\begin{aligned}
&\geq (2-\frac{2V_0}{k}C_2)\|w_n\|_\varepsilon^2\\
&\geq (2-\frac{2V_0}{k}C_2)\int_{\Omega_\varepsilon}\Big(|w_n|^2+|D_x^{-1}(w_n)_y|^2+V(\varepsilon x,\varepsilon y)w_n^2\Big)\,dx\,dy.
\end{aligned}
$$

Therefore,

$$
\begin{gathered}
w_n\to 0 \quad \text{in } X_\varepsilon,\\
w_n\to 0 \quad \text{in } L^p(\mathbb{R}^2),\\
w_n\to 0 \quad \text{in } L^p(\Omega_\varepsilon).
\end{gathered}
$$

Finally, by (A11)–(A14), we have

$$
\|w_n\|_\varepsilon^2=\int_{\mathbb{R}^2\setminus\Omega_\varepsilon} g(\varepsilon x,\varepsilon y,w_n)w_n\,dx\,dy+o_n(1)\leq \frac{V_0}{k}\|w_n\|_2^2+o_n(1). \tag{2.17}
$$

From the proof of Lemma 2.1, we can easy to check that there exists a number $d>0$ such that

$$
\|w_n\|_\varepsilon \geq d>0 \quad \text{for all } w_n\in\mathcal{N}_\varepsilon. \tag{2.18}
$$

Thus, there is a contradiction between (2.16) and (2.17). We obtain $a\neq 0$ and $\mu_n=o_n(1)$. From (2.13), we conclude that $\varphi_\varepsilon'(w_n)\to 0$, that is, $\{w_n\}$ is a (PS) sequence for $\varphi_\varepsilon$. This completes the proof. □

## 3. Multiplicity of solutions for the modified problem

3.1. **Autonomous problem.** Along all the section we shall assume that $\delta>0$ is small enough such that $M_\delta\subset\Omega$, where $\Omega$ is given in the condition (A9). We start by considering the limit problem associated with (2.10), namely, the problem

$$
\big(-u_{xx}+D_x^{-2}u_{yy}+V_0u-f(u)\big)_x=0, \quad \text{in } \mathbb{R}^2 \tag{3.1}
$$

The solutions of (3.1) are precisely critical points of the functional defined by

$$
\varphi_0(u)=\frac12\int_{\mathbb{R}^2}[|u_x|^2+|D_x^{-1}u_y|^2+V_0u^2]\,dx\,dy-\int_{\mathbb{R}^2}F(u)\,dx\,dy \tag{3.2}
$$

which is $C^1$ with Gateaux derivative

$$
\langle\varphi_0'(u),v\rangle=\int_{\mathbb{R}^2}\big(u_xv_x+D_x^{-1}u_yD_x^{-1}v_y+V_0uv\big)\,dx\,dy-\int_{\mathbb{R}^2}f(u)v\,dx\,dy, \tag{3.3}
$$

for all $u,v\in X_0$.

Let $\mathcal{N}_0$ be the Nehari manifold associated with $\varphi_0$ by

$$
\mathcal{N}_0=\{u\in X_0\setminus\{0\}:\langle\varphi_0'(u),u\rangle=0\}, \tag{3.4}
$$

where $X_0$ is defined as the Hilbert space $X_\varepsilon$ but endowed with the inner product

$$
(u,v)_0=\int_{\mathbb{R}^2}\big(u_xv_x+D_x^{-1}u_yD_x^{-1}v_y+V_0uv\big)\,dx\,dy, \tag{3.5}
$$

and the corresponding norm

$$
\|u\|_0=\Big(\int_{\mathbb{R}^2}\big(|u_x|^2+|D_x^{-1}u_y|^2+V_0u^2\big)\,dx\,dy\Big)^{1/2}. \tag{3.6}
$$

As in the previous section, the next lemma characterizes of the infimum of $\varphi_0$ over $\mathcal{N}_0$.

**Lemma 3.1.** *If the conditions* (A1), (A4), (A5), (A9) *hold, then, for each $u \in X_0$ with $u \neq 0$, there exists a unique $t_0 = t_0(u) > 0$ such that $t_0 u \in \mathcal{N}_0$ and $\varphi_0(t_0 u) = \max_{t\geq 0} \varphi_0(tu)$. Moreover, we have*

$$c_{V_0} = \inf_{u\in X_0\setminus\{0\}} \max_{t\geq 0} \varphi_0(tu) = \inf_{\mathcal{N}_0} \varphi_0. \tag{3.7}$$

*where $c_{V_0}$ is the minimax level of Mountain Pass Theorem applied to $\varphi_0$, namely*

$$c_{V_0} = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} \varphi_0(\eta(t)),$$

*where $\Gamma = \{\eta \in C([0,1], X_0) : \eta(0) = 0$ and $\varphi_0(\eta(1)) < 0\}$.*

The proof of the above lemma is similar to the proof of Lemma 2.2. We omit it. The next lemma allows us to assume that the weak limit of a $(PS)_{c_{V_0}}$ sequence is nontrivial.

**Lemma 3.2.** *If $\{w_n\}$ is bounded in $X_\varepsilon$ and there exist $R > 0$ and a sequence $(x_n, y_n) \in \mathbb{R}^2$ such that*

$$\limsup_{n\to\infty} \int_{B_R((x_n,y_n))} |w_n(x,y)|^2 \,dx\,dy = 0.$$

*Then $w_n \to 0$ strongly in $L^q(\mathbb{R}^2)$, for every $2 < q < 6$.*

*Proof.* First of all, we fix $q \in (2,6)$. Given $R > 0$ and $z \in \mathbb{R}^2$, by standard interpolation inequality and Sobolev embedding theorem, we obtain

$$\begin{aligned}
&\|w_n\|_{L^q(B_R(z))}\\
&\leq \|w_n\|^{1-\lambda}_{L^2(B_R(z))} \|w_n\|^{\lambda}_{L^6(B_R(z))}\\
&\leq C_3 \|w_n\|^{1-\lambda}_{L^2(B_R(z))} \Big( \int_{B_R(z)} [|(w_n)_x|^2 + |D_x^{-1}(w_n)_y|^2 + V(\varepsilon x, \varepsilon y) {w_n}^2] \,dx\,dy \Big)^{\lambda/2},
\end{aligned}$$

where $\frac{1-\lambda}{2} + \frac{\lambda}{6} = \frac{1}{q}$.

Now, covering $\mathbb{R}^2$ with balls of radius $R$, in such a way that each point of $\mathbb{R}^2$ is contained in at most 3 balls, we find that

$$\int_{\mathbb{R}^2} |w_n|^q \leq 3C_3 \sup_{z\in\mathbb{R}^2} \Big[ \int_{B_R(z)} {w_n}^2 \Big]^{\frac{(1-\lambda)q}{2}} \|w_n\|_\varepsilon^{\lambda q/2}.$$

Under the assumption of the lemma, we have $w_n \to 0$ in $L^q(\mathbb{R}^2)$. $\square$

**Lemma 3.3** ([10])**.** *Let $\{w_n\} \subset X_0$ be a $(PS)_{c_{V_0}}$ sequence for $\varphi_0$ and such that $w_n \rightharpoonup 0$ in $X_0$. Then, only one of the following alternatives holds.*

(i) $w_n \to 0$ in $X_0$, *or*
(ii) *there exist a sequence $\{(x_n, y_n)\} \subset \mathbb{R}^2$ and constants $R, \beta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R((x_n,y_n))} |w_n|^2 \,dx\,dy \geq \beta > 0.$$

**Lemma 3.4.** *Let $c_{V_0} > 0$ and $\{w_n\}$ be a $(PS)_{c_{V_0}}$ sequence for $\varphi_0$, then $\{w_n\}$ is bounded.*

*Proof.* Assume that $\{w_n\} \subset X_0$ is a $(PS)_{c_{V_0}}$ sequence for $\varphi_0$. Then

$$\varphi_0(w_n) \to c_{V_0} \quad \text{and} \quad \varphi_0'(w_n) \to 0.$$

Therefore, there exists $C_4 > 0$ large enough such that

$$C_4(1 + \|w_n\|_0) \geq \varphi_0(w_n) - \frac{1}{\alpha+1}\langle {\varphi_0}'(w_n), w_n\rangle, \quad \forall n \in \mathbb{N} \tag{3.8}$$

On the other hand, by conditions (A10) and (2.9), we have

$$\begin{aligned}
&\varphi_0(w_n) - \frac{1}{\alpha+1}{\varphi_0}'(w_n)w_n \\
&= (\frac{1}{2} - \frac{1}{\alpha+1})\|w_n\|_0^2 + \int_{\mathbb{R}^2}[\frac{1}{\alpha+1}f(w_n)w_n - F(w_n)]\,dx\,dy \\
&\geq (\frac{1}{2} - \frac{1}{\alpha+1})\|w_n\|_0^2.
\end{aligned}$$

The above inequality proves that $\{w_n\}$ is bounded in $X_0$. □

**Theorem 3.5.** *Suppose that* (A1), (A4), (A5), (A7), (A9), (A10) *are satisfied. Then problem* (3.1) *has a positive ground-state solution.*

*Proof.* Let $\{w_n\} \subset X_0$ be a $(PS)_{c_{V_0}}$ sequence for $\varphi_0$. By Lemma 3.4, we know that $\{w_n\}$ is bounded in $X_0$. Then, up to a subsequence, $w_n \rightharpoonup w$ weakly in $X_0$, $w_n \to w$ strongly in $L^p_{\rm loc}(\mathbb{R}^2), p \in [2, 6)$ and $w_n \to w$ a.e. in $\mathbb{R}^2$.

As in the proof of Lemma 2.1, we can easily prove that $\varphi_0$ satisfies the Mountain Pass Geometry. By the Mountain Pass Lemma (see [25]), there exists a Palais-Smale sequence $\{w_n\}$ for $\varphi_0$ at the mountain pass level $c_{V_0}$. Moreover, $w_n \rightharpoonup w$ in $X_0$ and $w$ is a critical point of $\varphi_0$. From Lemma 3.2, we know that $w \neq 0$ and $w \in \mathcal{N}_0$.

Next we prove that $w_n \to w$ strongly in $X_0$. From the semi-lower continuity of norm, we have

$$\lim_{n\to\infty} \|w_n\|_0 \geq \|w\|_0. \tag{3.9}$$

Observe that we must have the above equality hold. Otherwise, by Fatou's Lemma we obtain

$$\begin{aligned}
c_{V_0} &\leq \varphi_0(w) - \frac{1}{\alpha+1}\langle \varphi_0'(w), w\rangle \\
&= \Big(\frac{1}{2} - \frac{1}{\alpha+1}\Big)\|w\|_0^2 + \frac{1}{\alpha+1}\int_{\mathbb{R}^2}[f(w)w - (\alpha+1)F(w)]\,dx\,dy \\
&< \liminf_{n\to\infty}\big\{(\frac{1}{2} - \frac{1}{\alpha+1})\|w_n\|_0^2 + \frac{1}{\alpha+1}\int_{\mathbb{R}^2}[f(w_n)w_n - (\alpha+1)F(w_n)]\,dx\,dy\big\} \\
&= \liminf_{n\to\infty}\Big(\varphi_0(w_n) - \frac{1}{\alpha+1}\langle \varphi_0'(w_n), w_n\rangle\Big) = c_{V_0}.
\end{aligned}$$

which is a contradiction. Thus, we conclude that up to a subsequence,

$$\lim_{n\to\infty} \|w_n\|_0 = \|w\|_0. \tag{3.10}$$

So, the Brezis-Lieb Lemma [6] implies that $w_n \to w$ in $X_0$.

Last, we prove that the solution $w$ is nonnegative. From (A7) and using $-w^-$ as a testing function, we have

$$\varphi_0'(w)(-w^-) = \int_{\mathbb{R}^2}\Big(|w_x^-|^2 + |D_x^{-1}w_y^{-1}|^2 + V_0|w^-|^2\Big)\,dx\,dy = \|w^-\|_0^2 = 0. \tag{3.11}$$

where $w^- = \max\{-w, 0\}$. This implies that $w \geq 0$ in $\mathbb{R}^2$ is a nonnegative weak solution of (3.1) and the proof is complete. □

Then we consider $\delta > 0$ such that $M_\delta \subset \Omega$ and choose $\eta \in C_0^\infty(\mathbb{R}^+, [0,1])$ satisfying

$$\eta(s) = \begin{cases} 1, & \text{if } 0 \le s \le \delta/2, \\ 0, & \text{if } s > \delta. \end{cases}$$

For each $z \in M = \{x \in \Omega : V(x) = V_0\}$, let $\nu \in \mathbb{R}^2$, we define

$$\psi_{\varepsilon,z}(\nu) = \eta(|(\varepsilon\nu - z)|)w(\frac{\varepsilon\nu - z}{\varepsilon})$$

and $t_\varepsilon > 0$ satisfying

$$\max_{t\ge 0} \varphi_\varepsilon(t\psi_{\varepsilon,z}) = \varphi_\varepsilon(t_\varepsilon\psi_{\varepsilon,z}),$$

where $w$ is a solution of (3.1) such that $\varphi_0(w) = c_{V_0}$.

We define $\phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ by

$$\phi_\varepsilon(z) = t_\varepsilon\psi_{\varepsilon,z}.$$

**Lemma 3.6.** *The function $\phi_\varepsilon$ satisfies*

$$\lim_{\varepsilon\to 0} \varphi_\varepsilon(\phi_\varepsilon(z)) = c_{V_0} \quad \textit{uniformly in } z \in M.$$

*Proof.* Arguing by contradiction, we suppose that there exist $\delta > 0$, $\{z_n\} \subset M$, and $\varepsilon_n \to 0$, such that

$$|\varphi_{\varepsilon_n}(\phi_{\varepsilon_n}(z_n)) - c_{V_0}| \ge \delta. \tag{3.12}$$

Notice that for each $n \in \mathbb{N}$ and for all $\nu \in B_{\delta/\varepsilon_n}(\frac{z_n}{\varepsilon_n})$, we have $\varepsilon_n\nu \in B_\delta(z_n)$. By using the change of variables $\bar{\nu} := (\varepsilon_n\nu - z_n)/\varepsilon_n$, one easily has

$$\varepsilon_n\bar{\nu} + z_n \in B_\delta(z_n) \subset M_\delta \subset \Omega. \tag{3.13}$$

and

$$\begin{aligned} \varphi_{\varepsilon_n}(\phi_{\varepsilon_n}(z_n)) &= \varphi_{\varepsilon_n}(t_{\varepsilon_n}\psi_{\varepsilon_n,z_n}) \\ &= \frac{t_{\varepsilon_n}^2}{2}\|\psi_{\varepsilon_n,z_n}\|_\varepsilon^2 - \int_{\mathbb{R}^2} G(\varepsilon_n x, \varepsilon_n y, t_{\varepsilon_n}\psi_{\varepsilon_n,z_n})\,dx\,dy \\ &= \frac{t_{\varepsilon_n}^2}{2}\|\eta(|\varepsilon_n\bar{\nu}|)w(\bar{\nu})\|_\varepsilon^2 - \int_{\mathbb{R}^2} F(t_{\varepsilon_n}\eta(|\varepsilon_n\bar{\nu}|)w(\bar{\nu}))\,dx\,dy. \end{aligned} \tag{3.14}$$

Since $\phi_{\varepsilon_n}(z_n) \in \mathcal{N}_{\varepsilon_n}$ and $g = f$ in $\Omega$, we have

$$\begin{aligned} &t_{\varepsilon_n}^2\|\eta(|\varepsilon_n\bar{\nu}|)w(\bar{\nu})\|_{\varepsilon_n}^2 \\ &= \int_{\mathbb{R}^2} g\left(\varepsilon_n\bar{\nu} + z_n, t_{\varepsilon_n}\eta(|\varepsilon_n\bar{\nu}|)w(\bar{\nu})\right) t_{\varepsilon_n}\eta(|\varepsilon_n|)w(\bar{\nu})\,dx\,dy \\ &= \int_{\mathbb{R}^2} f\left(t_{\varepsilon_n}\eta(|\varepsilon_n\bar{\nu}|)w(\bar{\nu})\right) t_{\varepsilon_n}\eta(|\varepsilon_n|)w(\bar{\nu})\,dx\,dy. \end{aligned} \tag{3.15}$$

By (A5), if $t_{\varepsilon_n} \to \infty$, we obtain

$$\begin{aligned} \|\psi_{\varepsilon_n,z_n}\|_{\varepsilon_n}^2 &= \int_{\mathbb{R}^2} \frac{g(\varepsilon_n x, \varepsilon_n y, t_{\varepsilon_n}\psi_{\varepsilon_n,z_n})}{t_{\varepsilon_n}\psi_{\varepsilon_n,z_n}}\psi_{\varepsilon_n,z_n}^2\,dx\,dy \\ &> \int_{B_{\delta/2}(0)} C_0(t_{\varepsilon_n}\psi_{\varepsilon_n,z_n})^{q-2}\psi_{\varepsilon_n,z_n}^2\,dx\,dy \\ &= \int_{B_{\delta/2}(0)} C_0 t_{\varepsilon_n}^{q-2}(\eta(|\varepsilon_n\bar{\nu}|)w(\bar{\nu}))^q\,dx\,dy \\ &\ge C_0 t_{\varepsilon_n}^{q-2}c^q|B_{\delta/2}(0)| \to \infty. \end{aligned} \tag{3.16}$$

as $t_{\varepsilon_n} \to \infty$. Where $c = \inf_{u \in B_{\delta/2}(0)} w(u)$, and $q \in (2,6)$ is given in (A5). But the left side of the above inequality is bounded as $t_{\varepsilon_n} \to \infty$. This yields a contradiction. Hence, $t_{\varepsilon_n} \to t_0$ with $t_0 \geq 0$. Form (3.15) and (A4), we see that $t_0 > 0$. Now we prove a claim.

**Claim.** $t_0 = 1$**.** By using Lebesgue's theorem, we can verify that

$$\lim_{n\to\infty} \|\psi_{\varepsilon_n,z_n}\|^2_{\varepsilon_n} = \|w\|^2_0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^2} \psi_{\varepsilon_n,z_n} = \int_{\mathbb{R}^2} w\,dx\,dy,$$
$$\lim_{n\to\infty} \int_{\mathbb{R}^2} f(\psi_{\varepsilon_n,z_n})\psi_{\varepsilon_n,z_n} = \int_{\mathbb{R}^2} f(w)w\,dx\,dy.$$

Therefore, passing to limit in equality (3.15), we have

$$\begin{aligned}
\|w(\bar{\nu})\|^2_{\varepsilon} &= \lim_{n\to\infty} \int_{\mathbb{R}^2} \frac{g(\varepsilon_n\bar{\nu}, t_{\varepsilon_n}\psi_{\varepsilon_n,z_n})}{t_{\varepsilon_n}\psi_{\varepsilon_n,z_n}} \psi^2_{\varepsilon_n,z_n}\,dx\,dy \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^2} \frac{g(\varepsilon_n\bar{\nu}, t_{\varepsilon_n}\eta(|\varepsilon_n\bar{\nu}|)w)(\eta(|\varepsilon_n\bar{\nu}|)w(\bar{\nu}))^2}{t_{\varepsilon_n}\eta(|\varepsilon_n\bar{\nu}|)w(\bar{\nu})}\,dx\,dy \\
&= \int_{\mathbb{R}^2} \frac{f(t_0 w)}{t_0 w} w^2\,dx\,dy.
\end{aligned} \tag{3.17}$$

Since $w$ is a positive ground state of (1.2), we have that $\langle I'_{\varepsilon}(w), w\rangle = 0$, namely

$$\|w\|^2_{\varepsilon} = \int_{\mathbb{R}^2} f(w)w\,dx\,dy. \tag{3.18}$$

Combining (3.17) and (3.18), we have

$$t_0^{\alpha-1} \int_{\mathbb{R}^2} \frac{f(t_0 w)}{(t_0 w)^{\alpha}} w^{\alpha+1}\,dx\,dy = \int_{\mathbb{R}^2} \frac{f(w)}{w^{\alpha}} w^{\alpha+1}\,dx\,dy. \tag{3.19}$$

Arguing as in the proof of uniqueness in Lemma 2.2, we see that $t_0 = 1$ and we obtain the claim.

Letting $n \to \infty$ in (3.14), we obtain $\lim_{\varepsilon_n\to 0} \varphi_{\varepsilon_n}(\phi_{\varepsilon_n}(z_n)) = \varphi_0(w) = c_{V_0}$, which contradicts to (3.12). This completes the proof. $\square$

3.2. **Multiplicity of solutions for** (2.10)**.** In this subsection we will relate the number of solutions of (2.10) to the topology of the set $M$. For this, the next compactness result is fundamental for showing that the solutions of the modified problem are solutions of the original problem.

**Proposition 3.7** ([1])**.** *Let $\varepsilon_n \to 0$ and $\{w_n\} \subset \mathcal{N}_{\varepsilon_n}$ be such that $\varphi_{\varepsilon_n}(w_n) \to c_{V_0}$. Then there exists a sequence $\{\tilde{z}_n\} \subset \mathbb{R}^2$ such that $v_n(x) = w_n(x + \tilde{z}_n)$ has a convergent subsequence in $X_0$. Moreover, up to a subsequence, $z_n := \varepsilon_n\tilde{z}_n \to z_0 \in M$.*

We now consider the following subset of the Nehari mainfold

$$\mathcal{N}_{\varepsilon}^{c_{V_0}+h(\varepsilon)} = \{w \in \mathcal{N}_{\varepsilon} : \varphi_{\varepsilon}(w) \leq c_{V_0} + h(\varepsilon)\},$$

where $h(\varepsilon) = |\varphi_{\varepsilon}(\phi_{\varepsilon}(z)) - c_{V_0}|$ is such that $h(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$. Thus, there is a number $\hat{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \hat{\varepsilon})$, we have $\phi_{\varepsilon}(z) \in \mathcal{N}_{\varepsilon}^{c_{V_0}+h(\varepsilon)}$ and $\mathcal{N}_{\varepsilon}^{c_{V_0}+h(\varepsilon)} \neq \emptyset$.

Let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. We define $\chi : \mathbb{R}^2 \to \mathbb{R}^2$ as $\chi(x,y) = (x,y)$ for $|(x,y)| \le \rho$ and $\chi(x,y) = \rho(x,y)/|(x,y)|$ for $|(x,y)| \ge \rho$. Let us consider the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^2$ be given by

$$\beta_\varepsilon(w) = \frac{\int_{\mathbb{R}^2} \chi(\varepsilon x, \varepsilon y) w^2(x,y)\,dx\,dy}{\int_{\mathbb{R}^2} w^2(x,y)\,dx\,dy}.$$

Since $M \subset B_\rho(0)$, by the definition of $\chi$ and the Lebesgue's theorem, we have

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\phi_\varepsilon(z)) = z \quad \text{uniformly in } z \in M. \tag{3.20}$$

**Lemma 3.8.** *For any $\delta > 0$, it holds that*

$$\lim_{\varepsilon\to 0} \sup_{w\in\mathcal{N}_\varepsilon^{c_{V_0}+h(\varepsilon)}} \operatorname{dist}(\beta_\varepsilon(w), M_\delta) = 0.$$

The proof of the above lemma is similar to the proof of [13, Lemma 3.10]. We omit its proof.

**Theorem 3.9.** *Assume that conditions* (A1), (A9), (A4), (A5), (A7), (A10) *hold. Then for each $\delta > 0$, there exists $\widehat{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \widehat{\varepsilon}_\delta)$, problem* (2.10) *have at least* $\operatorname{cat}_{M_\delta}(M)$ *positive solutions.*

*Proof.* From Lemmas 3.6 and 3.8, and (3.20), we see that the continuous mappings below is well defined for $\varepsilon \in (0, \widehat{\varepsilon})$,

$$M \xrightarrow{\phi_\varepsilon} \mathcal{N}_\varepsilon^{c_{V_0}+h(\varepsilon)} \xrightarrow{\beta_\varepsilon} M_\delta.$$

From (3.20), we can choose a function $\theta(\varepsilon, z)$ with $|\theta(\varepsilon, z)| < \frac{\delta}{2}$ uniformly in $z \in M$, such that

$$\beta_\varepsilon(\phi_\varepsilon(z)) = z + \theta(\varepsilon, z) \quad \text{for all } \varepsilon \in (0, \hat{\varepsilon}), z \in M.$$

We define $H(t,z) = z + (1-t)\theta(\varepsilon, z)$. Then $H : [0,1] \times M \to M_\delta$ is continuous. Obviously, $H(0,z) = \beta_\varepsilon(\phi_\varepsilon(z)), H(1,z) = z$ for all $z \in M$. That is, $H(t,z)$ is a homotopy between $\beta_\varepsilon \circ \phi_\varepsilon$ and the inclusion map $Id : M \to M_\delta$.

From the proof of [20, Theorem 1.1], it is easy to check that there exists a subset $\mathcal{A} \subset \mathcal{N}_\varepsilon$ and we select suitably $\bar{c} > c_{V_0} + h(\varepsilon)$ such that

(i) $\mathcal{A}$ is not contractible in $\mathcal{N}_\varepsilon^{c_{V_0}+h(\varepsilon)}$,
(ii) $\mathcal{A}$ is contractible in $\mathcal{N}_\varepsilon^{\bar{c}}$.

The above properties imply that there exists a critical level between $c_{V_0} + h(\varepsilon)$ and $\bar{c}$. Therefore, there exists a number $\widehat{\varepsilon}_\delta > 0$ small, then for $\varepsilon \in (0, \widehat{\varepsilon}_\delta)$, it follows from Lemma 2.5 that the functional $\varphi_\varepsilon$ restricted to $\mathcal{N}_\varepsilon$ satisfies the (PS) condition in the interval $(c_{V_0}, c_{V_0} + h(\varepsilon))$. Notice that the well known properties of the category in [23] guarantee

$$\operatorname{cat}(\mathcal{N}_\varepsilon^{c_{V_0}+h(\varepsilon)}) \ge \operatorname{cat}_{M_\delta}(M),$$

and the Ljusternik-Schnirelmann theory ensures the existence of at least $\operatorname{cat}(\mathcal{N}_\varepsilon^{c_{V_0}+h(\varepsilon)})$ critical points of $\varphi_\varepsilon$ in $\mathcal{N}_\varepsilon^{c_{V_0}+h(\varepsilon)}$. Therefore, we conclude that $\varphi_\varepsilon$ admits at least $\operatorname{cat}_{M_\delta}(M)$ critical points in $\mathcal{N}_\varepsilon^{c_{V_0}+h(\varepsilon)}$. Moreover, from the proof of Theorem 3.5, we infer that these solutions of problem (2.10) are positive by a similar argument. $\square$

## 4. Proof of Theorem 1.1

In this section we shall prove our main result. The idea is to show that the solutions obtained in Theorems 3.5 and 3.9 satisfy the estimate $u_\varepsilon \le a, \forall x \in \mathbb{R}^2 \setminus \Omega$ for $\varepsilon$ small enough. This fact implies that these solutions are indeed solutions of the original problem (1.2). The following lemma plays a fundamental role in the study of behavior of the maximum points of the solutions.

**Lemma 4.1.** *Let $\varepsilon_n \to 0^+$ and $w_n \in \mathcal{N}_{\varepsilon_n}^{c_{V_0}+h(\varepsilon_n)}$ be a solution of problem* (2.10). *Then, up to a subsequence, there exists a sequence $(x_n, y_n) \subset \mathbb{R}^2$ such that $\psi_n(x,y) := w_n(x_n + x, y_n + y)$ satisfies that $w_n(x,y) = \psi_n(x,y) \in L^\infty(\mathbb{R}^2)$, and there exists $C_5 > 0$ such that*

$$\|w_n\|_{L^\infty(\mathbb{R}^2)} \le C_5, \quad \text{for all } n \in N, \tag{4.1}$$

*where $(x_n, y_n) = \tilde{z}_n$ is given in Proposition 3.7.*

The proof of the above lemma is similar to the proof of [13, Lemma 4.1]. We omit it.

*Proof of Theorem 1.1.* At first, we fix a number $\delta > 0$ small such that $M_\delta \subset \Omega$. Similar to the proof of [13, Theorem 1.1], there exists $\widetilde{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \widetilde{\varepsilon}_\delta)$ and $b > 0$, then for any solution $w_\varepsilon \in \mathcal{N}_\varepsilon^{c_{V_0}+h(\varepsilon)}$ of problem (2.10), it holds

$$\|w_\varepsilon\|_{L^\infty(\mathbb{R}^2 \setminus \Omega_\varepsilon)} < b. \tag{4.2}$$

Let $\widehat{\varepsilon}_\delta$ be given in Theorem 3.9 and $\varepsilon := \min\{\widehat{\varepsilon}_\delta, \widetilde{\varepsilon}_\delta\}$. From Theorem 3.9, we can know that there exists $\operatorname{cat}_{M_\delta}(M)$ nontrivial solutions of problem (2.10) in $\mathcal{N}_\varepsilon^{c_{V_0}+h(\varepsilon)}$. If $w \in X_\varepsilon$ is one of these solutions, then $w \in \mathcal{N}_\varepsilon^{c_{V_0}+h(\varepsilon)}$, it follows from (4.2) and the definition of $g$ that $g(\cdot, w) = f(w)$. Thus $w$ is also a solution of the original problem (1.2). Then (1.2) has at least $\operatorname{cat}_{M_\delta}(M)$ nontrivial solutions. The proof of Theorem 1.1 is complete. □

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