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EVOLUTION EQUATIONS ON TIME-DEPENDENT LEBESGUE SPACES WITH VARIABLE EXPONENTS

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Abstract. We extend the results in Kloeden-Simsen [CPAA 2014] to $p(x,t)$-Laplacian problems on time-dependent Lebesgue spaces with variable exponents. We study the equation

$$\begin{aligned}&\frac{\partial u_\lambda}{\partial t}(t) - \operatorname{div}\big(D_\lambda(t,x)|\nabla u_\lambda(t)|^{p(x,t)-2}\nabla u_\lambda(t)\big) + |u_\lambda(t)|^{p(x,t)-2}u_\lambda(t)\\ &= B(t,u_\lambda(t))\end{aligned}$$

on a bounded smooth domain Ω in $\mathbb{R}^n$, $n \geq 1$, with a homogeneous Neumann boundary condition, where the exponent $p(\cdot) \in C(\bar{\Omega}\times[\tau,T],\mathbb{R}^+)$ satisfies $\min p(x,t) > 2$, and $\lambda \in [0,\infty)$ is a parameter.

We establish the existence and upper semicontinuity of pullback attractors for this equation under the assumption, amongst others, that B is globally Lipschitz in its second variable and $D_\lambda \in L^\infty([\tau,T]\times\Omega,\mathbb{R}^+)$ is bounded from above and below, monotonically nonincreasing in time and continuous in the parameter λ.

1. Introduction

We consider the problem

$$\begin{gathered}\begin{aligned}&\frac{\partial u_\lambda}{\partial t}(t) - \operatorname{div}\big(D_\lambda(t,x)|\nabla u_\lambda(t)|^{p(x,t)-2}\nabla u_\lambda(t)\big) + |u_\lambda(t)|^{p(x,t)-2}u_\lambda(t)\\ &= B(t,u_\lambda(t))\end{aligned}\\ u_\lambda(\tau) = u_{\tau\lambda},\end{gathered} \tag{1.1}$$

on a bounded smooth domain Ω in $\mathbb{R}^n$ for $n \geq 1$ with a homogeneous Neumann boundary condition. The exponent $p(\cdot) \in C(\bar{\Omega}\times[\tau,T],\mathbb{R})$ satisfies

$$p^+ := \max_{(x,t)\in\bar{\Omega}\times[\tau,T]} p(x,t) \geq p^- := \min_{(x,t)\in\bar{\Omega}\times[\tau,T]} p(x,t) > 2$$

and the initial condition $u_\lambda(\tau) \in H := L^2(\Omega)$. For the mapping $B : [\tau,T]\times H \to H$ we use the following assumptions:

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(A1) there exists $L \geq 0$ such that

$$\|B(t,x_1) - B(t,x_2)\|_H \leq L\|x_1 - x_2\|_H$$

for all $t \in [\tau, T]$ and $x_1, x_2 \in H$;

(A2) for all $x \in H$ the mapping $t \mapsto B(t,x)$ belongs to $L^2(\tau, T; H)$;

(A3) the function $t \mapsto \|B(t,0)\|_H$ is nondecreasing, absolutely continuous and bounded on compact subsets of $\mathbb{R}$.

For each $\lambda \in [0,\infty)$, the mapping $D_\lambda : [\tau, T] \times \Omega \to \mathbb{R}$ belongs to $L^\infty([\tau,T] \times \Omega)$ and we use the following assumptions:

(A4) there are positive constants, β and M independent of λ such that $0 < \beta \leq D_\lambda(t,x) \leq M$ for almost all $(t,x) \in [\tau,T] \times \Omega$;

(A5) $D_\lambda \to D_{\lambda_1}$ in $L^\infty([\tau,T] \times \Omega)$ as $\lambda \to \lambda_1$;

(A6) $D_\lambda(t,x) \geq D_\lambda(s,x)$ for each $x \in \Omega$ and $t \leq s$ in $[0,T]$.

PDEs with variable exponents have application in electrorheological fluids (see [9, 17, 18]), image process (see [8, 12]), flow in porous media [2, 3], magnetostatics [5], and capillarity phenomena [4].

In this article, from the point of view of $\mathcal{D}$-pullback attractor theory [13, 15, 16], we study the asymptotic behavior and analyze the sensitivity of this nonlinear non-autonomous problem for large time varying the diffusion coefficients D_λ.

The study of existence of attractors for parabolic problems with spatially variable exponents is a very recent research issue. To the best of our knowledge, the first results for autonomous problems were published in [19] and the first result on existence of a pullback attractor for a non-autonomous problem were published in [14]. The main motivation of this work is that the exponents also depends on time, differently from the previous works [14, 19] which considered only spatially variable exponents.

This article is organized as follows. In Section 2 we present the operator which depends on time and show some of its properties. In Section 3 we prove existence and uniqueness of the solution of (1.1). Section 4 is used to prove the existence of the $\mathcal{D}$-pullback attractors. Section 5 is devoted to prove the upper semicontinuity of pullback attractors.

2. Operator A^λ and its properties

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded smooth domain. In this work we will use the following notation

$$L^{p(\cdot,t)}(\Omega) := \left\{u : \Omega \to \mathbb{R} : u \text{ is measurable}, \int_\Omega |u(x)|^{p(x,t)}\,dx < \infty\right\}.$$

We define $\rho^t(u) := \int_\Omega |u(x)|^{p(x,t)}\,dx$ and

$$\|u\|_{p(\cdot,t)} := \inf\left\{\lambda > 0 : \rho^t\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

for $u \in L^{p(\cdot,t)}(\Omega)$. The generalized Sobolev space, defined as

$$W^{1,p(\cdot,t)}(\Omega) = \left\{u \in L^{p(\cdot,t)}(\Omega) : |\nabla u| \in L^{p(\cdot,t)}(\Omega)\right\},$$

is a Banach Space with the norm

$$\|u\|_{W^{1,p(\cdot,t)}(\Omega)} := \|u\|_{p(\cdot,t)} + \|\nabla u\|_{p(\cdot,t)}.$$

Let us consider $H := L^2(\Omega)$, $Y_t := L^{p(\cdot,t)}(\Omega)$ and $X_t := W^{1,p(\cdot,t)}(\Omega)$ with $p(x,t) \in [p^-, p^+] \subset (2,\infty)$ for all $(x,t) \in \Omega \times \mathbb{R}$. Then $X_t \subset H \subset X_t^*$ with continuous and dense embeddings.

Consider the operator $A^\lambda(t)u : X_t \to \mathbb{R}$ such that to each $u \in X_t$ associate the following element of X_t^*,

$$
\begin{aligned}
&A^\lambda(t)u(v)\\
&:= \int_\Omega D_\lambda(t,x)|\nabla u(x)|^{p(x,t)-2}\nabla u(x)\nabla v(x)\,dx + \int_\Omega |u(x)|^{p(x,t)-2}u(x)v(x)\,dx.
\end{aligned}
$$

Lemma 2.1 ([11])**.** *For $u \in L^{p(\cdot,t)}(\Omega)$ we have:*

(i) $\|u\|_{p(\cdot,t)} < 1$ $(= 1; > 1)$ *if and only if* $\rho^t(u) < 1$ $(= 1; > 1)$*;*

(ii) *If* $\|u\|_{p(\cdot,t)} > 1$*, then* $\|u\|_{p(\cdot,t)}^{p^-} \le \rho^t(u) \le \|u\|_{p(\cdot,t)}^{p^+}$*;*

(iii) *If* $\|u\|_{p(\cdot,t)} < 1$*, then* $\|u\|_{p(\cdot,t)}^{p^+} \le \rho^t(u) \le \|u\|_{p(\cdot,t)}^{p^-}$*.*

Lemma 2.2 ([1])**.** *Let λ, μ be arbitrary nonnegative numbers. For every positive $\alpha, \beta, \alpha \ge \beta$, it holds*

$$
\lambda^\alpha + \mu^\beta \ge \frac{1}{2^\alpha}\begin{cases} (\lambda+\mu)^\alpha, & \text{if } \lambda+\mu < 1,\\ (\lambda+\mu)^\beta, & \text{if } \lambda+\mu \ge 1. \end{cases} \tag{2.1}
$$

Using the previous lemmas we can obtain the following estimates.

Lemma 2.3. *Let $u \in X_t$. For each $t \ge 0$ we have*

$$
\langle A^\lambda(t)u, u\rangle_{X_t^*,X_t} \ge \frac{\min\{1,\beta\}}{2^{p^+}}\begin{cases} \|u\|_{X_t^{p^+}}, & \text{if } \|u\|_{X_t} < 1\\ \|u\|_{X_t^{p^-}}, & \text{if } \|u\|_{X_t} \ge 1. \end{cases} \tag{2.2}
$$

Proof. For an arbitrary $u \in X_t$, we denote $\lambda_t = \|\nabla u\|_{p(\cdot,t)}$ and $\mu_t = \|u\|_{p(\cdot,t)}$. From Lemmas 2.1 and 2.2 we have

$$
\begin{aligned}
\langle A^\lambda(t)u, u\rangle_{X_t^*,X_t} &= \int_\Omega D_\lambda|\nabla u|^{p(x,t)}\,dx + \int_\Omega |u|^{p(x,t)}\,dx\\
&\ge \beta\rho^t(\nabla u) + \rho^t(u)\\
&\ge \{\|\nabla u\|_{p(\cdot,t)}^{p+}, \|\nabla u\|_{p(\cdot,t)}^{p-}\} + \min\{\|u\|_{p(\cdot,t)}^{p+}, \|u\|_{p(\cdot,t)}^{p-}\}\\
&\ge \min\{1,\beta\}\big(\min\{\lambda_t^{p+}, \lambda_t^{p-}\} + \min\{\mu_t^{p+}, \mu_t^{p-}\}\big)\\
&\ge \frac{\min\{1,\beta\}}{2^{p+}}\begin{cases} (\lambda_t+\mu_t)^{p^+}, & \text{if } \lambda_t+\mu_t < 1\\ (\lambda_t+\mu_t)^{p^-}, & \text{if } \lambda_t+\mu_t \ge 1 \end{cases}\\
&= \frac{\min\{1,\beta\}}{2^{p+}}\begin{cases} \|u\|_{X_t}^{p^+}, & \text{if } \|u\|_{X_t} < 1\\ \|u\|_{X_t}^{p^-}, & \text{if } \|u\|_{X_t} \ge 1. \end{cases}
\end{aligned}
$$
□

Lemma 2.4. *The operator $A^\lambda(t) : X_t \to X_t^*$ is monotone for each $t \in [\tau, T]$.*

Proof. By [10],

$$
(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta) \ge \big(\frac{1}{2}\big)^p|\xi-\eta|^p, \quad p \ge 2,\ \xi,\eta \in \mathbb{R}^N. \tag{2.3}
$$

Fix $t \in [\tau, T]$ and let $u, v \in X_t$. Using (2.3) for each fixed $x \in \Omega$, we obtain

$$
\langle A^\lambda(t)u - A^\lambda(t)v, u-v\rangle_{X_t^*,X_t}
$$

$$
\begin{aligned}
&= \int_\Omega D_\lambda(t,x)\big(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\big)\cdot(\nabla u - \nabla v)\,dx \\
&\quad + \int_\Omega \Big(|u|^{p(x,t)-2}u - |v|^{p(x,t)-2}v\Big)(u-v)\,dx \\
&\geq \beta \int_\Omega \big(\frac{1}{2}\big)^{p(x,t)}|\nabla u(x) - \nabla v(x)|^{p(x,t)}\,dx \\
&\quad + \int_\Omega \big(\frac{1}{2}\big)^{p(x,t)}|u(x) - v(x)|^{p(x,t)}\,dx \geq 0.
\end{aligned}
$$

The proof is complete. □

Remark 2.5. The operator $A^\lambda(t) : X_t \to X_t^*$ is coercive and hemicontinuous for each $t \in [\tau, T]$.

We will show that $A_H^\lambda(t)$ is the subdifferential of the convex, proper, and lower semicontinuous map $\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)} : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$, given by

$$
\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u) := \begin{cases} \big[\int_\Omega \frac{D_\lambda(t,x)}{p(x,t)}|\nabla u|^{p(x,t)}\,dx + \int_\Omega \frac{1}{p(x,t)}|u|^{p(x,t)}\,dx\big] & \text{if } u \in W^{1,p(\cdot,t)}(\Omega) \\ +\infty, & \text{otherwise.} \end{cases}
$$

Lemma 2.6. *The map $\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}$ is convex and proper.*

Proof. Let $u \in X_t$. Then, $u \in Y_t = L^{p(\cdot,t)}(\Omega)$ and $\nabla u \in Y_t$. So,

$$
\begin{aligned}
&\int_\Omega \frac{D_\lambda(t,x)}{p(x,t)}|\nabla u|^{p(x,t)}\,dx + \int_\Omega \frac{1}{p(x,t)}|u|^{p(x,t)}\,dx \\
&\leq \frac{1}{2}\big[M\int_\Omega |\nabla u|^{p(x,t)}\,dx + \int_\Omega |u|^{p(x,t)}\,dx\big] < \infty.
\end{aligned}
$$

Therefore $\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}$ is proper. Since the application γ^p is convex for a $\gamma > 0$ given, $u, v \in X_t$, and $0 \leq \lambda \leq 1$ we have

$$
\begin{aligned}
&\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(\lambda u + (1-\lambda)v) \\
&= \int_\Omega \frac{D_\lambda(t,x)}{p(x,t)}|\nabla(\lambda u + (1-\lambda)v)|^{p(x,t)}\,dx + \int_\Omega \frac{1}{p(x,t)}|\lambda u + (1-\lambda)v|^{p(x,t)}\,dx \\
&\leq \int_\Omega \frac{D_\lambda(t,x)}{p(x,t)}\Big(\lambda|\nabla u|^{p(x,t)} + (1-\lambda)|\nabla v|^{p(x,t)}\Big)\,dx \\
&\quad + \int_\Omega \frac{1}{p(x,t)}\Big(\lambda|u|^{p(x,t)} + (1-\lambda)|v|^{p(x,t)}\Big)\,dx \\
&= \lambda\int_\Omega \frac{D_\lambda(t,x)}{p(x,t)}|\nabla u|^{p(x,t)}\,dx + \lambda\int_\Omega \frac{1}{p(x,t)}|u|^{p(x,t)}\,dx \\
&\quad + (1-\lambda)\int_\Omega \frac{D_\lambda(t,x)}{p(x,t)}|\nabla v|^{p(x,t)}\,dx + (1-\lambda)\int_\Omega \frac{1}{p(x,t)}|v|^{p(x,t)}\,dx \\
&= \lambda\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u) + (1-\lambda)\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(v).
\end{aligned}
$$

Therefore $\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}$ is convex. □

Lemma 2.7. *The map $\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}$ is lower semicontinuous.*

Proof. Let (u_n) be a sequence such that $u_n \to u$ in H. We have to show that

$$\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u) \le \liminf_{n\to\infty} \varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u_n) \quad \text{if } u_n \to u \text{ in } H.$$

If $\liminf_{n\to\infty} \varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u_n) = +\infty$, then

$$\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u) \le +\infty = \liminf_{n\to\infty} \varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u_n).$$

On the other hand, if $\liminf_{n\to\infty} \varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u_n) = a < +\infty$ then there is a subsequence $(u_{n_j}) \subset X_t$ of (u_n) such that

$$\lim_{j\to\infty} \varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u_{n_j}) = \lim_{j\to\infty} \Big(\int_\Omega \frac{D_\lambda(t,x)}{p(x,t)} |\nabla u_{n_j}|^{p(x,t)}\,dx + \int_\Omega \frac{1}{p(x,t)} |u_{n_j}|^{p(x,t)}\,dx \Big) = a.$$

Since $\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u_{n_j}) \to a$ as $j \to \infty$ we have that $\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u_{n_j})$ is bounded, i.e., there exists $\delta > 0$ such that $|\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u_{n_j})| \le \delta$, for all $j \in \mathbb{N}$. We have

$$\begin{aligned}
\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u_{n_j}) &\ge \int_\Omega \frac{\beta}{p(x,t)} |\nabla u_{n_j}|^{p(x,t)}\,dx + \int_\Omega \frac{1}{p(x,t)} |u_{n_j}|^{p(x,t)}\,dx \\
&\ge \int_\Omega \frac{1}{p(x,t)} |u_{n_j}|^{p(x,t)}\,dx \\
&\ge \frac{1}{p^+} \int_\Omega |u_{n_j}|^{p(x,t)}\,dx \\
&\ge \frac{1}{p^+} \rho^t(u_{n_j}).
\end{aligned}$$

Consequently,

$$\frac{1}{p^+} \rho^t(u_{n_j}) \le \varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u_{n_j}) \le \delta.$$

Then

$$\rho^t(u_{n_j}) \le p^+\delta. \tag{2.4}$$

Similarly we have that

$$\rho^t(\nabla u_{n_j}) \le \frac{p^+\delta}{\beta}. \tag{2.5}$$

Using this inequality,(2.4), and Lemma 2.1, we obtain

$$\|u_{n_j}\|_{p(\cdot,t)} \le \begin{cases} (p^+\delta)^{1/p^-}, & \text{if } \|u_{n_j}\|_{p(\cdot,t)} \ge 1, \\ (p^+\delta)^{1/p^+}, & \text{if } \|u_{n_j}\|_{p(\cdot,t)} < 1. \end{cases}$$

and

$$\|\nabla u_{n_j}\|_{p(\cdot,t)} \le \begin{cases} \big(\frac{p^+\delta}{\beta}\big)^{1/p^-}, & \text{if } \|\nabla u_{n_j}\|_{p(\cdot,t)} \ge 1, \\ \big(\frac{p^+\delta}{\beta}\big)^{1/p^+}, & \text{if } \|\nabla u_{n_j}\|_{p(\cdot,t)} < 1. \end{cases}$$

Therefore, $\|u_{n_j}\|_{X_t}$ is a bounded sequence in the reflexive Banach space X_t. So, (u_{n_j}) has a subsequence (which we still denote by (u_{n_j})) such that $u_{n_j} \rightharpoonup v$ in X_t for some $v \in X_t$. As $H^* \subset X_t^*$ we have $u_{n_j} \rightharpoonup v$ in H and by the uniqueness of the weak limit $u = v \in X_t$. Considering the subdifferential $\partial\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}$ of $\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}$ we have

$$\langle \partial\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u), u_{n_j} - u \rangle_{X_t^*, X_t} \le \varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u_{n_j}) - \varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u)$$

for all $j \in \mathbb{N}$. We observe that $\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}$ is Gateaux differentiable at u. So, it follows from [6, Example 1, p. 54], that $u \in \mathcal{D}(\partial\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)})$ and $\partial\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u)$ consists of a single element, namely the Gateaux differential of $\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}$ at u, i.e., $\nabla_u\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u) = \partial\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u)$. As $u_{n_j} \rightharpoonup u$ in X_t and $\emptyset \neq \partial\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u) \in X_t^*$ we obtain

$$\langle \partial\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u), u_{n_j} - u\rangle_{X_t^*,X_t} \to 0$$

as $j \to \infty$. Therefore,

$$\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u) \le \lim_{j\to\infty} \varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u_{n_j}) = a = \liminf_{n\to\infty} \varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u_n). \qquad \square$$

Theorem 2.8. *The realization $A_H^\lambda(t)$ of $A^\lambda(t)$ in H is the subdifferential $\partial\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}$ of $\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}$.*

Proof. The realization $A_H^\lambda(t)$ of $A^\lambda(t)$ in H and $\partial\varphi_{p(\cdot)}^t$ are both maximal monotone operators in H, so it is sufficient to show that for any $u \in H$,

$$A_H^\lambda(t)u \subset \partial\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u).$$

Let $u \in \mathcal{D}(A_H^\lambda(t)) := \{u \in X_t; A^\lambda(t)u \in H\}$ and $v := A_H^\lambda(t)u = A^\lambda(t)u$. So, for all $\xi \in X_t$ we have

$$\begin{aligned}
&\langle v, \xi - u\rangle_{X_t^*,X_t}\\
&= \langle A^\lambda(t)u, \xi - u\rangle_{X_t^*,X_t}\\
&= \int_\Omega D_\lambda(t,x)|\nabla u|^{p(x,t)-2}\nabla u\cdot(\nabla\xi - \nabla u)\,dx + \int_\Omega |u|^{p(x,t)-2}u(\xi - u)\,dx\\
&= \int_\Omega D_\lambda(t,x)|\nabla u|^{p(x,t)-2}\nabla u\cdot\nabla\xi\,dx - \int_\Omega D_\lambda(t,x)|\nabla u|^{p(x,t)}\,dx\\
&\quad + \int_\Omega |u|^{p(x,t)-2}u\xi\,dx - \int_\Omega |u|^{p(x,t)}\,dx.
\end{aligned}$$

Considering $q(x,t)$ such that $\frac{1}{p(x,t)} + \frac{1}{q(x,t)} = 1$, we have

$$\begin{aligned}
&\langle v, \xi - u\rangle_{X_t^*,X_t} + \int_\Omega D_\lambda(t,x)|\nabla u|^{p(x,t)}\,dx + \int_\Omega |u|^{p(x,t)}\,dx\\
&= \int_\Omega D_\lambda(t,x)|\nabla u|^{p(x,t)-2}\nabla u\nabla\xi\,dx + \int_\Omega |u|^{p(x,t)-2}u\xi\,dx\\
&\le \int_\Omega D_\lambda(t,x)|\nabla u|^{p(x,t)-1}|\nabla\xi|dx + \int_\Omega |u|^{p(x,t)-1}|\xi|dx\\
&\le \int_\Omega \frac{D_\lambda(t,x)}{q(x,t)}|\nabla u|^{(p(x,t)-1)q(x,t)} + \frac{D_\lambda(t,x)}{p(x,t)}|\nabla\xi|^{p(x,t)}\,dx\\
&\quad + \int_\Omega \frac{1}{q(x,t)}|u|^{(p(x,t)-1)q(x,t)} + \frac{1}{p(x,t)}|\xi|^{p(x,t)}\,dx\\
&= \int_\Omega \frac{D_\lambda(t,x)}{q(x,t)}|\nabla u|^{p(x,t)}\,dx + \int_\Omega \frac{D_\lambda(t,x)}{p(x,t)}|\nabla\xi|^{p(x,t)}\,dx\\
&\quad + \int_\Omega \frac{1}{q(x,t)}|u|^{p(x,t)}\,dx + \int_\Omega \frac{1}{p(x,t)}|\xi|^{p(x,t)}\,dx.
\end{aligned}$$

Then

$$\begin{aligned}
&\langle v, \xi - u\rangle_{X_t^*, X_t} + \int_\Omega D_\lambda(t,x)\Big(1 - \frac{1}{q(x,t)}\Big)|\nabla u|^{p(x,t)}\,dx \\
&+ \int_\Omega \Big(1 - \frac{1}{q(x,t)}\Big)|u|^{p(x,t)}\,dx \\
&\le \int_\Omega \frac{D_\lambda(t,x)}{p(x,t)}|\nabla \xi|^{p(x,t)}\,dx + \int_\Omega \frac{1}{p(x,t)}|\xi|^{p(x,t)}\,dx.
\end{aligned}$$

So, we conclude that

$$\langle v, \xi - u\rangle_{X_t^*, X_t} \le \varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(\xi) - \varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u), \tag{2.6}$$

for all $\xi \in X_t$. If $\xi \in H - X_t$, then $\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(\xi) = \infty$ and consequently (2.6) holds. Therefore, $A_H^\lambda(t)(u) = v \in \partial\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(u)$. □

3. Existence of solutions

We consider the problem

$$\begin{gathered}
\frac{du}{dt}(t) + \partial\varphi^t u(t) = f(t), \quad t > \tau, \\
u(\tau) = u_\tau \in H.
\end{gathered} \tag{3.1}$$

The following definition was introduced by Yotsutani as [21, Definition 2.1].

Definition 3.1. A function $u : [\tau, T] \to H$ is called a strong solution of (3.1) on $[\tau, T]$ if the following holds:

- (i) u is in $C([\tau, T]; H)$,
- (ii) u is strongly absolutely continuous on any compact subset of (τ, T),
- (iii) $u(t)$ is in $D(\varphi^t)$ for a.e. $t \in [\tau, T]$ and satisfies (3.1) for a.e. $t \in [\tau, T]$.

For $T > \tau$ we introduce the following assumptions:

- (A7) There is a set $\tau \notin Z \subset [\tau, T]$ of zero measure such that φ^t is a lower semicontinuous proper convex function from H into $(-\infty, \infty]$ with a non-empty effective domain for each $t \in [\tau, T] - Z$;
- (A8) For any positive integer r there exist a constant $K_r > 0$, an absolutely continuous function $g_r : [\tau, T] \to \mathbb{R}$ with $g_r' \in L^\beta(\tau, T)$ and a function of bounded variation $h_r : [\tau, T] \to \mathbb{R}$ such that if $t \in [\tau, T] - Z$, $w \in \mathcal{D}(\varphi^t)$ with $|w| \le r$ and $s \in [t, T] - Z$ then there exists an element $\tilde{w} \in \mathcal{D}(\varphi^s)$ satisfying

$$|\tilde{w} - w| \le |g_r(s) - g_r(t)|(\varphi^t(w) + K_r)^\alpha, \tag{3.2}$$

$$\varphi^s(\tilde{w}) \le \varphi^t(w) + |h_r(s) - h_r(t)|(\varphi^t(w) + K_r) \tag{3.3}$$

where α is some fixed constant $0 \le \alpha \le 1$ and

$$\beta := \begin{cases} 2 & \text{if } 0 \le \alpha \le \frac{1}{2}, \\ \frac{1}{1-\alpha} & \text{if } \frac{1}{2} \le \alpha \le 1. \end{cases}$$

Theorem 3.2 ([21]). *Suppose that* (A7), (A8) *are satisfied. Then, for each* $f \in L^2(\tau, T; H)$ *and* $u_\tau \in \overline{\mathcal{D}(\varphi^\tau)}$ *equation* (3.1) *has a unique strong solution* u *on* $[\tau, T]$ *with* $u(\tau) = u_\tau$.

Using the monotonicity of the operator and the Gronwall Lemma we obtain the following

Lemma 3.3. *If $f, g \in L^2(\tau, T; H)$ and u, v are the solutions of the equations*

$$\begin{gathered}\frac{du}{dt}(t) + \partial\varphi^t u(t) = f(t),\\ u(\tau) = u_\tau \in H\end{gathered}$$

and

$$\begin{gathered}\frac{dv}{dt}(t) + \partial\varphi^t v(t) = g(t),\\ v(\tau) = v_\tau \in H,\end{gathered}$$

then for $\tau \le s \le t \le T$, we have

$$\|u(t) - v(t)\|_H \le \|u(s) - v(s)\|_H + \int_s^t \|f(r) - g(r)\|_H dr.$$

Using the monotonicity of the operator, Assumptions (A1) and (A2), Theorem 3.2, Lemma 3.3, proceeding as in [20] we obtain the following result.

Theorem 3.4. *If $B : [\tau, T] \times H \to H$ satisfies Assumptions* (A1), (A2) *and $u_\tau \in \overline{\mathcal{D}(\varphi^\tau)}$, then there exists a unique $u \in C([\tau, T]; H)$, such that*

$$\frac{du}{dt}(t) + \partial\varphi^t u(t) = B(t, u(t))$$

a.e. on $[\tau, T]$ and $u(\tau) = u_\tau$.

Let us consider now the problem with our specific operator

$$\begin{gathered}\frac{du}{dt}(t) + \partial\varphi^{D_\lambda(t,\cdot)}_{p(\cdot,t)} u(t) = f(t), \quad t > \tau,\\ u(\tau) = u_\tau \in H = \overline{\mathcal{D}(\varphi^{D_\lambda(\tau,\cdot)}_{p(\cdot,\tau)})}.\end{gathered} \tag{3.4}$$

Theorem 3.5. *For each $f \in L^2(\tau, T; H)$ and $u_\tau \in H$ equation* (3.4) *has a unique strong solution u on $[\tau, T]$ with $u(\tau) = u_\tau$.*

Proof. Taking Z as the empty set, $\varphi^{D_\lambda(t,\cdot)}_{p(\cdot,t)}$ is lower semicontinuous proper convex function for each $t \in [\tau, T]$. Consider r a positive integer, $K_r := r$ and $\alpha := \frac{1}{2}$. We define $g_r : [\tau, T] \to \mathbb{R}$ with $g_r(t) := t + r$, and $h_r(t) := r$. We have that g_r is an absolutely continuous function $g_r' = 1 \in L^2(\tau; T)$ and h_r is a bounded variation function. For all $t \in [\tau, T], w \in \mathcal{D}(\varphi^{D_\lambda(t,\cdot)}_{p(\cdot,t)}) = X_t$ with $\|w\| \le r$ and $s \in [t, T]$. Consider the element $\tilde{w} := w \in X_s = \mathcal{D}(\varphi^{D_\lambda(s,\cdot)}_{p(\cdot,s)})$. We will check that $\tilde{w}$ satisfies (3.2) and (3.3). Note that

$$\int_\Omega \frac{D_\lambda(t,x)}{p(x,t)} |\nabla w|^{p(x,t)}\, dx + \int_\Omega \frac{1}{p(x,t)} |w|^{p(x,t)}\, dx \ge 0$$

Thus,

$$\begin{aligned}|\tilde{w} - w| = 0 &\le |s - t| \Big(\int_\Omega \frac{D_\lambda(t,x)}{p(x,t)} |\nabla w|^{p(x,t)}\, dx + \int_\Omega \frac{1}{p(x,t)} |w|^{p(x,t)}\, dx + r\Big)^{1/2}\\ &= |g_r(s) - g_r(t)| (\varphi^{D_\lambda(t,\cdot)}_{p(\cdot,t)}(w) + K_r)^\alpha.\end{aligned}$$

Now, note that

$$\begin{aligned}
\varphi_{p(\cdot,s)}^{D_\lambda(s,\cdot)}(\tilde{w}) &= \int_\Omega \frac{D_\lambda(t,x)}{p(x,t)}|\nabla \tilde{w}|^{p(x,t)}\,dx + \int_\Omega \frac{1}{p(x,t)}|\tilde{w}|^{p(x,t)}\,dx \\
&= \int_\Omega \frac{D_\lambda(t,x)}{p(x,t)}|\nabla w|^{p(x,t)}\,dx + \int_\Omega \frac{1}{p(x,t)}|w|^{p(x,t)}\,dx \\
&\le \int_\Omega \frac{D_\lambda(t,x)}{p(x,t)}|\nabla w|^{p(x,t)}\,dx + \int_\Omega \frac{1}{p(x,t)}|w|^{p(x,t)}\,dx \\
&= \varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(w).
\end{aligned}$$

Thus,

$$\varphi_{p(\cdot,s)}^{D_\lambda(s,\cdot)}(\tilde{w}) \le \varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(w) + |h_r(s)-h_r(t)|(\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}(w) + K_r).$$

Then from Theorem 3.2 we obtain the existence of a global solution to (3.4). □

Theorem 3.6. *For each $\lambda \in [0,\infty)$, problem* (1.1) *has a unique global strong solution u_λ whenever $u_{\tau\lambda} \in H$.*

The above theorem follows from Theorem 3.4 with $\partial\varphi_{p(\cdot,t)}^{D_\lambda(t,\cdot)}$.

4. Existence of pullback attractors

We begin this section recalling some definitions and results on $\mathcal{D}$-pullback attractors theory.

Definition 4.1 ([13])**.** Let $\{Y_t\}_{t\in\mathbb{R}}$ be a family of nonempty metric spaces. A family of operators $\{U(t;\tau)\}_{t\ge\tau}$, with $U(t;\tau): Y_\tau \to Y_t$, which satisfies

(i) $U(\tau;\tau) = I_\tau$ (Identity operator), for all $\tau \in \mathbb{R}$;

(ii) $U(t;\tau) = U(t;s)U(s;\tau)$, for all $\tau \le s \le t$, is called an evolution process in $\{Y_t\}_{t\in\mathbb{R}}$. Furthermore, if for any sequences $x_\tau \to x$ in Y_τ and $y_t = U(t;\tau)x_\tau \to y$ in Y_t we have $y = U(t;\tau)x$, then $U(t;\tau)$ is called closed.

Definition 4.2 ([13])**.** Let $\{Y_t\}_{t\in\mathbb{R}}$ be a family of metric spaces. The universal $\mathcal{D}$ with respect to the family of spaces $\{Y_t\}_{t\in\mathbb{R}}$ is defined as

$$\mathcal{D} = \{\mathbb{D} = \{D(t)\}_{t\in\mathbb{R}}; D(t) \subset Y_t \text{ is nonempty, } \forall t \in \mathbb{R}\}.$$

In particular, the universal $\mathcal{D}$ is called inclusion closed whenever for any $\mathbb{D} \in \mathcal{D}$ and $\mathcal{C} = \{C(t)\}_{t\in\mathbb{R}}$ such that $C(t) \subset D(t)$ for all $t \in \mathbb{R}$, then $\mathcal{C} \in \mathcal{D}$.

Definition 4.3 ([13])**.** Let $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}}$ be a family of time-dependent nonempty sets. If B satisfies: for any $t \in \mathbb{R}$ and $\mathbb{D} \in \mathcal{D}$, there exists $\tau(D;t) \le t$, such that $U(t;\tau)D(\tau) \subset B(t)$ holds when $\tau \le \tau(D;t)$, then $\mathcal{B}$ is called the pullback $\mathcal{D}$-absorbing family of the process $U(t;\tau): Y_\tau \to Y_t$.

In addition, $\mathcal{B}$ is called the uniformly pullback $\mathcal{D}$-absorbing, if for any $\mathbb{D} \in \mathcal{D}$, there exists the positive constant $e(\mathcal{D})$ which only depends on $\mathcal{D}$, such that $U(t;\tau)D(\tau) \subset B(t)$ for any $\tau \le t - e(\mathcal{D})$ and $t \in \mathbb{R}$.

Definition 4.4 ([13])**.** For any given time-dependent family $\mathbb{D} = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$, the process $U(t;\tau)$ is called pullback $\mathbb{D}$-asymptotically compact in $\{Y_t\}_{t\in\mathbb{R}}$, if for any $t \in \mathbb{R}$, any sequences $\{\tau_n\} \subset (-\infty,t]$ which satisfies $\tau_n \to -\infty$ as $n \to \infty$ and any sequences $y_n \in D(\tau_n)$, such that the sequence $\{U(t;\tau_n)y_n\}$ is relatively compact in Y_t.

Moreover, if for any $\mathbb{D} \in \mathcal{D}$, the process $U(t;\tau)$ is pullback $\mathbb{D}$-asymptotically compact in $\{Y_t\}_{t\in\mathbb{R}}$, then we may call that this process is pullback $\mathcal{D}$-asymptotically compact in $\{Y_t\}_{t\in\mathbb{R}}$.

Lemma 4.5 ([13])**.** *Let $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$ be the pullback $\mathcal{D}$-absorbing family of $U(t;\tau)$. The process $U(t;\tau)$ is pullback $\mathcal{D}$-asymptotically compact in $\{Y_t\}_{t\in\mathbb{R}}$, if $B(t)$ is compact in Y_t for any $t \in \mathbb{R}$.*

Definition 4.6 ([16])**.** For any time-dependent family $\mathbb{D} = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$, the pullback ω-limit set with respect to the process $U(t;\tau)$ in the family of spaces $\{Y_t\}_{t\in\mathbb{R}}$ is defined as

$$\omega(\mathbb{D};t) := \cap_{s\le t}\overline{\cup_{\tau\le s}U(t;\tau)D(\tau)}^{Y_t}.$$

Definition 4.7 ([16, 13])**.** A family of sets $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ is said to be the pullback $\mathcal{D}$-attractor for the process $U(t;\tau) : Y_\tau \to Y_t$, if

(i) For any $t \in \mathbb{R}$, $A(t)$ is a nonempty compact subset of Y_t;
(ii) $\mathcal{A}$ pullback attracts every $\mathbb{D} \in \mathcal{D}$,

$$\lim_{\tau\to-\infty} \operatorname{dist}_{Y_t}(U(t;\tau)D(\tau), A(t)) = 0,$$

for all $t \in \mathbb{R}$; and
(iii) $\mathcal{A}$ is invariant, i.e., $U(t;\tau)A(\tau) = A(t)$ for all $t \ge \tau$.

Moreover, if for any pullback $\mathcal{D}$-attracting family $\mathcal{C} = \{C(t)\}_{t\in\mathbb{R}}$ composed of nonempty closed set family, there is $A(t) \subset C(t)$ for any $t \in \mathbb{R}$, then it is said that the pullback $\mathcal{D}$-attractor has the property of minimality.

The following theorem will be used to obtain the existence of the pullback $\mathcal{D}$-attractors.

Theorem 4.8 ([15])**.** *Let $U(t;\tau) : Y_\tau \to Y_t$ be the closed process in the family of time-dependent metric spaces $\{Y_t\}_{t\in\mathbb{R}}$, $\mathcal{D}$ is the universal (with respect to $\{Y_t\}_{t\in\mathbb{R}}$). If the process $U(t;\tau)$ possesses the pullback $\mathcal{D}$-absorbing families $\mathcal{B}_0 = \{B_0(t)\}_{t\in\mathbb{R}}$ and it is pullback $\mathcal{B}_0$-asymptotically compact. Then the family of sets $\mathcal{A}_\mathcal{D} = \{A(t)\}_{t\in\mathbb{R}}$ is the minimal pullback $\mathcal{D}$-attractor of the process $U(t;\tau)$, where the sections $A(t)$ are given as follows*

$$A(t) = \overline{\cup_{\mathbb{D}\in\mathcal{D}}\omega(\mathbb{D}, t)}^{Y_t}, \ \forall\, t \in \mathbb{R}.$$

If the pullback $\mathcal{D}$-absorbing family $\mathcal{B}_0$ is in $\mathcal{D}$, then

$$A(t) = \omega(\mathcal{B}_0; t) \subset \overline{\mathcal{B}_0(t)}^{Y_t}, \quad \forall t \in \mathbb{R}.$$

Moreover, if $B_0(t)$ is a closed subset in Y_t for any $t \in \mathbb{R}$, and the universal $\mathcal{D}$ is inclusion closed, then the pullback $\mathcal{D}$-attractor $\mathcal{A}_\mathcal{D}$ belongs to $\mathcal{D}$.

Definition 4.9. A global (or entire) solution of a process $U(\cdot,\cdot)$ is a function $\xi : \mathbb{R} \to \{Y_t\}_{t\in\mathbb{R}}$ such that for each $\ell \in \mathbb{R}$ $\xi(\ell) \in Y_\ell$ and $U(t,s)\xi(s) = \xi(t)$ for all $t \ge s$.

Using the invariance of the pullback attractor and proceeding as in [7, Lemma 1.10, page 9] we obtain the following result.

Theorem 4.10. *The pullback $\mathcal{D}$-attractor in Theorem 4.8 consists of a collection of global solutions.*

Now consider problem (1.1) but with the initial datum $u_{0\lambda} \in Y_\tau = L^{p(\cdot,\tau)}(\Omega) \subset H$,

$$\begin{gathered}\frac{\partial u_\lambda}{\partial t}(t) - \operatorname{div}\Big(D_\lambda(t,x)|\nabla u_\lambda(t)|^{p(x,t)-2}\nabla u_\lambda(t)\Big) + |u_\lambda(t)|^{p(x,t)-2}u_\lambda(t) \\ = B(t,u_\lambda(t)) \\ u_\lambda(\tau) = u_{\tau\lambda} \in Y_\tau.\end{gathered} \tag{4.1}$$

Consider $U_\lambda(t;\tau) : Y_\tau \to Y_t$, where $U_\lambda(t;\tau) = u_\lambda(t)$, the global solution of (4.1). Moreover consider the universal

$$\mathcal{D} = \{\mathbb{D} = \{D(t)\}_{t\in\mathbb{R}}; D(t) \subset Y_t \text{ is nonempty}, \ \forall t \in \mathbb{R}\}.$$

Proceeding analogously as in [14] we obtain the following two uniform estimates.

Theorem 4.11. *Let u_λ be a solution of* (4.1). *Then there exist a constant T_1 and a nondecreasing function $B_1 : \mathbb{R} \to \mathbb{R}$ such that*

$$\|u_\lambda(t)\|_H \le B_1(t), \quad \forall\, t \ge T_1 + \tau$$

and $\lambda \in [0,\infty)$.

Theorem 4.12. *Let $u_\lambda(\cdot) \in C([\tau,\infty); H)$ be the global solution of* (4.1). *Then there exist a constant $T_2 > 0$ and a nondecreasing function $B_2 : \mathbb{R} \to \mathbb{R}$ such that*

$$\|u_\lambda(t)\|_{X_t} \le B_2(t), \quad \forall\, t \ge T_2 + \tau,\ \lambda \in [0,\infty).$$

Theorem 4.13. *The evolution process $\{U_\lambda(t;\tau)\}_{t\ge\tau}$ associated with problem* (4.1) *in $\{Y_t\}_{t\in\mathbb{R}}$ has a pullback $\mathcal{D}$-attractor $\{\mathcal{A}^\lambda(t) : t \in \mathbb{R}\} =: \mathcal{A}^\lambda \in \mathcal{D}$.*

Proof. If $x_\tau \to x$ in Y_τ and $y_t := U_\lambda(t;\tau)x_\tau \to y$ in Y_t we have that $x_\tau \to x$ in H. Using the monotonicity of the main operator and that B is Lipschitz, it is easy to show that $y_t \to U_\lambda(t;\tau)x$ in H. Since $y_t \to y$ in Y_t implies that $y_t \to y$ in H, then by the uniqueness of the limit we conclude that $y = U_\lambda(t;\tau)x$. So, the process is closed.

Now consider $\mathcal{B}_0 = \{B_0(t)\}_{t\in\mathbb{R}}$, $B_0(t) = \overline{B_{X_t}(0, B_2(t))}^{Y_t}$. Observe that the sets $B_0(t)$ are compact in Y_t. Theorem 4.12 shows that $\mathcal{B}_0$ is a pullback $\mathcal{D}$-absorbing family of the process $U_\lambda(t;\tau) : Y_\tau \to Y_t$. Then, by Lemma 4.5, $U_\lambda(t;\tau)$ is pullback $\mathcal{B}_0$-asymptotically compact in $\{Y_t\}_{t\in\mathbb{R}}$. Thus, the existence of the pullback $\mathcal{D}$-attractor follows from Theorem 4.8. $\square$

Remark 4.14. (i) By Theorem 4.10 the pullback $\mathcal{D}$-attractor in the previous theorem consists of a collection of global solutions and by Theorem 4.8, the sections of the pullback attractor are characterized as

$$\mathcal{A}^\lambda(t) = \omega_\lambda(\mathcal{B}_0; t) \subset \overline{\mathcal{B}_0(t)}^{Y_t},\ \forall\, t \in \mathbb{R}.$$

(ii) Using the invariance of the pullback attractors and Theorem 4.12, we have that $\mathcal{A}^\lambda(t) \subset X_t$.

As a consequence of Theorem 4.12, we have the following corollary.

Corollary 4.15. $\overline{\cup_{\lambda\in[0,\infty)}\mathcal{A}^\lambda(\tau)}^{Y_t}$ *is a compact subset of Y_t and* $\overline{\cup_{\lambda\in[0,\infty)}\mathcal{A}^\lambda(\tau)}^{H}$ *is a compact subset of H for each $\tau \in \mathbb{R}$.*

5. Robustness of the pullback attractors

Consider a family of functions $D_\lambda \in L^\infty([\tau,T]\times\Omega)$ with $0<\beta\le D_\lambda(t,x)\le M$ in $[\tau,T]\times\Omega$, $\lambda\in[0,\infty)$, $D_\lambda\to D_{\lambda_1}$ in $L^\infty([\tau,T]\times\Omega)$ as $\lambda\to\lambda_1$.

Our objective in this section is to prove that the family of pullback attractors behaves as upper semicontinuously with respect to positive finite diffusion parameters. Proceeding as in the proof of [14, Theorem 4.1] we obtain the following.

Theorem 5.1. *Let $\{U_\lambda(t,\tau): t\ge\tau\}$ be the evolution process generated by the problem (4.1). If $\{u_{\tau\lambda}:\lambda\in[0,\infty)\}$ is a bounded set in X_τ and $u_{\tau\lambda}\to u_{\tau\lambda_1}$ in H as $\lambda\to\lambda_1$, then $U_\lambda(t,\tau)u_{\tau\lambda}\to U_{\lambda_1}(t,\tau)u_{\tau\lambda_1}$ in H as $\lambda\to\lambda_1$, uniformly for t in compact subsets of $\mathbb{R}$.*

Theorem 5.2. *The family of pullback attractors $\{\mathcal{A}^\lambda(t): t\in\mathbb{R}\}$, $\lambda\in[0,\infty)$ is upper semicontinuous at λ_1 in the topology of H.*

Proof. We will prove that for each $t\in\mathbb{R}$,
$$\operatorname{dist}_H\big(\mathcal{A}^\lambda(t),\mathcal{A}^{\lambda_1}(t)\big)\to 0 \quad \text{as } \lambda\to\lambda_1.$$
For $t\in\mathbb{R}$ and $\epsilon>0$, let $\tau\in\mathbb{R}$ be such that
$$\operatorname{dist}_{Y_t}\big(U_{\lambda_1}(t,\tau)B(\tau),\mathcal{A}^{\lambda_1}(t)\big)<\frac{\epsilon}{3},$$
where $\cup_{\lambda\in[0,\infty)}\mathcal{A}^\lambda(\tau)\subset B(\tau)$ and $B(\tau)$ is a nonempty set in $X_\tau\subset Y_\tau$ (see Theorem 4.12). Once $Y_t\subset H$, we have
$$\operatorname{dist}_{H}\big(U_{\lambda_1}(t,\tau)B(\tau),\mathcal{A}^{\lambda_1}(t)\big)<\frac{\epsilon}{3}.$$
Using the invariance of the pullback attractors, Theorems 4.12 and 5.1, there exists $\delta=\delta(\epsilon)>0$ such that
$$\sup_{\psi_\lambda\in\mathcal{A}^\lambda(\tau)}\|U_\lambda(t,\tau)\psi_\lambda-U_{\lambda_1}(t,\tau)\psi_\lambda\|_H<\frac{\epsilon}{3}$$
for all $|\lambda-\lambda_1|<\delta$. Then
$$\begin{aligned}
&\operatorname{dist}_H\big(\mathcal{A}^\lambda(t),\mathcal{A}^{\lambda_1}(t)\big)\\
&=\operatorname{dist}_H\big(U_\lambda(t,\tau)\mathcal{A}^\lambda(\tau),\mathcal{A}^{\lambda_1}(t)\big)\\
&=\sup_{\psi_\lambda\in\mathcal{A}^\lambda(\tau)}\operatorname{dist}_H\big(U_\lambda(t,\tau)\psi_\lambda,\mathcal{A}^{\lambda_1}(t)\big)\\
&\le\sup_{\psi_\lambda\in\mathcal{A}^\lambda(\tau)}\big\{\operatorname{dist}_H\left(U_\lambda(t,\tau)\psi_\lambda,U_{\lambda_1}(t,\tau)\psi_\lambda\right)+\operatorname{dist}_H\big(U_{\lambda_1}(t,\tau)\psi_\lambda,\mathcal{A}^{\lambda_1}(t)\big)\big\}\\
&\le\frac{\epsilon}{3}+\frac{\epsilon}{3}<\epsilon,
\end{aligned}$$
for all $|\lambda-\lambda_1|<\delta$, showing the upper semicontinuity as desired. □

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