

**EXISTENCE AND UNIQUENESS RESULTS FOR
FOURTH-ORDER FOUR-POINT BVP ARISING IN BRIDGE
DESIGN IN THE PRESENCE OF REVERSE ORDERED UPPER
AND LOWER SOLUTIONS**

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ABSTRACT. In this article, we establish the existence of solutions for a fourth-order four-point non-linear boundary value problem (BVP) which arises in bridge design,

$$\begin{aligned} -y^{(4)}(s) - \lambda y''(s) &= \mathcal{F}(s, y(s)), \quad s \in (0, 1), \\ y(0) &= 0, \quad y(1) = \delta_1 y(\eta_1) + \delta_2 y(\eta_2), \\ y''(0) &= 0, \quad y''(1) = \delta_1 y''(\eta_1) + \delta_2 y''(\eta_2), \end{aligned}$$

where $\mathcal{F} \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, $\delta_1, \delta_2 > 0$, $0 < \eta_1 \leq \eta_2 < 1$, $\lambda = \zeta_1 + \zeta_2$, where ζ_1 and ζ_2 are the real constants. We have explored all gathered $0 < \zeta_1 < \zeta_2$, $\zeta_1 < 0 < \zeta_2$, and $\zeta_1 < \zeta_2 < 0$. We extend the monotone iterative technique and establish the existence results with reverse ordered upper and lower solutions to fourth-order four-point non-linear BVPs.

1. INTRODUCTION

Higher order boundary value problems (BVP) play a vital role in studying various branches of science and engineering, e.g., suspension bridge [18, 13]. The suspension bridge is identified as a beam of length l_b with fixed ends which are supported in equilibrium position and is given as the solution of the steady state equation

$$EIy^{(4)} + \zeta y^+ = W(s), \tag{1.1}$$

$$y(0) = y(l_b) = y''(0) = y''(l_b) = 0, \tag{1.2}$$

where, E is Youngs' modulus, I is moment of inertia, ζ is spring constant, $W(s)$ is weight per unit length, and $y(s)$ denotes downward deflection. y^+ denotes the y , if y is positive, and zero if y is negative.

There have been extensive studies on fourth-order BVP via different techniques such as fixed point theorem [2, 6, 14, 19, 21], upper and lower solutions (UL solutions) method [12, 29], monotone iterative (MI) method [20, 22], etc. The fact of sign-constancy of Green's function for these class of BVPs can be used in the frame of Azbelev W-transform [1, 4, 5] for equation of the forth order (1.1). The

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functional differential equation of second order can be obtained and then analysis of positivity of Green's function of obtained functional differential equation can be achieved. For an alternative approach one can refer to [3].

The idea and realizations of the monotone iterative technique appeared first in the classical work by Chaplygin [11]. Adventures of monotone iterative technique were explained then by Luzin [24]. For the existence of solutions, method of UL solutions is extensively used to develop MI technique on the second order BVP [16, 17, 30, 33, 35, 36, 37, 38, 39]. There are also several research articles available on higher order two-point BVP with MI technique [7, 8, 26, 32, 34, 42]. To the best of our knowledge only few works are there on fourth-order four-point BVP with monotone iterative technique [12, 25, 31, 43]. By using monotone iterative technique Chen et al. [12] studied the existence of solution of fourth-order four-point BVP with derivative independent non-linear function. The fourth-ordered four-point BVP with derivative dependent non-linear function is also studied in [25, 31, 43].

In this article, we develop MI technique to obtain existence of solution for the four-point non-linear BVP

$$\begin{aligned} Ly &\equiv \left(-\frac{d^{(4)}}{ds^{(4)}} - \lambda \frac{d^{(2)}}{ds^{(2)}} \right) y = \mathcal{F}(s, y(s)), \quad 0 < s < 1, \\ B_0(y) &\equiv (y(0), y''(0)) = (0, 0), \\ B_1(y) &\equiv y(1) - \delta_1 y(\eta_1) - \delta_2 y(\eta_2) = 0, \\ B_2(y) &\equiv y''(1) - \delta_1 y''(\eta_1) - \delta_2 y''(\eta_2) = 0, \end{aligned} \tag{1.3}$$

where $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, $\delta_1, \delta_2 > 0$, and $0 < \eta_1 \leq \eta_2 < 1$ are constants. The parameter $\lambda = \zeta_1 + \zeta_2$, where ζ_1, ζ_2 are any real constants. A function y is said to be a solution of (1.3), if $y \in C^4[0, 1]$, and satisfies (1.3) for all $s \in [0, 1]$. The above ODE represents the model of the stationary state of the deflection of an elastic beam [15, 10, 23].

To establish the existence of solution we use method of UL solutions and develop MI technique in reverse order case. For this purpose, we obtain Green's function, solution of corresponding linear problem and its sign. Finding Green's function and its sign for fourth-order four-point BVP is a very difficult task. To accomplish this, we introduce a linear operator and make this task less difficult. Since the existence of UL solutions ensure the existence of a solution between them. However, even for simple boundary conditions in higher order BVP, the usage of UL solutions is significantly dependent on the sign of the corresponding linear operators [9].

Vrabel [40] considered the following fourth-order two-point BVP

$$y^{(4)}(s) + \lambda y^{(2)}(s) + \zeta y(s) = \mathcal{F}(s, y(s)), \quad 0 < s < 1, \tag{1.4}$$

$$y(0) = y''(0) = y(1) = y''(1) = 0, \tag{1.5}$$

where $\lambda = \zeta_1 + \zeta_2$ and $\zeta = \zeta_1 \zeta_2$; $\zeta_1, \zeta_2 \in \mathbb{R}$ such that $\zeta_1 < \zeta_2 < 0$ and \mathcal{F} is a continuous and monotone decreasing with respect to y . To establish existence results he formulated method of UL solution. Ma et al. [28] extended this theory for the problem (1.4)-(1.5), where $\zeta_1 < 0 < \zeta_2 < \pi^2$. They established method of UL solution by introducing linear operator. Ma et al. [27] further generalized method of UL solution for fourth-order BVP (1.4)-(1.5), where $\zeta_1, \zeta_2 \in (0, \pi^2)$. In the above articles Vrabel [40] and Ma et al. [27, 28] have only focused on positivity of Green's

function. Wang et al. [41] considered BVP (1.4) with the boundary conditions

$$y'(0) = y'''(0) = y'(1) = y'''(1) = 0, \quad (1.6)$$

where the non-linear function $\mathcal{F}(s, y(s)) = \beta g(s, y(s))$, with $\beta > 0$. They discussed the existence of positive solutions for more general condition than imposed by Vrabel [40] and Ma et al. [28, 27]. Here, Wang et al. [41] established their result in three gathered, where $0 < \zeta_1 < \zeta_2$, $\zeta_1 < 0 < \zeta_2$ and $\zeta_1 < \zeta_2 < 0$.

Inspired by above articles we study the existence of solution of fourth-order four-point non-linear BVP (1.3). This article generalizes and improves the result of Vrabel [40], Ma et al. [27, 28], and Wang et al. [41] to a class of four-point BVP for all three gathered, when $0 < \zeta_1 < \zeta_2$, $\zeta_1 < 0 < \zeta_2$ and $\zeta_1 < \zeta_2 < 0$.

This article, is divided into 6 sections. In section 2, we develop approximating schemes for corresponding linear BVP and give some important assumptions which we need through out this article. In sections 3, 4, and 5, we study Green's function, anti-maximum principle and existence theorem for three gathered $0 < \zeta_1 < \zeta_2$, $\zeta_1 < 0 < \zeta_2$, and $\zeta_1 < \zeta_2 < 0$, respectively. In these sections, we also validate our technique by constructing some numerical illustrations. Section 6, is devoted to conclusions.

2. PRELIMINARIES

In this section, we develop iterative scheme and impose conditions which we need through out this article. To study the existence of solution we first linearize BVP (1.3) and consider corresponding nonhomogeneous linear BVP

$$(L - \zeta)y(s) = h(s), \quad 0 < s < 1, \quad (2.1)$$

$$B_0(y) = (0, 0), \quad B_1(y) = c_1, \quad B_2(y) = c_2, \quad (2.2)$$

where B_0, B_1, B_2 are B.C. defined in (1.3), $\zeta = \zeta_1\zeta_2$, $h(s) = \mathcal{F}(s, y(s)) - \zeta y(s) \in C[0, 1]$, and c_1, c_2 are any real constants.

Now we define two iterative sequences with initial guesses of $l_0(s)$ and $u_0(s)$ as follows

$$\begin{aligned} (L - \zeta)l_{n+1}(s) &= \mathcal{F}(s, l_n(s)) - \zeta l_n(s), \quad 0 < s < 1, \quad n \in \mathbb{N}, \\ B_0(l_{n+1}, l''_{n+1}) &= (0, 0), \quad B_1(l_{n+1}) = 0, \quad B_2(l_{n+1}) = 0. \end{aligned} \quad (2.3)$$

$$\begin{aligned} (L - \zeta)u_{n+1}(s) &= \mathcal{F}(s, u_n(s)) - \zeta u_n(s), \quad 0 < s < 1, \quad n \in \mathbb{N}, \\ B_0(u_{n+1}, l''_{n+1}) &= (0, 0), \quad B_1(u_{n+1}) = 0, \quad B_2(u_{n+1}) = 0. \end{aligned} \quad (2.4)$$

Assumptions: We assume the following conditions on non-linear term \mathcal{F}

- (A1) Let $D_{\mathcal{F}} := \{(s, y) \in [0, 1] \times \mathbb{R} : u_0 \leq y \leq l_0\}$, there exists a pair of UL solutions $l_0(s)$ and $u_0(s)$ such that $u_0(s) \leq l_0(s)$.
- (A2) The function $\mathcal{F} : D_{\mathcal{F}} \rightarrow \mathbb{R}$ is continuous on $D_{\mathcal{F}}$.
- (A3) There exists a constant $M \geq 0$ in the region $D_{\mathcal{F}}$ such that for all $(s, y_i) \in D_{\mathcal{F}}$, where $i = 1, 2$,

$$\text{if } \zeta > 0, \quad y_1 \leq y_2 \Rightarrow \mathcal{F}(s, y_2) - \mathcal{F}(s, y_1) \leq M(y_2 - y_1). \quad (2.5)$$

$$\text{if } \zeta < 0, \quad y_1 \leq y_2 \Rightarrow \mathcal{F}(s, y_2) - \mathcal{F}(s, y_1) \leq -M(y_2 - y_1). \quad (2.6)$$

3. REVERSE ORDER MI TECHNIQUE WHEN $0 < \zeta_1 < \zeta_2$

In this section, first we obtain Green's function and anti-maximum principle (AMP) for corresponding linear BVP (2.1)-(2.2). Based on AMP we discuss the qualitative properties of linear BVP (2.1)-(2.2) and define UL solutions for non-linear BVP (1.3) when $0 < \zeta_1 < \zeta_2$. Consequently, we conclude the existence results for the non-linear BVP (1.3).

3.1. Linear BVP. Let $E = C[0, 1]$ be the Banach space of continuous function defined on $[0, 1]$, with its usual normal $\|\cdot\|$. Denote

$$\zeta_1 = r^2 \quad \text{and} \quad \zeta_2 = m^2, \quad (3.1)$$

with some $r, m > 0$. We substitute the above values of (3.1) in the linear BVP (2.1)-(2.2) and then to obtain Green's function we consider a corresponding homogeneous linear BVP

$$y^{(4)}(s) + (m^2 + r^2)y^{(2)}(s) + r^2m^2y(s) = 0, \quad 0 < s < 1, \quad (3.2)$$

$$B_0(y) = (0, 0), \quad B_1(y) = 0, \quad B_2(y) = 0. \quad (3.3)$$

We define $L^* : D(L^*) \rightarrow E$, a linear operator such that

$$L^*y := y^{(4)}(s) + (m^2 + r^2)y^{(2)}(s) + r^2m^2y(s), \quad y \in D(L^*), \quad (3.4)$$

with domain

$$D(L^*) := \{y \in C^4[0, 1] : B_0(y) = (0, 0), B_1(y) = 0, B_2(y) = 0\}. \quad (3.5)$$

To construct $G(s, x)$ for the BVP (3.2)-(3.3), let us define two linear operators L_1 and L_2 as follows

$$L_1y := y''(t) + r^2y(t), \quad y \in D(L_1), \quad (3.6)$$

$$L_2y := y''(t) + m^2y(t), \quad y \in D(L_2), \quad (3.7)$$

where

$$D(L_1) := \{y \in C^2[0, 1] : y(0) = 0, B_1(y) = 0\}, \quad (3.8)$$

$$D(L_2) := \{y \in C^2[0, 1] : y(0) = 0, B_1(y) = 0\}. \quad (3.9)$$

Lemma 3.1. *Assume that $r, m \in (0, \pi/2)$ and that the following assumption is fulfilled*

$$D_l = \delta_1 \sin(\eta_1 l) + \delta_2 \sin(\eta_2 l) - \sin(l) \neq 0, \quad \text{where } l = m \text{ or } r. \quad (3.10)$$

Then $G(s, x) : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ of linear fourth-order BVP (3.2)-(3.3) is given by

$$G(s, x) = \int_0^1 G_m(s, t)G_r(t, x)dt, \quad (s, x) \in [0, 1] \times [0, 1], \quad (3.11)$$

where $G_r(t, x)$ and $G_m(t, x)$ are Green's function of (3.6)-(3.8) and (3.7)-(3.9), respectively, given as

$$G_l(t, x) = \frac{1}{lD_l} \begin{cases} \sin(lt)(\delta_1 \sin(l(x - \eta_1)) + \delta_2 \sin(l(x - \eta_2)) + \sin(l - lx)) \\ \quad \text{if } 0 \leq t \leq x \leq \eta_1; \\ \sin(lx)(\delta_1 \sin(l(t - \eta_1)) + \delta_2 \sin(l(t - \eta_2)) + \sin l(1 - t)) \\ \quad \text{if } 0 \leq x \leq t \leq \eta_1; \\ \sin(lt)(\delta_2 \sin(l(x - \eta_2)) + \sin l(1 - x)) \quad \text{if } \eta_1 \leq t \leq x \leq \eta_2; \\ -\delta_1 \sin(\eta_1 l) \sin(l(x - t)) + \sin(lx)(\delta_2 \sin(l(t - \eta_2)) + \sin l(1 - t)) \\ \quad \text{if } \eta_1 \leq x \leq t \leq \eta_2; \\ \sin l(1 - x) \sin(lt) \quad \text{if } \eta_2 \leq t \leq x \leq 1; \\ \sin(lt)(\sin l(1 - x) + D_l \cos(lx)) - D_l \sin(lx) \cos(lt) \\ \quad \text{if } \eta_2 \leq x \leq t \leq 1. \end{cases} \quad (3.12)$$

Proof. Since L^* is a linear operator, it can be easily proved that $L^*y = L_2(L_1)y$ and hence the Green's function of (3.2)-(3.3) is

$$G(s, x) = \int_0^1 G_m(s, t)G_r(t, x)dt, \quad (s, x) \in [0, 1] \times [0, 1]. \quad (3.13)$$

For brevity, we skip the proof of (3.13). For details of the proof we refer the articles of Ma et al. [27] and to Wang et al. [41].

In equation (3.13), $G_r(t, x)$ and $G_m(t, x)$ are Green's functions of (3.6)-(3.8) and (3.7)-(3.9), respectively, given as

$$G_l(t, x) = \begin{cases} a_1 \sin(lt) + a_2 \cos(lt), & 0 \leq t \leq x \leq \eta_1, \\ a_3 \sin(lt) + a_4 \cos(lt), & 0 \leq x \leq t \leq \eta_1, \\ a_5 \sin(lt) + a_6 \cos(lt), & \eta_1 \leq t \leq x \leq \eta_2, \\ a_7 \sin(lt) + a_8 \cos(lt), & \eta_1 \leq x \leq t \leq \eta_2, \\ a_9 \sin(lt) + a_{10} \cos(lt), & \eta_2 \leq t \leq x \leq 1, \\ a_{11} \sin(lt) + a_{12} \cos(lt), & \eta_2 \leq x \leq t \leq 1, \end{cases} \quad (3.14)$$

where

$$\begin{aligned} a_1 &= a_5 = a_9 = 0, & a_3 &= a_7 = a_{11} = -\frac{\sin(lx)}{l}, \\ a_2 &= \frac{1}{lD_l}(\delta_1 \sin(l(x - \eta_1)) + \delta_2 \sin(l(x - \eta_2)) + \sin l(1 - x)), \\ a_4 &= \frac{-1}{lD_l}(\sin(lx)(-\delta_1 \cos(\eta_1 l) - \delta_2 \cos(\eta_2 l) + \cos(l))), \\ a_6 &= \frac{1}{lD_l}(\delta_2 \sin(l(x - \eta_2)) + \sin l(1 - x)), \\ a_8 &= \frac{1}{lD_l}(\delta_1 \sin(\eta_1 l) \cos(lx) + \sin(lx)(\delta_2 \cos(\eta_2 l) - \cos(l))), \\ a_{10} &= \frac{1}{lD_l}(\sin l(1 - x)), \end{aligned}$$

$$a_{12} = \frac{1}{lD_l}(\sin l(1-x) + D_l \cos(lx)).$$

Substituting the above values of a_i , $i = 1, 2, \dots, 12$ in (3.14), we obtain the Green's function $G_l(t, x)$ of (3.6) and (3.8) and (3.7) and (3.9) that is given in equation (3.12). \square

Lemma 3.2. *Assume that $r, m \in (0, \frac{\pi}{2})$. Then $G(s, x)$ of fourth-order BVP (3.2)-(3.3) given by expression (3.11) is nonnegative on $[0, 1] \times [0, 1]$ if and only if $0 < \delta_1 + \delta_2 < 1$.*

Proof. From equation (3.11) we observe that sign of $G(s, x)$ of (3.2)-(3.3) follows from sign of Green's function $G_m(s, t)$ and $G_r(t, x)$ given by equation (3.12).

Given that $\delta_1 + \delta_2 < 1$, applying properties of $\sin x$, we have

$$\begin{aligned} -(\delta_1 + \delta_2) \sin l(x - \eta_2) &\leq \sin l(1 - x), \\ \sin l(1 - x) + \delta_2 \sin l(x - \eta_2) &\geq 0, \quad \text{and} \quad D_l < 0. \end{aligned}$$

Using the above inequalities in (3.12), we obtain that

$$G_l(t, x) \leq 0, \quad (t, x) \in [0, 1] \times [0, 1].$$

Hence from (3.11), we have $G(s, x) \geq 0$ on $[0, 1] \times [0, 1]$. In a similar fashion we can prove that the converse is also true. \square

Lemma 3.3. *Assume that $r, m \in (0, \pi/2)$ such that $m^2 - r^2 \neq 0$ and $D_l \neq 0$. Then the solution $y \in C^4[0, 1]$ of linear fourth-order non homogeneous BVP (2.1)-(2.2) is*

$$y(s) = \frac{N_y(s)}{(m^2 - r^2)D_m D_r} - \int_0^1 G(s, x)h(x)dx, \quad (3.15)$$

where $N_y(s) = D_r(c_2 + c_1 r^2) \sin(ms) - D_m(c_2 + c_1 m^2) \sin(rs)$ and the Green's function $G : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is given by equation (3.13).

Proof. Let $\tilde{y}_1(s) \in C^2[0, 1]$ be the solution of BVP (2.1)-(2.2), where $h(s) = 0$ and $c_1, c_2 \neq 0$. Hence, we have

$$\tilde{y}_1(s) = \frac{N_y(s)}{(m^2 - r^2)D_m D_r}, \quad (3.16)$$

where $m^2 - r^2, D_l \neq 0$. Again, let $\tilde{y}_2(s) \in C^2[0, 1]$ be the solution of BVP (2.1)-(2.2), where $h(s) \neq 0$ and $c_1 = c_2 = 0$. Hence, we have

$$\tilde{y}_2(s) = - \int_0^1 G(s, x)h(x)dx. \quad (3.17)$$

Now, the solution $y \in C^4[0, 1]$ of linear fourth-order non homogeneous BVP (2.1)-(2.2) can be written as

$$y(s) = \tilde{y}_1(s) + \tilde{y}_2(s).$$

Substituting the values of $\tilde{y}_1(s)$ and $\tilde{y}_2(s)$ in the above expression we obtain the final result. \square

Anti-maximum principle.

Proposition 3.4. *Let $r, m \in (0, \frac{\pi}{2})$ such that $m^2 - r^2 > 0$, $D_l \neq 0$, and $0 < \delta_1 + \delta_2 < 1$. Further let $h(s) \geq 0$ on $s \in [0, 1]$, and the constants $c_1, c_2 \leq 0$, then the solution $y \in C^4[0, 1]$ of linear fourth-order non homogeneous BVP (2.1)-(2.2) given by (3.15) is nonpositive for all $s \in [0, 1]$.*

Proof. Given that $h(s) \geq 0$, from Lemma 3.2, we have $G(s, x) \geq 0$. Now to prove that $y(s) \leq 0$, it remains to prove that $\tilde{y}_1(s) \leq 0$, defined by equation (3.16).

Clearly, $m^2 - r^2 > 0$ and $D_m, D_r < 0$, if $0 < \delta_1 + \delta_2 < 1$, hence it is sufficient to prove that $N_y(s) \leq 0$. Using properties of $\sin s$, we deduce that

$$N_y(s) \leq \sin(rs)(c_2(D_r - D_m) + c_1(r^2 D_r - m^2 D_m)).$$

Applying $r < m$, we obtain $(D_r - D_m) \geq 0$, $r^2 D_r - m^2 D_m \geq 0$, and since $c_1, c_2 \leq 0$, we obtain the desired result, i.e., $y(s) \leq 0$. \square

3.2. Non-linear BVP. In this subsection, we establish MI technique in RO case to solve the four-point fourth-order non-linear BVP (1.3). To do so, we first introduce the concepts of UL solutions $u(s)$ and $l(s)$, respectively for the BVP (1.3) such that $u(s) \leq l(s)$.

Definition 3.5. A function $l(s) \in C^4[0, 1]$ is called lower solution of fourth-order non-linear BVP (1.3), if it satisfies

$$\begin{aligned} Ll(s) &\leq \mathcal{F}(s, l(s)), \quad 0 < s < 1, \\ B_0(l) &= (0, 0), \quad B_1(l) \geq 0, \quad B_2(l) \geq 0. \end{aligned} \tag{3.18}$$

Definition 3.6. A function $u(s) \in C^4[0, 1]$ is called upper solution of fourth-order non-linear BVP (1.3), if it satisfies

$$\begin{aligned} Lu(s) &\geq \mathcal{F}(s, u(s)), \quad 0 < s < 1, \\ B_0(u) &= (0, 0), \quad B_1(u) \leq 0, \quad B_2(u) \leq 0. \end{aligned} \tag{3.19}$$

Theorem 3.7. *Assume $0 < \zeta_1 < \zeta_2$ and $0 < \delta_1 + \delta_2 < 1$. Let $l_0(s)$ and $u_0(s) \in C^4[0, 1]$ exist such that $u_0(s) \leq l_0(s)$ satisfy (3.18) and (3.19), respectively. If the non-linear function \mathcal{F} is such that it satisfies (A1)–(A3), then (2.1)–(2.2) has at least one solution in the region $D_{\mathcal{F}}$. Further, if there exist a constant $\zeta > 0$ such that $\zeta - M_1 \geq 0$, then the sequences $u_n(s)$ generated by (2.4), with initial iterate $u_0(s)$ converge monotonically non-decreasing and uniformly towards a solution $w_2(s)$ of fourth-order BVP (2.1)–(2.2).*

Similarly, using $l_0(s)$ as an initial iterates leads to a non increasing sequence $l_n(s)$ generated by (2.3) converging monotonically decreasing and uniformly towards a solution $w_1(s)$ of fourth-order BVP (2.1)–(2.2). Every solution $y(s)$ in $D_{\mathcal{F}}$ must satisfy

$$w_2(s) \leq y(s) \leq w_1(s).$$

Proof. We prove this theorem in three steps by using the principle of mathematical induction.

Step 1: For $n = 0$, we have that $u(s) = u_0(s)$ satisfies inequality (3.19) and from equation (2.4), we have

$$\begin{aligned} (L - \zeta)(u_0 - u_1)(s) &\geq 0, \\ B_0(u_0 - u_1) &= (0, 0), \quad B_1(u_0 - u_1) \leq 0, \quad B_2(u_0 - u_1) \leq 0. \end{aligned}$$

From the anti-maximum principle 3.4, we obtain $u_0 \leq u_1$.

Step 2: Let us assume that $u_n \leq u_{n+1}$. Since $\mathcal{F}(s, y)$ satisfies (2.5), we have

$$\mathcal{F}(s, u_{n+1}) - \mathcal{F}(s, u_n) \leq M(u_{n+1} - u_n).$$

Now from (2.4), in view of $\zeta - M \geq 0$, we arrive at

$$\begin{aligned} Lu_{n+1}(s) &\geq (\zeta - M)(u_{n+1} - u_n) + \mathcal{F}(s, u_{n+1}) \geq \mathcal{F}(s, u_{n+1}), \\ B_0(u_{n+1} - u_n) &= (0, 0), \\ B_1(u_{n+1} - u_n) &= 0, \quad B_2(u_{n+1} - u_n) = 0. \end{aligned} \tag{3.20}$$

Using (3.20) with $n = 0$, in (2.4) for $n = 1$, we obtain

$$\begin{aligned} (L - \zeta)(u_1 - u_2)(s) &\geq 0, \\ B_0(u_1 - u_2) &= (0, 0), \quad B_1(u_1 - u_2) = 0, \quad B_2(u_1 - u_2) = 0. \end{aligned}$$

From anti-maximum principle 3.4, we obtain $u_1 \leq u_2$.

Step 3: To prove $u_1 \leq l_0$, we use (3.18) and (2.4) for $n = 0$. Also, in view of $u_0 \leq l_0$ using inequality (2.5) we have

$$\begin{aligned} (L - \zeta)(u_1 - l_0)(s) &\geq 0, \\ B_0(u_1 - l_0) &= (0, 0), \quad B_1(u_1 - l_0) \leq 0, \quad B_2(u_1 - l_0) \leq 0. \end{aligned}$$

Hence, $u_1 \leq l_0$.

Step 4: Now by assuming $u_{n+1} \geq u_n$ and $u_{n+1} \leq l_0$, we show that $u_{n+2} \geq u_{n+1}$ and $u_{n+2} \leq l_0$. Using inequality (3.20) in (2.4) for $n = n+1$ we can easily prove that $u_{n+2} \geq u_{n+1}$. Now in view of $\zeta - M \geq 0$, applying inequality (2.5) for $u_{n+1} \leq l_0$, we have

$$\mathcal{F}(s, l_0) - \zeta l_0 \leq \mathcal{F}(s, u_{n+1}) - \zeta u_{n+1}. \tag{3.21}$$

Using (3.21) in equation (2.4) for $n = n+1$, we obtain

$$\begin{aligned} (L - \zeta)(u_{n+2} - l_0)(s) &\geq 0, \\ B_0(u_{n+2} - l_0) &= (0, 0), \quad B_1(u_{n+2} - l_0) \leq 0, \quad B_2(u_{n+2} - l_0) \leq 0. \end{aligned}$$

From the anti-maximum principle 3.4, we obtain $u_{n+2} \leq l_0$. Hence

$$u(s) = u_0 \leq u_1 \leq \dots \leq u_n \leq u_{n+1} \leq \dots \leq l_0, \tag{3.22}$$

Step 5: Similarly, we deduce that

$$u_0 \leq \dots \leq l_{n+1} \leq l_n \leq \dots \leq l_1 \leq l_0 = l(s). \tag{3.23}$$

Step 6: Finally, by assuming that $u_n \leq l_n$ we show that $u_{n+1} \leq l_{n+1}$. Subtracting equation (2.3) and (2.4) and applying Lipschitz condition we obtain,

$$\begin{aligned} (L - \zeta)(u_{n+1} - l_{n+1})(s) &\geq (\zeta - M)(u_n - l_n)(s) \geq 0, \\ B_0(u_{n+1} - l_{n+1}) &= (0, 0), \\ B_1(u_{n+1} - l_{n+1}) &= 0, \quad B_2(u_{n+1} - l_{n+1}) = 0. \end{aligned}$$

Hence, $u_{n+1} \leq l_{n+1}$.

Thus we arrive at, the sequences l_n and u_n such that

$$u_0 \leq u_1 \leq \dots \leq u_n \leq u_{n+1} \leq \dots \leq l_{n+1} \leq l_n \leq \dots \leq l_1 \leq l_0. \tag{3.24}$$

Using Dini's theorem we prove that the sequences of UL solutions are uniformly convergent. Let

$$w_1(s) = \lim_{n \rightarrow \infty} l_n(s), \quad \text{and} \quad w_2(s) = \lim_{n \rightarrow \infty} u_n(s). \quad (3.25)$$

Taking limit as $n \rightarrow \infty$ on both sides of solution of (2.3), we obtain

$$\lim_{n \rightarrow \infty} l_n(s) = \lim_{n \rightarrow \infty} \left(\tilde{y}_1(s) - \int_0^1 G(s, x) (\mathcal{F}(x, l_n(x)) - \zeta l_n(x)) dx \right).$$

Then

$$w_1(s) = \tilde{y}_1(s) - \int_0^1 G(s, x) (\mathcal{F}(x, l(x)) - \zeta l(x)) dx.$$

Similarly, we deduce that

$$w_2(s) = \tilde{y}_1(s) - \int_0^1 G(s, x) (\mathcal{F}(x, u(x)) - \zeta u(x)) dx,$$

where $\tilde{y}_1(s)$ is given by (3.16). These are the solutions of fourth-order linear BVP (2.1)-(2.2). Any solution $y(s)$ in $D_{\mathcal{F}}$ can play the role of $l_0(s)$ and $u_0(s)$, hence we obtain

$$w_2(s) \leq y(s) \leq w_1(s). \quad \square$$

Theorem 3.8 (Uniqueness). *Let $\zeta > 0$. Suppose that $\mathcal{F}(s, y)$ satisfies conditions (A1), (A2) and there is a constant $0 < M_3 < \pi^2/4$ such that*

$$\mathcal{F}(s, y_1) - \mathcal{F}(s, y_2) \geq M_3(y_1 - y_2). \quad (3.26)$$

Then the non-linear BVP (1.3) has unique solution.

Proof. Let $y = y_1 - y_2$. Then y satisfies

$$L(y_1 - y_2) - M_3(y_1 - y_2) \geq 0.$$

Applying anti-maximum principle 3.4, we obtain $y_1 \leq y_2$. Similarly we can prove $y_1 \geq y_2$ by taking $y = y_2 - y_1$. Hence we have the required result as $y_2 = y_1$. Hence we obtain unique solution y of non-linear BVP (1.3). \square

3.3. Numerical illustrations.

Example 3.9 (Reverse order case). Consider the four-point non-linear BVP

$$\begin{aligned} -y^{(4)}(s) - \lambda y^{(2)}(s) - \zeta y(s) &= \frac{e^s - 1}{15} y^3 + \sin(2s), \\ y(0) = 0, \quad y(1) &= 0.5y(0.2) + 0.4y(0.5), \\ y''(0) = 0, \quad y''(1) &= 0.5y''(0.2) + 0.4y''(0.5), \end{aligned} \quad (3.27)$$

where $\lambda = \zeta_1 + \zeta_2 = 3/2$.

We define initial lower solution

$$l_0(s) = \frac{s}{2} \left(1 + \frac{1}{3} s^3 \right)$$

and initial upper solution

$$u_0(s) = -s \left(2 + \frac{1}{2} s^2 \right)$$

such that $u_0(s) \leq l_0(s)$. With the help of (P_3) we obtain Lipschitz constant $M = 0.23865$. Now using $\zeta - M \geq 0$ we obtain, $\zeta \geq 0.23865$, the range for the convergence

of iterative sequences of UL solution of non-linear BVP (3.27). In Figure 1, we can see that for $\zeta = 1/2$, where $\zeta_1 = 1/2$, $\zeta_2 = 1$ such that $\zeta_1 + \zeta_2 = 3/2$, the sequences of UL solution $l_n(s)$ and $u_n(s)$, $n = 0, 1, 2$, are monotonically converging to the solution of non-linear BVP (3.27) for suitable choices of ζ_1, ζ_2 .

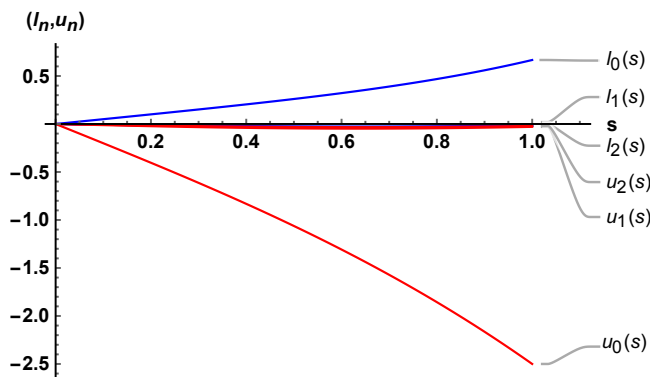


FIGURE 1. Plots of sequences of UL solution $\zeta = 1/2$, $n = 3$.

4. REVERSE ORDER MI TECHNIQUE WHEN $\zeta_1 < 0 < \zeta_2$

In this section, we construct solution of BVP (2.1)-(2.2) and AMP. Further, we establish MI technique in reverse order case to solve the four-point fourth-order non-linear BVP (1.3).

Let

$$\zeta_1 = -r^2 \quad \text{and} \quad \zeta_2 = m^2, \quad (4.1)$$

with some $r, m > 0$. We put the above values of (4.1) in the linear BVP (2.1)-(2.2) and then to obtain Green's function let us consider a corresponding homogeneous linear BVP

$$\begin{aligned} y^{(4)}(s) + (m^2 - r^2)y^{(2)}(s) - r^2m^2y(s) &= 0, \quad 0 < s < 1, \\ B_0(y) = (0, 0), \quad B_1(y) = 0, \quad B_2(y) = 0. \end{aligned} \quad (4.2)$$

Define $L^* : D(L^*) \rightarrow E$, such that

$$L^*y := y^{(4)}(s) + (m^2 - r^2)y^{(2)}(s) - r^2m^2y(s), \quad y \in D(L), \quad (4.3)$$

with domain

$$D(L^*) := \{y \in C^4[0, 1] : B_0(y) = (0, 0), B_1(y) = 0, B_2(y) = 0\}. \quad (4.4)$$

To construct $G(s, x)$ for the BVP (4.2) let us first define

$$L_1y := y''(t) - r^2y(t), \quad y \in D(L_1), \quad (4.5)$$

$$L_2y := y''(t) + m^2y(t), \quad y \in D(L_2), \quad (4.6)$$

where L_1, L_2 are linear operators and

$$D(L_1) := \{y \in C^2[0, 1] : y(0) = 0, B_1(y) = 0\}, \quad (4.7)$$

$$D(L_2) := \{y \in C^2[0, 1] : y(0) = 0, B_1(y) = 0\}. \quad (4.8)$$

Lemma 4.1. *Assume that $m \in (0, \pi/2)$ and $r \in (0, \infty)$. Also assume that*

$$D'_r = \delta_1 \sinh(\eta_1 r) + \delta_2 \sinh(\eta_2 r) - \sinh(r) \neq 0, \quad \text{and} \quad D_m \neq 0, \quad (4.9)$$

where D_m is given by (3.10). Then the $G(s, x) : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ of linear fourth-order BVP (4.2) is expressed as

$$G(s, x) = \int_0^1 G_m(s, t)G'_r(t, x)dt, \quad (s, x) \in [0, 1] \times [0, 1], \quad (4.10)$$

where $G'_r(t, x)$ and $G_m(t, x)$ are Green's functions of (4.5), (4.7) and (4.6), (4.8), respectively. The Green's function $G_m(t, x)$ is given by (3.12) and $G'_r(t, x)$ is

$$G'_r(t, x) = \frac{1}{rD'_r} \begin{cases} \sinh(rt)(\delta_1 \sinh(r(x - \eta_1)) + \delta_2 \sinh(r(x - \eta_2)) + \sinh r(1 - x)) \\ \text{quad if } 0 \leq t \leq x \leq \eta_1 \\ \sinh(rx)(\delta_1 \sinh(r(t - \eta_1)) + \delta_2 \sinh(r(t - \eta_2)) + \sinh r(1 - t)) \\ \text{if } 0 \leq x \leq t \leq \eta_1, \\ \sinh(rt)(\delta_2 \sinh(r(x - \eta_2)) + \sinh r(1 - x)) \quad \text{if } \eta_1 \leq t \leq x \leq \eta_2, \\ -\delta_1 \sinh(\eta_1 r) \sinh(r(x - t)) + \sinh(rx)(\delta_2 \sinh(r(t - \eta_2)) \\ + \sinh r(1 - t)) \quad \text{if } \eta_1 \leq x \leq t \leq \eta_2, \\ \sinh r(1 - x) \sinh(rt) \quad \text{if } \eta_2 \leq t \leq x \leq 1, \\ \sinh r(1 - x) \sinh(rt) - D_r \sinh(r(x - t)) \quad \text{if } \eta_2 \leq x \leq t \leq 1. \end{cases} \quad (4.11)$$

For a proof of the above lemma, see the proof of lemma 3.1.

Lemma 4.2. *Assume that $m \in (0, \frac{\pi}{2})$ and $r \in (0, \infty)$. Then $G(s, x)$ of BVP (4.2) given by expression (4.10) is nonnegative on $[0, 1] \times [0, 1]$ if and only if $0 < \delta_1 + \delta_2 < 1$.*

The proof of the above lemma is similar to the proof of lemma 3.2. We omit it.

Lemma 4.3. *Assume that $r \in (0, \infty)$ and $m \in (0, \frac{\pi}{2})$ such that $D_m D'_r \neq 0$. Then the solution $y \in C^4[0, 1]$ of the linear fourth-order non homogeneous BVP (2.1)-(2.2) is*

$$y(s) = \frac{-1}{(m^2 + r^2)} \left[\frac{\sin(ms)(c_1 r^2 - c_2)}{D_m} + \frac{\sinh(rs)(c_1 m^2 + c_2)}{D'_r} \right] - \int_0^1 G(s, x)h(x)dx, \quad (4.12)$$

where the $G(s, x)$ is given by (4.10).

The above lemma follows from Lemma 3.3.

Proposition 4.4 (Anti-maximum principle). *Let $r \in (0, \infty)$ and $m \in (0, \frac{\pi}{2})$ such that $D_m D'_r \neq 0$. Further, assume that $h(s) \geq 0$ and the $c_1 \leq 0, c_2 \geq 0$, and $\delta_1 + \delta_2 < 1$. Then the solution $y \in C^4[0, 1]$ of the linear fourth-order non homogeneous BVP (2.1)-(2.2) given by (4.12) is nonpositive for all $s \in [0, 1]$.*

Proof. Since $\delta_1 + \delta_2 < 1$, we obtain that D_m and $D'_r < 0$. Also since $c_1 \leq 0$ and $c_2 \geq 0$, the result can be concluded easily. □

In this case the UL solutions are defined as follows.

Definition 4.5 (Lower solution). A function $l(s)$ is known as lower solution of fourth-order non-linear BVP (1.3), if $l(s) \in C^4[0, 1]$ and it satisfies the following conditions

$$\begin{aligned} Ll(s) &\leq \mathcal{F}(s, l(s)), \quad 0 < s < 1, \\ B_0(l) &= (0, 0), \quad B_1(l) \geq 0, \quad B_2(l) \leq 0. \end{aligned} \quad (4.13)$$

Definition 4.6 (Upper solution). A function $u(s)$ is known as upper solution of the fourth-order non-linear BVP (1.3), if $u(s) \in C^4[0, 1]$ and it satisfies the following conditions

$$\begin{aligned} Lu(s) &\geq \mathcal{F}(s, u(s)), \quad 0 < s < 1, \\ B_0(u) &= (0, 0), \quad B_1(u) \leq 0, \quad B_2(u) \geq 0. \end{aligned} \quad (4.14)$$

Theorem 4.7. Assume $\zeta_1 < 0 < \zeta_2$ and $\delta_1 + \delta_2 < 1$. Also assume there exist $l_0(s)$ and $u_0(s) \in C^4[0, 1]$ such that $u_0(s) \leq l_0(s)$ satisfying (4.13) and (4.14), respectively. If the non-linear function \mathcal{F} is such that it satisfies (A1)–(A3), then (2.1)–(2.2) has at least one solution in the region $D_{\mathcal{F}}$. Further, if there exists a constant $\zeta > 0$ such that $\zeta + M \geq 0$, then the monotonically non decreasing sequence $u_n(s)$ generated by (2.4), with initial iterate $u_0(s)$ converges uniformly towards a solution $w_2(s)$ of the fourth-order BVP (2.1)–(2.2).

Similarly, using $l_0(s)$ as an initial iterate leads to a monotonically non-increasing sequence $l_n(s)$ generated by (2.3) converging uniformly towards a solution $w_1(s)$ of fourth-order BVP (2.1)–(2.2). Any solution $y(s)$ in $D_{\mathcal{F}}$ must satisfy

$$w_2(s) \leq y(s) \leq w_1(s).$$

The proof of the above theorem follows from the proof of Theorem 3.7.

Theorem 4.8 (Uniqueness). Let $\zeta < 0$. Suppose that $\mathcal{F}(s, y)$ satisfies (A1), (A2) and that there is a constant $M_3 > 0$ such that

$$\mathcal{F}(s, y_1) - \mathcal{F}(s, y_2) \geq -M_3(y_1 - y_2), \quad (4.15)$$

then the non-linear BVP (1.3) has unique solution.

The proof of the above theorem follows from proof of theorem 3.8.

4.1. Numerical illustrations.

Example 4.9 (Reverse order case). Consider the four-point non-linear BVP

$$\begin{aligned} -y^{(4)}(s) - \lambda y^{(2)}(s) - \zeta y(s) &= -11e + 25e^{-y}, \\ y(0) = 0, \quad y(1) &= 0.4y(0.7) + 0.4y(0.8), \\ y''(0) = 0, \quad y''(1) &= 0.4y''(0.7) + 0.4y''(0.8), \end{aligned} \quad (4.16)$$

where $\lambda = \zeta_1 + \zeta_2 = -1/2$.

We define initial lower solution $l_0(s) = s(4 - 3s^2 + s^3)$ and initial upper solution $u_0(s) = \frac{s^4}{40}(s-2)$ such that $u_0(s) \leq l_0(s)$. With the help of (A3) we obtain Lipschitz constants $M = 3.126$. Now using $\zeta + M \geq 0$ we obtain, $\zeta \geq -3.126$, the range for the convergence of iterative sequences of UL solution of non-linear BVP (4.16). We can see in figure 2, for $\zeta = -1/2$, where $\zeta_1 = -1$, $\zeta_2 = 1/2$ such that $\zeta_1 + \zeta_2 = -1/2$ and $n = 6$ the sequences of UL solution monotonically converge to the solution for suitable choices of ζ_1 and ζ_2 .

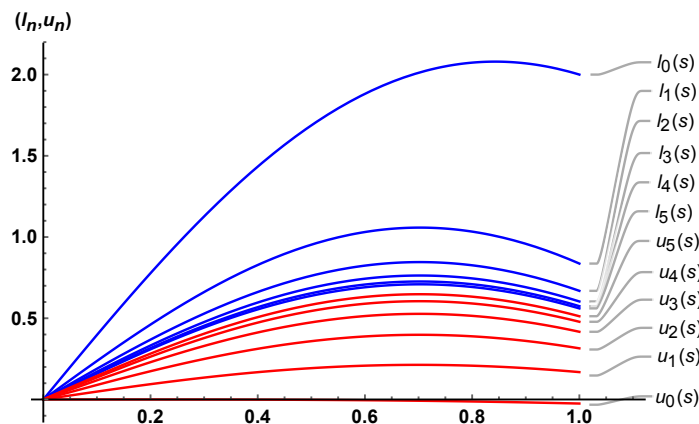


FIGURE 2. Plots of sequences of UL solution $\zeta = -1/2, n = 6$.

5. REVERSE ORDER MI TECHNIQUE WHEN $\zeta_1 < \zeta_2 < 0$

Let

$$\zeta_1 = -r^2 \quad \text{and} \quad \zeta_2 = -m^2, \tag{5.1}$$

with some $r, m > 0$. Let us consider a corresponding fourth-order four-point linear BVP with non homogeneous boundary conditions

$$\begin{aligned} y^{(4)}(s) - (m^2 + r^2)y^{(2)}(s) + r^2m^2y(s) &= 0, \quad 0 < s < 1, \\ B_0(y) = (0, 0), \quad B_1(y) = 0, \quad B_2(y) &= 0. \end{aligned} \tag{5.2}$$

Define $L^* : D(L^*) \rightarrow E$,

$$L^*y := y^{(4)}(s) - (m^2 + r^2)y^{(2)}(s) + r^2m^2y(s), \quad y \in D(L), \tag{5.3}$$

with domain

$$D(L^*) := \{y \in C^4[0, 1] : B_0(y) = (0, 0), B_1(y) = 0, B_2(y) = 0\}. \tag{5.4}$$

To construct $G(s, t)$ for the BVP (5.2) let us first define

$$L_1y := y''(t) - r^2y(t), \quad y \in D(L_1), \tag{5.5}$$

$$L_2y := y''(t) - m^2y(t), \quad y \in D(L_2), \tag{5.6}$$

where

$$D(L_1) := \{y \in C^2[0, 1] : y(0) = 0, B_1(y) = 0\}, \tag{5.7}$$

$$D(L_2) := \{y \in C^2[0, 1] : y(0) = 0, B_1(y) = 0\}. \tag{5.8}$$

Lemma 5.1. Assume that $m, r \in (0, \infty)$. Also assume that

$$D'_l = \delta_1 \sinh(\eta_1 r) + \delta_2 \sinh(\eta_2 r) - \sinh(r) \neq 0, \quad l = m \text{ or } r. \tag{5.9}$$

Then the $G(s, x) : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ of linear fourth-order BVP (5.2) is expressed as

$$G(s, x) = \int_0^1 G'_m(s, t)G'_r(t, x)dt, \quad (s, x) \in [0, 1] \times [0, 1], \tag{5.10}$$

where $G'_l(t, x), l = m/r$, is Green's function of (5.5) and (5.6) with boundary condition (5.7) and (5.8) is given by (4.11), where $r = m$.

Lemma 5.2. Assume that $r, m \in (0, \infty)$. Then $G(s, x)$ of BVP (5.2) given by (5.10) is nonnegative on $[0, 1] \times [0, 1]$ if and only if $0 < \delta_1 + \delta_2 < 1$.

Lemma 5.3. Assume that $m, r \in (0, \infty)$ such that $D_m D'_r (m^2 - r^2) \neq 0$. Then the solution $y \in C^4([0, 1])$ of linear fourth-order non homogeneous BVP (2.1)-(2.2) is

$$y(s) = \frac{1}{(m^2 - r^2)} \left[\frac{\sinh(ms)(c_1 r^2 - c_2)}{D'_m} + \frac{\sinh(rs)(c_2 - c_1 m^2)}{D'_r} \right] - \int_0^1 G(s, x) h(x) dx, \quad (5.11)$$

where $G(s, x)$ is given by (5.10).

Remark 5.4. Since $m^2 < r^2$ we have $(m^2 - r^2) < 0$. Also since we have $D'_l < 0$, for both $l = m$ and r , we obtain that $D'_r < D'_m$.

Anti-maximum principle.

Proposition 5.5. Let $m, r \in (0, \infty)$ such that $(m^2 - r^2) < 0$. Further $h(s) \geq 0$ and the constants $c_1, c_2 \geq 0$ and $0 < \delta_1 + \delta_2 < 1$ then the solution $y \in C^4[0, 1]$ of linear fourth-order non homogeneous BVP (2.1)-(2.2) given by (5.11) is non-negative for all $s \in [0, 1]$.

Proof. Since $\delta_1 + \delta_2 < 1$ and $m^2 < r^2$ we deduce that $(m^2 - r^2) < 0$, $D'_l < 0$ and $D'_r < D'_m$. Using properties of $\sinh s$ and $c_1, c_2 \geq 0$ we obtain the desired result. \square

In this case lower solution $l(s)$ and upper solution $u(s)$ such that $u(s) \leq l(s)$ are defined as follows

Definition 5.6 (Lower solution). A function $l(s)$ is known as lower solution of fourth-order non-linear BVP (1.3), if $l(s) \in C^4[0, 1]$ and it satisfies the following conditions

$$\begin{aligned} Ll(s) &\leq \mathcal{F}(s, l(s)), \quad 0 < s < 1, \\ B_0(l) &= (0, 0), \quad B_1(l) \leq 0, \quad B_2(l) \leq 0. \end{aligned} \quad (5.12)$$

Definition 5.7 (Upper solution). A function $u(s)$ is known as an upper solution of the fourth-order non-linear BVP (1.3), if $u(s) \in C^4[0, 1]$ and it satisfies the following conditions

$$\begin{aligned} Lu(s) &\geq \mathcal{F}(s, u(s)), \quad 0 < s < 1, \\ B_0(u) &= (0, 0), \quad B_1(u) \geq 0, \quad B_2(u) \geq 0. \end{aligned} \quad (5.13)$$

Theorem 5.8 (Reverse order). Assume $\zeta_1 < \zeta_2 < 0$ and $\delta_1 + \delta_2 < 1$. Let $l_0(s)$ and $u_0(s) \in C^4[0, 1]$ exist such that $u_0(s) \leq l_0(s)$ satisfying (5.12) and (5.13), respectively. If the non-linear function \mathcal{F} is such that it satisfies (A1)–(A3), then (2.1)-(2.2) has at least one solution in the region $D_{\mathcal{F}}$. Further, if there exists a constant $\zeta > 0$ such that $\zeta - M \geq 0$. Then the monotonically non decreasing sequence $u_n(s)$ generated by (2.4), with initial iterate $u_0(s)$ converges uniformly towards a solution $w_2(s)$ of the fourth-order BVP (2.1)-(2.2).

Similarly, using $l_0(s)$ as an initial iterates leads to a non increasing sequence $l_n(s)$ generated by (2.3) converging monotonically decreasing and uniformly towards a solution $w_1(s)$ of the fourth-order BVP (2.1)-(2.2). Every solution $y(s)$ in $D_{\mathcal{F}}$ must satisfy

$$w_2(s) \leq y(s) \leq w_1(s).$$

The proof of the above theorem follows from the proof of Theorem 3.7.

Theorem 5.9 (Uniqueness). *Let $\zeta > 0$. Suppose that $\mathcal{F}(s, y)$ satisfies (A1), (A2) and that there is a constant $M_3 > 0$ such that*

$$\mathcal{F}(s, y_1) - \mathcal{F}(s, y_2) \geq M_3(y_1 - y_2). \tag{5.14}$$

Then the non-linear BVP (1.3) has unique solution.

The proof of the above theorem follows from the proof of Theorem 3.8.

5.1. Numerical illustrations.

Example 5.10 (Reverse order case). Consider the four-point non-linear BVP

$$\begin{aligned} -y^{(4)}(s) - \lambda y^{(2)}(s) - \zeta y(s) &= \frac{(e-1)}{50}y^2 + \frac{1}{25}\sin(s), \\ y(0) = 0, \quad y(1) &= 0.3y(0.2) + 0.6y(0.5), \\ y''(0) = 0, \quad y''(1) &= 0.3y''(0.2) + 0.6y''(0.5), \end{aligned} \tag{5.15}$$

where $\lambda = -3$.

We define initial lower solution $l_0(s) = \frac{5s}{4} - s^3$ and initial upper solution $u_0(s) = -\frac{5s}{4} + s^3$ such that $u_0(s) \leq l_0(s)$. With the help of (P_3) we obtain Lipschitz constant $M = 0.0369$. Now using $\zeta - M \geq 0$ we obtain, $\zeta \geq 0.0369$, the range for the convergence of iterative sequences of UL solution of the non-linear BVP (5.15). Hence, we can see in figure 3, for $\zeta = 2$ where $\zeta_1 = -2, \zeta_2 = -1$ such that $\zeta_1 + \zeta_2 = -3$ and $n = 3$, that the sequences of UL solution converges monotonically to the solution of NLBVP (5.15) for suitable choices of ζ_1 and ζ_2 .

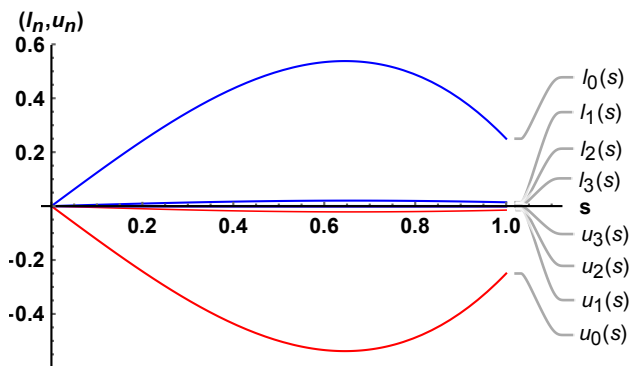


FIGURE 3. Plots of sequences of UL solution $\zeta = 2, n = 3$

6. CONCLUSION

In this article, we developed an MI technique in the reverse order case to establish existence of a unique solution. We have studied existence results in three gathered when $0 < \zeta_1 < \zeta_2, \zeta_1 < 0 < \zeta_2,$ and $\zeta_1 < \zeta_2 < 0$. We need to assume one sided Lipschitz condition on the non-linear function \mathcal{F} to construct monotone sequences of UL solutions. Based on anti-maximum principle, we observe that for all the three gathered we need to define an appropriate form of UL solutions which paves the way for the establishment of MI technique. To validate our results we have constructed examples in each case.

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