EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS FOR FOURTH-ORDER ELLIPTIC PROBLEMS

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ABSTRACT. This article studies a fourth-order elliptic problem with and without an eigenvalue parameter. New criteria for the existence and nonexistence of positive solution are established under some sublinear conditions which involve the principal eigenvalues of the corresponding linear problems. The interesting point is that the nonlinear term f is involved in the second-order derivative explicitly.

1. INTRODUCTION

Consider the fourth-order elliptic problem

$$
\begin{aligned}
\Delta^2 u &= \lambda f(u, -\Delta u) \quad \text{in } \Omega, \\
u &= \Delta u = 0 \quad \text{on } \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where $(\Delta)^2 u = \Delta(\Delta u)$ denotes the biharmonic operator, $\lambda > 0$ is a parameter, Ω is a smooth bounded domain in $\mathbb{R}^n$ ($n \geq 2$), and the nonlinear term satisfies

(A1) $f : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous.

Fourth-order elliptic problems belong to an open problem raised by Lion in [22, Section 4.2 (c)]. They are important questions for understanding related higher order problems, and they have important applications in the study of traveling waves in suspension bridges [6] and in static deflection of a bending beams [15]. Fourth-order elliptic problems have attracted the interest of many mathematicians; see for example [5, 7, 8, 10, 11, 12, 13, 16, 19, 20, 21, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38] and the references cited therein. In particular, Abid-Baraket [1] studied the existence of singular solution to the biharmonic elliptic problem

$$
\begin{aligned}
\Delta^2 u &= u^p \quad \text{in } \Omega, \\
u &= \Delta u = 0 \quad \text{on } \partial\Omega,
\end{aligned}
\tag{1.2}
$$

where Ω is a subset of $\mathbb{R}^n$ ($n \geq 5$) with a smooth boundary. Let Σ be a compact submanifold of Ω without boundary of dimension $(n - m)$ and $4 < m < n$. When $p > \frac{m}{m-4}$ and close enough to this value, the authors verified that problem (1.2) admits at least one solution which is singular on Σ.

Guo-Wei-Zhou [14] considered the existence, uniqueness, asymptotic behavior and further qualitative properties of singular radial solutions of the biharmonic equation

$$\Delta^2 u = u^p \quad \text{in } \mathbb{R}^n \backslash \{0\},$$

$$u > 0, \quad \text{and} \quad \lim_{|x| \to 0} u(x) = +\infty,$$

where $n \geq 5$ and $\frac{n}{n-4} < p < \frac{n+4}{n-4}$. In addition, the authors also constructed positive weak solutions with a prescribed singular set for problem (1.2).

Let Ω be the unit ball in $\mathbb{R}^n$ $(n \geq 5)$ and $\frac{\partial u}{\partial \mathbf{n}}$ denote the differentiation with respect to the exterior unit normal. Arioli-Gazzola-Grunau-Mitidieri [4] studied the fourth-order elliptic problem

$$\Delta^2 u = \lambda e^u \quad \text{in } \Omega,$$
$$u = \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\lambda \geq 0$ is a parameter. For $5 \leq n \leq 16$, the authors proved the existence of singular solutions for problem (1.3) by means of computer assistance.

Liu-Wang [23] used a variant version of Mountain Pass Theorem to demonstrate the existence and nonexistence of positive solution for the fourth-order elliptic problem

$$\Delta^2 u = f(x, u) \quad \text{in } \Omega,$$
$$u = \Delta u = 0 \quad \text{on } \partial\Omega,$$

where Ω denotes a smooth bounded domain in $\mathbb{R}^n$ $(n > 4)$.

Recently, Feng [9] studied the Navier boundary value problem

$$\Delta^2 u = \lambda f(x, u) \quad \text{in } \Omega,$$
$$u = \Delta u = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\lambda \neq 0$ is a parameter, Ω is a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 2)$. The author derived some criteria for the existence, multiplicity and nonexistence of positive solutions to (1.4) by applying fixed point theorems in a cone.

However, to our best knowledge, there are almost no papers studying the fourth-order elliptic problem when the nonlinear term f is involved with the second-order derivative explicitly. In this article, we do some research on this problem.

More precisely, this article has the following features. Firstly, comparing with [1, 4, 9, 14, 23], we discuss the fourth-order elliptic problem when the nonlinear term f is involved with the second-order derivative explicitly. Secondly, we find some new sublinear conditions, which involve the principle eigenvalues of the corresponding linear systems and do not appear in [1, 4, 9, 14, 23]. In addition, we are going to employ a simpler method, i.e. the theory of fixed points on cones to demonstrate the existence of positive solution for fourth-order elliptic problems, which is completely different from that used in [1, 4, 14, 23].

The article is organized as follows. In Section 2, we use a well-known fixed point theorem for completely continuous operators to prove existence results of positive solution to problem (1.1) without an eigenvalue parameter. Section 3 is devoted to analyzing the existence and nonexistence results of positive solution to problem (1.1) with an eigenvalue parameter. Finally, we give some comments on higher order elliptic problems and second-order elliptic systems in Section 4.

2. EXISTENCE OF A POSITIVE SOLUTION WITHOUT AN EIGENVALUE PARAMETER

In this section, we demonstrate a general result of existence of positive solution to (1.1). Conclusions to be demonstrated in this section are true for all $\lambda > 0$. Hence, we may suppose that $\lambda = 1$ for simplicity and so study

$$\begin{aligned}
\Delta^2 u &= f(u, -\Delta u) \quad \text{in } \Omega, \\
u &= \Delta u = 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{2.1}$$

Letting $-\Delta u = v$, one can transform the fourth-order elliptic problem (2.1) into the second-order elliptic system

$$\begin{aligned}
-\Delta u &= v \quad \text{in } \Omega, \\
-\Delta v &= f(u, v) \quad \text{in } \Omega, \\
u &= 0 = v \quad \text{on } \partial\Omega.
\end{aligned} \tag{2.2}$$

From this system, we derive that

$$u(x) = \int_{\bar{\Omega}} G(x, y) v(y) \, dy, \tag{2.3}$$

$$v(x) = \int_{\bar{\Omega}} G(x, y) f(u(y), v(y)) \, dy, \tag{2.4}$$

where $G(x, y)$ denotes the Green's function of $-\Delta$ on Ω, which satisfies

$$0 \le G(x, y) \le C|x - y|^{2-n},$$

where $n \ge 3$, and C is a constant, which depends only on Ω. In addition, for $x, y \in \Omega, x \ne y$, we find that

$$0 \le G(x, y) \le \frac{1}{4\pi|x - y|}, \quad n = 3,$$

$$0 \le G(x, y) \le \frac{1}{2\pi} \ln \frac{d}{|x - y|}, \quad n = 2,$$

where d is the diameter of Ω.

Let the maximum norm in $\mathbb{R}^2$ be defined by $|\mathbf{u}| = \max\{|u|, |v|\}$, where $\mathbf{u} = (u, v) \in \mathbb{R}^2$.

We also let $E = C(\bar{\Omega})^2$ denote the real Banach space of continuous functions with the norm

$$\|\mathbf{u}\| = \max\{\|u\|_0, \ \|v\|_0\},$$

where $\|\cdot\|_0$ denotes the supremum norm of the real Banach space $C(\bar{\Omega})$, and

$$\mathbf{u}(x) = (u(x), v(x)) \quad \text{for } x \in \bar{\Omega}.$$

We define a positive cone in E as

$$K = \{\mathbf{u} \in E : u(x) \ge 0, \ v(x) \ge 0, \ x \in \bar{\Omega}\}. \tag{2.5}$$

For $\varrho > 0$, we also define

$$K_\varrho = \{\mathbf{u} : \mathbf{u} \in K, \ \|\mathbf{u}\| < \varrho\}, \ \partial K_\varrho = \{\mathbf{u} : \mathbf{u} \in K, \ \|\mathbf{u}\| = \varrho\}$$
$$\bar{K}_\varrho = \{\mathbf{u} \in K : \|\mathbf{u}\| \le \varrho\}.$$

Let

$$K_1 = \{w \in C(\bar{\Omega}) : w(x) \ge 0, \ x \in \bar{\Omega}\}.$$

For $u, v \in K_1$, we define $T_1, T_2 : K_1 \to C(\bar{\Omega})$ as

$$T_1 u(x) = \int_{\bar{\Omega}} G(x, y) v(y) \, dy, \tag{2.6}$$

$$T_2 v(x) = \int_{\bar{\Omega}} G(x, y) f(u(y), v(y)) \, dy. \tag{2.7}$$

From the continuity of f, it is clear that $T_1, T_2 : K_1 \to K_1$ are completely continuous.

On the other hand, it is well known that system (2.2) is equivalent to the fixed point equation

$$\mathbf{u}(x) = (T_1 u(x), T_2 v(x)) := T\mathbf{u}(x) \quad \text{for } x \in \bar{\Omega}. \tag{2.8}$$

Since $T_1, T_2 : K_1 \to K_1$ are completely continuous, so we derive that $T : K \to K$ is completely continuous.

We define a linear integral operator G by

$$G u(x) = \int_{\bar{\Omega}} G(x, y) u(y) \, dy. \tag{2.9}$$

We also define a function $e(x)$ on $\bar{\Omega}$ by

$$e(x) = \int_{\bar{\Omega}} G(x, y) \, dy, \ \forall x \in \bar{\Omega}.$$

Now we consider the integral operator G defined in (2.9). One can find in Krasnosel'skii [17] and Amann [2] that the linear integral operator G is e-positive, i.e. for any $v > \theta$, there exist $\xi = \xi(v) > 0$ and $\zeta = \zeta(v) > 0$ such that

$$\xi e \leq G v \leq \zeta e.$$

From this and the well-known Krein-Rutman theorem [18, Theorem 6.2], it is not difficult to see that $\mu_1 \in (0, +\infty)$ and that there exists $\varphi_1 \in K \backslash \{0\}$ so that

$$\varphi_1 = \mu_1 G \varphi_1, \tag{2.10}$$

where $\mu_1 = \frac{1}{r(G)}$ and $r(G)$ denotes the spectral radius of G.

For a function f, we define

$$f_0 = \lim_{|\mathbf{u}| \to 0^+} \frac{f(\mathbf{u})}{|\mathbf{u}|}, \quad f_\infty^\alpha = \lim_{|\mathbf{u}| \to +\infty} \frac{f(\mathbf{u})}{|\mathbf{u}|^\alpha},$$

where $0 < \alpha < 1$.

Theorem 2.1. *Let μ_1 and φ_1 be defined as in (2.10). Under assumption (A1), if $f_0 > \mu_1$ and $f_\infty^\alpha < \mu_1$, then system (2.1) admits at least one positive solution in K.*

Next, we shall apply the following fixed point theorem for completely continuous operators to demonstrate Theorem 2.1.

Lemma 2.2 ([3, Theorem 12.3]). *Let P be a cone in a real Banach space E. Assume Ω_1, Ω_2 are bounded open sets in E with $\theta \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$. If $A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous such that either*

(i) *there exists a $u_0 > 0$ such that $u - Au \neq tu_0$ for all $u \in P \cap \partial\Omega_2$, $t \geq 0$; $Au \neq \mu u$ for all $u \in P \cap \partial\Omega_1$, $\mu \geq 1$, or*

(ii) *there exists a $u_0 > 0$ such that $u - Au \neq tu_0$ for all $u \in P \cap \partial\Omega_1$, $t \geq 0$; $Au \neq \mu u$ for all $u \in P \cap \partial\Omega_2$, $\mu \geq 1$.*

Then A has at least one fixed point in $P \cap (\Omega_2 \backslash \bar{\Omega}_1)$.

Proof of Theorem 2.1. Assume that there is $r_1 > 0$ such that

$$\mathbf{u} - T\mathbf{u} \neq \theta, \quad \forall \mathbf{u} \in K, \ 0 < \|\mathbf{u}\| \leq r_1. \tag{2.11}$$

If not, then there is $\mathbf{u} \in \partial K_{r_1}$ such that $T\mathbf{u} = \mathbf{u}$. Considering $f_0 > \mu_1$, there are $\varepsilon > 0$ and $r^* > 0$ such that

$$f(\mathbf{u}) \geq (\varepsilon + \mu_1)|\mathbf{u}|, \quad \forall \mathbf{u} \in \mathbb{R}_+^2 \text{ with } |\mathbf{u}| \in [0, r^*]. \tag{2.12}$$

Suppose that $\varphi = (\varphi_1, \varphi_1)$, where φ_1 is defined as (2.10). We demonstrate that

$$\mathbf{u} - T\mathbf{u} \neq \zeta\varphi \quad \forall \mathbf{u} \in \partial K_r, \ \zeta \geq 0, \tag{2.13}$$

where $0 < r < \min\{r_1, r^*\}$. If not, then there are $\mathbf{u} \in \partial K_r$ and $\zeta \geq 0$ so that $\mathbf{u} - T\mathbf{u} = \zeta\varphi$. Then (2.11) indicates that $\zeta > 0$ and

$$v_0 = \zeta\varphi_1 + T_2 v_0 \geq \zeta\varphi_1.$$

Set

$$\zeta^* = \sup\{\zeta | v_0 \geq \zeta\varphi_1\}. \tag{2.14}$$

Then we obtain that $0 < \zeta \leq \zeta^* < +\infty$ and $v_0 \geq \zeta^*\varphi_1$.

So, for each $x \in \bar{\Omega}$ and $\mathbf{u} \in \partial K_r$, we derive from (2.7), (2.10), (2.12), and (2.14) that

$$\begin{aligned}
v_0(x) &= \int_{\bar{\Omega}} G(x, y) f(\mathbf{u}(y)) \, dy + \zeta\varphi_1(x) \\
&\geq \int_{\bar{\Omega}} G(x, y)(\varepsilon + \mu_1) v_0(y) \, dy \\
&\geq \int_{\bar{\Omega}} G(x, y)(\varepsilon + \mu_1)\zeta^*\varphi_1(y) \, dy \\
&= (\varepsilon + \mu_1)\zeta^* \int_{\bar{\Omega}} G(x, y)\varphi_1(y) \, dy \\
&= (\varepsilon + \mu_1)\zeta^* \frac{\varphi_1(x)}{\mu_1}.
\end{aligned} \tag{2.15}$$

So we derive that

$$\zeta^* \geq (\varepsilon + \mu_1)\zeta^* \frac{1}{\mu_1} > \zeta^*,$$

which is a contradiction. Thus, (2.13) holds.

Next, turning to $f_\infty^\alpha < \mu_1$, there are $\varepsilon_1 > 0$ and $r_2 > 0$ so that

$$f(\mathbf{u}) \leq (\mu_1 - \varepsilon_1)|\mathbf{u}|^\alpha, \quad \forall \mathbf{u} \in \mathbb{R}_+^2 \text{ with } |\mathbf{u}| \geq r_2.$$

From the continuity of f, there is $L_1 > 0$ such that

$$f(\mathbf{u}) \leq L_1, \quad \forall \mathbf{u} \in \mathbb{R}_+^2 \text{ with } |\mathbf{u}| \leq r_2.$$

Hence we obtain that

$$f(\mathbf{u}) \leq (\mu_1 - \varepsilon_1)|\mathbf{u}|^\alpha + L_1, \quad \forall \mathbf{u} \in \mathbb{R}_+^2. \tag{2.16}$$

Moreover, it is not difficult to see that $v \leq |v|$.

Let $L_2 = \max\{1, |\mathbf{u}| - (\mu_1 - \varepsilon_1)|\mathbf{u}|^\alpha\}$. Then

$$v \leq L_2 + (\mu_1 - \varepsilon_1)|\mathbf{u}|^\alpha, \quad \forall u, v \in \mathbb{R}_+. \tag{2.17}$$

Assume that R is large enough $(R > r)$ so that

$$\frac{L\|\phi\|_0}{R} + \frac{(\mu_1 - \varepsilon_1)\|\phi\|_0}{R^{1-\alpha}} < 1, \tag{2.18}$$

where $L = \max\{L_1, L_2\}$, and $\phi \in C^2(\bar{\Omega})$ satisfies

$$\begin{aligned} -\Delta\phi &= 1 \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.19}$$

We claim that for all $\mathbf{u} \in \partial K_R$, we have that

$$\mu \geq 1 \Rightarrow T\mathbf{u} \neq \mu\mathbf{u}. \tag{2.20}$$

As a matter of fact, if there are $\mathbf{u} \in \partial K_R$ and $\mu \geq 1$ so that $T\mathbf{u} = \mu\mathbf{u}$, then it follows from (2.7) and (2.16) that

$$\begin{aligned} \mu v(x) &= \int_{\bar{\Omega}} G(x,y) f(\mathbf{u}(y))\, dy \\ &\leq \int_{\bar{\Omega}} G(x,y)((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_1)\, dy \\ &= ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_1) \int_{\bar{\Omega}} G(x,y)\, dy \\ &\leq ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_1)\|\phi\|_0. \end{aligned} \tag{2.21}$$

Similarly, from (2.6) and (2.17) it follows that

$$\begin{aligned} \mu u(x) &= \int_{\bar{\Omega}} G(x,y) v(y))\, dy \\ &\leq \int_{\bar{\Omega}} G(x,y)((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_2)\, dy \\ &= ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_2) \int_{\bar{\Omega}} G(x,y)\, dy \\ &\leq ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_2)\|\phi\|_0. \end{aligned} \tag{2.22}$$

Taking the maximum in (2.20) and (2.21), we have

$$\begin{aligned} \mu\|v\|_0 &\leq ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_1)\|\phi\|_0, \\ \mu\|u\|_0 &\leq ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_2)\|\phi\|_0. \end{aligned}$$

This indicates that

$$\mu\|\mathbf{u}\| \leq ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L)\|\phi\|_0.$$

Hence it follows from (2.18) that

$$\mu \leq \frac{L\|\phi\|_0}{R} + \frac{(\mu_1 - \varepsilon_1)\|\phi\|_0}{R^{1-\alpha}} < 1,$$

which contradicts $\mu \geq 1$. So (2.20) holds.

Applying (ii) of Lemma 2.2 to (2.13) and (2.20) yields that T possesses a fixed point $\mathbf{u}$ in $K_R \backslash \bar{K}_r$ with $r < \|\mathbf{u}\| < R$. It follows that system (2.1) admits at least one positive solution $\mathbf{u}$ with $r < \|\mathbf{u}\| < R$. This completes the proof. $\qquad\square$

One of the contributions of Theorem 2.1 is to use a simple method, i.e. the theory of fixed points for completely continuous operators to prove the existence of positive solution for biharmonic problems.

Lemma 2.3. *Let* $\|G\|_0$ *denote the norm of the linear integral operator*

$$Gu(x) = \int_{\bar{\Omega}} G(x,y) u(y)\, dy. \tag{2.23}$$

Then G maps $C(\bar{\Omega})$ into $C(\bar{\Omega})$.

Proof. Let R be large enough $(R > r)$ such that

$$\frac{L\|G\|_0}{R} + \frac{(\mu_1 - \varepsilon_1)\|G\|_0}{R^{1-\alpha}} < 1, \tag{2.24}$$

where L is defined as in (2.18).

If there are $\mathbf{u} \in \partial K_R$ and $\mu \geq 1$ so that $T\mathbf{u} = \mu\mathbf{u}$, then it follows from (2.7) and (2.16) that

$$
\begin{aligned}
\mu v(x) &= \int_{\bar{\Omega}} G(x, y) f(\mathbf{u}(y)) \, dy \\
&\leq \int_{\bar{\Omega}} G(x, y)((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_1) \, dy \\
&= ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_1) \int_{\bar{\Omega}} G(x, y) \, dy \\
&\leq ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_1)\|G\|_0.
\end{aligned}
\tag{2.25}
$$

Similarly, from (2.6) and (2.17) it follows that

$$
\begin{aligned}
\mu u(x) &= \int_{\bar{\Omega}} G(x, y) v(y)) \, dy \\
&\leq \int_{\bar{\Omega}} G(x, y)((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_2) \, dy \\
&= ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_2) \int_{\bar{\Omega}} G(x, y) \, dy \\
&\leq ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_2)\|G\|_0.
\end{aligned}
\tag{2.26}
$$

Taking the maximum in (2.25) and (2.26), we have

$$\mu\|v\|_0 \leq ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_1)\|G\|_0,$$
$$\mu\|u\|_0 \leq ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L_2)\|G\|_0.$$

This indicates that

$$\mu\|\mathbf{u}\| \leq ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha + L)\|G\|_0.$$

Hence it follows from (2.24) that

$$\mu \leq \frac{L\|G\|_0}{R} + \frac{(\mu_1 - \varepsilon_1)\|G\|_0}{R^{1-\alpha}} < 1,$$

which contradicts $\mu \geq 1$. So (2.20) holds.   □

Now we use the following assumptions:

(A2) there exist $\varepsilon_2 > 0$ and $r_3 > 0$ such that

$$f(\mathbf{u}) \geq (\varepsilon_2 + \mu_1)v, \quad \forall \mathbf{u} \in \mathbb{R}_+^2 \text{ with } |\mathbf{u}| \in [0, r_3];$$

(A3) for $0 < \alpha < 1$, there exist $\varepsilon_3 > 0$ and $r_4 > 0$ such that

$$f(\mathbf{u}) \leq (\mu_1 - \varepsilon_3)\mathbf{u}^\alpha, \quad \forall \mathbf{u} \in \mathbb{R}_+^2 \text{ with } |\mathbf{u}| \geq r_4,$$

From the proof of Theorem 2.1, we can derive the following results.

Corollary 2.4. *Under assumptions* (A1)–(A3), *system* (2.1) *admits at least one positive solution in* K.

Corollary 2.5. *Under assumption* (A1), *if* $f_0 > \mu_1$ *and* $f_\infty^\alpha = 0$, *then system* (2.1) *admits at least one positive solution in* K.

Corollary 2.6. *Under assumption* (A1), *if* $f_0 = \infty$ *and* $f_\infty^\alpha < \mu_1$, *then system* (2.1) *admits at least one positive solution in* K.

Corollary 2.7. *Under assumption* (A1), *if* $f_0 = \infty$ *and* $f_\infty^\alpha = 0$, *then system* (2.1) *admits at least one positive solution in* K.

It is easy to see that the conditions in Corollary 2.7 does not depend on the μ_1 defined in (2.10).

3. EXISTENCE OF POSITIVE SOLUTION TO (1.1)

Letting $-\Delta u = v$, one can transform the biharmonic problem (1.1) into the second-order elliptic system

$$\begin{aligned} -\Delta u &= v \quad \text{in } \Omega, \\ -\Delta v &= \lambda f(u, v) \quad \text{in } \Omega, \\ u &= 0 = v \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

From (3.1), we derive that

$$u(x) = \int_{\bar{\Omega}} G(x, y) v(y) \, dy, \tag{3.2}$$

$$v(x) = \lambda \int_{\bar{\Omega}} G(x, y) f(u(y), v(y)) \, dy, \tag{3.3}$$

where $G(x, y)$ denotes the Green's function of $-\Delta$ on Ω.

Because, for $x, y \in \bar{\Omega} \subset \mathbb{R}^n$ ($n \geq 2$), $G(x, y)$ is nonnegative, continuous (when $x \neq y$) and symmetric, there must exist three points x_0, y_0 and z_0 with $x_0 \neq y_0 \neq z_0$, which are interior points of Ω, so that

$$\begin{aligned} G(x_0, y_0) &= G(y_0, x_0) > 0; \\ G(x_0, z_0) &= G(z_0, x_0) > 0; \\ G(z_0, y_0) &= G(y_0, z_0) > 0. \end{aligned}$$

So there exist $\tau_1, \tau_2, \tau_3 > 0$ and three disjoint small closed balls $B_1, B_2, B_3 \subset \Omega$ such that

$$\begin{aligned} G(x, y) &\geq \tau_1, \quad \forall (x, y) \in (B_1 \times B_2) \cup (B_2 \times B_1), \\ G(y, z) &\geq \tau_2, \quad \forall (y, z) \in (B_2 \times B_3) \cup (B_3 \times B_2), \\ G(x, z) &\geq \tau_3, \quad \forall (x, z) \in (B_1 \times B_3) \cup (B_3 \times B_1). \end{aligned} \tag{3.4}$$

Here

$$\begin{aligned} B_1 &= \{x \in \Omega : |x - x_0| \leq \delta\}, \\ B_2 &= \{x \in \Omega : |x - y_0| \leq \delta\}, \\ B_3 &= \{x \in \Omega : |x - z_0| \leq \delta\}, \end{aligned} \tag{3.5}$$

where $\delta > 0$ is small enough. It is not difficult to see that $\text{meas } B_1 = \text{meas } B_2 = \text{meas } B_3$.

Let $|\cdot|$ denote the maximum norm in $\mathbb{R}^2$ defined by $|\mathbf{u}| = \max\{|u|, |v|\}$, where $\mathbf{u} = (u, v) \in \mathbb{R}^2$. We also let $E = C(\bar{\Omega})^2$ denote the real Banach space of continuous functions with norm

$$\|\mathbf{u}\| = \max\{\|u\|_0, \ \|v\|_0\},$$

where $\|\cdot\|_0$ denotes the supremum norm of the real Banach space $C(\bar{\Omega})$, and

$$\mathbf{u}(x) = (u(x), v(x)) \text{ for } x \in \bar{\Omega}.$$

We define a positive cone in E as

$$K = \{\mathbf{u} \in E : \ u(x) \geq 0, \ v(x) \geq 0, \ x \in \bar{\Omega}\}. \tag{3.6}$$

For $\varrho > 0$, we also define

$$K_\varrho = \{\mathbf{u} \in K : \|\mathbf{u}\| < \varrho\}, \quad \partial K_\varrho = \{\mathbf{u} \in K : \|\mathbf{u}\| = \varrho\},$$
$$\bar{K}_\varrho = \{\mathbf{u} \in K : \|\mathbf{u}\| \leq \varrho\}.$$

Let

$$K_1 = \{u \in C(\bar{\Omega}) : \ u(x) \geq 0, \ x \in \bar{\Omega}\}.$$

For $u, v \in K_1$, we define $T_1, T_3 : K_1 \to C(\bar{\Omega})$ as

$$T_1 u(x) = \int_{\bar{\Omega}} G(x, y) v(y) \, dy, \tag{3.7}$$

$$T_3 v(x) = \lambda \int_{\bar{\Omega}} G(x, y) f(u(y), v(y)) \, dy. \tag{3.8}$$

Under assumption (A1), it is clear that $T_1, T_3 : K_1 \to K_1$ are completely continuous. On the other hand, it is well known that system (3.1) is equivalent to the following fixed point equation:

$$\mathbf{u}(x) = (T_1 u(x), T_3 v(x)) := \bar{T}\mathbf{u}(x) \quad \text{for } x \in \bar{\Omega}. \tag{3.9}$$

Since $T_1, T_3 : K_1 \to K_1$ are completely continuous, we derive that $\bar{T} : K \to K$ is completely continuous.

Let G denote the linear integral operator defined as in (2.9). For each function f, we define

$$f_0 = \lim_{|\mathbf{u}| \to 0^+} \frac{f(\mathbf{u})}{|\mathbf{u}|}, \quad f_\infty^\alpha = \lim_{|\mathbf{u}| \to +\infty} \frac{f(\mathbf{u})}{|\mathbf{u}|^\alpha},$$

where $0 < \alpha < 1$.

Theorem 3.1. *Let μ_1 and φ_1 be defined as in (2.10). Under assumption (A1), if $f_0 > \mu_1$ and $f_\infty^\alpha < \mu_1$, then system (3.1) admits at least one positive solution in K for each*

$$\lambda \geq \frac{M}{\tau_1 \mu_1 \int_{B_2} \varphi_1(z) dz},$$

where

$$M = \max_{x \in B_1} \varphi_1(x). \tag{3.10}$$

Proof. Assume that there is $r_1 > 0$ such that

$$\mathbf{u} - T\mathbf{u} \neq \theta, \quad \forall \mathbf{u} \in K, \ 0 < \|\mathbf{u}\| \leq r_1. \tag{3.11}$$

If not, then there is $\mathbf{u} \in \partial K_{r_1}$ so that $T\mathbf{u} = \mathbf{u}$.

Considering $f_0 > \mu_1$, there are $\varepsilon > 0$ and $r^* > 0$ such that

$$f(\mathbf{u}) \geq (\varepsilon + \mu_1)|\mathbf{u}|, \quad \forall \mathbf{u} \in \mathbb{R}_+^2 \text{ with } |\mathbf{u}| \in [0, r^*]. \tag{3.12}$$

Suppose that $\varphi = (\varphi_1, \varphi_1)$, where φ_1 is defined by (2.10). We demonstrate that

$$\mathbf{u} - T\mathbf{u} \neq \zeta\varphi \quad \forall \mathbf{u} \in \partial K_r, \ \zeta \geq 0, \tag{3.13}$$

where $0 < r < \min\{r_1, r^*\}$. If not, there are $\mathbf{u} \in \partial K_r$ and $\zeta \geq 0$ so that $\mathbf{u} - T\mathbf{u} = \zeta\varphi$. Then (3.11) indicates that $\zeta > 0$ and

$$v_0 = \zeta\varphi_1 + T_1 v_0 \geq \zeta\varphi_1. \tag{3.14}$$

Set
$$\zeta^* = \sup\{\zeta | v_0 \geq \zeta\varphi_1\}.$$
Then we obtain that $0 < \zeta \leq \zeta^* < +\infty$ and $v_0 \geq \zeta^*\varphi_1$.

So, for any $x \in B_1$ and $\mathbf{u} \in \partial K_r$, we derive from (3.4), (3.8), (3.9), (3.12), and (3.14) that

$$
\begin{aligned}
v_0(x) &= \lambda \int_{\bar{\Omega}} G(x,y)f(\mathbf{u}(y))\,dy + \zeta\varphi_1(x) \\
&\geq \lambda \int_{\bar{\Omega}} G(x,y)(\varepsilon + \mu_1)v_0(y)\,dy \\
&\geq \lambda \int_{\bar{\Omega}} G(x,y)(\varepsilon + \mu_1)\zeta^*\varphi_1(y)\,dy \\
&= \lambda(\varepsilon + \mu_1)\zeta^* \int_{\bar{\Omega}} G(x,y)\varphi_1(y)\,dy \\
&\geq \lambda(\varepsilon + \mu_1)\zeta^* \int_{B_2} G(x,y)\varphi_1(y)\,dy \\
&\geq \lambda\tau_1(\varepsilon + \mu_1)\zeta^* \int_{B_2} \varphi_1(y)\,dy \\
&\geq \lambda\tau_1(\varepsilon + \mu_1)\zeta^* \int_{B_2} \varphi_1(y)\,dy \times \frac{\varphi_1(x)}{M} \\
&\geq \frac{M}{\tau_1\mu_1 \int_{B_2}\varphi_1(z)dz}\tau_1(\varepsilon + \mu_1)\zeta^* \int_{B_2} \varphi_1(y)\,dy \times \frac{\varphi_1(x)}{M} \\
&= (\varepsilon + \mu_1)\zeta^*\frac{\varphi_1(x)}{\mu_1}.
\end{aligned}
\tag{3.15}
$$

So we derive that
$$\zeta^* \geq (\varepsilon + \mu_1)\zeta^*\frac{1}{\mu_1} > \zeta^*,$$
which is a contradiction. Thus, (3.13) holds.

Next, turning to $f_\infty^\alpha < \mu_1$, there are $\varepsilon_1 > 0$ and $r_2 > 0$ such that
$$f(\mathbf{u}) \leq (\mu_1 - \varepsilon_1)|\mathbf{u}|^\alpha, \quad \forall \mathbf{u} \in \mathbb{R}_+^2 \text{ with } |\mathbf{u}| \geq r_2.$$
By the continuity of f, there is $L_1 > 0$ such that
$$f(\mathbf{u}) \leq L_1, \quad \forall \mathbf{u} \in \mathbb{R}_+^2 \text{ with } |\mathbf{u}| \leq r_2.$$
Hence we obtain that
$$f(\mathbf{u}) \leq (\mu_1 - \varepsilon_1)|\mathbf{u}|^\alpha + L_1, \quad \forall \mathbf{u} \in \mathbb{R}_+^2. \tag{3.16}$$
Moreover, it is not difficult to see that
$$u \leq |u| < |u| + (\mu_1 - \varepsilon_1)\|\mathbf{u}\|^\alpha.$$
Therefore, there is a constant $L_2 > 0$ small enough such that
$$u \leq L_2 + (\mu_1 - \varepsilon_1)\|\mathbf{u}\|. \tag{3.17}$$
Assume that R is large enough $(R > r)$ such that
$$\frac{(\lambda + 1)L\|\phi\|_0}{R} + \frac{(\lambda + 1)(\mu_1 - \varepsilon_1)\|\phi\|_0}{R^{1-\alpha}} < 1, \tag{3.18}$$
where ϕ satisfies (2.19).

We claim that for all $\mathbf{u} \in \partial K_R$, we have that

$$\mu \geq 1 \Rightarrow T\mathbf{u} \neq \mu\mathbf{u}. \tag{3.19}$$

As a matter of fact, if there are $\mathbf{u} \in \partial K_R$ and $\mu \geq 1$ so that $T\mathbf{u} = \mu\mathbf{u}$, then it follows from (3.8), (3.9) and (3.16) that

$$\begin{aligned}
\mu v(x) &= \lambda \int_{\bar{\Omega}} G(x,y) f(\mathbf{u}(y)) \, dy \\
&\leq \lambda \int_{\bar{\Omega}} G(x,y)((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^{\alpha} + L_1) \, dy \\
&= \lambda((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^{\alpha} + L_1) \int_{\bar{\Omega}} G(x,y) \, dy \\
&\leq \lambda((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^{\alpha} + L_1)\|\phi\|_0.
\end{aligned} \tag{3.20}$$

Similarly, from (3.7), (3.9) and (3.17) it follows that

$$\begin{aligned}
\mu u(x) &= \int_{\bar{\Omega}} G(x,y) v(y)) \, dy \\
&\leq \int_{\bar{\Omega}} G(x,y)((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^{\alpha} + L_2) \, dy \\
&= ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^{\alpha} + L_2) \int_{\bar{\Omega}} G(x,y) \, dy \\
&\leq ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^{\alpha} + L_2)\|\phi\|_0.
\end{aligned} \tag{3.21}$$

Taking the maximum in (3.20) and (3.21), we have

$$\mu\|v\|_0 \leq \lambda((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^{\alpha} + L_1)\|\phi\|_0,$$
$$\mu\|u\|_0 \leq ((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^{\alpha} + L_2)\|\phi\|_0.$$

This indicates that

$$\mu\|\mathbf{u}\| \leq (\lambda + 1)((\mu_1 - \varepsilon_1)\|\mathbf{u}\|^{\alpha} + L)\|\phi\|_0,$$

where $L = \max\{L_1, L_2\}$. Hence it follows from (3.18) that

$$\mu \leq \frac{(\lambda + 1)L\|\phi\|_0}{R} + \frac{(\lambda + 1)(\mu_1 - \varepsilon_1)\|\phi\|_0}{R^{1-\alpha}} < 1,$$

which contradicts $\mu \geq 1$. So (3.19) holds.

Applying (ii) of Lemma 2.2 to (3.13) and (3.19) yields that T possesses a fixed point $\mathbf{u}$ in $K_R \backslash \bar{K}_r$ with $r < \|\mathbf{u}\| < R$. It follows that system (3.1) admits at least one positive solution $\mathbf{u}$ with $r < \|\mathbf{u}\| < R$. This completes the proof. $\square$

In the proof of Theorem 3.1, we use a new technique to prove that (3.15) holds, which is different from that used in proving (2.15). From the proof of Theorem 3.1, we can derive the following results.

Corollary 3.2. *If* (A1)–(A3) *hold, then system* (3.1) *admits at least one positive solution in K for each*

$$\lambda \geq \frac{M}{\tau_1 \mu_1 \int_{B_2} \varphi_1(z) dz},$$

where M is defined in (3.10).

Corollary 3.3. *Under assumption $(\mathring{A}1)$, if $f_0 > \mu_1$ and $f_\infty^\alpha = 0$, then system (3.1) admits at least one positive solution in K for each*

$$\lambda \geq \frac{M}{\tau_1 \mu_1 \int_{B_2} \varphi_1(z) dz},$$

where M is defined in (3.10).

Corollary 3.4. *Under assumption (A1), if $f_0 = \infty$ and $f_\infty^\alpha < \mu_1$, then system (3.1) admits at least one positive solution in K for each $\lambda \geq \lambda_0$, where $\lambda_0 > 0$ is a finite number.*

Corollary 3.5. *Under assumption (A1), if $f_0 = \infty$ and $f_\infty^\alpha = 0$, then system (3.1) admits at least one positive solution in K for each $\lambda \geq \lambda_0$, where λ_0 is defined in Corollary 3.4.*

Let

$$f_\infty = \lim_{|\mathbf{u}| \to +\infty} \frac{f(\mathbf{u})}{|\mathbf{u}|}.$$

Theorem 3.6. *Under assumption (A1), if $f_0 > \mu_1$ and $f_\infty > \mu_1$, then system (3.1) admits no positive solution for any $\lambda > \bar{\lambda}$, where $\bar{\lambda} > 0$ is a finite number.*

Proof. Considering (A1), if $f_0 > \mu_1$ and $f_\infty > \mu_1$, then there are positive numbers ε_3, ε_4, h_1, and h_2 such that $h_1 < h_2$ and for $\mathbf{u} \in \mathbb{R}_+^2$ and $|\mathbf{u}| \leq h_1$, we obtain

$$f(\mathbf{u}) \geq (\mu_1 + \varepsilon_3)|\mathbf{u}|, \tag{3.22}$$

and for $\mathbf{u} \in \mathbb{R}_+^2$, $|\mathbf{u}| \geq h_2$, we obtain

$$f(\mathbf{u}) \geq (\mu_1 + \varepsilon_4)|\mathbf{u}|. \tag{3.23}$$

Letting

$$\mu^* = \min\left\{ \mu_1 + \varepsilon_3, \mu_1 + \varepsilon_4, \min\left\{ \frac{f(\mathbf{u})}{|\mathbf{u}|} : |\mathbf{u}| \in [h_1, h_2] \right\} \right\} > 0,$$

we have

$$f(\mathbf{u}) \geq \mu^*|\mathbf{u}|, \quad \mathbf{u} \in \mathbb{R}_+^2. \tag{3.24}$$

Let $\mathbf{u}$ be a positive solution of system (3.1). We will demonstrate that this leads to a contradiction for $\lambda > \bar{\lambda} = \frac{1}{\mu^* \tau_1 \ \text{meas} B_1}$.

In deed, for $x \in B_1$ and $\lambda > \bar{\lambda}$, we derive from (3.4), (3.8), (3.9), and (3.24) that

$$\begin{aligned}
v(x) &= \lambda \int_{\bar{\Omega}} G(x, y) f(u(y), v(y)) \, dy \\
&\geq \lambda \int_{\bar{\Omega}} G(x, y)(\mu^* + \varepsilon_5)\|\mathbf{u}\| dy \\
&= \lambda \mu^* \|\mathbf{u}\| \int_{\bar{\Omega}} G(x, y) \, dy \\
&\geq \lambda \mu^* \|\mathbf{u}\| \int_{B_2} G(x, y) \, dy \\
&\geq \lambda \mu^* \|\mathbf{u}\| \tau_1 \ \text{meas} B_2 \\
&> \bar{\lambda} \mu^* \|\mathbf{u}\| \tau_1 \ \text{meas} B_2 \\
&= \|\mathbf{u}\|.
\end{aligned}$$

This indicates that $\|\mathbf{u}\| > \|\mathbf{u}\|$, which is a contradiction. $\qquad\square$

Corollary 3.7. *Under assumption* (A1), *if $f_0 > 0$ and $f_\infty > 0$, then system* (3.1) *admits no positive solution for any $\lambda > \bar{\lambda}_0$, where $\bar{\lambda}_0 > 0$ is a finite number.*

4. Comments on higher-order elliptic equations

It is well known that one can change a higher-order elliptic equation into a second-order elliptic system. Thus we can employ some of the methods used for studying second-order elliptic systems. However, we find that it is more difficult to deal with higher-order elliptic equations than second-order elliptic systems. The main difficulty lies in the right-hand side of the equations. As an example, we consider the second-order elliptic system

$$
\begin{aligned}
-\Delta u_1 &= \lambda f_1(u_1, u_2) \quad \text{in } \Omega, \\
-\Delta u_2 &= \lambda f_2(u_1, u_2) \quad \text{in } \Omega, \\
u_1 &= u_2 = 0 \quad \text{on } \partial\Omega,
\end{aligned}
\tag{4.1}
$$

where Ω denotes a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter, and for each $i \in \{1, 2\}$, f_i satisfies:

(A4) $f_i : \mathbb{R}_+^2 \to \mathbb{R}_+$ is continuous.

It follows from (4.1) that for $i \in \{1, 2\}$,

$$
u_i(x) = \lambda \int_{\bar{\Omega}} G(x, y) f_i(u_1(y), u_2(y)) \, dy,
\tag{4.2}
$$

where $G(x, y)$ denotes the Green's function of $-\Delta$ on Ω.

Now we define a positive cone in $E = C(\bar{\Omega})^2$ as (3.6). Let

$$
K_1 = \{u \in C(\bar{\Omega}) : u(x) \geq 0, \ x \in \bar{\Omega}\}.
$$

For $u_1, u_2 \in K_1$ and $i \in \{1, 2\}$, we define $\bar{T}_i : K_1 \to C(\bar{\Omega})$ as

$$
\bar{T}_i u_i(x) = \lambda \int_{\bar{\Omega}} G(x, y) f_i(u_1(y), u_2(y)) \, dy.
\tag{4.3}
$$

Under assumption (A4), it is clear that $\bar{T}_i : K_1 \to K_1$ is completely continuous for $i \in \{1, 2\}$. On the other hand, it is well known that system (4.1) is equivalent to the fixed point equation

$$
\mathbf{u}(x) = (\bar{T}_1 u(x), \bar{T}_2 v(x)) := \hat{T}\mathbf{u}(x) \quad \text{for } x \in \bar{\Omega},
\tag{4.4}
$$

where $\mathbf{u}(x) = (u_1(x), u_2(x))$. Since $\bar{T}_1, \bar{T}_2 : K_1 \to K_1$ are completely continuous, we derive that $\hat{T} : K \to K$ is completely continuous.

For each $i \in \{1, 2\}$, let

$$
(f_i)_\infty = \lim_{|\mathbf{u}| \to +\infty} \frac{f_i(\mathbf{u})}{|\mathbf{u}|}, \quad (f_i)_0 = \lim_{|\mathbf{u}| \to 0^+} \frac{f_i(\mathbf{u})}{|\mathbf{u}|},
$$

where $\mathbf{u} = (u_1, u_2)$.

Theorem 4.1. *Under assumption* (A4), *if $(f_i)_0 < \mu_1$ and $(f_i)_\infty < \mu_1$ for $i \in \{1, 2\}$, then system* (4.1) *admits no positive solution for any $\lambda < \lambda^*$, where $\lambda^* > 0$ is a finite number.*

Proof. Considering (A4), if $(f_i)_0 < \mu_1$ and $(f_i)_\infty < \mu_1$ for each $i \in \{1, 2\}$, then there are positive numbers ε_6, ε_7, h_3 and h_4 such that $h_3 < h_4$ and for $\mathbf{u} \in \mathbb{R}_+^2$ and $|\mathbf{u}| \leq h_3$, we derive

$$
f_i(\mathbf{u}) \leq (\mu_1 - \varepsilon_6)|\mathbf{u}|,
\tag{4.5}
$$

and for $\mathbf{u} \in \mathbb{R}_+^2$ and $|\mathbf{u}| \geq h_2$, we derive

$$f_i(\mathbf{u}) \leq (\mu_1 - \varepsilon_7)|\mathbf{u}|. \tag{4.6}$$

Letting

$$\mu^{**} = \max\left\{\mu_1 - \varepsilon_6, \mu_1 - \varepsilon_7, \ \max\left\{\frac{f_i(\mathbf{u})}{|\mathbf{u}|} : |\mathbf{u}| \in [h_1, h_2]\right\}\right\} > 0,$$

we have

$$f_i(\mathbf{u}) \leq \mu^{**}|\mathbf{u}|, \ \mathbf{u} \in \mathbb{R}_+^2. \tag{4.7}$$

Let $\mathbf{u}$ be a positive solution of system (4.1). We will demonstrate that this leads to a contradiction for $\lambda < \hat{\lambda} = \frac{1}{\mu^{**}\|\phi\|_0}$.

Indeed for $x \in \bar{\Omega}$ and $\lambda < \hat{\lambda}$, we derive from (4.3), (4.4) and (4.7) that

$$
\begin{aligned}
u_i(x) &= \lambda \int_{\bar{\Omega}} G(x,y) f_i(u_1(y), u_2(y)) \, dy \\
&\leq \lambda \int_{\bar{\Omega}} G(x,y) \mu^{**} \|\mathbf{u}\| dy \\
&= \lambda \mu^{**} \|\mathbf{u}\| \int_{\bar{\Omega}} G(x,y) \, dy \\
&\leq \lambda \mu^{**} \|\mathbf{u}\| \|\phi\|_0 \\
&< \hat{\lambda} \mu^{**} \|\mathbf{u}\| \|\phi\|_0 \\
&= \|\mathbf{u}\|.
\end{aligned}
$$

This indicates that $\|\mathbf{u}\| < \|\mathbf{u}\|$, which is a contradiction. $\qquad \square$

Similar to the proof of Theorem 4.1, we can prove the following result.

Theorem 4.2. *Under assumption* (A4), *if* $(f_i)_0 > \mu_1$ *and* $(f_i)_\infty > \mu_1$ *for* $i \in \{1, 2\}$, *then system* (4.1) *admits no positive solution for any* $\lambda > \lambda^{**}$, *where* $\lambda^{**} > 0$ *is a finite number.*

For each $i \in \{1, 2\}$, let

$$(f_i)_\infty^\alpha = \lim_{|\mathbf{u}| \to +\infty} \frac{f_i(\mathbf{u})}{|\mathbf{u}|^\alpha},$$

where $\mathbf{u} = (u_1, u_2)$. Then, similar to the proof of Theorem 3.1, we can prove the following result.

Theorem 4.3. *Let* μ_1 *and* φ_1 *be defined as in* (2.10). *Under assumption* (A4), *if there exists* $i_0 \in \{1, 2\}$ *such that* $(f_{i_0})_0 > \mu_1$, *and* $(f_i)_\infty^\alpha < \mu_1$ *for each* $i \in \{1, 2\}$, *then system* (4.1) *admits at least one positive solution in* K *for each*

$$\lambda \geq \frac{M}{\tau_1 \mu_1 \int_{B_2} \varphi_1(z) dz},$$

where τ_1, B_2 *and* M *are respectively defined in* (3.4), (3.5), *and* (3.10).

We believe the following conclusion also holds, but we do not have a proof right now.

Theorem 4.4. *Under assumption* (A1), *if* $f_0 < \mu_1$ *and* $f_\infty < \mu_1$, *then system* (3.1) *admits no positive solution for any* $\lambda < \bar{\lambda}_0$, *where* $\bar{\lambda}_0 > 0$ *is a finite number.*

Theorem 4.4 is similar to Theorem 4.1. Since Theorem 4.1 is related to system (4.1), we can demonstrate that Theorem 4.1 is correct. But Theorem 4.4 is related to system (3.1), which comes from a fourth-order elliptic problem (1.1), hence we can not give a proof right now.

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