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NORMALIZED SOLUTIONS FOR SOBOLEV CRITICAL SCHRÖDINGER-BOPP-PODOLSKY SYSTEMS

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Abstract. We study the Sobolev critical Schrödinger-Bopp-Podolsky system

$$\begin{gathered} -\Delta u + \phi u = \lambda u + \mu |u|^{p-2}u + |u|^4 u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = 4\pi u^2 \quad \text{in } \mathbb{R}^3, \end{gathered}$$

under the mass constraint

$$\int_{\mathbb{R}^3} u^2\,dx = c$$

for some prescribed $c > 0$, where $2 < p < 8/3$, $\mu > 0$ is a parameter, and $\lambda \in \mathbb{R}$ is a Lagrange multiplier. By developing a constraint minimizing approach, we show that the above system admits a local minimizer. Furthermore, we establish the existence of normalized ground state solutions.

1. Introduction

We consider the Schrödinger-Bopp-Podolsky system

$$\begin{gathered} -\Delta u + \phi u = \lambda u + \mu |u|^{p-2}u + |u|^4 u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = 4\pi u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.1}$$

where $u, \phi : \mathbb{R}^3 \to \mathbb{R}$, $\mu > 0$, $\lambda \in \mathbb{R}$ and $2 < p < 8/3$. System (1.1) was suggested as a model to describe solitary waves for nonlinear Schrödinger equation coupled with an electromagnetic field in the Bopp-Podolsky electromagnetic theory [11, 38]. The functions u and ϕ denote the modulus of the wave function and the electrostatic potential, respectively. The Bopp-Podolsky theory is a second-order gauge theory of the electromagnetic field, which was developed by Bopp [11] and Podolsky [38] independently to solve the alleged infinity problem in classical Maxwell theory. For more physical applications, we refer the reader to [9, 12, 13, 16, 17, 21, 23, 30] and the references therein.

In the recent decades, considerable attention has been given to the Schrödinger-Bopp-Podolsky system from quite a few scientific fields. Siciliano-D'Avenia [22] studied a Schrödinger-Bopp-Podolsky system with Sobolev subcritical growth,

$$\begin{gathered} -\Delta u + \omega u + q^2\phi u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.2}$$

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where $a > 0$, $\omega > 0$, $q \neq 0$, and $p \in (2,3]$. They obtained the existence and nonexistence results depending on the various ranges of p and q and showed that, in the radial case, those solutions tend to the solutions of the classical Schrödinger-Poisson equation as $a \to 0$. Silva-Siciliano [41] proved that the system has no solutions for large q and has two radial solutions for small q. They also presented qualitative properties about the energy level of the solutions and a variational characterization of these extremal values of q. Wang-Chen-Liu [44] established the existence, multiplicity and asymptotic behavior of solutions for the Schrödinger-Bopp-Podolsky system with general nonlinearities. Figueiredo-Siciliano [24] proved the existence and multiplicity of solutions for the Schrödinger-Bopp-Podolsky system under positive potentials.

Chen-Tang [19] studied a critical Schrödinger-Bopp-Podolsky system with a subcritical perturbation,

$$\begin{gathered} -\Delta u + V(x)u + \phi u = \mu f(u) + u^5 \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.3}$$

where $a > 0$, V, and f are continuous functions, and $\int_0^t f(s)\,ds \geq t^p$ for $p \in (2,6)$ and $t \geq 0$. They showed that system (1.3) admits ground state solutions under certain conditions of V and f. Using different variational techniques, Li-Pucci-Tang [31] obtained the existence of a nontrivial ground state solution for (1.3) when $f(u) = |u|^{p-1}u$ and its limit system in the sense that $V(x) \to V_\infty \in \mathbb{R}^+$ as $|x| \to +\infty$. Subsequently, Hu-Wu-Tang [26] established the existence of least energy sign-changing solutions of (1.3). For more recent results, we refer to [14, 34, 36, 37, 39, 47, 48].

Note that the papers mentioned above on system (1.2) assume $\omega \in \mathbb{R}$ as a fixed parameter to study nontrivial solutions. Alternatively, we can search for solutions with the prescribed L^2-norm for system (1.1). This approach seems to be meaningful from the physical point of view because of the conservation of mass. In the present study, we focus on finding normalized solutions to (1.1), i.e. a couple $(u,\lambda) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ satisfies (1.1) together with the normalization condition

$$\int_{\mathbb{R}^3} |u|^2\,dx = c \tag{1.4}$$

for a priori given $c > 0$. As we know, for each $u \in H^1(\mathbb{R}^3)$, there exists a unique solution $\phi = \phi_u$ to the second equation of (1.1) satisfying

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1 - e^{-|x-y|}}{|x-y|} u^2(y)\,dy.$$

Then, system (1.1) is reduced into an equivalent integro-differential form

$$-\Delta u + \phi_u u = \lambda u + \mu|u|^{p-2}u + |u|^4 u \quad \text{in } \mathbb{R}^3. \tag{1.5}$$

It is standard that for any $c > 0$, a solution of (1.5)-(1.4) can be regarded as a critical point of the corresponding Energy functional

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx - \frac{\mu}{p}\int_{\mathbb{R}^3} |u|^p\,dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6\,dx,$$

restricted to

$$S(c) := \big\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} u^2\,dx = c\big\}.$$

Then the parameter $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. It is easy to verify that $I(u)$ is a well-defined and C^1 functional on $H^1(\mathbb{R}^3)$. Recently, there are numerous contributions flourishing within this topic, for instance, see [2, 3, 4, 5, 10, 18, 20, 27, 28, 29, 42, 43, 45].

Definition 1.1. We say that $\tilde{u} \in S(c)$ is a ground state solution of (1.5) if it is a solution having minimal energy among all the solutions which belong to $S(c)$, i.e.,

$$dI|_{S(c)}(\tilde{u}) = 0, \quad I(\tilde{u}) = \inf\{I(u) : dI|_{S(c)}(u) = 0,\ u \in S(c)\}.$$

Afonso-Siciliano [1] considered the existence of normalized solutions for the Schrödinger-Bopp-Podolsky system in bounded domains under Neumann boundary conditions. He-Li-Chen [25] investigated the following system,

$$\begin{gathered}
-\Delta u + \omega u + \phi u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3,\\
-\Delta \phi + a^2\Delta^2\phi = 4\pi u^2 \quad \text{in } \mathbb{R}^3,\\
\|u\|_{L^2}^2 = \rho,
\end{gathered} \tag{1.6}$$

where $\omega \in \mathbb{R}$, $a > 0$, and $\rho > 0$. By the minimizing method, they obtained the existence of normalized solutions for $\omega > 0$, $a = 1$, and $p \in (2, \frac{10}{3})$, in which the corresponding functional is bounded from below on $S(c)$. Ramos-Siciliano [40] proved that if $2 < p < 3$, $\rho > 0$ is sufficiently small or if $3 < p < \frac{10}{3}$, $\rho > 0$ is sufficiently large, then system (1.6) admits a least energy solution. Moreover, in the case of $2 < p < \frac{14}{5}$ and $\rho > 0$ small enough, the least energy solutions are radially symmetric up to translation and converge to a least energy solution of the Schrödinger-Poisson-Slater system under the same L^2-norm constraint.

We remark that the above papers do not involve the L^2-supcritical case where $p > \frac{10}{3}$. Since in such a situation, the classical methods for dealing with L^2-supercritical problems fail due to the fact that ϕ_u is not homogeneous, which is difficult for us to make use of the scaling of type $t \mapsto t^\alpha u(t^\beta \cdot)$ for $\alpha, \beta \in \mathbb{R}$ and $t > 0$. Moreover, the appearance of the term $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-|x-y|}u^2(x)u^2(y)\,dx\,dy$ in the corresponding Pohozaev identity is another obstacle to deal with. It is worthy to note that in the case of $a = 0$, this problem reduces to the well-known Schrödinger-Poisson-Slater system

$$\begin{gathered}
-\Delta u + b_1(|x|^{-1} * |u|^2)u + \lambda u = b_2|u|^{p-2}u \quad \text{in } \mathbb{R}^3,\\
\|u\|_{L^2}^2 = c,
\end{gathered} \tag{1.7}$$

where $b_1,\ b_2 \in \mathbb{R}$, and $p \in (\frac{10}{3}, 6]$. Bellazzini-Jeanjean-Luo [6] proved that if $b_1, b_2 > 0$, then (1.7) admits a solution of mountain pass type for $c > 0$ sufficiently small by using a mountain-pass argument. Recently, this result has been developed by Jeanjean-Le [29] under different assumptions on b_1, b_2, and p. Li-Zhang [32] investigated system (1.1) with a Sobolev critical term. For $p \in (\frac{10}{3}, 6)$, by applying a mountain-pass argument, they established the existence of positive normalized ground state solutions to (1.1) for large $\mu > 0$ and small $c > 0$. For $p \in (2, \frac{10}{3}]$, by combining the mountain pass theorem with Lebesgue dominated convergence theorem, they proved the existence of normalized ground state solutions for large $\mu > 0$ and small $c > 0$.

Because of the critical term $|u|^4u$, it is not difficult to check that $I|_{S(c)}$ is unbounded from below. However, as we see, the combined action of L^2 subcritical term $\mu|u|^{p-2}u$ and the nonlocal term $\phi_u u$ creates a geometry of local minima of

I on $S(c)$ for $c > 0$ small enough. Based on [28, 43], there is a natural question whether $I|_{S(c)}$ has a critical point which is a local minimizer in the case where $p \in (2, \frac{8}{3})$. This constitutes the main motivation of this study and our goal is to make an effort to find a positive answer to this question.

For $u \in S(c)$, we set

$$u^t(x) := t^{3/2}u(tx), \quad t > 0,\ x \in \mathbb{R}^3.$$

A direct calculation leads to

$$\begin{aligned} \Phi_u(t) := I(u^t) = \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{t}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1 - e^{-\frac{1}{t}|x-y|}}{|x-y|}u^2(x)u^2(y)\,dx\,dy \\ - \frac{\mu\, t^{\frac{3(p-2)}{2}}}{p}\int_{\mathbb{R}^3}|u|^p\,dx - \frac{t^6}{6}\int_{\mathbb{R}^3}|u|^6\,dx, \end{aligned} \tag{1.8}$$

which is the so-called fiber map and plays an important role in the discussion of the geometrical structure of the functional I. At this stage, we introduce the related Pohozaev manifold defined by

$$\Lambda(c) := \{u \in S(c) : Q(u) = 0\},$$

where

$$\begin{aligned} Q(u) &:= \frac{d}{dt}\Big|_{t=1} I(u^t) \\ &= \int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{1}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1 - e^{-|x-y|}}{|x-y|}u^2(x)u^2(y)\,dx\,dy \\ &\quad - \frac{1}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-|x-y|}u^2(x)u^2(y)\,dx\,dy - \frac{3\mu\,(p-2)}{2p}\int_{\mathbb{R}^3}|u|^p\,dx \\ &\quad - \int_{\mathbb{R}^3}|u|^6\,dx. \end{aligned} \tag{1.9}$$

As mentioned earlier, for any fixed $\mu > 0$, we can find a $c_0 = c_0(\mu) > 0$ such that, for any $c \in (0, c_0)$ there exists an open set $V(c) \subset S(c)$ with the property

$$m(c) := \inf_{u \in V(c)} I(u) < 0 < \inf_{u \in \partial V(c)} I(u),$$

where

$$V(c) := \{u \in S(c) : \|\nabla u\|_2^2 < \rho_0\},\ \partial V(c) := \{u \in S(c) : \|\nabla u\|_2^2 = \rho_0\}$$

for a suitable $\rho_0 > 0$ only depending on $c_0 > 0$.

In the process of minimization, the key difficulty is the lack of compactness of the bounded minimizing sequence $\{u_n\} \subset V(c)$ and the most critical step is to prove the strong subadditivity inequality

$$m(c) < m(c_1) + m(c_2) \text{ for all } 0 < c_1, c_2 < c, \tag{1.10}$$

which is a sufficient condition to ensure that any minimizing sequence on $V(c)$ is relatively compact. Moreover, (1.10) is a stronger version of the so-called weak subadditivity inequality

$$m(c) \le m(c_1) + m(c_2) \quad \text{for all } 0 < c_1, c_2 < c. \tag{1.11}$$

However, because of $p \in (2, \frac{8}{3})$ and the existence of the nonlocal term $\int_{\mathbb{R}^3} \phi_u u^2\,dx$, the method introduced in [28, Lemma 2.6] becomes invalid. In fact, if we do the scaling $v = \sqrt{\theta} u$, it is impossible to obtain that

$$m(\theta\alpha) \leq \theta m(\alpha), \quad \theta > 1,\ \alpha > 0.$$

Following [7], we introduce the condition

$$\text{the function } c \to \frac{m(c)}{c} \text{ is strictly decreasing.} \tag{1.12}$$

From this assumption it follows

$$\frac{c_1}{c} m(c) < m(c_1), \quad \frac{c - c_1}{c} < m(c - c_1).$$

That is,

$$m(c) = \frac{c_1}{c} m(c) + \frac{c - c_1}{c} m(c) < m(c_1) + m(c - c_1),$$

whenever $0 < c_1 < c$. However, verifying condition (1.12) is not easy since the function $c \to \frac{m(c)}{c}$ has oscillatory behavior, even in a neighborhood of the origin. To overcome this difficulty, we adapt the techniques developed by Bellazzini-Siciliano [7, 8] as to finding sufficient conditions to avoid dichotomy.

As we see, the presence of the term $\int_{\mathbb{R}^3} \phi_u u^2\,dx$ has a significant impact on the geometry of $\Phi_u(t)$ and on the existence of ground state solutions. The existence of a normalized ground state solution for the nonlinear Schrödinger equation with a Sobolev critical term and a L^2-subcritical perturbation was discussed in [28], where the local minima of the constraint functional is exactly a ground state. However, it is not trivial for our case due to the complex structure of the fiber map. For this situation, we turn our attention to the Pohozaev manifold $\Lambda(c)$ which contains the solutions of (1.1), see Lemma 2.3, to search for a ground state by taking the minimization in the set of solutions. Our main result reads as follows.

Theorem 1.2. *Let $p \in (2, 8/3)$. For any $\mu > 0$ there exists a $c_0 = c_0(\mu) > 0$ such that, for any $c \in (0, c_0)$, $I(u)$ restricted to $S(c)$ has a critical point u_c at a negative level $m(c) < 0$ which is an interior local minimizer of $I(u)$ in the set $V(c)$. Moreover, system* (1.1) *admits a ground state solution on $S(c)$.*

Before concluding this section, we would like to summarize new features in this study.

- The approach which we use for Theorem 1.2 distinguishes from those described in the literature, for example, see [32, Theorem 1.2]. In fact, our arguments are based on the minimizing method instead of the mountain pass theorem. Moreover, our arguments work for all $\mu > 0$ and we do not need to assume the range of the Lagrange multiplier λ in the first step.
- To show that the minimizing sequences for $m(c)$ are relatively compact, we make use of the strong subadditivity inequality (1.10). We can not just take the same steps as shown in [28] because of the presence of the term $\int_{\mathbb{R}^3} \phi_u u\,dx$. Therefore, we turn to develop another method to ensure the inequality (1.10) to be true. Namely, we take into account how to guarantee the condition (1.12).
- The exponential term in ϕ_u makes the fiber map $\Phi_u(t)$ more complicated since it exists in the first and second derivatives. Thus we cannot follow [28] directly to draw a conclusion that any ground state is contained in

$V(c)$. To show the existence of the ground state solution, we take a series of solutions in $\Lambda(c)$ and obtain the bounded Palais-Smale sequences which are the minimizers of $I(u)$ on $S(c)$.

This article is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used later. In Section 3, we prove our main result by clarifying the local minima structure and showing the convergence, up to a translation, of all minimizing sequences for the functional $I(u)$ on $V(c)$.

2. Preliminary results

Throughout this paper, for any $1 \le s < \infty$, we denote by $L^s(\mathbb{R}^3)$ the usual Lebesgue space with norm $\|u\|_s^s := \int_{\mathbb{R}^3} |u|^s \, dx$. We use $C_0^\infty(\mathbb{R}^3)$ to denote the space of the functions infinitely differentiable with compact support in $\mathbb{R}^3$. We denote by $C_1, C_2, \dots$ the positive constants whose values possibly vary from line to line. The open ball in $\mathbb{R}^3$ is denoted by $B(x, R)$ with the center at x and the radius R.

We start with the Hardy-Littlewood-Sobolev inequality [33]:

$$\Big| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\lambda} \, dx \, dy \Big| \le C(N, \lambda, p, q) \|f\|_p \|g\|_q, \tag{2.1}$$

where $f \in L^p(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N), p, q > 1$, $0 < \lambda < N$, and $\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{N} = 2$.

The following Gagliardo-Nirenberg inequality can be found in [46]:

$$\|u\|_p \le K_{GN}^{1/p} \|\nabla u\|_2^\tau \|u\|_2^{1-\tau}, \tag{2.2}$$

where $N \ge 3$, $p \in [2, \frac{2N}{N-2}]$, and $\tau = N(\frac{1}{2} - \frac{1}{p})$.

We recall the optimal Sobolev constant $\mathcal{S} > 0$, see [15], which is

$$\mathcal{S} = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^3), \, u \neq 0} \frac{\|\nabla u\|_2^2}{\|u\|_6^2},$$

where

$$\mathcal{D}^{1,2}(\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$$

is the completion of $C_0^\infty(\mathbb{R}^3)$ with the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = \|\nabla u\|_2.$$

The Hilbert space defined by

$$\mathcal{D} := \{u \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \Delta u \in L^2(\mathbb{R}^3)\}$$

is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}}^2 = \|\Delta u\|_2^2 + \|\nabla u\|_2^2.$$

It is easy to show that $\mathcal{D}$ is continuously embedded into $\mathcal{D}^{1,2}(\mathbb{R}^3)$, see [22].

Lemma 2.1. [22, Lemma 3.4] *For every $u \in H^1(\mathbb{R}^3)$ we have*

- (i) *for every $y \in \mathbb{R}^3$, $\phi_{u(\cdot+y)} = \phi_u(\cdot + y)$;*
- (ii) *$\phi_u \ge 0$;*
- (iii) *$\phi_u \in \mathcal{D}$;*
- (iv) *$\|\phi_u\|_6 \le C\|u\|_{H^1(\mathbb{R}^3)}^2$; and*
- (v) *if $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, then $\phi_{v_n} \rightharpoonup \phi_v$ in $\mathcal{D}$.*

Lemma 2.2. *Let $u \in S(c)$. Then we have*

(i) *There exists a constant* $K_H > 0$ *such that*
$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}\,dx\,dy \le K_H\|\nabla u\|_2\, c^{3/2}.$$
(ii) *There exists a constant* $K_{GN} > 0$ *such that*
$$\|u\|_p^p \le K_{GN}\|\nabla u\|_2^{\frac{3(p-2)}{2}}\, c^{\frac{6-p}{4}}.$$

Proof. From (2.1) and (2.2) it follows that
$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}\,dx\,dy \le K_1\|u\|_{12/5}^4 \le K_H\|\nabla u\|_2\|u\|_2^3,$$
which implies (i). In view of (2.2), we derive
$$\|u\|_p^p \le K_{GN}\|\nabla u\|_2^{\frac{3(p-2)}{2}}\|u\|_2^{\frac{6-p}{2}},$$
which leads to (ii). □

Lemma 2.3. *Let* $\mu > 0$ *and* $2 < p < 6$. *If* $(u,\lambda) \in H^1(\mathbb{R}^3)\times\mathbb{R}$ *weakly solves*
$$-\Delta u + \phi_u u = \lambda u + \mu|u|^{p-2}u + |u|^4u, \tag{2.3}$$
then $Q(u) = 0$, *where* $Q(u)$ *is defined by* (1.9).

Proof. Using the Pohozaev identity in [19, Lemma 4.2] yields
$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx + \frac{5}{4}\int_{\mathbb{R}^3}\phi_u u^2\,dx + \frac{1}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-|x-y|}u^2(x)u^2(y)\,dx\,dy\\ = \frac{3\lambda}{2}\int_{\mathbb{R}^3}|u|^2\,dx + \frac{3\mu}{p}\int_{\mathbb{R}^3}|u|^p\,dx + \frac{1}{2}\int_{\mathbb{R}^3}|u|^6\,dx = 0.\end{aligned} \tag{2.4}$$
Multiplying both sides of (2.3) by u and integrating on $\mathbb{R}^3$ leads to
$$\int_{\mathbb{R}^3}|\nabla u|^2\,dx + \int_{\mathbb{R}^3}\phi_u u^2\,dx = \lambda\int_{\mathbb{R}^3}|u|^2\,dx + \mu\int_{\mathbb{R}^3}|u|^p\,dx + \int_{\mathbb{R}^3}|u|^6\,dx. \tag{2.5}$$
By combining (2.4) and (2.5), we obtain
$$\begin{aligned}&\int_{\mathbb{R}^3}|\nabla u|^2\,dx + \frac{1}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1-e^{-|x-y|}}{|x-y|}u^2(x)u^2(y)\,dx\,dy\\ &= \frac{1}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-|x-y|}u^2(x)u^2(y)\,dx\,dy + \frac{3\mu\,(p-2)}{2p}\int_{\mathbb{R}^3}|u|^p\,dx + \int_{\mathbb{R}^3}|u|^6\,dx.\end{aligned}$$
Therefore, we arrive at the desired result. □

To ensure the condition (1.12), we define
$$A(u) := \int_{\mathbb{R}^3}|\nabla u|^2\,dx, \quad B(u) := \int_{\mathbb{R}^3}\phi_u u^2\,dx,$$
$$C(u) := \int_{\mathbb{R}^3}|u|^p\,dx, \quad D(u) := \int_{\mathbb{R}^3}|u|^6\,dx,$$
$$T(u) := \frac{1}{4}B(u) - \frac{\mu}{p}C(u) - \frac{1}{6}D(u).$$
Thus, the functional $I(u)$ can be simply re-written as
$$I(u) = \frac{1}{2}A(u) + T(u).$$
Next we recall two definitions introduced in [7].

Definition 2.4. Let $u \in H^1(\mathbb{R}^3)$ with $u \neq 0$. A continuous path $g_u : \theta \in \mathbb{R}^+ \mapsto g_u(\theta) \in H^1(\mathbb{R}^3)$ such that $g_u(1) = u$ is said to be a scaling path of u if $\Theta_{g_u}(\theta) := \|g_u(\theta)\|_2^2/\|u\|_2^2$ is differentiable and $\Theta'_{g_u}(1) \neq 0$. We denote by G_u the set of the scaling paths of u.

Definition 2.5. Let $u \neq 0$ be fixed and $g_u \in G_u$. We say that the scaling path g_u is admissible for the functional I if

$$h_{g_u}(\theta) := I(g_u(\theta)) - \Theta_{g_u}(\theta)I(u),\ \theta \geq 0$$

is a differentiable function.

The following lemma is regarding the splitting properties of the term T, see [25, Lemma 2.8] and [22, Lemma B.2].

Lemma 2.6. *If $p \in (2, 10/3)$, we let $\{u_n\} \subset V(c)$ be a minimizing sequence for $m(c)$ such that $u_n \rightharpoonup u \neq 0$. Then T satisfies the following properties:*

- (i) $T(u_n - u) + T(u) = T(u_n) + o_n(1)$*; and*
- (ii) $T(\alpha_n(u_n - u)) - T(u_n - u) = o_n(1)$*, where* $\alpha_n = \|u_n\|_2^2 - \|u\|_2^2/\|u_n - u\|_2^2$.

The following proposition provides us a useful criterion for the condition (1.12), [7, Theorem 2.1].

Proposition 2.7. *Let $T \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ satisfy Lemma 2.6 (i) and (ii). Assume that for every $c > 0$, all the minimizing sequences $\{u_n\}$ for $m(c)$ have a weak limit up to translations different from zero. Assume that* (1.11) *and the following conditions hold*

$$-\infty < m(c) < 0, \quad \text{for all } c > 0\ (I(0) = 0), \tag{2.6}$$

$$c \mapsto m(c) \quad \text{is continuous}, \tag{2.7}$$

$$\lim_{c \to 0} \frac{m(c)}{c} = 0. \tag{2.8}$$

Then, for each $c > 0$, the set

$$M(c) = \cup_{\tilde{c} \in (0,c]} \{u \in S(\tilde{c}) : I(u) = m(\tilde{c})\}$$

is nonempty. In addition, if

$$\forall u \in M(c),\ \exists g_u \in G_u \text{ is admissible such that } \frac{d}{d\theta} h_{g_u}(\theta)|_{\theta=1} \neq 0, \tag{2.9}$$

then the condition (1.12) *holds. Moreover, if $\{u_n\}$ is a minimizing sequence weakly convergent to a certain u (necessarily $\neq 0$), then $\|u_n - u\|_{H^1(\mathbb{R}^3)} \to 0$ and $I(u) = m(c)$.*

3. Proof of Theorem 1.2

Throughout the whole section, we assume that $2 < p < \frac{8}{3}$, from which we have $0 < \frac{3(p-2)}{2} < 1$. Note that

$$I(u) \geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu K_{GN}}{p} c^{\frac{6-p}{4}} \|\nabla u\|_2^{\frac{3(p-2)}{2}} - \frac{1}{6\mathcal{S}^3}\|\nabla u\|_2^6 \tag{3.1}$$

for any $u \in S(c)$. Now we consider the function $h : \mathbb{R}^+ \to \mathbb{R}$, defined by

$$h_c(t) := \frac{1}{2}t^2 - \frac{\mu K_{GN}}{p} c^{\frac{6-p}{4}} t^{\frac{3(p-2)}{2}} - \frac{1}{6\mathcal{S}^3} t^6$$

$$= t^2\Big[\frac{1}{2} - \frac{\mu K_{GN}}{p} c^{\frac{6-p}{4}} t^{\frac{3(p-2)}{2}-2} - \frac{1}{6\mathcal{S}^3} t^4\Big].$$

Since $\mu > 0$ and $\frac{3(p-2)}{2} < 1$, we have $h_c(0^+) = 0^-$ and $h_c(+\infty) = -\infty$. Moreover, the following properties hold for h_c.

Lemma 3.1 ([28, Lemma 2.1]). *For any $\mu > 0$ there exist a $c_0 = c_0(\mu) > 0$ and $\rho_0 := \rho_{c_0} > 0$ such that $h_{c_0}(\sqrt{\rho_0}) = 0$ and $h_c(\sqrt{\rho_0}) > 0$ hold for any $c \in (0, c_0)$, where c_0 and ρ_0 are explicitly given by*

$$c_0 := \Big(\frac{1}{2K}\Big)^{3/2} > 0,$$

with

$$K := \frac{\mu}{p} K_{GN}\Big[-\frac{3(3p-10)\mu K_{GN}\mathcal{S}^3}{4p}\Big]^{\frac{3p-10}{3(6-p)}} + \frac{1}{6\mathcal{S}^3}\Big[-\frac{3(3p-10)\mu K_{GN}\mathcal{S}^3}{4p}\Big]^{\frac{8}{3(6-p)}} > 0,$$

$$\rho_0 := \Big[-\frac{3(3p-10)\mu K_{GN}\mathcal{S}^3}{4p}\Big]^{\frac{4}{3(6-p)}} c_0^{1/3}.$$

Lemma 3.2 ([28, Lemma 2.2]). *Let $(c_1, \rho_1) \in (0,\infty)\times(0,\infty)$ satisfy $h_{c_1}(\sqrt{\rho_1}) \geq 0$. Then for any $c_2 \in (0, c_1]$ there holds*

$$h_{c_2}(\sqrt{\rho_2}) \geq 0, \quad \text{if } \rho_2 \in \Big[\frac{c_2}{c_1}\rho_1, \rho_1\Big].$$

Remark 3.3. For $p \in (2, 10/3)$, from the expression of $h_c(t)$ we can deduce that $h_c(0^+) = 0^-$ and $h_c(+\infty) = -\infty$, which means that Lemma 3.1 also holds in such a case. However, taking into account the geometrical structure of the fiber map $\Phi_u(t)$, we have to reduce the range of p to $p \in (2, \frac{8}{3})$, see Lemma 3.6 below.

Remark 3.4. According to Lemmas 3.1 and 3.2, it is not difficult to see that $h_{c_0}(\sqrt{\rho_0}) = 0$ and $h_c(\sqrt{\rho_0}) > 0$ for all $c \in (0, c_0)$.

Lemma 3.5. *For $c \in (0, c_0)$, $I(u)$ restricted to $\Lambda(c)$ is coercive on $H^1(\mathbb{R}^3)$. Namely, if $\{u_n\} \subset H^1(\mathbb{R}^3)$ satisfies $\|u_n\|_{H^1(\mathbb{R}^3)} \to +\infty$, then $I(u_n) \to +\infty$.*

Proof. Let $u \in \Lambda(c)$. Taking into account $Q(u) = 0$, we have

$$\begin{aligned} &A(u) + \frac{1}{4}B(u) - \frac{1}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-|x-y|}u^2(x)u^2(y)\,dx\,dy \\ &= \frac{3\mu(p-2)}{2p}C(u) + D(u). \end{aligned} \tag{3.2}$$

From Lemma 2.2 (i), there exists a constant $C_1 > 0$ such that

$$B(u) \leq C_1\|\nabla u\|_2 c^{3/2}.$$

In view of (3.2), there exists a constant $C_2 > 0$ such that

$$D(u) \leq A(u) + C_2\|\nabla u\|_2 c^{3/2}.$$

This together with Lemma 2.2 (ii), for some $C_3 > 0$, leads to

$$\begin{aligned} I(u) &= \frac{1}{2}A(u) + \frac{1}{4}B(u) - \frac{\mu}{p}C(u) - \frac{1}{6}D(u) \\ &\geq \frac{1}{2}A(u) - \frac{\mu}{p}C(u) - \frac{1}{6}A(u) - \frac{C_2}{6}A(u)^{\frac{1}{2}}c^{3/2} \end{aligned}$$

$$\geq \frac{1}{3}A(u) - C_3 A(u)^{\frac{3(p-2)}{4}} c^{\frac{6-p}{4}} - \frac{C_2}{6} A(u)^{\frac{1}{2}} c^{3/2},$$

from which we e complete the proof. $\square$

Define

$$B_{\rho_0} := \{u \in H^1(\mathbb{R}^3) : \|\nabla u\|_2^2 < \rho_0\}, \quad V(c) := S(c) \cap B_{\rho_0}$$

and consider a minimization problem:

$$m(c) = \inf_{u \in V(c)} I(u), \text{ for any } c \in (0, c_0).$$

Lemma 3.6. *Let $c \in (0, c_0)$. Then the following three assertions hold.*

(i) $m(c) = \inf_{u \in V(c)} I(u) < 0 < \inf_{u \in \partial V(c)} I(u)$.
(ii) *The function $c \mapsto m(c)$ is a continuous mapping.*
(iii) *For all $\alpha \in (0, c)$, we have $m(c) \leq m(\alpha) + m(c - \alpha)$.*

Proof. (i) For any $u \in \partial V(c)$ we have $\|\nabla u\|_2^2 = \rho_0$. From (3.1) it follows that

$$I(u) \geq h_c(\|\nabla u\|_2) = h_c(\sqrt{\rho_0}) > 0.$$

Taking into account $\frac{3(p-2)}{2} < 1$, we have $\Phi_u(t) \to 0^-$ as $t \to 0$. Therefore, there exists $t_0 < 1$ small enough such that $\|\nabla u^{t_0}\|_2^2 = t_0^2 \|\nabla u\|_2^2 < \rho_0$ and $I(u^{t_0}) = \Phi_u(t_0) < 0$, which means $m(c) < 0$.

(ii) For any $c \in (0, c_0)$, let $\{c_n\} \subset (0, c_0)$ satisfy $c_n \to c$ as $n \to \infty$. Recall the definition of $m(c_n) < 0$. For any $\epsilon > 0$ small enough, there exists $\{u_n\} \subset V(c_n)$ such that

$$I(u_n) \leq m(c_n) + \epsilon, \ I(u_n) < 0.$$

Let $v_n := \big(\frac{c}{c_n}\big)^{\sqrt{1/2}} u_n$. Then $v_n \in S(c)$ and by similar arguments as described in [28, Lemma 2.6], we see that $v_n \in V(c)$. Furthermore, we have

$$\begin{aligned}
m(c) &\leq I(v_n) \\
&= \frac{1}{2}\frac{c}{c_n} A(u_n) + \frac{1}{4}(\frac{c}{c_n})^2 B(u_n) - \frac{\mu}{p}(\frac{c}{c_n})^{\frac{p}{2}} C(u_n) - \frac{1}{6}(\frac{c}{c_n})^3 D(u_n) \\
&= I(u_n) + o_n(1) \\
&\leq m(c_n) + \epsilon + o_n(1).
\end{aligned}$$

Similarly, for any $\epsilon > 0$ there exists $u \in V(c)$ such that

$$I(u) \leq m(c) + \epsilon, \ I(u) < 0.$$

Let $v_n := \big(\frac{c_n}{c}\big)^{\sqrt{1/2}} u$. Then $v_n \in V(c_n)$. Processing in an analogous manner, we can obtain

$$m(c_n) \leq I(u_n) = [I(u_n) - I(u)] + I(u) \leq m(c) + \epsilon + o_n(1).$$

In view of $\epsilon > 0$ being arbitrary, we have $m(c_n) \to m(c)$ as $n \to \infty$ which implies (ii).

(iii) By the fact that $C_0^\infty(\mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3)$, for any $\epsilon > 0$ there exist $u_1 \in C_0^\infty(\mathbb{R}^3) \cap V(\alpha)$ and $u_2 \in C_0^\infty(\mathbb{R}^3) \cap V(c - \alpha)$ satisfying

$$I(u_1) \leq m(\alpha) + \frac{\epsilon}{2}, \quad I(u_2) \leq m(c - \alpha) + \frac{\epsilon}{2}, \tag{3.3}$$

$$I(u_1) < 0, \quad I(u_2) < 0. \tag{3.4}$$

Moreover, for any $n \in \mathbb{N}$, by a translation, we may assume that

$$\operatorname{dist}(\operatorname{supp} u_1, \operatorname{supp} u_2) > n.$$

By Lemma 3.2, we have $h_\alpha(\sqrt{\rho}) \geq 0$ for any $\rho \in [\frac{\alpha}{c}\rho_0, \rho_0]$ and $h_{c-\alpha}(\sqrt{\rho}) \geq 0$ for each $\rho \in [\frac{c-\alpha}{c}\rho_0, \rho_0]$. Hence, we can deduce from (3.4) that

$$\|\nabla u_1\|_2^2 < \frac{\alpha}{c}\rho_0, \quad \|\nabla u_2\|_2^2 < \frac{c-\alpha}{c}\rho_0.$$

Let $u = u_1 + u_2$. It is easy to verify that $\|u\|_2^2 = \|u_1\|_2^2 + \|u_2\|_2^2$ and $A(u) = A(u_1) + A(u_2)$. Thus we have

$$\|u\|_2^2 = c, \ \|\nabla u\|_2^2 < \rho_0,$$

which means $u \in V(c)$.

Notice that

$$\begin{aligned}
|u(x)|^2|u(y)|^2 &= |u_1(x) + u_2(x)|^2|u_1(y) + u_2(y)|^2 \\
&= |u_1(x)|^2|u_1(y)|^2 + |u_1(x)|^2|u_2(y)|^2 + |u_2(x)|^2|u_1(y)|^2 \\
&\quad + |u_2(x)|^2|u_2(y)|^2 + 2|u_1(x)|^2u_1(y)u_2(y) + 2|u_2(x)|^2u_1(y)u_2(y) \\
&\quad + 2|u_1(y)|^2u_1(x)u_2(x) + 2|u_2(y)|^2u_1(x)u_2(x) \\
&\quad + 4u_1(x)u_2(x)u_1(y)u_2(y).
\end{aligned}$$

Then we can deduce that

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-e^{-|x-y|}}{|x-y|}|u_1(x)|^2|u_2(y)|^2\,dx\,dy &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-e^{-|x-y|}}{|x-y|}|u_1(y)|^2|u_2(x)|^2\,dx\,dy \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_1(y)|^2|u_2(x)|^2}{|x-y|}\,dx\,dy \\
&= \int_{\operatorname{supp} u_1}\int_{\operatorname{supp} u_2} \frac{|u_1(y)|^2|u_2(x)|^2}{|x-y|}\,dx\,dy \\
&\leq \frac{\alpha(c-\alpha)}{n},
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-e^{-|x-y|}}{|x-y|}|u_1(x)|^2|u_1(y)||u_2(y)|\,dx\,dy \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-e^{-|x-y|}}{|x-y|}|u_1(y)|^2|u_1(x)||u_2(x)|\,dx\,dy \leq \frac{\alpha c}{2n},
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-e^{-|x-y|}}{|x-y|}|u_2(x)|^2|u_1(y)||u_2(y)|\,dx\,dy \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-e^{-|x-y|}}{|x-y|}|u_2(y)|^2|u_1(x)||u_2(x)|\,dx\,dy \leq \frac{(c-\alpha)c}{2n},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-e^{-|x-y|}}{|x-y|}|u_1(x)||u_2(x)||u_1(y)||u_2(y)|\,dx\,dy \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_1(x)||u_2(x)||u_1(y)||u_2(y)|}{|x-y|}\,dx\,dy \leq \frac{c^2}{4n}.
\end{aligned}$$

Therefore,
$$B(u) = B(u_1) + B(u_2) + o_n(1).$$
Clearly, it holds
$$C(u) = C(u_1) + C(u_2), \quad D(u) = D(u_1) + D(u_2).$$
Therefore, from (3.3) it follows that
$$\begin{aligned} m(c) &\leq I(u) = I(u_1) + I(u_2) + o_n(1) \\ &\leq m(\alpha) + m(c-\alpha) + \epsilon + o_n(1). \end{aligned}$$
Since $\epsilon > 0$ is arbitrary, we have $m(c) \leq m(\alpha) + m(c-\alpha)$. Consequently, we have arrived at (iii). $\square$

Lemma 3.7. *Let $\{v_n\} \subset B_{\rho_0}$ satisfy $\|v_n\|_p \to 0$. Then there exists a $\beta_0 > 0$ such that*
$$I(v_n) \geq \beta_0 \|\nabla v_n\|_2^2 + o_n(1).$$

Proof. By a direct calculation, we have
$$\begin{aligned} I(v_n) &\geq \frac{1}{2}\|\nabla v_n\|_2^2 - \frac{1}{6}\|v_n\|_6^6 + o_n(1) \\ &\geq \frac{1}{2}\|\nabla v_n\|_2^2 - \frac{1}{6\mathcal{S}^3}\|\nabla v_n\|_2^6 + o_n(1) \\ &\geq \|\nabla v_n\|_2^2 [\frac{1}{2} - \frac{1}{6\mathcal{S}^3}\rho_0^2] + o_n(1). \end{aligned}$$
Since $h_{c_0}(\sqrt{\rho_0}) = 0$, we obtain
$$\beta_0 := \frac{1}{2} - \frac{1}{6\mathcal{S}^3}\rho_0^2 = \frac{\mu K_{GN}}{p} c_0^{\frac{6-p}{4}} \rho_0^{\frac{3(p-2)}{4}-1} > 0. \qquad \square$$

Lemma 3.8. *For any $c \in (0, c_0)$, let $\{u_n\} \subset V(c)$ be a minimizing sequence for $m(c)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$. Then $u \neq 0$.*

Proof. To show that there exist a $\beta_1 > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that for some $R > 0$ it holds
$$\int_{B(y_n,R)} |u_n|^2 \, dx \geq \beta_1 > 0, \tag{3.5}$$
we argue by contradiction that (3.5) does not hold. According to [35, Lemma I.1], for $2 < p < 6$ we have
$$\|u_n\|_{L^p(\mathbb{R}^3)} \to 0 \quad \text{as } n \to \infty.$$
Then, from Lemma 3.7 it follows that $I(u_n) \geq o_n(1)$. This contradicts the fact that $m(c) < 0$.

From (3.5), we know that there exist some $C > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that
$$\int_{B(0,R)} |u_n(\cdot - y_n)|^2 \, dx \geq C > 0.$$
Because of the Rellich compactness theorem, we have
$$u_n(x - y_n) \rightharpoonup u \neq 0 \text{ in } H^1(\mathbb{R}^3),$$
which enables us to arrive at the desired result. $\square$

Next we are going to verify all conditions described in Proposition 2.7 for presenting the proof of Theorem 1.2. In view of Lemmas 2.6, 3.6, and 3.8, it suffices to prove (2.8) and (2.9).

Lemma 3.9. *Assume that $c \in (0, c_0)$. Then the function $c \mapsto m(c)$ satisfies* (2.8).

Proof. Since $m(c) < 0$, we have

$$\frac{\tilde{m}(c)}{c} \le \frac{m(c)}{c} < 0,$$

where

$$\tilde{m}(c) := \inf_{u \in V(c)} \tilde{I}(u), \quad \tilde{I}(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \frac{\mu}{p}\int_{\mathbb{R}^3} |u|^p \, dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 \, dx.$$

Thus, it suffices to show that $\frac{\tilde{m}(c)}{c} \to 0$ as $c \to 0$. Indeed, $\tilde{I}(u)$ is the functional associated with the following Schrödinger equation with combined nonlinearities

$$-\Delta u = \lambda u + \mu |u|^{p-2}u + |u|^4 u \text{ in } \mathbb{R}^3,$$

with the normalized condition $\|u\|_2^2 = c$. According to [28, Theorem 1.2], for any $c \in (0, c_0)$ there exists $u_c \in V(c)$ such that $\tilde{m}(c) = \tilde{I}(u_c) < 0$. Moreover, we know that the sequence $\{u_c\}_{c \in (0,c_0)}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $c \to 0$ and u_c satisfies the following equation in the weak sense

$$-\Delta u_c = \lambda_c u_c + \mu |u_c|^{p-2} u_c + |u_c|^4 u_c \text{ in } \mathbb{R}^3, \tag{3.6}$$

from which we deduce that

$$\begin{aligned}
\frac{\lambda_c}{2} &= \frac{\int_{\mathbb{R}^3} |\nabla u_c|^2 \, dx - \mu \int_{\mathbb{R}^3} |u_c|^p \, dx - \int_{\mathbb{R}^3} |u_c|^6 \, dx}{2\int_{\mathbb{R}^3} |u_c|^2 \, dx} \\
&\le \frac{\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_c|^2 \, dx - \frac{\mu}{p}\int_{\mathbb{R}^3} |u_c|^p \, dx - \frac{1}{6}\int_{\mathbb{R}^3} |u_c|^6 \, dx}{\int_{\mathbb{R}^3} |u_c|^2 \, dx} \\
&= \frac{\tilde{I}(u_c)}{c} = \frac{\tilde{m}(c)}{c} < 0.
\end{aligned}$$

To show that $\lim_{c \to 0} \lambda_c = 0$, we argue by contradiction: assume that there exists a sequence $c_n \to 0$ such that $\lambda_{c_n} < -C$ for some $C \in (0, 1)$. Since the minimizers $u_n := u_{c_n}$ satisfies (3.6), there exist some $C_1, C_2 > 0$ such that

$$\begin{aligned}
C\|u_n\|_{H^1(\mathbb{R}^3)}^2 &\le \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx + C\int_{\mathbb{R}^3} |u_n|^2 \, dx \\
&< \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx - \lambda_{c_n}\int_{\mathbb{R}^3} |u_n|^2 \, dx \\
&= \mu\int_{\mathbb{R}^3} |u_n|^p \, dx + \int_{\mathbb{R}^3} |u_n|^6 \, dx \\
&\le C_1\|u_n\|_{H^1(\mathbb{R}^3)}^p + C_2\|u_n\|_{H^1(\mathbb{R}^3)}^6.
\end{aligned}$$

This indicates that there exists $C_3 > 0$ such that $\|\nabla u_n\|_2^2 > C_3 > 0$ because $p > 2$. Hence, from Remark 3.4 it follows that

$$\begin{aligned}
0 &> \tilde{I}(u_n) \\
&\ge \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{\mu K_{GN}}{p} c_n^{\frac{6-p}{4}} \|\nabla u_n\|_2^{\frac{3(p-2)}{2}} - \frac{1}{6\mathcal{S}^3}\|\nabla u_n\|_2^6
\end{aligned}$$

$$
\begin{aligned}
&= \|\nabla u_n\|_2^2\Big(\frac{1}{2}-\frac{\mu K_{GN}}{p}c_n^{\frac{6-p}{4}}\|\nabla u_n\|_2^{\frac{3(p-2)}{2}-2}-\frac{1}{6\mathcal{S}^3}\|\nabla u_n\|_2^4\Big)\\
&\geq C_3\Big(\frac{1}{2}-\frac{\mu K_{GN}}{p}c_n^{\frac{6-p}{4}}\rho_0^{\frac{3(p-2)}{4}-1}-\frac{1}{6\mathcal{S}^3}\rho_0^2\Big)>0,
\end{aligned}
$$

which yields a contradiction. □

Lemma 3.10. *Let $c\in(0,c_0)$. Then the following strict subadditivity inequality holds,*

$$m(c)<m(c_1)+m(c_2),$$

where $c=c_1+c_2$ and $0<c_1,c_2<c$.

Proof. Note that from Proposition 2.7, we just need to show the condition (2.9) holds. For $u\in M(c)$, without loss of generality, we suppose that there exists some $\tilde c\in(0,c]$ such that $\|u\|_2^2=\tilde c$ and $I(u)=m(\tilde c)$. Since u is a minimizer of $I(u)$ on $V(\tilde c)$, we deduce from Lemma 2.3 that

$$A(u)+\frac{1}{4}B(u)-\frac{1}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{-|x-y|}u^2(x)u^2(y)\,dx\,dy-\frac{3\mu(p-2)}{2p}C(u)-D(u)=0. \tag{3.7}$$

For $u\neq 0$ we compute $h_{g_u}(\theta)$ by considering the family of scaling paths of u parameterized with $\beta\in\mathbb{R}$ given by $u_\theta(x):=\theta^{\frac{1+3\beta}{2}}u(\theta^\beta x)$. By a straightforward computation, we have

$$
\begin{gathered}
A(u_\theta)=\theta^{1+2\beta}A(u),\quad B(u_\theta)=\theta^{2+\beta}H(u)-\frac{\theta^{2+\beta}}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\frac{|x-y|}{\theta^\beta}}}{|x-y|}u^2(x)u^2(y)\,dx\,dy,\\
C(u_\theta)=\theta^{\frac{(1+3\beta)p}{2}-3\beta}C(u),\quad D(u_\theta)=\theta^{3(1+2\beta)}D(u),\ \|u_\theta\|_2^2=\theta\|u\|_2^2,
\end{gathered}
$$

where $H(u)=\frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}\,dx\,dy$.

Let $h_{g_u}(\theta)=f(\theta,u):=I(u_\theta)-\theta I(u)$. Then

$$
\begin{aligned}
h_{g_u}(\theta)&=f(\theta,u)\\
&=\frac{1}{2}(\theta^{1+2\beta}-\theta)A(u)+\frac{1}{4}\Big[\theta^{2+\beta}H(u)\\
&\quad-\frac{\theta^{2+\beta}}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\frac{|x-y|}{\theta^\beta}}}{|x-y|}u^2(x)u^2(y)\,dx\,dy-\theta B(u)\Big]\\
&\quad-\frac{\mu}{p}(\theta^{\frac{(1+3\beta)p}{2}-3\beta}-\theta)C(u)-\frac{1}{6}(\theta^{3(1+2\beta)}-\theta)D(u).
\end{aligned}
\tag{3.8}
$$

Moreover, we can deduce that

$$
\begin{aligned}
f'_\theta(\theta,u)&=\frac{1}{2}\left((1+2\beta)\theta^{2\beta}-1\right)A(u)+\frac{1}{4}\Big[(2+\beta)\theta^{1+\beta}H(u)\\
&\quad-\frac{(2+\beta)\theta^{1+\beta}}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\frac{|x-y|}{\theta^\beta}}}{|x-y|}u^2(x)u^2(y)\,dx\,dy\\
&\quad-\frac{\theta^{2+\beta}}{4\pi}\frac{\beta}{\theta^{\beta+1}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{-\frac{|x-y|}{\theta^\beta}}u^2(x)u^2(y)\,dx\,dy-B(u)\Big]\\
&\quad-\frac{\mu}{p}\Big((\frac{(1+3\beta)p}{2}-3\beta)\theta^{\frac{(1+3\beta)p}{2}-3\beta-1}-1\Big)C(u)\\
&\quad-\frac{1}{6}\left((3(1+2\beta))\theta^{3(1+2\beta)-1}-1\right)D(u),
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&f'_\theta(1,u)\\
&= \beta A(u) + \frac{1}{4}\Big[-\frac{\beta}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-|x-y|}u^2(x)u^2(y)\,dx\,dy + (1+\beta)B(u)\Big]\\
&\quad - \frac{\mu}{p}\big(\frac{(1+3\beta)p}{2} - 3\beta - 1\big)C(u) - \frac{1}{6}(3(1+2\beta)-1)D(u).
\end{aligned}
\tag{3.9}
$$

Now it remains to show that the admissible scaling path satisfies $h'_{g_u}(1) \neq 0$. Again, we process by way of contradiction: assume that there exists a sequence $\{u_n\} \subset M(c)$ with $\|u_n\|_2^2 = c_n \le c$ and $c_n \to 0$ such that for all $\beta \in \mathbb{R}$, we have $f'_\theta(1,u_n) = 0$. That is,

$$
\begin{aligned}
&\beta A(u_n) + \frac{1}{4}\Big[-\frac{\beta}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-|x-y|}u_n^2(x)u_n^2(y)\,dx\,dy + (1+\beta)B(u_n)\Big]\\
&- \frac{\mu}{p}\big(\frac{(1+3\beta)p}{2} - 3\beta - 1\big)C(u_n) - \frac{1}{6}(3(1+2\beta)-1)D(u_n) = 0.
\end{aligned}
\tag{3.10}
$$

Combining (3.7) and (3.10) yields

$$
\frac{1}{4}B(u_n) = \frac{\mu(p-2)}{2p}C(u_n) + \frac{1}{3}D(u_n), \tag{3.11}
$$

$$
B(u_n) = 2A(u_n) - \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-|x-y|}u_n^2(x)u_n^2(y)\,dx\,dy. \tag{3.12}
$$

Moreover, from the continuity of $m(c)$ and the Gagliardo-Nirenberg inequality (2.2), we have

$$
\begin{gathered}
I(u_n) = m(c_n) \to 0,\\
A(u_n), B(u_n), C(u_n), D(u_n) \to 0.
\end{gathered}
\tag{3.13}
$$

We need to consider three cases.

Case 1: $2 < p < 12/5$. From the Hardy-Littlewood-Sobolev inequality, the interpolation inequality, the Sobolev embedding theorem and (3.11), it follows that

$$
\begin{aligned}
B(u_n) &= \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-e^{-|x-y|}}{|x-y|}u_n^2(x)u_n^2(y)\,dx\,dy\\
&\le \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|}\,dx\,dy\\
&\le C\|u_n\|_{12/5}^4\\
&\le C\|u_n\|_p^{\frac{6p}{6-p}}\|u_n\|_6^{\frac{12-5p}{3p}}\\
&= CC(u_n)^{\frac{6}{6-p}}A(u_n)^{\frac{12-5p}{6p}}\\
&\le C_1 B(u_n)^{\frac{6}{6-p}}A(u_n)^{\frac{12-5p}{6p}}.
\end{aligned}
$$

This leads to

$$
1 \le C_1 B(u_n)^{\frac{p}{6-p}}A(u_n)^{\frac{12-5p}{6p}},
$$

which yields a contradiction with (3.13).

Case 2: $p = \frac{12}{5}$. Due to (3.11), we obtain

$$
\|u_n\|_{12/5}^{12/5} \le \frac{3}{\mu}B(u_n) \le C_2\|u_n\|_{12/5}^4,
$$

which is impossible because of the fact $\|u_n\|_{12/5} \to 0$.

Case 3: $12/5 < p < 8/3$. From (3.11) it follows that

$$\|u_n\|_p^p \le \frac{p}{2\mu(p-2)} B(u_n) \le C_3\|u_n\|_{12/5}^4 \le C_3\|u_n\|_2^{\frac{2(5p-12)}{3(p-2)}} \|u_n\|_p^{\frac{2p}{3(p-2)}}.$$

This leads to

$$1 \le C_3 c_n^{\frac{5p-12}{3(p-2)}} \|u_n\|_p^{\frac{p(8-3p)}{3(p-2)}},$$

which yields another contradiction with (3.13).

All the hypotheses of Proposition 2.7 have been verified and thus we have arrived at the desired result. □

Proof of Theorem 1.2. For any $c \in (0, c_0)$, we assume that $\{u_n\} \subset V(c)$ satisfies $I(u_n) \to m(c)$. By Lemma 3.8, there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$u_n(x - y_n) \rightharpoonup u \ne 0 \text{ in } H^1(\mathbb{R}^3).$$

We start by showing that $w_n(x) := u_n(x - y_n) - u(x) \to 0$ in $H^1(\mathbb{R}^3)$. Clearly, we can see that

$$\begin{gathered}
\|u_n\|_2^2 = \|w_n\|_2^2 + \|u\|_2^2 + o_n(1), \\
\|\nabla u_n\|_2^2 = \|\nabla w_n\|_2^2 + \|\nabla u\|_2^2 + o_n(1), \\
I(u_n) = I(w_n) + I(u) + o_n(1).
\end{gathered}$$

The last equality holds because of the translational invariance. Then we claim that $\|w_n\|_2^2 \to 0$. Let $\|u\|_2^2 = c_1 > 0$. It suffices to show that $c_1 = c$. We assume by contradiction that $c_1 < c$. Since we have, for n large enough, $\|w_n\|_2^2 \le c$ and $\|\nabla w_n\|_2^2 \le \|\nabla u_n\|_2^2 < \rho_0$. Then $w_n \in V(\|w_n\|_2^2)$ and $I(w_n) \ge m(\|w_n\|_2^2)$, which implies that

$$m(c) = I(w_n) + I(u) + o_n(1) \ge m(\|w_n\|_2^2) + I(u) + o_n(1).$$

By Lemma 3.6 (ii), we have

$$m(c) \ge m(c - c_1) + I(u).$$

Moreover, we see that $u \in V(c_1)$. Then $I(u) \ge m(c_1)$, and from from Lemma 3.10 it follows that

$$m(c) \ge m(c - c_1) + m(c_1) > m(c),$$

which is impossible. Hence the claim follows and $\|u\|_2^2 = c$.

Now we show that $\|\nabla w_n\|_2^2 \to 0$. Since $u \ne 0$, we have $\|\nabla w_n\|_2^2 \le \|\nabla u_n\|_2^2 < \rho_0$ for n large enough. Thus $\{w_n\} \subset B_{\rho_0}$ and $\{w_n\}$ is bounded in $H^1(\mathbb{R}^3)$. By Lemma 2.2 (ii), recalling $\|w_n\|_2^2 \to 0$, we obtain $\|w_n\|_p^p \to 0$. Then, from Lemma 3.7 we can deduce that

$$I(w_n) \ge \beta_0\|\nabla w_n\|_2^2 + o_n(1) \text{ for } \beta_0 > 0. \tag{3.14}$$

Since $u \in V(c)$, we obtain $I(u) \ge m(c)$. Then

$$I(u_n) = I(u) + I(w_n) + o_n(1) \to m(c),$$

which implies

$$I(w_n) \le o_n(1).$$

Taking into account (3.14), we can see that $\|\nabla w_n\|_2^2 \to 0$. Therefore, $u_n \to u$ holds in $H^1(\mathbb{R}^3)$. Moreover, u is a minimizer for I on $V(c)$.

By Lemma 2.3, we know that all the minimizers of the functional I restricted on $S(c)$ lie in $\Lambda(c)$. We define

$$\bar{m}(c) := \inf_{u \in \Lambda(c)} I(u).$$

By Lemma 3.5, I restricted to $\Lambda(c)$ is bounded from below, which means that $\bar{m}(c)$ is well-defined. By an analogous argument, we can see that Proposition 2.7 also holds for $\bar{m}(c)$. This indicates that any minimizing sequence $\{\bar{u}_n\}$ on $\Lambda(c)$ is relatively compact, i.e., $\bar{u}_n \to \bar{u}$ in $H^1(\mathbb{R}^3)$. It is easily seen that $\{\bar{u}_n\}$ is a bounded Palais-Smale sequence for I on $S(c)$, and thus $\bar{u}$ is a ground state solution of (1.1) on $S(c)$. $\square$

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References

[1] D. G. Afonso, G. Siciliano; Normalized solutions to a Schrödinger-Bopp-Podolsky system under Neumann boundary conditions, *Commun. Contemp. Math.*, **25** (2023), Paper No. 2150100, 20 pp.

[2] B. Bartosz, J. Mederski; Normalized ground states of the nonlinear Schrödinger equation with at least mass critical growth, *J. Funct. Anal.*, **280** (2021), Paper No. 108989, 26 pp.

[3] T. Bartsch, L. Jeanjean, N. Soave; Normalized solutions for a system of coupled cubic Schrödinger equations on $\mathbb{R}^3$, *J. Math. Pures Appl.*, **106** (2016), 583-614.

[4] T. Bartsch, N. Soave; A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems, *J. Funct. Anal.*, **272** (2017), 4998-5037.

[5] T. Bartsch, X. Zhong, W. Zou; Normalized solutions for a coupled Schrödinger system, *Math. Ann.*, **380** (2021), 1713-1740.

[6] J. Bellazzini, L. Jeanjean, T. Luo; Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations, *Proc. Lond. Math. Soc. (3)*, **107** (2013), 303-339.

[7] J. Bellazzini, G. Siciliano; Scaling properties of functionals and existence of constrained minimizers, *J. Funct. Anal.*, **261** (2011), 2486-2507.

[8] J. Bellazzini, G. Siciliano; Stable standing waves for a class of nonlinear Schrödinger-Poisson equations, *Z. Angew. Math. Phys.*, **62** (2011), 267-280.

[9] M. Bertin, B. Pimentel, C. Valcárcel, G. Zambrano; Hamilton-Jacobi formalism for Podolsky's electromagnetic theory on the null-plane, *J. Math. Phys.*, **58** (2017), 082902.

[10] D. Bhimani, T. Gou, H. Hajaiej; Normalized solutions to nonlinear Schrödinger equations with competing Hartree-type nonlinearities, Preprint arXiv: 2209.00429, (2022).

[11] F. Bopp; Eine lineare theorie des elektrons, *Ann. Phys.*, **430** (1940), 345-384.

[12] M. Born; On the quantum theory of the electromagnetic field, *Proc. Roy. Soc. London Ser. A.*, **143** (1934), 410-437.

[13] M. Born, L. Infeld; Foundations of the new field theory, *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, **144** (1934), 425-451.

[14] J. Borthwick, X. J. Chang, L. Jeanjean, N. Soave; Normalized solutions of L^2-supercritical NLS equations on noncompact metric graphs with localized nonlinearities, *Nonlinearity*, **36** (2023), 3776-3795.

[15] H. Brezis; Analyse fonctionnelle, *Collection Mathématiques Appliquées pour la Maitrise. [Collection of Applied Mathematics for the Master's Degree], Théorie et applications. [Theory and applications]*, Masson, Paris, 1983.

[16] R. Bufalo, B. Pimentel, D. Soto; Causal approach for the electron-positron scattering in generalized quantum electrodynamics, *Phys. Rev. D.*, **90** (2014), 085012.

[17] R. Bufalo, B. Pimentel, D. Soto; Normalizability analysis of the generalized quantum electrodynamics from the causal point of view, *Int. J. Mod. Phys. A.*, **32** (2017), 1750165.

[18] X. J. Chang, L. Jeanjean, N. Soave; Normalized solutions of L^2-supercritical NLS equations on compact metric graphs, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, DOI: 10.4171/AIHPC/8.

[19] S. Chen, X. Tang; On the critical Schrödinger-Bopp-Podolsky system with general nonlinearities, *Nonlinear Anal.*, **195** (2020), 111734, 25 pp.

[20] S. Cingolani, L. Jeanjean; Stationary waves with prescribed L^2-norm for the planar Schrödinger-Poisson system, *SIAM J. Math. Anal.*, **51** (2019), 3533-3568.

[21] R. Cuzinatto, C. de Melo, L. Medeiros, B. Pimentel, P. Pompeia; Bopp-Podolsky black holes and the no-hair theorem, *Eur. Phys. J. C.*, **78** (2018), 1-9.

[22] P. D'Avenia, G. Siciliano; Nonlinear Schrödinger equation in the Bopp-Podolsky electrodynamics: Solutions in the electrostatic case, *J. Differ. Equ.*, **267** (2019), 1025-1065.

[23] Z. Feng, Y. Su; Ground state solution to the biharmonic equation, *Z. Angew. Math. Phys.*, **73** (2022), Paper No. 15, 24 pp.

[24] G. M. Figueiredo, G. Siciliano; Multiple solutions for a Schrödinger-Bopp-Podolsky system with positive potentials, *Math. Nachr.*, **296** (2023), 2332-2351.

[25] C. He, L. Li, S. Chen; Normalized solutions for Schrödinger-Bopp-Podolsky system, Preprint arXiv: 2206.04008, (2022).

[26] Y. X. Hu, X. P. Wu, C. L. Tang; Existence of least-energy sign-changing solutions for the Schrödinger-Bopp-Podolsky system with critical growth, *Bull. Malays. Math. Sci. Soc.*, **46** (2023), 1-19.

[27] L. Jeanjean; Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal.*, **28** (1997), 1633-1659.

[28] L. Jeanjean, J. Jendrej, T. T. Le, N. Visciglia; Orbital stability of ground states for a Sobolev critical Schrödinger equation, *J. Math. Pures Appl.*, **164** (2022), 158-179.

[29] L. Jeanjean, T. T. Le; Multiple normalized solutions for a Sobolev critical Schrödinger-Poisson-Slater equation, *J. Differ. Equ.*, **303** (2021), 277-325.

[30] S. Kanesawa, S. Tomonaga; On a relativistically invariant formulation of the quantum theory of wave fields. V. Case of interacting electromagnetic and meson fields, *Progr. Theoret. Phys.*, **3** (1948), 101-113.

[31] L. Li, P. Pucci, X. Tang; Ground state solutions for the nonlinear Schrödinger-Bopp-Podolsky system with critical Sobolev exponent, *Adv. Nonlinear Stud.*, **20** (2020), 511-538.

[32] Y. Li, B. Zhang; Critical Schrödinger-Bopp-Podolsky System with prescribed mass, *J. Geom. Anal.*, **33** (2023), Paper No. 220, 27 pp.

[33] E. H. Lieb, M. Loss; Analysis, *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, (1997).

[34] X. Lin, S. Zheng; Mixed local and nonlocal Schrödinger-Poisson type system involving variable exponents, *Electron. J. Differential Equations*, **2022** (2022), Paper No. 81, 16 pp.

[35] P. Lions; The concentration-compactness principle in the calculus of variations. The locally compact case. II, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1** (1984), 223-283.

[36] B. Mascaro, G. Siciliano; Positive solutions for a Schrödinger-Bopp-Podolsky system, *Commun. Math.*, **31** (2023), 237-249.

[37] X. Q. Peng, G. Jia; Existence and concentration behavior of solutions for the logarithmic Schrödinger-Bopp-Podolsky system, *Z. Angew. Math. Phys.*, **72** (2021), Paper No. 198, 18 pp.

[38] B. Podolsky; A generalized electrodynamics, *Ann. Phys.*, **62** (1942), 68-71.

[39] S. Q. Qu, X. M. He; Multiplicity of high energy solutions for fractional Schrödinger-Poisson systems with critical frequency, *Electron. J. Differential Equations*, **2022** (2022), Paper No. 47, 21 pp.

[40] G. Ramos, G. Siciliano; Existence and limit behavior of least energy solutions to constrained Schrödinger-Bopp-Podolsky systems in $\mathbb{R}^3$, *Z. Angew. Math. Phys.*, **74** (2023), Paper No. 56, 17 pp.

[41] G. Siciliano, K. Silva; The fibering method approach for a non-linear Schrödinger equation coupled with the electromagnetic field, *Publ. Mat.*, **64** (2020), 373-390.

[42] N. Soave; Normalized ground states for the NLS equation with combined nonlinearities, *J. Differ. Equ.*, **269** (2020), 6941-6987.

[43] N. Soave; Normalized ground states for the NLS equation with combined nonlinearities: The Sobolev critical case, *J. Funct. Anal.*, **279** (2020), 108610, 43pp.

[44] L. X. Wang, H. B. Chen, S. L. Liu; Existence and multiplicity of sign-changing solutions for a Schrödinger-Bopp-Podolsky system, *Topol. Methods Nonlinear Anal.*, **59** (2022), 913-940.

[45] J. C. Wei, Y. Z. Wu; Normalized solutions for Schrödinger equations with critical Sobolev exponent and mixed nonlinearities, *J. Funct. Anal.*, **283** (2022), 109574, 46pp.

[46] M. I. Weinstein; Nonlinear Schrödinger equations and sharp interpolation estimates, *Comm. Math. Phys.*, **87** (1983), 567-576.

[47] Q. Zhang; Sign-changing solutions for Schrödinger-Bopp-Podolsky system with general nonlinearity, *Z. Angew. Math. Phys.*, **73** (2022), Paper No. 235, 15 pp.

[48] Y. Zhu, C. Chen, J. Chen; The Schrödinger-Bopp-Podolsky equation under the effect of nonlinearities, *Bull. Malays. Math. Sci. Soc.*, **44** (2021), 953-980.

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