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SPACE-TIME BEHAVIOR FOR RADIATIVE HYDRODYNAMICS MODEL WITH OR WITHOUT HEAT CONDUCTION

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ABSTRACT. We consider space-time behaviors of smooth solutions for the radiative hydrodynamics system with or without heat conduction in the whole space $\mathbb{R}^3$ by using Green's function method. This result exhibits the generalized Huygens' principle as the classical compressible Navier-Stokes equations [3, 26], which is different from the Hamer model for radiating gases in [36].

1. INTRODUCTION

In this paper, we are concerned with a radiation hydrodynamics model of the compressible Navier-Stokes equations coupled with an elliptic equation for radiative flux. Such model is used to describe the motion of viscous and heat-conducting fluids with radiative effects, and simulate supernova explosions, nonlinear stellar pulsation and stellar winds in astrophysics. In particular, we shall study the equations of radiation hydrodynamics [9],

$$\begin{aligned}
&\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0,\\
&(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P = \mu\operatorname{div}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) + \mu'\nabla\operatorname{div}\mathbf{u},\\
&(\rho E)_t + \operatorname{div}(\rho\mathbf{u}E + \mathbf{u}P) + \operatorname{div}\mathbf{q}\\
&= \kappa\Delta\theta + \operatorname{div}(\mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)\cdot\mathbf{u} + \mu'\operatorname{div}\mathbf{u}),\\
&-\nabla\operatorname{div}\mathbf{q} + \mathbf{q} + \nabla(\theta^4) = 0.
\end{aligned}\tag{1.1}$$

Here $\rho(x,t)$, $\mathbf{u}(x,t)\in\mathbb{R}^3$, $\mathbf{q}(x,t)\in\mathbb{R}^3$ and $\theta(x,t)$ are the fluid density, velocity, radiative heat flux and temperature, respectively. The total specific energy $E = e + \frac{1}{2}|\mathbf{u}|^2$ with the specific internal energy e. Without loss of generality, we consider the ideal polytropic gases for the system (1.1), that is, the pressure $P = R\rho\theta$ and the specific internal energy $e = C_v\theta$ with the positive constants R and C_v. Positive constants μ and μ' are coefficients of viscosity satisfying $\mu > 0$ and $2\mu + \mu' > 0$, and κ denotes the coefficient of heat conduction. We consider two cases: $\kappa > 0$ (with heat conduction) and $\kappa = 0$ (without heat conduction).

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We would like to mention the Hamer model of radiating gases [10],

$$\begin{aligned} &v_t + \sum_{i=1}^{n} f_i(v)_{x_i} + \operatorname{div}\mathbf{q} = 0,\\ &-\nabla \operatorname{div}\mathbf{q} + \mathbf{q} + \nabla v = 0, \end{aligned} \tag{1.2}$$

where v is a scalar unknown function. This simplified model provides a good approximation to the radiation hydrodynamics model. There are many important efforts for the one dimensional case, see [28] for the time-asymptotic behavior of solutions with discontinuous initial data, [31] for the asymptotic stability of the rarefaction wave, [32] for the stability of traveling waves and L^1-stability of constants, and [29] for the convergence rates toward the travelling waves. For the multi-dimensional case, [6] studied the relaxation limit of the initial value problem, and [4, 7, 8] focused on decay rates to the planar rarefaction waves. Last but not least, we shall refer to [36], where pointwise space-time estimate for the global classical solution of the Cauchy problem in any dimension $n \geq 1$ was given. In particular, they obtained the following estimate of v,

$$|v(x,t)| \leq C(1+t)^{-n/2}\Big((1+t)^{-n/2}(1+\frac{|x|^2}{1+t})^{-\gamma} + (1+\frac{|x-\mathbf{c}t|^2}{1+t})^{-\gamma}\Big), \tag{1.3}$$

with $\gamma = \min\{r, 3n/4\}$. Here the vector $\mathbf{c} = (f_1'(0), \cdots, f_n'(0))$, and r is from the pointwise assumption on the initial data $|v_0| \leq C\epsilon_0(1+|x|^2)^{-r}$ with $r > \frac{n}{2}$ and suitably small constant $\epsilon_0 > 0$. Additionally, (1.3) immediately gives the L^p estimate with $p \geq 1$.

For small perturbation problems of the typical fluid models with suitable dissipation, L^2-estimates can be derived by standard energy methods. However, the usual L^2-estimates can only exhibit the dissipative properties. On the other hand, pointwise space-time estimates can provide more information of the solution, even exhibit the wave propagations. In this field, there are also a few results. The pioneering works was Zeng [43] and Liu and Zeng [27] for one dimensional compressible fluid models. The isentropic compressible Navier-Stokes system in three dimensions was investigated by Hoff and Zumbrun [11, 12] and Liu and Wang [26] for the linear and nonlinear problems, respectively. In [26], they verified the generalized Huygens' principle through detailed analysis on Green's function and developing subtle nonlinear estimates of the convolutions. A hyperbolic-parabolic system obeys the generalized Huygens' principle in [26] implies that its pointwise space-time description contains both a diffusion wave (D-wave): $(1+t)^{-3/2}\big(1+\frac{|x|^2}{1+t}\big)^{-3/2}$ and a Huygens' wave (H-wave): $(1+t)^{-2}\big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-3/2}$ in $\mathbb{R}^3$. Obviously, the L^2-decay rates of these two diffusion waves are the same as the heat kernel. Recently, there are series of works on other compressible fluid models, for instance, the Navier-Stokes equations in [1, 20, 25], the damped Euler equation in [38], the Navier-Stokes-Poisson equations in [37, 39, 40, 41] and references therein.

We aim to obtain the pointwise description of the solution around the steady state for the Cauchy problem of (1.1) in this paper. The motivation is to find the difference between our pointwise result for the model (1.1) ($\kappa \geq 0$) and the pointwise result (1.3) for the simplified model (1.2). Since the conservation of the system will be used in deriving the pointwise space-time behavior of the solution to this nonlinear problem, the initial data for the system (1.1) is given as follows:

$$(\rho, \mathbf{m}, w)|_{t=0} = (\rho_0, \mathbf{m}_0, w_0)(x),\ x \in \mathbb{R}^3. \tag{1.4}$$

Now, we come back to some of results on the model (1.1) closely related to the topic in this paper, since there exactly are lots of literature devoted to the mathematical theory on radiative hydrodynamics. The system (1.1) can also be viewed as the Navier-Stokes-Fourier equations coupled with a parabolic equation with high order nonlinear term of the temperature θ. In one-dimensional case, the radiation takes a rather good effect on the system even when omitting the heat conduction and viscosity. In fact, the Cauchy problem of the system (1.1) guarantees a unique global smooth solution for the small perturbation. See Kawashima et al. [15, 16], Lin and Goudon [23] for global existence of classical solutions and Deng and Yang [2] for pointwise space-time behavior. In terms of the stability of three elementary waves, see Lattanzio et al. [18] and Lin et al. [24] for the shock wave, Lin et al. [22] for rarefaction wave and Wang and Xie [34] for viscous contact wave. He et al. [17] and Liao and Zhao [21] investigated the global existence and large-time behavior for the viscous radiative and reactive gas. For the stability of the composite of the elementary waves, we refer to Fan [5] about two viscous shock waves, Rohde et al. [30] about rarefaction and contact waves, and Xie [42] about viscous contact wave and rarefaction waves. Hong [13] gave the large-time behavior toward the combination of two rarefaction waves and viscous contact wave when there exist additional heat conduction and viscosity. We shall also refer to Li [19] for the formation of singularities of the large perturbation. In multi-dimensional case, Huang and Zhang [14] studied the asymptotic stability of planar rarefaction wave, Wan and Xu [33] considered radial symmetric classical solutions in an exterior domain and a bounded concentric annular domain. Finally, mostly close to our topic, Wang and Xie [35] obtained the existence of the classical global solution and the decay rate by energy method in H^4-framework. Recently, the condition on the regularity of initial data was relaxed to H^2-framework by Gong et al. in [9].

In deriving the pointwise description of the solution constructed in [35], the main difficulties include that deriving the representation of this Green's matrix in Fourier space, dealing with the singularity from Riesz operator in the Green's matrix for the pointwise estimates through finding suitable combinations for the singular components, and closing the ansatz for the nonlinear problem. In addition, compared with the non-isentropic Navier-Stokes equations in [3], the presence of nonlocal operator $(1-\Delta)^{-1}$ arising from the relation of $\mathbf{q}$ and θ in $(1.1)_4$ will bring some new differences when deriving the pointwise estimates for both the Green's function and the nonlinear coupling. See the details in Section 2 and Section 3. Our main result can be stated as follows.

Theorem 1.1. *Assume that $(\rho_0 - \bar{\rho}, \rho_0\mathbf{u}_0, w_0 - \bar{w}) \in H^5(\mathbb{R}^3)$, with $\varepsilon_0 := \|(\rho_0 - \bar{\rho}, \rho_0\mathbf{u}_0, w_0 - \bar{w})\|_{H^5(\mathbb{R}^3)}$ small and the constants $\bar{\rho} > 0$ and $\bar{w} > 0$. Then there is a unique global classical solution $(\rho, \rho\mathbf{u}, w)$ of the Cauchy problem* (1.1)-(1.4). *If further, the initial data satisfies the pointwise assumptions*

$$|\partial_x^\alpha(\rho_0 - \bar{\rho}, \rho_0\mathbf{u}_0, w_0 - \bar{w})| \le C\varepsilon_0(1+|x|^2)^{-r_1}, \quad r_1 > \frac{21}{10},\ |\alpha| \le 1. \tag{1.5}$$

Then for the base sound speed $c := \sqrt{(1+\frac{R}{C_v})\frac{R\bar{w}}{C_v\bar{\rho}}}$*, it holds for* $|\alpha| \le 1$ *that*

$$
\begin{aligned}
|\partial_x^\alpha(\rho-\bar{\rho},\rho\mathbf{u})| &\le C(1+t)^{-\frac{4+|\alpha|}{2}}\Big(1+\frac{(|x|-ct)^3}{1+t}\Big)^{-3/2} \\
&\quad + C(1+t)^{-\frac{3+|\alpha|}{2}}\Big(1+\frac{|x|^2}{1+t}\Big)^{-3/2}, \qquad (1.6)\\
|\partial_x^\alpha(w-\bar{w})| &\le C(1+t)^{-\frac{4+|\alpha|}{2}}\Big(\Big(1+\frac{(|x|-ct)^3}{1+t}\Big)^{-3/2}+\Big(1+\frac{|x|^2}{1+t}\Big)^{-3/2}\Big).
\end{aligned}
$$

The base sound speed $c = \sqrt{(1+\frac{R}{C_v})\frac{R\bar{w}}{C_v\bar{\rho}}}$ is the same as the non-isentropic Navier-Stokes systems in [3], which means that the radiation does not affect the propagation speed of the Huygens' wave. Note that (1.6) exhibiting the generalized Huygens' principle is completely different from (1.3) for the model (1.2).

From the spectral analysis in Section 2, we know that the system is linear stable when $\kappa > 0$ and $\kappa = 0$, and these two cases are almost the same when analyzing Green's function in different frequency parts. In other words, we can see that for radiative hydrodynamics, the radiation effect can do the same job as the heat conduction when deducing the space-time behavior of the solution.

As a byproduct, one get L^p-decay estimate when $|\alpha| \le 1$:

$$
\begin{aligned}
\|\partial_x^\alpha(\rho-\bar{\rho},\rho\mathbf{u})(\cdot,t)\|_{L^p(\mathbb{R}^3)} &\le \begin{cases} C(1+t)^{-\left(2-\frac{5}{2p}\right)-\frac{|\alpha|}{2}}, & 1<p\le 2,\\ C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}, & 2\le p\le\infty, \end{cases} \qquad (1.7)\\
\|\partial_x^\alpha(w-\bar{w})(\cdot,t)\|_{L^p(\mathbb{R}^3)} &\le C(1+t)^{-(2-\frac{1}{p})-\frac{|\alpha|}{2}}, \quad 1<p\le\infty.
\end{aligned}
$$

This article is structured as follows. Section 2 shows the representation of Green's function in Fourier space and giving the expansions of spectral and Green's function in high-medium-low frequencies, and then establishes the pointwise estimate of Green's function. The pointwise estimates of the solution for the nonlinear problem is given in Section 3. Some useful lemmas are stated in Appendix.

2. Green's function

In this section, we shall first derive the representation of Green's function in Fourier space, and then establish pointwise estimates of Green's function.

2.1. **Linearization and Green's function.**

Here we assume the steady state of the Cauchy problem (1.1)-(1.4) is $(\bar{\rho},0,\bar{w})$.

For simplicity, we use $(\rho,\mathbf{m},w)$ to denote the perturbation $(\rho-\bar{\rho},\mathbf{m},w-\bar{w})$. Then, the system (1.1) can be rewritten as

$$
\begin{gathered}
\rho_t + \operatorname{div}\mathbf{m} = 0,\\
\mathbf{m}_t + \frac{R}{C_v}\nabla w - \frac{\mu}{\bar{\rho}}\Delta\mathbf{m} - \frac{(\mu+\mu')}{\bar{\rho}}\nabla\operatorname{div}\mathbf{m} = F_1,\\
w_t + \frac{\bar{w}}{\bar{\rho}}(1+\frac{R}{C_v})\operatorname{div}\mathbf{m} + \frac{\kappa\bar{w}}{C_v\bar{\rho}^2}\Delta\rho + \frac{4\bar{w}^4}{C_v^4\bar{\rho}^5}\frac{\Delta}{1-\Delta}\rho \\
- \frac{\kappa}{C_v\bar{\rho}}\Delta w - \frac{4\bar{w}^3}{C_v^4\bar{\rho}^4}\frac{\Delta}{1-\Delta}w = F_2,
\end{gathered} \qquad (2.1)
$$

where the nonlinear terms are

$$
\begin{aligned}
F_1 &= -\operatorname{div}\frac{\mathbf{m}\otimes\mathbf{m}}{\rho+\bar\rho}+\frac{R}{2C_v}\nabla\frac{|\mathbf{m}|^2}{\rho+\bar\rho}-\mu\Delta\frac{\rho\mathbf{m}}{\bar\rho(\rho+\bar\rho)}-(\mu+\mu')\nabla\operatorname{div}\frac{\rho\mathbf{m}}{\bar\rho(\rho+\bar\rho)},\\
F_2 &= \frac{\kappa}{C_v\bar\rho^2}\Delta\frac{(\bar w\rho-\bar\rho w)\rho}{\rho+\bar\rho}-\frac{\kappa}{2C_v}\Delta\frac{|\mathbf{m}|^2}{(\rho+\bar\rho)^2}+(1+\frac{R}{C_v})\operatorname{div}\frac{(\bar w\rho-\bar\rho w)\mathbf{m}}{\bar\rho(\rho+\bar\rho)}\\
&\quad+\frac{R}{2C_v}\operatorname{div}\frac{|\mathbf{m}|^2\mathbf{m}}{(\rho+\bar\rho)^2}\\
&\quad+\frac{1}{C_v^4}\frac{\Delta}{1-\Delta}\frac{4\bar w^3(\bar w\rho-\bar\rho w)(\rho^4+4\bar\rho\rho^3+6\bar\rho^2\rho^2+4\bar\rho^3\rho)}{\bar\rho^5(\rho+\bar\rho)^4}\\
&\quad+\frac{1}{C_v^4}\frac{\Delta}{1-\Delta}\Big[\frac{w^4+4\bar w w^3+6\bar w^2w^2}{(\rho+\bar\rho)^4}-\frac{\bar w^4(\rho^4+4\bar\rho\rho^3+6\bar\rho^2\rho^2)}{\bar\rho^4(\rho+\bar\rho)^4}\Big]\\
&\quad+\frac{1}{C_v^4}\frac{\Delta}{1-\Delta}\Big[\frac{|\mathbf{m}|^8}{16(\rho+\bar\rho)^8}-\frac{(w+\bar w)|\mathbf{m}|^6}{2(\rho+\bar\rho)^7}+\frac{3(w+\bar w)^2|\mathbf{m}|^4}{2(\rho+\bar\rho)^6}\\
&\quad-\frac{2(w+\bar w)^3|\mathbf{m}|^2}{(\rho+\bar\rho)^5}\Big]+\operatorname{div}\Big[\mu(\nabla(\frac{\mathbf{m}}{\rho+\bar\rho})+\nabla(\frac{\mathbf{m}}{\rho+\bar\rho})^T)\cdot\frac{\mathbf{m}}{\rho+\bar\rho}\\
&\quad+\mu'\operatorname{div}(\frac{\mathbf{m}}{\rho+\bar\rho})\frac{\mathbf{m}}{\rho+\bar\rho}\Big].
\end{aligned}
\tag{2.2}
$$

We consider Green's function of the linearized system of (2.1) about the variables $(\rho,\mathbf{m},w)$. The linearized equation with the initial condition can be written as

$$
\begin{gathered}
\rho_t+\operatorname{div}\mathbf{m}=0,\\
\mathbf{m}_t+\frac{R}{C_v}\nabla w-\frac{\mu}{\bar\rho}\Delta\mathbf{m}-\frac{(\mu+\mu')}{\bar\rho}\nabla\operatorname{div}\mathbf{m}=0,\\
w_t+\frac{\bar w}{\bar\rho}(1+\frac{R}{C_v})\operatorname{div}\mathbf{m}+\frac{\kappa\bar w}{C_v\bar\rho^2}\Delta\rho+\frac{4\bar w^4}{C_v^4\bar\rho^5}\frac{\Delta}{1-\Delta}\rho\\
-\frac{\kappa}{C_v\bar\rho}\Delta w-\frac{4\bar w^3}{C_v^4\bar\rho^4}\frac{\Delta}{1-\Delta}w=0,\\
(\rho,\mathbf{m},w)|_{t=0}=(\rho_0,\mathbf{m}_0,w_0).
\end{gathered}
\tag{2.3}
$$

Set $U=(\rho,\mathbf{m},w)^T$, then

$$
\partial_tU=AU, \tag{2.4}
$$

where A is the differential operator corresponding to (2.3) and

$$
A=\begin{pmatrix}
0 & -\operatorname{div} & 0\\
0 & \frac{\mu}{\bar\rho}\Delta+\frac{\mu+\mu'}{\bar\rho}\nabla\operatorname{div} & -\frac{R}{C_v}\nabla\\
-\frac{\kappa\bar w}{C_v\bar\rho^2}\Delta-\frac{4\bar w^4}{C_v^4\bar\rho^5}\frac{\Delta}{1-\Delta} & -\frac{\bar w}{\bar\rho}(1+\frac{R}{C_v})\operatorname{div} & \frac{\kappa}{C_v\bar\rho}\Delta+\frac{4\bar w^3}{C_v^4\bar\rho^4}\frac{\Delta}{1-\Delta}
\end{pmatrix}.
$$

In what follows, we use $G(x,t)$ to denote the Green's function of (2.3), which is defined as

$$
\begin{gathered}
G_t=AG,\\
G(x,0)=\delta(x)I.
\end{gathered}
\tag{2.5}
$$

Here $\delta(x)$ is the Dirac delta function and I is the 5×5 identity matrix.

2.2. **Derivation of Green's function $\hat{G}(\xi,t)$.** In this subsection, we compute Green's function for the linearized system of (2.1) with the initial data

$$\hat{U}_0 := (\hat{\rho}_0, \hat{\mathbf{m}}_0, \hat{w}_0)^T = (\hat{\rho}_0, \hat{m}_{0,1}, \hat{m}_{0,2}, \hat{m}_{0,3}, \hat{w}_0)^T. \tag{2.6}$$

By employing the Fourier transform, it is obvious from (2.4) that

$$\partial_t \hat{U} = A_1 \hat{U}, \tag{2.7}$$

where

$$A_1 = \begin{pmatrix} 0 & -i\xi^T & 0 \\ 0 & -\frac{\mu}{\bar{\rho}}|\xi|^2 I - \frac{(\mu+\mu')}{\bar{\rho}}\xi\xi^T & -\frac{R}{C_v} i\xi \\ \frac{k\bar{w}}{C_v\bar{\rho}^2}|\xi|^2 + \frac{4\bar{w}^4|\xi|^2}{C_v^4\bar{\rho}^5(1+|\xi|^2)} & -\frac{\bar{w}}{\bar{\rho}}(1+\frac{R}{C_v})i\xi^T & -\frac{\kappa}{C_v\bar{\rho}}|\xi|^2 - \frac{4\bar{w}^3|\xi|^2}{C_v^4\bar{\rho}^4(1+|\xi|^2)} \end{pmatrix}.$$

After a direct computation, we can get the eigenvalues of A_1: λ_1, $\lambda_2, \lambda_3, -\frac{\mu}{\bar{\rho}}\xi^2$ (with multiplicity 2), and their corresponding right eigenvectors

$$\begin{pmatrix} g_1 \\ i\xi_1 \\ i\xi_2 \\ i\xi_3 \\ \nu_1 \end{pmatrix}, \quad \begin{pmatrix} g_2 \\ i\xi_1 \\ i\xi_2 \\ i\xi_3 \\ \nu_2 \end{pmatrix}, \quad \begin{pmatrix} g_3 \\ i\xi_1 \\ i\xi_2 \\ i\xi_3 \\ \nu_3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \xi_3 \\ 0 \\ -\xi_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ \xi_3 \\ -\xi_2 \\ 0 \end{pmatrix}, \tag{2.8}$$

with

$$g_1 := \frac{|\xi|^2}{\lambda_1}, \quad g_2 := \frac{|\xi|^2}{\lambda_2}, \quad g_3 := \frac{|\xi|^2}{\lambda_3},$$
$$\nu_1 := -\frac{C_v(2\mu+\mu')}{R\bar{\rho}}|\xi|^2 - \frac{C_v}{R}\lambda_1, \quad \nu_2 := -\frac{C_v(2\mu+\mu')}{R\bar{\rho}}|\xi|^2 - \frac{C_v}{R}\lambda_2,$$
$$\nu_3 := -\frac{C_v(2\mu+\mu')}{R\bar{\rho}}|\xi|^2 - \frac{C_v}{R}\lambda_3.$$

Here λ_1, λ_2, and λ_3 are the roots of the equation

$$\begin{aligned} &\lambda^3 + (\frac{\kappa}{C_v\bar{\rho}} + \frac{4\bar{w}^3}{C_v^4\bar{\rho}^4(1+|\xi|^2)} + \frac{2\mu+\mu'}{\bar{\rho}})|\xi|^2\lambda^2 + [\frac{\kappa(2\mu+\mu')}{C_v\bar{\rho}^2}|\xi|^4 \\ &+ \frac{4(2\mu+\mu')\bar{w}^3|\xi|^4}{C_v^4\bar{\rho}^5(1+|\xi|^2)} + \frac{(C_v+R)R\bar{w}}{C_v^2\bar{\rho}}|\xi|^2]\lambda + \frac{\kappa R\bar{w}}{C_v^2\bar{\rho}^2}|\xi|^4 + \frac{4R\bar{w}^4|\xi|^4}{C_v^5\bar{\rho}^5(1+|\xi|^2)} = 0. \end{aligned} \tag{2.9}$$

Then the solution of the Cauchy problem (2.6)-(2.7) can be written as

$$\begin{aligned} \begin{pmatrix} \hat{\rho} \\ \hat{m}_1 \\ \hat{m}_2 \\ \hat{m}_3 \\ \hat{w} \end{pmatrix} &= \left[A \begin{pmatrix} 0 \\ \xi_3 \\ 0 \\ -\xi_1 \\ 0 \end{pmatrix} + B \begin{pmatrix} 0 \\ 0 \\ \xi_3 \\ -\xi_2 \\ 0 \end{pmatrix} \right] \mathrm{e}^{-\frac{\mu}{\bar{\rho}}|\xi|^2 t} + C \begin{pmatrix} g_1 \\ i\xi_1 \\ i\xi_2 \\ i\xi_3 \\ \nu_1 \end{pmatrix} \mathrm{e}^{\lambda_1 t} \\ &\quad + D \begin{pmatrix} g_2 \\ i\xi_1 \\ i\xi_2 \\ i\xi_3 \\ \nu_2 \end{pmatrix} \mathrm{e}^{\lambda_2 t} + E \begin{pmatrix} g_3 \\ i\xi_1 \\ i\xi_2 \\ i\xi_3 \\ \nu_3 \end{pmatrix} \mathrm{e}^{\lambda_3 t}. \end{aligned} \tag{2.10}$$

By using the initial values, we have

$$A\begin{pmatrix}0\\ \xi_3\\ 0\\ -\xi_1\\ 0\end{pmatrix}+B\begin{pmatrix}0\\ 0\\ \xi_3\\ -\xi_2\\ 0\end{pmatrix}=\begin{pmatrix}0\\ (I-\frac{\xi\xi^\tau}{|\xi|^2})\hat{m}_{0,1}\\ (I-\frac{\xi\xi^\tau}{|\xi|^2})\hat{m}_{0,2}\\ (I-\frac{\xi\xi^\tau}{|\xi|^2})\hat{m}_{0,3}\\ 0\end{pmatrix}, \tag{2.11}$$

$$C\begin{pmatrix}g_1\\ i\xi\\ \nu_1\end{pmatrix}+D\begin{pmatrix}g_2\\ i\xi\\ \nu_2\end{pmatrix}+E\begin{pmatrix}g_3\\ i\xi\\ \nu_3\end{pmatrix}=\begin{pmatrix}\hat{\rho}_0\\ \frac{\xi\xi^T}{|\xi|^2}\hat{\mathbf{m}}_0\\ \hat{w}_0\end{pmatrix}, \tag{2.12}$$

with

$$\begin{gathered}C=C_1\hat{\rho}_0+C_2\frac{\xi^\tau\cdot\hat{\mathbf{m}}_0 i}{|\xi|^2}+C_3\hat{w}_0,\quad D=D_1\hat{\rho}_0+D_2\frac{\xi^\tau\cdot\hat{\mathbf{m}}_0 i}{|\xi|^2}+D_3\hat{w}_0,\\ E=E_1\hat{\rho}_0+E_2\frac{\xi^\tau\cdot\hat{\mathbf{m}}_0 i}{|\xi|^2}+E_3\hat{w}_0,\end{gathered} \tag{2.13}$$

and

$$\begin{gathered}C_1=\frac{\nu_3-\nu_2}{\Omega},\quad C_2=\frac{\nu_3g_2-\nu_2g_3}{\Omega},\quad C_3=\frac{g_2-g_3}{\Omega},\\ D_1=\frac{\nu_1-\nu_3}{\Omega},\quad D_2=\frac{\nu_1g_3-\nu_3g_1}{\Omega},\quad D_3=\frac{g_3-g_1}{\Omega},\\ E_1=\frac{\nu_2-\nu_1}{\Omega},\quad E_2=\frac{\nu_2g_1-\nu_1g_2}{\Omega},\quad E_3=\frac{g_1-g_2}{\Omega}.\end{gathered} \tag{2.14}$$

Here $\Omega=-\nu_3g_2+\nu_3g_1+\nu_1g_2+\nu_2g_3-\nu_2g_1-\nu_1g_3$.

Therefore, the Fourier transform of Green's function can be rewritten as

$$\hat{G}=\hat{\hat{G}}e^{-\frac{\mu}{\rho}|\xi|^2t}+\hat{G}^1e^{\lambda_1t}+\hat{G}^2e^{\lambda_2t}+\hat{G}^3e^{\lambda_3t}, \tag{2.15}$$

with

$$\hat{\hat{G}}=\begin{pmatrix}0 & 0_{1\times3} & 0\\ 0_{3\times1} & I-\frac{\xi\xi^\tau}{|\xi|^2} & 0_{3\times1}\\ 0 & 0_{1\times3} & 0\end{pmatrix},\quad \hat{G}^1=\begin{pmatrix}g_1C_1 & g_1C_2\frac{i\xi^\tau}{|\xi|^2} & g_1C_3\\ i\xi C_1 & -C_2\frac{\xi\xi^\tau}{|\xi|^2} & i\xi C_3\\ \nu_1C_1 & \nu_1C_2\frac{i\xi^\tau}{|\xi|^2} & \nu_1C_3\end{pmatrix}, \tag{2.16}$$

$$\hat{G}^2=\begin{pmatrix}g_2D_1 & g_2D_2\cdot\frac{i\xi^\tau}{|\xi|^2} & g_2D_3\\ i\xi D_1 & -D_2\frac{\xi\xi^\tau}{|\xi|^2} & i\xi D_3\\ \nu_2D_1 & \nu_2D_2\cdot\frac{i\xi^\tau}{|\xi|^2} & \nu_2D_3\end{pmatrix},\quad \hat{G}^3=\begin{pmatrix}g_3E_1 & g_3E_2\frac{i\xi^\tau}{|\xi|^2} & g_3E_3\\ i\xi E_1 & -E_2\frac{\xi\xi^\tau}{|\xi|^2} & i\xi E_3\\ \nu_3E_1 & \nu_3E_2\frac{i\xi^\tau}{|\xi|^2} & \nu_3E_3\end{pmatrix} \tag{2.17}$$

2.3. **Pointwise estimates of Green's function.** In this section, we shall derive the pointwise estimate of Green's function $G(x,t)$ by using the expression of $\hat{G}(\xi,t)$ in (2.15)-(2.17) together with the high-low frequency decomposition. To this end, we divide the Green's function as follows:

$$G(x,t):=\chi_1(D)G(x,t)+\chi_2(D)G(x,t)+\chi_3(D)G(x,t),$$

where

$$\chi_1(\xi)=\begin{cases}1, & |\xi|<\varepsilon_1,\\ 0, & |\xi|>2\varepsilon_1,\end{cases}\quad \chi_3(\xi)=\begin{cases}1, & |\xi|>K+1,\\ 0, & |\xi|<K,\end{cases}$$

be the smooth cut-off functions with $2\varepsilon_1<K$ and $\chi_2=1-\chi_1-\chi_3$.

Low frequency part. Three roots of the characteristic equation (2.9) in low frequency can be estimated by using the implicit function theorem as in [20], and we omit the proof for simplicity.

Lemma 2.1. *For sufficiently small $|\xi|$, λ_1 is real and $\lambda_{2,3}$ are complex conjugate, and*

$$\lambda_1 = -\frac{\frac{\kappa}{\bar\rho} + \frac{4\bar w^3}{C_v^3\bar\rho^4}}{C_v + R}|\xi|^2 + \sum_{j=2}^{\infty} a_j|\xi|^{2j}; \tag{2.18}$$

$$\begin{aligned}\lambda_{2,3} = &-\Big[\frac{2\mu+\mu'}{2\bar\rho} + \frac{R(\frac{\kappa}{2C_v\bar\rho} + \frac{2\bar w^3}{C_v^4\bar\rho^4})}{C_v+R}\Big]|\xi|^2 \\ &+ \sum_{j=2}^{\infty}\bar a_{2j}|\xi|^{2j} \pm i(c|\xi| + \sum_{j=2}^{\infty}\bar a_{2j-1}|\xi|^{2j-1}),\end{aligned} \tag{2.19}$$

where the base sound speed $c = \sqrt{(1+\frac{R}{C_v})\frac{R\bar w}{C_v\bar\rho}}$, and the coefficients a_j, $\bar a_{2j}$ and $\bar a_{2j-1}$ are real.

Subsequently, one has the following for the components in (2.14) after a direct computation.

Lemma 2.2. *For sufficiently small $|\xi|$, we have the following expansions:*

$$g_1 = -\frac{C_v+R}{\frac{\kappa}{\bar\rho} + \frac{4\bar w^3}{C_v^3\bar\rho^4}} + \sum_{j=1}^{\infty} b_{2j}|\xi|^{2j}, \quad \nu_1 = \frac{C_v}{R}\Big(\frac{\frac{\kappa}{\bar\rho} + \frac{4\bar w^3}{C_v^3\bar\rho^4}}{C_v+R} - \frac{2\mu+\mu'}{\bar\rho}\Big)|\xi|^2 + \sum_{j=2}^{\infty} c_{2j}|\xi|^{2j},$$

$$g_{2,3} = -\frac{\frac{2\mu+\mu'}{2\bar\rho} + \frac{R(\frac{\kappa}{2C_v\bar\rho} + \frac{2\bar w^3}{C_v^4\bar\rho^4})}{C_v+R}}{c^2}|\xi|^2 + \sum_{j=2}^{\infty}\tilde b_{2j}|\xi|^{2j} \mp i(\frac{1}{c}|\xi| + \sum_{j=2}^{\infty}\tilde b_{2j-1}|\xi|^{2j-1}),$$

$$\nu_{2,3} = \Big[\frac{\frac{\kappa}{2\bar\rho} + \frac{2\bar w^3}{C_v^3\bar\rho^4}}{C_v+R} - \frac{C_v(2\mu+\mu')}{2R\bar\rho}\Big]|\xi|^2 + \sum_{j=2}^{\infty}\tilde c_{2j}|\xi|^{2j} \mp \frac{C_v}{R}i(c|\xi| + \sum_{j=2}^{\infty}\tilde c_{2j-1}|\xi|^{2j-1}),$$

$$\Omega = -i\frac{2c}{R}\frac{C_v(C_v+R)}{\frac{\kappa}{\bar\rho} + \frac{4\bar w^3}{C_v^3\bar\rho^4}}|\xi| + i\sum_{j=2}^{\infty}\breve c|\xi|^{2j-1},$$

$$C_1 = -\frac{\frac{\kappa}{\bar\rho} + \frac{4\bar w^3}{C_v^3\bar\rho^4}}{C_v+R} + \dots, \quad C_2 = \frac{R(\frac{\kappa}{\bar\rho} + \frac{4\bar w^3}{C_v^3\bar\rho^4})^2}{c^2C_v(C_v+R)^2}|\xi|^2 + \cdots,$$

$$C_3 = \frac{R(\frac{\kappa}{\bar\rho} + \frac{4\bar w^3}{C_v^3\bar\rho^4})}{c^2C_v(C_v+R)}|\xi|^2 + \cdots,$$

$$D_1 = \frac{\frac{\kappa}{\bar\rho} + \frac{4\bar w^3}{C_v^3\bar\rho^4}}{2(C_v+R)} + \cdots, \quad D_2 = -\frac{1}{2} + \cdots, \quad D_3 = i\frac{1}{2c}\frac{R}{C_v}\frac{1}{|\xi|} + \cdots,$$

$$E_1 = \frac{\frac{\kappa}{\bar\rho} + \frac{4\bar w^3}{C_v^3\bar\rho^4}}{2(C_v+R)} + \cdots, \quad E_2 = -\frac{1}{2} + \cdots, \quad E_3 = -i\frac{1}{2c}\frac{R}{C_v}\frac{1}{|\xi|} + \cdots.$$

Here and below, "$\cdots$" denote the reminding terms, which do not affect the results. Lemma 2.2 and the expression of $\hat G, \hat{\hat G}, \hat G^1, \hat G^2, \hat G^3$ immediately yield the following result.

Lemma 2.3. *For sufficiently small $|\xi|$, we have the following:*

$$
\begin{gathered}
\hat{G}_{11}=\mathrm{e}^{\lambda_1 t}-i\frac{\frac{\kappa}{2\bar\rho}+\frac{2\bar w^3}{C_v^3\bar\rho^4}}{c(C_v+R)}|\xi|(\mathrm{e}^{\lambda_2 t}-\mathrm{e}^{\lambda_3 t})+\cdots,\\
\hat{G}_{12}=-i\frac{R(\frac{\kappa}{\bar\rho}+\frac{4\bar w^3}{C_v^3\bar\rho^4})}{c^2C_v(C_v+R)}\xi^T\mathrm{e}^{\lambda_1 t}-\frac{1}{2c}\frac{\xi^T}{|\xi|}(\mathrm{e}^{\lambda_2 t}-\mathrm{e}^{\lambda_3 t})+\cdots,\\
\hat{G}_{13}=-\frac{R}{c^2C_v}\mathrm{e}^{\lambda_1 t}+\frac{R}{2c^2C_v}(\mathrm{e}^{\lambda_2 t}+\mathrm{e}^{\lambda_3 t})+\cdots,\\
\hat{G}_{21}=-i\frac{(\frac{\kappa}{\bar\rho}+\frac{4\bar w^3}{C_v^3\bar\rho^4})}{C_v+R}\xi\mathrm{e}^{\lambda_1 t}+i\frac{(\frac{\kappa}{\bar\rho}+\frac{4\bar w^3}{C_v^3\bar\rho^4})}{2(C_v+R)}\xi(\mathrm{e}^{\lambda_2 t}+\mathrm{e}^{\lambda_3 t})+\cdots,\\
\hat{G}_{22}=(I-\frac{\xi\xi^T}{|\xi|^2})\mathrm{e}^{-\frac{\mu}{\bar\rho}|\xi|^2 t}-\frac{R(\frac{\kappa}{\bar\rho}+\frac{4\bar w^3}{C_v^3\bar\rho^4})^2}{c^2C_v(C_v+R)^2}\xi\xi^T\mathrm{e}^{\lambda_1 t}+\frac{1}{2}\frac{\xi\xi^T}{|\xi|^2}(\mathrm{e}^{\lambda_2 t}+\mathrm{e}^{\lambda_3 t})+\cdots,\\
\hat{G}_{23}=i\frac{R(\frac{\kappa}{\bar\rho}+\frac{4\bar w^3}{C_v^3\bar\rho^4})}{c^2C_v(C_v+R)^2}\xi\mathrm{e}^{\lambda_1 t}-\frac{R}{2cC_v}\frac{\xi}{|\xi|}(\mathrm{e}^{\lambda_2 t}-\mathrm{e}^{\lambda_3 t})+\cdots,\\
\begin{aligned}
\hat{G}_{31}=&\frac{C_v(\frac{\kappa}{\bar\rho}+\frac{4\bar w^3}{C_v^3\bar\rho^4})}{R(C_v+R)}(\frac{2\mu+\mu'}{\bar\rho}-\frac{\frac{\kappa}{\bar\rho}+\frac{4\bar w^3}{C_v^3\bar\rho^4}}{C_v+R})|\xi|^2\mathrm{e}^{\lambda_1 t}\\
&-i\frac{cC_v(\frac{\kappa}{\bar\rho}+\frac{4\bar w^3}{C_v^3\bar\rho^4})}{2R(C_v+R)}|\xi|(\mathrm{e}^{\lambda_2 t}-\mathrm{e}^{\lambda_3 t})+\cdots,
\end{aligned}\\
\hat{G}_{32}=i\frac{(\frac{\kappa}{\bar\rho}+\frac{4\bar w^3}{C_v^3\bar\rho^4})^2}{c^2(C_v+R)^2}(\frac{\frac{\kappa}{\bar\rho}+\frac{4\bar w^3}{C_v^3\bar\rho^4}}{C_v+R}-\frac{2\mu+\mu'}{\bar\rho})|\xi|^2\xi^T\mathrm{e}^{\lambda_1 t}-\frac{cC_v}{2R}\frac{\xi^T}{|\xi|}(\mathrm{e}^{\lambda_2 t}-\mathrm{e}^{\lambda_3 t})+\cdots,\\
\hat{G}_{33}=\frac{\frac{\kappa}{\bar\rho}+\frac{4\bar w^3}{C_v^3\bar\rho^4}}{c^2(C_v+R)}(\frac{\frac{\kappa}{\bar\rho}+\frac{4\bar w^3}{C_v^3\bar\rho^4}}{C_v+R}-\frac{2\mu+\mu'}{\bar\rho})|\xi|^2\mathrm{e}^{\lambda_1 t}+\frac{1}{2}(\mathrm{e}^{\lambda_2 t}+\mathrm{e}^{\lambda_3 t})+\cdots.
\end{gathered}
$$

In fact, one can deal with the leading term in the low frequency part since the rest terms just have the faster temporal decay rate. Indeed, the Huygens' wave is arising from the low frequency part based on the above frequency analysis. Although the low frequency part is basically the same as the non-isentropic Navier-Stokes equations in [3], for completeness, we still take a typical leading term in $\hat{G}_{22}^l$ for example. In particular, one has

$$
\begin{aligned}
&[\frac{1}{2}(e^{\lambda_2 t}+e^{\lambda_3 t})\frac{\xi\xi^T}{|\xi|^2}+e^{-\frac{\mu}{\bar\rho}|\xi|^2 t}(I-\frac{\xi\xi^T}{|\xi|^2})]\\
&=\cos(c|\xi|t)\frac{\xi\xi^T}{|\xi|^2}\left[e^{-\theta_1|\xi|^2 t+\mathcal{O}(|\xi|^4)t}\cos(|\xi|\beta(|\xi|^2)t)\right]\\
&\quad-\frac{\sin(|\xi|\beta(|\xi|^2)t)}{|\xi|}\left[|\xi|\sin(|\xi|\beta(|\xi|^2)t)\right]\frac{\xi\xi^T}{|\xi|^2}e^{-\theta_1|\xi|^2 t+\mathcal{O}(|\xi|^4)t}+e^{-\frac{\mu}{\bar\rho}|\xi|^2 t}(I-\frac{\xi\xi^T}{|\xi|^2})\\
&=\underbrace{(\cos(c|\xi|t)-1)\frac{\xi\xi^T}{|\xi|^2}e^{-\theta_1|\xi|^2 t}}_{I_1}+\underbrace{\frac{\xi\xi^T}{|\xi|^2}\left(e^{-\theta_1|\xi|^2 t}-e^{-\frac{\mu}{\bar\rho}|\xi|^2 t}\right)}_{I_2}+e^{-\frac{\mu}{\bar\rho}|\xi|^2 t}I\\
&\quad+\underbrace{\cos(c|\xi|t)\frac{\xi\xi^T}{|\xi|^2}e^{-\theta_1|\xi|^2 t}(\cos(|\xi|\beta(|\xi|^2)t)-1)}_{I_3}
\end{aligned}
$$

$$+\underbrace{\cos(c|\xi|t)\frac{\xi\xi^T}{|\xi|^2}\big(e^{\mathcal{O}(|\xi|^4)t}-1\big)\cos(|\xi|\beta(|\xi|^2)t)e^{-\theta_1|\xi|^2t}}_{I_4}$$
$$-\underbrace{\frac{\sin(|\xi|\beta(|\xi|^2)t)}{|\xi|}\frac{|\xi|\sin(|\xi|\beta(|\xi|^2)t)}{|\xi|^2}\xi\xi^Te^{-\theta_1|\xi|^2t+\mathcal{O}(|\xi|^4)t}}_{I_5},$$

where $\beta(\cdot)$ is analytic and the constant $\theta_1>0$ based on Lemma 2.1. I_1 and I_2 are corresponding to Riesz waves I and II in [3, 20] for both the isentropic and non-isentropic compressible Navier-Stokes system. Their pointwise space-time descriptions are

$$\begin{gathered}|I_1|\le C\Big((1+t)^{-3/2}\big(1+\frac{|x|^2}{1+t}\big)^{-3/2}+(1+t)^{-2}\big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-N}\Big),\\ |I_2|\le C(1+t)^{-3/2}\big(1+\frac{|x|^2}{1+t}\big)^{-N},\end{gathered}\tag{2.20}$$

with an arbitrarily large integer N. Next we consider $\frac{\xi\xi^T}{|\xi|^2}e^{-\theta_1|\xi|^2t}(\cos(|\xi|\beta(|\xi|^2)t)-1)$ in I_3, $\frac{\xi\xi^T}{|\xi|^2}\big(e^{\mathcal{O}(|\xi|^4)t}-1\big)\cos(|\xi|\beta(|\xi|^2)t)e^{-\theta_1|\xi|^2t}$ in I_4 and $\big[|\xi|\sin(|\xi|\beta(|\xi|^2)t)\big]\frac{\xi\xi^T}{|\xi|^2}$ in I_5. Because of their analyticities and the faster decay rates, after a direct computation for these rest terms as in [20, Lemma 5.4], the inverse Fourier transform of the first two terms can be bounded by $C(1+t)^{-3/2}\big(1+\frac{|x|^2}{1+t}\big)^{-N}$, and inverse Fourier transform of the last term can be bounded by $C(1+t)^{-2}\big(1+\frac{|x|^2}{1+t}\big)^{-N}$. Recall the standard convolution estimates in [3, 20]: If

$$|\partial_x^\alpha f(x,t)|\le(1+t)^{-\frac{|\alpha|+n+k}{2}}\big(1+\frac{|x|^2}{1+t}\big)^{-N},x\in\mathbb{R}^n,$$

for all $N>0$, then

$$\begin{gathered}|\partial_x^\alpha\mathbf{w}_t*_xf(x,t)|\le C(1+t)^{-\frac{|\alpha|+n+k}{2}}(1+t)^{-\frac{n-1}{4}}\big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-N},\\ |\partial_x^\alpha\mathbf{w}*_xf(x,t)|\le C(1+t)^{-\frac{|\alpha|+n+k-1}{2}}(1+t)^{-\frac{n-1}{4}}\big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-N}.\end{gathered}\tag{2.21}$$

Here $\mathbf{w}_t=\cos(c|\xi|t)$ and $\mathbf{w}=\frac{\sin(c|\xi|t)}{|\xi|}$ are the Fourier transform of wave operators. Hence, one has

$$|\mathcal{F}^{-1}(I_3,I_4,I_5)|\le C(1+t)^{-2}\big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-N}.\tag{2.22}$$

Obtaining estimates for the other entries in Green's matrix in the low frequency part is much easier, and one can refer to [3, 20].

Proposition 2.4. *The low frequency of Green's function $G(x,t)$ for the system $(2.3)_{1,2,3}$ in 3 dimensional space has the following estimates for $t>0$ and $|\alpha|\ge0$:*

$$\begin{gathered}|\partial_x^\alpha(\chi_1(D)G_{ij})(x,t)|\le C(1+t)^{-\frac{4+|\alpha|}{2}}\big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-N}\\ \textit{for } (i,j)\ne(2,2),(3,1),(3,3),\end{gathered}$$

$$
\begin{aligned}
&|\partial_x^\alpha(\chi_1(D)G_{22})(x,t)| \\
&\le C(1+t)^{-\frac{3+|\alpha|}{2}}\big(1+\frac{|x|^2}{1+t}\big)^{-\frac{3+|\alpha|}{2}}+C(1+t)^{-\frac{4+|\alpha|}{2}}\big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-N},
\end{aligned}
$$

$$
\begin{aligned}
&|\partial_x^\alpha(\chi_1(D)(G_{31},G_{33}))(x,t)| \\
&\le C(1+t)^{-\frac{4+|\alpha|}{2}}\Big(\big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-N}+\big(1+\frac{|x|^2}{1+t}\big)^{-N}\Big).
\end{aligned}
$$

Note that the above estimates are crucial for us to show the different pointwise estimates between $(\rho,\mathbf{m})$ and w in (1.6).

High frequency part. In the same way, when $|\xi|\gg 1$ one can obtain the following expansions for $\lambda_1,\lambda_2,\lambda_3$.

Lemma 2.5. *For $|\xi|\gg 1$, λ_1, λ_2 and λ_3 are all real. Furthermore, when $\kappa>0$,*

$$
\begin{gathered}
\lambda_1=-\frac{R\bar w}{C_v(2\mu+\mu')}+\sum_{j=1}^\infty d_j^1|\xi|^{-2j};\\
\lambda_2=-\frac{\kappa}{C_v\bar\rho}|\xi|^2+\frac{R^2\bar w}{C_v[\kappa-C_v(2\mu+\mu')]}-\frac{4\bar w^3}{C_v^4\bar\rho^4}+\sum_{j=1}^\infty d_j^2|\xi|^{-2j};\\
\lambda_3=-\frac{2\mu+\mu'}{\bar\rho}|\xi|^2+\frac{R}{C_v}\frac{\bar w[\kappa-(C_v+R)(2\mu+\mu')]}{(2\mu+\mu')[\kappa-C_v(2\mu+\mu')]}+\sum_{j=1}^\infty d_j^3|\xi|^{-2j},
\end{gathered}
$$

where d_j^1,d_j^2,d_j^3 are real constants. When $\kappa=0$, we have

$$
\begin{aligned}
\lambda_1=\frac12\Big[&-\frac{4\bar w^3}{C_v^4\bar\rho^4}-\frac{(C_v+R)R\bar w}{C_v^2(2\mu+\mu')}\\
&+\sqrt{\big(\frac{4\bar w^3}{C_v^4\bar\rho^4}+\frac{(R-C_v)R\bar w}{C_v^2(2\mu+\mu')}\big)^2+\frac{4R^3\bar w^2}{C_v^3(2\mu+\mu')^2}}\Big]+\sum_{j=1}^\infty d_j^4|\xi|^{-2j},\\
\lambda_2=\frac12\Big[&-\frac{4\bar w^3}{C_v^4\bar\rho^4}-\frac{(C_v+R)R\bar w}{C_v^2(2\mu+\mu')}\\
&-\sqrt{\big(\frac{4\bar w^3}{C_v^4\bar\rho^4}+\frac{(R-C_v)R\bar w}{C_v^2(2\mu+\mu')}\big)^2+\frac{4R^3\bar w^2}{C_v^3(2\mu+\mu')^2}}\Big]+\sum_{j=1}^\infty d_j^5|\xi|^{-2j},
\end{aligned}
$$

$$
\lambda_3=-\frac{2\mu+\mu'}{\bar\rho}|\xi|^2+\frac{R\bar w(C_v+R)}{C_v^2(2\mu+\mu')}+\sum_{j=1}^\infty d_j^6|\xi|^{-2j},
$$

where d_j^4,d_j^5,d_j^6 are real constants.

As a result, one has the following result.

Lemma 2.6. *For sufficiently large $|\xi|$, when $\kappa>0$, we have the following:*

$$
\begin{gathered}
g_1=-\frac{C_v(2\mu+\mu')}{R\bar w}|\xi|^2+\sum_{j=0}^\infty g_{2j}|\xi|^{-2j},\\
\nu_1=-\frac{C_v(2\mu+\mu')}{R\bar\rho}|\xi|^2+\frac{\bar w}{2\mu+\mu'}+\sum_{j=1}^\infty \nu_{2j}|\xi|^{-2j},
\end{gathered}
$$

$$g_2=-\frac{C_v\bar\rho}{\kappa}+\sum_{j=1}^\infty\tilde g_{2j}|\xi|^{-2j},\quad \nu_2=\frac{\kappa-C_v(2\mu+\mu')}{R\bar\rho}|\xi|^2+\sum_{j=0}^\infty\bar\nu_{2j}|\xi|^{-2j},$$
$$g_3=-\frac{\bar\rho}{2\mu+\mu'}+\sum_{j=1}^\infty\tilde{\tilde g}_{2j}|\xi|^{-2j},$$
$$\nu_3=\frac{\bar w[\kappa-(C_v+R)(2\mu+\mu')]}{(2\mu+\mu')[\kappa-C_v(2\mu+\mu')]}+\sum_{j=1}^\infty\bar{\bar\nu}_{2j}|\xi|^{-2j},$$
$$\Omega=\frac{[\kappa-C_v(2\mu+\mu')]C_v(2\mu+\mu')}{R^2\bar\rho\bar w}|\xi|^4+\sum_{j=-1}^\infty\Omega_{2j}|\xi|^{-2j},$$

which implies

$$C_1=-\frac{R\bar w|\xi|^{-2}}{C_v(2\mu+\mu')}+\cdots,\quad C_2=\frac{R\bar\rho\bar w|\xi|^{-2}}{C_v(2\mu+\mu')^2}+\cdots,$$
$$C_3=\frac{R^2\bar\rho^2\bar w|\xi|^{-4}}{\kappa C_v(2\mu+\mu')^2}+\cdots,\quad D_1=\frac{R\bar w|\xi|^{-2}}{C_v(2\mu+\mu')-\kappa}+\cdots,$$
$$D_2=\frac{R^2\bar\rho\bar w|\xi|^{-2}}{[\kappa-C_v(2\mu+\mu')]^2}+\cdots,\quad D_3=\frac{R\bar\rho|\xi|^{-2}}{\kappa-C_v(2\mu+\mu')}+\cdots,$$
$$E_1=\frac{\kappa R\bar w|\xi|^{-2}}{C_v(2\mu+\mu')[\kappa-C_v(2\mu+\mu')]}+\cdots,\quad E_2=-1+\cdots,$$
$$E_3=\frac{R\bar\rho|\xi|^{-2}}{C_v(2\mu+\mu')-\kappa}+\cdots.$$

When $\kappa=0$, denoting

$$a_1=\frac12\Big[-\frac{4\bar w^3}{C_v^4\bar\rho^4}-\frac{(C_v+R)R\bar w}{C_v^2(2\mu+\mu')}+\sqrt{\Big(\frac{4\bar w^3}{C_v^4\bar\rho^4}+\frac{(R-C_v)R\bar w}{C_v^2(2\mu+\mu')}\Big)^2+\frac{4R^3\bar w^2}{C_v^3(2\mu+\mu')^2}}\Big],$$
$$a_2=\frac12\Big[-\frac{4\bar w^3}{C_v^4\bar\rho^4}-\frac{(C_v+R)R\bar w}{C_v^2(2\mu+\mu')}-\sqrt{\Big(\frac{4\bar w^3}{C_v^4\bar\rho^4}+\frac{(R-C_v)R\bar w}{C_v^2(2\mu+\mu')}\Big)^2+\frac{4R^3\bar w^2}{C_v^3(2\mu+\mu')^2}}\Big],$$

we obtain

$$g_1=\frac{1}{a_1}|\xi|^2+\sum_{j=0}^\infty g_{2j}|\xi|^{-2j},\quad \nu_1=-\frac{C_v(2\mu+\mu')}{R\bar\rho}|\xi|^2-\frac{C_v}{R}a_1+\sum_{j=1}^\infty\nu_{2j}|\xi|^{-2j},$$
$$g_2=\frac{1}{a_2}|\xi|^2+\sum_{j=0}^\infty\tilde g_{2j}|\xi|^{-2j},\quad \nu_2=-\frac{C_v(2\mu+\mu')}{R\bar\rho}|\xi|^2-\frac{C_v}{R}a_2+\sum_{j=1}^\infty\bar\nu_{2j}|\xi|^{-2j},$$
$$g_3=-\frac{\bar\rho}{2\mu+\mu'}+\sum_{j=1}^\infty\tilde{\tilde g}_{2j}|\xi|^{-2j},\quad \nu_3=-\frac{(C_v+R)\bar w}{C_v(2\mu+\mu')}+\sum_{j=1}^\infty\bar{\bar\nu}_{2j}|\xi|^{-2j},$$
$$\Omega=\frac{C_v(2\mu+\mu')(a_2-a_1)}{R\bar\rho a_1a_2}|\xi|^4+\sum_{j=-1}^\infty\Omega_{2j}|\xi|^{-2j},$$

which implies

$$C_1=\frac{a_1a_2|\xi|^{-2}}{a_2-a_1}+\cdots,\quad C_2=\frac{\big[\frac{(C_v+R)\bar w}{C_v(2\mu+\mu')a_2}+\frac{C_v}{R}\big]|\xi|^{-2}}{\frac{C_v(2\mu+\mu')(a_1-a_2)}{R\bar\rho a_1a_2}}+\cdots,$$

$$C_3 = \frac{R\bar{\rho}a_1|\xi|^{-2}}{C_v(2\mu+\mu')(a_2-a_1)} + \cdots, \quad D_1 = \frac{a_1a_2|\xi|^{-2}}{a_1-a_2} + \cdots,$$
$$D_2 = \frac{[\frac{(C_v+R)\bar{w}}{C_v(2\mu+\mu')a_1} + \frac{C_v}{R}]|\xi|^{-2}}{\frac{C_v(2\mu+\mu')(a_2-a_1)}{R\bar{\rho}a_1a_2}} + \cdots, \quad D_3 = \frac{R\bar{\rho}a_2|\xi|^{-2}}{C_v(2\mu+\mu')(a_1-a_2)} + \cdots,$$
$$E_1 = -\frac{\bar{\rho}a_1a_2|\xi|^{-4}}{2\mu+\mu'} + \cdots, \quad E_2 = -1 + \cdots, \quad E_3 = \frac{R\bar{\rho}|\xi|^{-2}}{C_v(2\mu+\mu')} + \cdots.$$

Here all of the above coefficients in Sigma summation symbols are real constants.

Lemma 2.7. *For sufficiently large $|\xi|$, when $\kappa > 0$, it holds*

$$\begin{aligned}
\hat{G}_{11} &= \mathrm{e}^{\lambda_1 t} + \frac{C_vR\bar{\rho}\bar{w}}{\kappa[\kappa - C_v(2\mu+\mu')]}\frac{1}{|\xi|^2}\mathrm{e}^{\lambda_2 t} \\
&\quad - \frac{\kappa R\bar{\rho}\bar{w}}{C_v(2\mu+\mu')^2[\kappa - C_v(2\mu+\mu')]}\frac{1}{|\xi|^2}\mathrm{e}^{\lambda_3 t} + \cdots,
\end{aligned}$$
$$\hat{G}_{12} = -\frac{\bar{\rho}}{2\mu+\mu'}\frac{i\xi^T}{|\xi|^2}\mathrm{e}^{\lambda_1 t} - \frac{C_vR^2\bar{\rho}^2\bar{w}}{\kappa[\kappa - C_v(2\mu+\mu')]^2}\frac{i\xi^T}{|\xi|^4}\mathrm{e}^{\lambda_2 t} + \frac{\bar{\rho}}{2\mu+\mu'}\frac{i\xi^T}{|\xi|^2}\mathrm{e}^{\lambda_3 t} + \cdots,$$
$$\begin{aligned}
\hat{G}_{13} &= -\frac{R\bar{\rho}^2}{\kappa(2\mu+\mu')}\frac{1}{|\xi|^2}\mathrm{e}^{\lambda_1 t} - \frac{RC_v\bar{\rho}^2}{\kappa[\kappa - C_v(2\mu+\mu')}\frac{1}{|\xi|^2}\mathrm{e}^{\lambda_2 t} \\
&\quad + \frac{R\bar{\rho}^2}{(2\mu+\mu')[\kappa - C_v(2\mu+\mu')]}\frac{1}{|\xi|^2}\mathrm{e}^{\lambda_3 t} + \cdots,
\end{aligned}$$
$$\begin{aligned}
\hat{G}_{21} &= \frac{-R\bar{w}}{C_v(2\mu+\mu')}\frac{i\xi}{|\xi|^2}\mathrm{e}^{\lambda_1 t} - \frac{R\bar{w}}{\kappa - C_v(2\mu+\mu')}\frac{i\xi}{|\xi|^2}\mathrm{e}^{\lambda_2 t} \\
&\quad + \frac{\kappa R\bar{w}}{C_v(2\mu+\mu')[\kappa - C_v(2\mu+\mu')]}\frac{i\xi}{|\xi|^2}\mathrm{e}^{\lambda_3 t} + \cdots,
\end{aligned}$$
$$\begin{aligned}
\hat{G}_{22} &= (I - \frac{\xi\xi^T}{|\xi|^2})\mathrm{e}^{-\frac{\mu}{\bar{\rho}}|\xi|^2 t} - \frac{R\bar{\rho}\bar{w}}{C_v(2\mu+\mu')^2}\frac{\xi\xi^T}{|\xi|^4}\mathrm{e}^{\lambda_1 t} - \frac{R^2\bar{\rho}\bar{w}}{[\kappa - C_v(2\mu+\mu')]^2}\frac{\xi\xi^T}{|\xi|^4}\mathrm{e}^{\lambda_2 t} \\
&\quad + \frac{\xi\xi^T}{|\xi|^2}\mathrm{e}^{\lambda_3 t} + \cdots,
\end{aligned}$$
$$\hat{G}_{23} = \frac{R^2\bar{\rho}^2\bar{w}}{\kappa C_v(2\mu+\mu')^2}\frac{i\xi}{|\xi|^4}\mathrm{e}^{\lambda_1 t} + \frac{R\bar{\rho}}{\kappa - C_v(2\mu+\mu')}\frac{i\xi}{|\xi|^2}(\mathrm{e}^{\lambda_2 t} - \mathrm{e}^{\lambda_3 t}) + \cdots,$$
$$\hat{G}_{31} = \frac{\bar{w}}{\bar{\rho}}\mathrm{e}^{\lambda_1 t} + \frac{\bar{w}}{\bar{\rho}}\mathrm{e}^{\lambda_2 t} - \frac{\kappa R\bar{w}^2[\kappa - (C_v+R)(2\mu+\mu')]}{C_v(2\mu+\mu')^2[\kappa - C_v(2\mu+\mu')]^2}\frac{1}{|\xi|^2}\mathrm{e}^{\lambda_3 t} + \cdots,$$
$$\begin{aligned}
\hat{G}_{32} &= -\frac{\bar{w}}{2\mu+\mu'}\frac{i\xi^T}{|\xi|^2}\mathrm{e}^{\lambda_1 t} + \frac{R\bar{w}}{\kappa - C_v(2\mu+\mu')}\frac{i\xi^T}{|\xi|^2}\mathrm{e}^{\lambda_2 t} \\
&\quad + \frac{\bar{w}[\kappa - (C_v+R)2\mu+\mu']}{(2\mu+\mu')[\kappa - C_v(2\mu+\mu')]}\frac{i\xi^T}{|\xi|^2}\mathrm{e}^{\lambda_3 t} + \cdots,
\end{aligned}$$
$$\hat{G}_{33} = -\frac{R\bar{\rho}\bar{w}}{\kappa(2\mu+\mu')}\frac{1}{|\xi|^2}\mathrm{e}^{\lambda_1 t} + \mathrm{e}^{\lambda_2 t} + \frac{R\bar{\rho}\bar{w}[\kappa - (C_v+R)(2\mu+\mu')]}{(2\mu+\mu')[\kappa - C_v(2\mu+\mu')]^2}\frac{1}{|\xi|^2}\mathrm{e}^{\lambda_3 t} + \cdots.$$

When $\kappa = 0$, we have

$$\hat{G}_{11} = \frac{a_2}{a_2-a_1}\mathrm{e}^{\lambda_1 t} + \frac{a_1}{a_1-a_2}\mathrm{e}^{\lambda_2 t} + \frac{\bar{\rho}^2a_1a_2}{(2\mu+\mu')^2}\frac{1}{|\xi|^4}\mathrm{e}^{\lambda_3 t} + \cdots,$$

$$\hat{G}_{12} = \frac{\left[\frac{(C_v+R)\bar{w}}{C_v(2\mu+\mu')a_2} + \frac{C_v}{R}\right]}{\frac{C_v(2\mu+\mu')(a_1-a_2)}{R\bar{\rho}a_2}} e^{\lambda_1 t} + \frac{\left[\frac{(C_v+R)\bar{w}}{C_v(2\mu+\mu')a_2} + \frac{C_v}{R}\right]}{\frac{C_v(2\mu+\mu')(a_2-a_1)}{R\bar{\rho}a_1}} e^{\lambda_2 t} + \frac{\bar{\rho}}{2\mu+\mu'} \frac{i\xi^T}{|\xi|^2} e^{\lambda_3 t} + \cdots,$$

$$\begin{aligned} \hat{G}_{13} &= \frac{R\bar{\rho}}{C_v(2\mu+\mu')} \frac{1}{a_2-a_1} e^{\lambda_1 t} + \frac{R\bar{\rho}}{C_v(2\mu+\mu')} \frac{1}{a_1-a_2} e^{\lambda_2 t} \\ &\quad - \frac{R\bar{\rho}^2}{C_v(2\mu+\mu')^2} \frac{1}{|\xi|^2} e^{\lambda_3 t} + \cdots, \end{aligned}$$

$$\hat{G}_{21} = \frac{a_1 a_2}{a_2-a_1} \frac{i\xi}{|\xi|^2} e^{\lambda_1 t} - \frac{a_1 a_2}{a_2-a_1} \frac{i\xi}{|\xi|^2} e^{\lambda_2 t} - \frac{\bar{\rho} a_1 a_2}{2\mu+\mu'} \frac{i\xi}{|\xi|^4} e^{\lambda_3 t} + \cdots,$$

$$\begin{aligned} \hat{G}_{22} &= (I - \frac{\xi\xi^T}{|\xi|^2}) e^{-\frac{\mu}{\bar{\rho}}|\xi|^2 t} + \frac{\frac{(C_v+R)\bar{w}}{C_v(2\mu+\mu')a_2} + \frac{C_v}{R}}{\frac{C_v(2\mu+\mu')(a_2-a_1)}{R\bar{\rho}a_1a_2}} \frac{\xi\xi^T}{|\xi|^4} e^{\lambda_1 t} - \frac{\frac{(C_v+R)\bar{w}}{C_v(2\mu+\mu')a_1} + \frac{C_v}{R}}{\frac{C_v(2\mu+\mu')(a_2-a_1)}{R\bar{\rho}a_1a_2}} \frac{\xi\xi^T}{|\xi|^4} e^{\lambda_2 t} \\ &\quad + \frac{\xi\xi^T}{|\xi|^2} e^{\lambda_3 t} + \cdots, \end{aligned}$$

$$\begin{aligned} \hat{G}_{23} &= \frac{R\bar{\rho}a_1}{C_v(2\mu+\mu')(a_2-a_1)} \frac{i\xi}{|\xi|^2} e^{\lambda_1 t} \\ &\quad + \frac{R\bar{\rho}a_2}{C_v(2\mu+\mu')(a_1-a_2)} \frac{i\xi}{|\xi|^2} e^{\lambda_2 t} + \frac{R\bar{\rho}}{C_v(2\mu+\mu')} \frac{i\xi}{|\xi|^2} e^{\lambda_3 t} + \cdots, \end{aligned}$$

$$\begin{aligned} \hat{G}_{31} &= -\frac{C_v(2\mu+\mu')a_1a_2}{R\bar{\rho}a_2 - a_1} e^{\lambda_1 t} + \frac{C_v(2\mu+\mu')a_1a_2}{R\bar{\rho}a_2 - a_1} e^{\lambda_2 t} \\ &\quad - \frac{(C_v+R)\bar{w}\bar{\rho}a_1a_2}{C_v(2\mu+\mu')^2} \frac{1}{|\xi|^4} e^{\lambda_3 t} + \cdots, \end{aligned}$$

$$\begin{aligned} \hat{G}_{32} &= \frac{\frac{(C_v+R)\bar{w}}{C_v(2\mu+\mu')a_2} + \frac{C_v}{R}}{\frac{(a_2-a_1)}{a_1a_2}} \frac{i\xi^T}{|\xi|^2} e^{\lambda_1 t} - \frac{\frac{(C_v+R)\bar{w}}{C_v(2\mu+\mu')a_1} + \frac{C_v}{R}}{\frac{(a_2-a_1)}{a_1a_2}} \frac{i\xi^T}{|\xi|^2} e^{\lambda_2 t} \\ &\quad + \frac{(C_v+R)\bar{w}}{C_v(2\mu+\mu')} \frac{i\xi^T}{|\xi|^2} e^{\lambda_3 t} + \cdots, \end{aligned}$$

$$\hat{G}_{33} = \frac{a_1}{a_1-a_2} e^{\lambda_1 t} - \frac{a_2}{a_1-a_2} e^{\lambda_2 t} - \frac{(C_v+R)R\bar{w}\bar{\rho}}{C_v^2(2\mu+\mu')^2} \frac{1}{|\xi|^2} e^{\lambda_3 t} + \cdots.$$

From Lemmas 2.5–2.7, we can find that there basically exist two kinds of singular components in the high frequency part of Green's function. One is like the heat kernel $t^{-3/2}e^{-\frac{|x|^2}{Ct}}$ arising from the term $e^{-C|\xi|^2 t}$ in Fourier space. The second one is like a Dirac δ-function or some δ-like functions, which is rising from the term like $|\xi|^{-\beta}e^{-Ct}$ in Fourier space and the integer $\beta \ge 1$. Thus, from Lemma 4.2, we have the following description for the high frequency part.

Proposition 2.8. *There exists a constant $C > 0$ such that the high frequency part satisfies*

$$|\partial_x^\alpha(\chi_3(D)G_{ij} - G_S)(x,t)| \le Ce^{-t/C}(1+|x|^2)^{-N},$$

for all integer $N > 0$. Here the singular parts $G_S(x,t)$ satisfy

$$G_S(x,t) = e^{-t/C}\big[C_1 t^{-\frac{3+|\alpha|}{2}} e^{-\frac{|x|^2}{Ct}} + C_2\delta(x)\big]. \tag{2.23}$$

Middle frequency part. The analysis for the middle frequency part is partially based on the idea in Li [20]. In particular, we derive the following estimates for the eigenvalues.

Lemma 2.9. *When $\eta < |\xi| \le K$ with two fixed positive constants η and K, there exists a constant $b > 0$ such that*

$$\mathrm{Re}(\lambda_1(|\xi|), \lambda_2(|\xi|), \lambda_3(|\xi|)) \le -b. \tag{2.24}$$

Proof. Note first that

$$\begin{aligned}
\lambda_1 + \lambda_2 + \lambda_3 &= -(\frac{\kappa}{C_v\bar\rho} + \frac{4\bar w^3}{C_v^4\bar\rho^4(1+|\xi|^2)} + \frac{2\mu+\mu'}{\bar\rho})|\xi|^2, \\
&\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\
&= \frac{\kappa(2\mu+\mu')}{C_v\bar\rho^2}|\xi|^4 + \frac{4(2\mu+\mu')\bar w^3|\xi|^4}{C_v^4\bar\rho^5(1+|\xi|^2)} + \frac{(C_v+R)R\bar w}{C_v^2\bar\rho}|\xi|^2, \\
\lambda_1\lambda_2\lambda_3 &= -\frac{\kappa R\bar w}{C_v^2\bar\rho^2}|\xi|^4 - \frac{4R\bar w^4|\xi|^4}{C_v^5\bar\rho^5(1+|\xi|^2)}.
\end{aligned} \tag{2.25}$$

We will prove (2.24) by two steps:

Step 1. Suppose there is a real root λ_1 and $\lambda_1 > 0$. Combining $(2.25)_1$ and $(2.25)_2$, one has

$$\begin{aligned}
\lambda_2\lambda_3 = &\frac{\kappa(2\mu+\mu')}{C_v\bar\rho^2}|\xi|^4 + \frac{4(2\mu+\mu')\bar w^3|\xi|^4}{C_v^4\bar\rho^5(1+|\xi|^2)} + \frac{(C_v+R)R\bar w}{C_v^2\bar\rho}|\xi|^2 \\
&+ (\frac{\kappa}{C_v\bar\rho} + \frac{4\bar w^3}{C_v^4\bar\rho^4(1+|\xi|^2)} + \frac{2\mu+\mu'}{\bar\rho})|\xi|^2\lambda_1 + \lambda_1^2.
\end{aligned} \tag{2.26}$$

On the other hand, by $(2.25)_3$ we have

$$\lambda_2\lambda_3 = -(\frac{\kappa R\bar w}{C_v^2\bar\rho^2}|\xi|^4 + \frac{4R\bar w^4|\xi|^4}{C_v^5\bar\rho^5(1+|\xi|^2)})/\lambda_1. \tag{2.27}$$

This yields a contradiction since the signs of $\lambda_2\lambda_3$ in (2.26) and (2.27) are opposite. Accordingly, the assumption is not true.

Step 2. Suppose that $\lambda_1 < 0$ and there is a pair of conjugate imaginary roots λ_2 and λ_3 with $\lambda_2 + \lambda_3 > 0$. From $(2.25)_1$, one has

$$\lambda_2 + \lambda_3 = -(\frac{\kappa}{C_v\bar\rho} + \frac{4\bar w^3}{C_v^4\bar\rho^4(1+|\xi|^2)} + \frac{2\mu+\mu'}{\bar\rho})|\xi|^2 - \lambda_1 > 0, \tag{2.28}$$

$$\lambda_1 < -(\frac{\kappa}{C_v\bar\rho} + \frac{4\bar w^3}{C_v^4\bar\rho^4(1+|\xi|^2)} + \frac{2\mu+\mu'}{\bar\rho})|\xi|^2. \tag{2.29}$$

Then, combining $(2.25)_2$, $(2.25)_3$, and (2.29), a routine computation gives rise to

$$\begin{aligned}
&\lambda_2 + \lambda_3 \\
&= \frac{\lambda_1[\frac{\kappa(2\mu+\mu')}{C_v\bar\rho^2}|\xi|^4 + \frac{4\bar w^3(2\mu+\mu')}{C_v^4\bar\rho^5(1+|\xi|^2)}|\xi|^4 + \frac{(C_v+R)R\bar w}{C_v^2\bar\rho}|\xi|^2] + \frac{\kappa R\bar w}{C_v^2\bar\rho^2}|\xi|^4 + \frac{4R\bar w^4|\xi|^4}{C_v^5\bar\rho^5(1+|\xi|^2)}}{\lambda_1^2} \\
&\le \Big(-(\frac{\kappa}{C_v\bar\rho} + \frac{4\bar w^3}{C_v^4\bar\rho^4(1+|\xi|^2)} + \frac{2\mu+\mu'}{\bar\rho})|\xi|^2[\frac{\kappa(2\mu+\mu')}{C_v\bar\rho^2}|\xi|^4 + \frac{4\bar w^3(2\mu+\mu')}{C_v^4\bar\rho^5(1+|\xi|^2)}|\xi|^4 \\
&\quad + \frac{R^2\bar w}{C_v^2\bar\rho}|\xi|^2] - \frac{R\bar w(2\mu+\mu')}{C_v\bar\rho^2}|\xi|^4\Big)/\lambda_1^2 < 0.
\end{aligned}$$

Obviously, this contradicts (2.28). Therefore, $\lambda_2 + \lambda_3 < 0$, i.e., $\mathrm{Re}(\lambda_2) < 0$ and $\mathrm{Re}(\lambda_3) < 0$. This completes the proof. □

Lemma 2.10. *The Green function $\hat{G}(\xi,t)$ is analytic when $|\xi|^2 \geq \delta$, where δ is any fixed positive constant.*

Proof. We shall only present the proof for $\hat{G}_{11}(\xi,t)$, since the other entries in the $\hat{G}(\xi,t)$ can be treated similarly. First, from Lemmas 2.1 and 2.5, we can induce that (2.9) has not repeated roots. Then, we can see that

$$\begin{aligned}
&\hat{G}_{11}(\xi,\lambda_1,\lambda_2,\lambda_3,t)\\
&=\hat{G}_{11}^{1,m}e^{\lambda_1 t}+\hat{G}_{11}^{2,m}e^{\lambda_2 t}+\hat{G}_{11}^{3,m}e^{\lambda_3 t}\\
&=g_1C_1e^{\lambda_1 t}+g_2D_1e^{\lambda_2 t}+g_3E_1e^{\lambda_3 t}\\
&=g_1\frac{\nu_3-\nu_2}{\Omega}e^{\lambda_1 t}+g_2\frac{\nu_1-\nu_3}{\Omega}e^{\lambda_2 t}+g_3\frac{\nu_2-\nu_1}{\Omega}e^{\lambda_3 t}\\
&=\frac{g_1\Omega(\nu_3-\nu_2)e^{\lambda_1 t}+g_2\Omega(\nu_1-\nu_3)e^{\lambda_2 t}+g_3\Omega(\nu_2-\nu_1)e^{\lambda_3 t}}{\Omega^2}\\
&=\frac{A(\xi,\lambda_1,\lambda_2,\lambda_3,t)}{(\nu_2g_3-\nu_3g_2+\nu_3g_1-\nu_1g_3+\nu_1g_2-\nu_2g_1)^2},
\end{aligned}\tag{2.30}$$

where

$$\begin{aligned}
&A(\xi,\lambda_1,\lambda_2,\lambda_3,t)\\
&=g_1(\nu_2g_3-\nu_3g_2+\nu_3g_1-\nu_1g_3+\nu_1g_2-\nu_2g_1)(\nu_3-\nu_2)e^{\lambda_1 t}\\
&\quad+g_2(\nu_2g_3-\nu_3g_2+\nu_3g_1-\nu_1g_3+\nu_1g_2-\nu_2g_1)(\nu_1-\nu_3)e^{\lambda_2 t}\\
&\quad+g_3(\nu_2g_3-\nu_3g_2+\nu_3g_1-\nu_1g_3+\nu_1g_2-\nu_2g_1)(\nu_2-\nu_1)e^{\lambda_3 t}\\
&=g_1(\nu_2g_3-\nu_3g_2+\nu_3g_1-\nu_1g_3+\nu_1g_2-\nu_2g_1)(\nu_3-\nu_2)\sum_{n=1}^{\infty}\frac{(\lambda_1 t)^n}{n!}\\
&\quad+g_2(\nu_2g_3-\nu_3g_2+\nu_3g_1-\nu_1g_3+\nu_1g_2-\nu_2g_1)(\nu_1-\nu_3)\sum_{n=1}^{\infty}\frac{(\lambda_2 t)^n}{n!}\\
&\quad+g_3(\nu_2g_3-\nu_3g_2+\nu_3g_1-\nu_1g_3+\nu_1g_2-\nu_2g_1)(\nu_2-\nu_1)\sum_{n=1}^{\infty}\frac{(\lambda_3 t)^n}{n!}.
\end{aligned}$$

We claim that g_i and ν_i $(i=1,2,3)$ are symmetric about λ_i $(i=1,2,3)$. It is easy to obtain that the numerator is a symmetric power series in λ_i $(i=1,2,3)$. In fact, for instance, exchanging λ_1 and λ_2, the first term of A becomes the second term, the second term becomes the first term, and the last term is still itself. Thus, $A(\xi,\lambda_1,\lambda_2,\lambda_3,t)=A(\xi,\lambda_2,\lambda_1,\lambda_3,t)$. Then, we have $\hat{G}_{11}(\xi,\lambda_1,\lambda_2,\lambda_3,t)=\hat{G}_{11}(\xi,\lambda_2,\lambda_1,\lambda_3,t)$.

It is well known that every symmetric polynomial can be written as a power sum of the elementary symmetric polynomials $\lambda_1+\lambda_2+\lambda_3$, $\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3$, and $\lambda_1\lambda_2\lambda_3$. From (2.25), we know that the numerator can be written as a power series in $|\xi|^2$ and therefore is entire in $|\xi|^2$. Similarly, we notice that the denominator is a symmetric polynomial in λ_i $(i=1,2,3)$, so it can be written as an polynomial in $|\xi|^2$. Therefore, $\hat{G}_{11}(\xi,t)$ must be analytic when $|\xi|^2\geq\delta>0$. Thus, we have completed the proof. □

Lemmas 2.9 and 2.10 immediately yield the following result.

Proposition 2.11. *There exists a constant $b_1 > 0$ such that*

$$|\partial_x^\alpha(\chi_2(D)G(x,t))| \le Ce^{-b_1 t}(1+\frac{|x|^2}{1+t})^{-N},$$

where N can be arbitratily large.

In summary, from Propositions 2.4–2.11, we can have the following pointwise description involving the Huygens' wave and the diffusion wave.

Theorem 2.12. *The Green function $G(x,t)$ for system $(2.3)_{1,2,3}$ in the 3 dimensional space has the following estimates for $t > 0$:*

$$\begin{aligned}
&|\partial_x^\alpha(G_{ij}-G_S)(x,t))| \\
&\le C(1+t)^{-\frac{4+|\alpha|}{2}}\big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-N} \quad \text{for } (i,j)\ne(2,2),(3,1),(3,3), \\
&|\partial_x^\alpha(G_{22}-G_S)(x,t)\big)| \\
&\le C(1+t)^{-\frac{3+|\alpha|}{2}}\big(1+\frac{|x|^2}{1+t}\big)^{-\frac{3+|\alpha|}{2}} + C(1+t)^{-\frac{4+|\alpha|}{2}}\big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-N}, \\
&|\partial_x^\alpha((G_{31}-G_S,G_{33}-G_S))(x,t))| \\
&\le C(1+t)^{-\frac{4+|\alpha|}{2}}\Big(\big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-N} + \big(1+\frac{|x|^2}{1+t}\big)^{-N}\Big).
\end{aligned}$$

Here $N > 0$ is an arbitrary large constant and G_S is defined in Proposition 2.8.

3. Pointwise estimates for the nonlinear system

First, by using Duhamel's principle, we represent the solution $(\rho, \mathbf{m}, w)$ for the nonlinear problem (2.1).

$$\partial_x^\alpha\begin{pmatrix}\rho\\ \mathbf{m}\\ w\end{pmatrix} = \partial_x^\alpha G *_x U_0 + \int_0^t \partial_x^\alpha G *_x \begin{pmatrix}0\\ F_1\\ F_2\end{pmatrix}(\cdot,s)ds, \tag{3.1}$$

where the initial data $U_0 := (\rho_0, \mathbf{m}_0, \omega_0)^T$, and the nonlinear terms F_1, F_2 are defined in (2.2).

Initial propagation. Let $(\breve{\rho}, \breve{\mathbf{m}}, \breve{w})$ denote the linear part of the solution in (3.1). Theorem 2.12, the initial condition (1.5), and the representation (3.1) yield the linear estimates

$$\begin{gathered}
|\partial_x^\alpha(\breve{\rho},\breve{\mathbf{m}})| \le 2C\epsilon\Big((1+t)^{-\frac{3+|\alpha|}{2}}\big(1+\frac{|x|^2}{1+t}\big)^{-3/2} + (1+t)^{-2}\big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-3/2}\Big), \\
|\partial_x^\alpha\breve{w}| \le 2C\epsilon(1+t)^{-\frac{4+|\alpha|}{2}}\Big(\big(1+\frac{|x|^2}{1+t}\big)^{-3/2} + \big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-3/2}\Big).
\end{gathered} \tag{3.2}$$

Here we have used the convolution estimate in Lemma 4.3 for the initial propagation, and the different pointwise estimates for the first two rows and the last row of Green's function in Theorem 2.12.

Nonlinear Coupling. According to the above initial propagation, we should give the ansatz for the nonlinear problem when $|\alpha| \leq 1$,

$$
\begin{aligned}
&|\partial_x^\alpha(\rho,\mathbf{m})| \\
&\leq 2C\epsilon\Big((1+t)^{-\frac{3+|\alpha|}{2}}\big(1+\frac{|x|^2}{1+t}\big)^{-3/2}+(1+t)^{-2}\big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-3/2}\Big), \\
&|\partial_x^\alpha w| \leq 2C\epsilon(1+t)^{-\frac{4+|\alpha|}{2}}\Big(\big(1+\frac{|x|^2}{1+t}\big)^{-3/2}+\big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-3/2}\Big).
\end{aligned}
\tag{3.3}
$$

Now, we substitute (3.3) into the representation of the solution $(\rho,\mathbf{m},w)$ in (3.1) to close the ansatz. To this end, we also split Green's function into the regular term $G_{ij}-G_S$ and the singular term G_S. For the convolution between $G_{ij}-G_S$ and the nonlinear terms, one can put all of the derivatives on $G_{ij}-G_S$ and use the nonlinear convolution estimates in Lemma 4.4 to obtain the corresponding estimate as in the ansatz (3.2). We emphasize that although there exists a nonlocal operator $\frac{\Delta}{1-\Delta}$ in some nonlinear terms of F_2, it actually does not affect the result. In fact, one can put this operator onto Green's function by integration by parts, and it's easy to see that it is harmless for the pointwise estimates of Green's function in all of frequency parts.

Next, we consider the nonlinear convolution between the singular part of Green's function and the nonlinear term. We only take the nonlinear estimate of the momentum $\mathbf{m}$ for example, since from this one can see why we need H^5-framework for this quasi-linear problem. In fact, when estimating $\partial_x^k\mathbf{m}$, one will encounter the term $\int_0^t G_S(\cdot,t-\tau)*_x\partial_x^k[\partial_x^2(\rho\mathbf{m})](\cdot,\tau)d\tau$. Noting that G_S is like $\delta(x)$ with exponential decay rate, one has to put the derivative on the nonlinear term. That is, $\int_0^t\delta(\cdot,t-\tau)*_x\partial_x^k[\partial_x^2(\rho\mathbf{m})](\cdot,\tau)d\tau$. As a result, one can close the ansatz only when $|k|\geq 1$. Indeed, when $|k|=1$, we should use the pointwise information of $\partial_x^3(\rho\mathbf{m})$, and hence it also requires $\partial_x^3(\rho,m)\in L^\infty(\mathbb{R}^3)$. This together with Sobolev inequality yields that we can close the ansatz in H^5-framework.

Finally, by using the smallness of ϵ and the continuity, one can close the ansatz (3.2) and hence proves Theorem 1.1.

4. Appendix

Some useful lemmas are given here. The first one is used to derive the pointwise estimates of Green's function in the low frequency.

Lemma 4.1 ([38])**.** *If there exists a constant $C>0$ such that when $|\xi|\leq 1$, $\hat f(\xi,t)$ satisfies*

$$
|\partial_\xi^\beta(\xi^\alpha\hat f(\xi,t))| \leq C(|\xi|^{(|\alpha|-|\beta|)_+}+|\xi|^{|\alpha|}t^{|\beta|/2})(1+(t|\xi|^2))^a\exp(-b|\xi|^2t),
$$

for some constant $b>0$, each fixed integer a and any multi-indexes α,β with $|\beta|\leq 2N$, then

$$
|\partial_x^\alpha f(x,t)| \leq C_N(1+t)^{-(n+|\alpha|)}\Big(1+\frac{|x|^2}{1+t}\Big)^{-N}, \tag{4.1}
$$

where N is a positive constant and can be arbitrarily large.

The second lemma describes the singular part of the high frequency.

Lemma 4.2 ([38])**.** *If $\operatorname{supp}\hat f(\xi)\subset O_K=:\{\xi,|\xi|\geq K>0\}$, and $\hat f(\xi)$ satisfies*

$$
|\partial_\xi^\beta\hat f(\xi)| \leq C|\xi|^{-|\beta|-1} \quad (\text{or } |D_\xi^\beta\hat f(\xi)|\leq C|\xi|^{-|\beta|}),
$$

then there exist distributions $f_1(x), f_2(x)$ *and a constant* C_0 *such that*

$$f(x) = f_1(x) + f_2(x) + C_0\delta(x) \quad (\textit{or } f(x) = f_1(x) + f_2(x) + C_0\partial_x\delta(x)),$$

where $\delta(x)$ *is the Dirac function. Furthermore, for any* $|\alpha| \geq 0$ *and any positive integer* N*, we have*

$$|\partial_x^\alpha f_1(x)| \leq C(1+|x|^2)^{-N}, \quad \|f_2\|_{L^1} \leq C, \quad \operatorname{supp} f_2(x) \subset \{x; |x| < \eta_0 \ll 1\}.$$

The next two lemmas are often used to deal with initial propagation and nonlinear coupling, respectively. We also state several typical cases for completeness.

Lemma 4.3 ([40])**.** *There exists a constant* $C > 0$ *such that for* $n_1, n_2 > 3/2$ *and* $n_3 = \min\{n_1, n_2\}$*, we have*

$$\int_{\mathbb{R}^3} \big(1 + \frac{|x-y|^2}{1+t}\big)^{-n_1}\big(1+|y|^2\big)^{-n_2} dy \leq C\Big(1 + \frac{|x|^2}{1+t}\Big)^{-n_3};$$

and for $N \geq r_1 > 21.10$*, we have*

$$\int_{\mathbb{R}^3} \big(1 + \frac{(|x-y|-ct)^2}{1+t}\big)^{-N}\big(1+|y|^2\big)^{-r_1} dy \leq C\Big(1 + \frac{(|x|-ct)^2}{1+t}\Big)^{-3/2}.$$

Lemma 4.4 ([25])**.** *There exists a constant* $C > 0$ *such that*

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} (1+t-s)^{-2}\Big(1 + \frac{|x-y|^2}{1+t-s}\Big)^{-2}(1+s)^{-3}\Big(1+\frac{|y|^2}{1+s}\Big)^{-3} dy\, ds \\
&\leq C(1+t)^{-2}\big(1+\frac{|x|^2}{1+t}\big)^{-3/2},
\end{aligned}$$

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} (1+t-s)^{-2}\Big(1 + \frac{|x-y|^2}{1+t-s}\Big)^{-2}(1+s)^{-4}\Big(1+\frac{(|y|-cs)^2}{1+s}\Big)^{-3} dy\, ds \\
&\leq C(1+t)^{-2}\Big(\big(1+\frac{|x|^2}{1+t}\big)^{-3/2} + \big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-3/2}\Big),
\end{aligned}$$

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} (1+t-s)^{-5/2}\Big(1 + \frac{(|x-y|-c(t-s))^2}{1+t-s}\Big)^{-N} \\
&\times (1+s)^{-4}\Big(1+\frac{(|y|-cs)^2}{1+s}\Big)^{-3} dy\, ds \\
&\leq C(1+t)^{-2}\Big(\big(1+\frac{|x|^2}{1+t}\big)^{-3/2} + \big(1+\frac{(|x|-ct)^2}{1+t}\big)^{-3/2}\Big),
\end{aligned}$$

where the constant $N > 0$ *can be arbitrarily large.*

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