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ABSTRACT DEGENERATE VOLTERRA INCLUSIONS IN LOCALLY CONVEX SPACES

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Abstract. In this article, we analyze the abstract degenerate Volterra integro-differential equations in sequentially complete locally convex spaces by using multivalued linear operators and vector-valued Laplace transform. We follow the method which is based on the use of (a, k)-regularized C-resolvent families generated by multivalued linear operators and which suggests a very general way of approaching abstract Volterra equations. Among many other themes, we consider the Hille-Yosida type theorems for (a, k)-regularized C-resolvent families, differential and analytical properties of (a, k)-regularized C-resolvent families, the generalized variation of parameters formula, and subordination principles. We also introduce and analyze the class of (a, k)-regularized (C_1, C_2)-existence and uniqueness families. The main purpose of third section, which can be viewed of some independent interest, is to introduce a relatively simple and new theoretical concept useful in the analysis of operational properties of Laplace transform of non-continuous functions with values in sequentially complete locally convex spaces. This concept coincides with the classical concept of vector-valued Laplace transform in the case that X is a Banach space.

1. Introduction and preliminaries

The main aim of this paper is to analyze the abstract degenerate Volterra integro-differential equations in sequentially complete locally convex spaces by using multivalued linear operators (cf. [68] and [36] for a comprehensive survey of results on abstract non-degenerate Volterra equations), as well as to introduce a new theoretical approach to the Laplace transform of functions with values in sequentially complete locally convex spaces. To outline the motivation of our research, let us mention that there exists only a few published papers in the existing literature treating the abstract degenerate Volterra equations ([16], [18]-[21], [31]) and the abstract degenerate fractional inclusions associated with the use of Caputo fractional derivatives ([45]-[49], [51]). In this paper, we make an attempt to perform the first systematic exploration of abstract degenerate Volterra equations and abstract degenerate fractional differential equations in locally convex spaces, contributing also to the theories of abstract degenerate differential equations of first and second order

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(for pioneering results about semigroups of operators in locally convex spaces, we refer the reader to the papers [33, 34, 82]). A great number of our results seems to be new even in the Bahach space setting.

The organization and main ideas of this paper can be briefly described as follows. In the second section of paper, we will take a preliminary and incomplete look at the multivalued linear operators in locally convex spaces; for more details, we refer the reader to the monographs [9, 17]. We introduce the notion of a C-resolvent of a multivalued linear operator, reconsider the assertions from [17, Chapter I] and state a generalization of [36, Proposition 2.1.14] for C-resolvents of multivalued linear operators. Following the approach of Knuckles and Neubrander [32], we introduce the notion of a relatively closed multivalued linear operator in locally convex space. The generalized resolvent equations continue to hold in our framework.

As mentioned in [36, Section 1.2], only a few noteworthy facts has been said about the Laplace transform of functions with values in sequentially complete locally convex spaces. In Section 3, we propose a new theoretical approach to the Laplace transform of functions with values in sequentially complete locally convex spaces. This concept extends the corresponding one introduced by Xiao and Liang ([80], 1997), and coincides with the classical concept of vector-valued Laplace transform in the case that the state space X is one of Banach's [1]. Concerning the integration of functions with values in sequentially complete locally convex spaces, we follow the approach of Martinez and Sanz (cf. [61, pp. 99-102] for more details); for Pettis integration in locally convex spaces and some applications to abstract differential inclusions of first order, we refer the reader to [28, 29, 56]. Once we have proved the formula for partial integration in Theorem 3.1, we have an open door to consider various operational properties of Laplace transform by using the methods already known in the Banach space case. The non-possibility of establishing Fubini-Tonelli theorem in this concept of integration additionally hinders our research and does not able us to fully transfer some assertions from the Banach space case to the general locally convex space case; for example, in Theorem 3.3(vi) we consider the Laplace transform of finite convolution product and there it is almost inevitable to impose the condition that the function $f(t)$ is continuous.

A large number of research papers, starting presumably with that of Yagi [81], written over the last twenty five years, have concerned applications of multivalued linear operators to abstract degenerate differential equations (cf. [8], [13], [17] and [63]-[65] for the primary source of information on this subject). In Section 4, we analyze the abstract degenerate Volterra inclusion

$$\mathcal{B}u(t) \subseteq \mathcal{A}\int_0^t a(t-s)u(s)\,ds + \mathcal{F}(t), \quad t \in [0,\tau), \tag{1.1}$$

where $a \in L^1_{\rm loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : X \to P(Y)$ are given multivalued linear operators acting between sequentially complete locally convex spaces X and Y, and $\mathcal{F} : X \to P(Y)$ is a given mutivalued mapping, as well as the fractional Sobolev inclusions

$$\begin{gathered} \mathbf{D}_t^\alpha Bu(t) \in \mathcal{A}u(t) + \mathcal{F}(t), \quad t \geq 0, \\ (Bu)^{(j)}(0) = Bx_j, \quad 0 \leq j \leq \lceil \alpha \rceil - 1, \end{gathered} \tag{1.2}$$

where we assume that $B = \mathcal{B}$ is single-valued, and

$$\begin{gathered} \mathcal{B}\mathbf{D}_t^\alpha u(t) \subseteq \mathcal{A}u(t) + \mathcal{F}(t), \quad t \geq 0, \\ u^{(j)}(0) = x_j, \quad 0 \leq j \leq \lceil \alpha \rceil - 1. \end{gathered} \tag{1.3}$$

Here, $\mathbf{D}_t^\alpha u(t)$ denotes the Caputo fractional derivative of function $u(t)$. We define various types of solutions of problems (1.1), (1.2) and (1.3). In Theorems 4.3 and 4.5, we reconsider the main results of research of Kim [31], while in Theorem 4.6 we prove an extension of [32, Theorem 3.5] for abstract degenerate fractional differential inclusions. Subordination principles are clarified in Theorem 4.8 and Theorem 4.9 following the methods proposed by Prüss [68, Section 4] and Bazhlekova [5, Section 3] (cf. [22] and [42]-[46] for similar results known in degenerate case).

Following the old ideas of deLaubenfels [11], in Section 5 we introduce and analyze the class of (a, k)-regularized (C_1, C_2)-existence and uniqueness families (cf. [36, Section 2.8] for non-degenerate case). Later on, we single out the class of (a, k)-regularized C-resolvent families for special considerations. We focus our attention on the analysis of Hille-Yosida's type theorems for (a, k)-regularized C-resolvent families generated by multivalued linear operators (as in all previous researches of non-degenerate case, we introduce the notion of a subgenerator of an (a, k)-regularized C-resolvent family and investigate the most important properties of subgenerators; our analysis is based on the use of vector-valued Laplace transform). It is well known (see e.g. [17, Theorem 2.4], [32, Theorem 3.6] and [31, p. 169]) that Hille-Yosida's type estimates for the resolvent of a multivalued operator $\mathcal{A}$ immediately implies that $\mathcal{A}$ is single-valued in a certain sense. In part (ii) of Theorem 5.12, we will prove a similar assertion provided that the Hille-Yosida condition (5.17) below holds. For the validity of Theorem 5.12(ii), we have found the condition $k(0) \neq 0$ very important to be satisfied; in other words, the existence of above-mentioned single-valued branch of $\mathcal{A}$ can be proved exactly in non-convoluted or non-integrated case, so that we have arrived to a diametrically opposite conclusion to that stated on l. 7-13, p. 169 of [31]. Nevertheless, the existence or non-existence of such a single-valued branch of $\mathcal{A}$ is not sufficient for obtaining a fairly complete information on the well-posedness of inclusion (1.1) with $\mathcal{B} = I$ (the reading of papers [31, 32] has strongly influenced us to write this paper, and compared with the results of [31], here we do not need the assumption that $a(t)$ is a normalized function of local bounded variation). In the remainder of Section 5, we enquire into the possibility to extend the most important results from [36, Section 2.1, Section 2.2] to (a, k)-regularized C-resolvent families generated by multivalued linear operators, and present several examples and possible applications of our abstract theoretical results. We clarify the complex characterization theorem for the generation of exponentially equicontinuous (a, k)-regularized C-resolvent families, the generalized variation of parameters formula, and subordination principles; in two separate subsections, we analyze differential and analytical properties of (a, k)-regularized C-resolvent families as well as the case in which some of the regularizing operators C and C_2 is not injective. We provide several illustrative examples, including applications to fractional Maxwell's equations, fractional linearized Benney-Luke equation and backward Poisson heat equation.

Because of some similarity with our previous researches of non-degenerate case, we have decided to write this paper in a half-expository manner, including only the

most relevant details of proofs of our structural results. The author would like to express his appreciation and sincere thanks to Prof. Vladimir Fedorov (Chelyabinsk, Russia) and Prof. Rodrigo Ponce (Talca, Chile) for many stimulating and enlightening discussions during the research.

We use the standard terminology throughout the paper. By X we denote a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short. If Y is also an SCLCS over the same field of scalars as X, then we denote by $L(X,Y)$ the space consisting of all continuous linear mappings from X into Y; $L(X) \equiv L(X,X)$. By $\circledast_X$ ($\circledast$, if there is no risk for confusion), we denote the fundamental system of seminorms which defines the topology of X; the fundamental system of seminorms which defines the topology on an arbitrary SCLCS Z is denoted by $\circledast_Z$. The symbol I_X (I_Y) denotes the identity operator on X (Y); if there is no risk for confusion, then we also write I in place of I_X. By X^* we denote the dual space of X. Let $0 < \tau \leq \infty$. A strongly continuous operator family $(W(t))_{t\in[0,\tau)} \subseteq L(X,Y)$ is said to be locally equicontinuous if and only if, for every $T \in (0,\tau)$ and for every $p \in \circledast_Y$, there exist $q_p \in \circledast_X$ and $c_p > 0$ such that $p(W(t)x) \leq c_p q_p(x)$, $x \in X$, $t \in [0,T]$; the notions of equicontinuity of $(W(t))_{t\in[0,\tau)}$ and the exponential equicontinuity of $(W(t))_{t\geq 0}$ are defined similarly. Notice that $(W(t))_{t\in[0,\tau)}$ is automatically locally equicontinuous in case that the space X is barreled ([62]).

By $\mathcal{B}$ we denote the family consisting of all bounded subsets of X. Define $p_{\mathbb{B}}(T) := \sup_{x\in\mathbb{B}} p(Tx)$, $p \in \circledast_Y$, $\mathbb{B} \in \mathcal{B}$, $T \in L(X,Y)$. Then $p_{\mathbb{B}}(\cdot)$ is a seminorm on $L(X,Y)$ and the system $(p_{\mathbb{B}})_{(p,\mathbb{B})\in\circledast_Y\times\mathcal{B}}$ induces the Hausdorff locally convex topology on $L(X,Y)$. If Y is continuously embedded in X, we will use the notation $Y \hookrightarrow X$. Suppose that A is a closed linear operator acting on X. Then we denote the domain, kernel space and range of A by $D(A)$, $N(A)$ and $R(A)$, respectively. Since no confusion seems likely, we will identify A with its graph. Set $p_A(x) := p(x) + p(Ax)$, $x \in D(A)$, $p \in \circledast$. Then the calibration $(p_A)_{p\in\circledast}$ induces the Hausdorff sequentially complete locally convex topology on $D(A)$; we denote this space simply by $[D(A)]$.

Suppose that V is a general topological vector space (the consistent and stable theory of abstract degenerate Volterra integro-differential equations in non-locally convex spaces has not been yet created; see [26] for some results established in non-degenerate case). As it is well-known, a function $f : \Omega \to V$, where Ω is an open non-empty subset of $\mathbb{C}$, is said to be analytic if it is locally expressible in a neighborhood of any point $z \in \Omega$ by a uniformly convergent power series with coefficients in V. The reader may consult [1], [36, Section 1.1] and references cited there for the basic information about vector-valued analytic functions. In our framework, the analyticity of a mapping $f : \Omega \to X$ is equivalent with its weak analyticity.

A function $f : [0,T] \to X$, where $0 < T < \infty$, is said to be Hölder continuous with the exponent $r \in (0,1]$ if for each $p \in \circledast_X$ there exists $M \geq 1$ such that $p(f(t) - f(s)) \leq M|t-s|^r$, provided $0 \leq t, s \leq T$, while a function $f : [0,\infty) \to X$ is said to be locally Hölder continuous with the exponent r if its restriction on any finite interval $[0,T]$ is Hölder continuous with the same exponent. By $AC_{\mathrm{loc}}([0,\infty))$ we denote the space consisting of all functions $f : [0,\infty) \to X$ whose restriction on any finite interval $[0,T]$ ($T > 0$) is absolutely continuous.

Let $0 < \tau \leq \infty$ and $a \in L^1_{\rm loc}([0,\tau))$. Then we say that the function $a(t)$ is a kernel on $[0,\tau)$ if for each $f \in C([0,\tau))$ the assumption $\int_0^t a(t-s)f(s)\,ds = 0$, $t \in [0,\tau)$ implies $f(t)=0$, $t \in [0,\tau)$. Given $s \in \mathbb{R}$ in advance, set $\lfloor s\rfloor := \sup\{l \in \mathbb{Z} : l \leq s\}$ and $\lceil s\rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ ($\zeta > 0$, $t > 0$), $g_0(t) :=$ δ-distribution and, by common consent, $0^\zeta := 0$. For any angle $\alpha \in (0,\pi]$, we define $\Sigma_\alpha := \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \alpha\}$. Set $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \Re\lambda > 0\}$.

Now we repeat some basic facts and definitions about integration of functions with values in SCLCSs. Unless stated otherwise, by Ω we denote a locally compact, separable metric space and by μ we denote a locally finite Borel measure defined on Ω. A function $f : \Omega \to X$ is said to be μ-measurable if and only if there exists a sequence (f_n) in X^Ω of simple functions (cf. [36, Definition 1.1.1(i)] for the notion) such that $\lim_{n\to\infty} f_n(t) = f(t)$ for a.e. $t \in \Omega$.

Definition 1.1. Let $K \subseteq \Omega$ be a compact set, and let a function $f : K \to X$ be strongly measurable. Then it is said that $f(\cdot)$ is (μ-)integrable if there is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions such that $\lim_{n\to\infty} f_n(t) = f(t)$ a.e. $t \in K$ and for all $\epsilon > 0$ and each $p \in \circledast$ there is a number $n_0 = n_0(\epsilon, p)$ such that

$$\int_K p\big(f_n - f_m\big)\,d\mu \leq \epsilon \quad (m,\ n \geq n_0). \tag{1.4}$$

In this case we define

$$\int_K f\,d\mu := \lim_{n\to\infty} \int_K f_n\,d\mu.$$

From (1.4), we have that $(p(f_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $L^1(K,\mu)$, so that the limit $p(f) = \lim_{n\to\infty} p(f_n)$ is μ-integrable. Similarly we can prove that each function $p(f_n - f)$ is μ-integrable and the sequence of its corresponding integrals converges to zero. Recall that every continuous function $f : K \to X$ is μ-integrable.

Definition 1.2. (i) A function $f : \Omega \to X$ is said to be locally μ-integrable if, for every compact set $K \subseteq \Omega$, the restriction $f_{|K} : K \to X$ is μ-integrable.

(ii) A function $f : \Omega \to X$ is said to be μ-integrable if it is locally integrable and if additionally

$$\int_\Omega p(f)\,d\mu < \infty, \quad p \in \circledast. \tag{1.5}$$

If this is the case, we define

$$\int_\Omega f\,d\mu := \lim_{n\to\infty} \int_{K_n} f\,d\mu,$$

with $(K_n)_{n\in\mathbb{N}}$ being an expansive sequence of compact subsets of Ω with the property that $\bigcup_{n\in\mathbb{N}} K_n = \Omega$.

The above definition does not depend on the choice of sequence $(K_n)_{n\in\mathbb{N}}$. Moreover,

$$p\Big(\int_\Omega f\,d\mu\Big) \leq \int_\Omega p(f)\,d\mu, \quad p \in \circledast. \tag{1.6}$$

It is not difficult to verify that the μ-integrability of a function $f : K \to X$, resp. $f : \Omega \to X$, implies that for each $x^* \in X^*$, one has:

$$\Big\langle x^*, \int_K f\,d\mu\Big\rangle = \int_K \big\langle x^*, f\big\rangle\,d\mu, \quad \text{resp. } \Big\langle x^*, \int_\Omega f\,d\mu\Big\rangle = \int_\Omega \big\langle x^*, f\big\rangle\,d\mu. \tag{1.7}$$

Definition 1.2 is equivalent with the definition of Bochner integral, provided that X is a Banach space. Furthermore, every continuous function $f : \Omega \to X$ satisfying (1.5) is μ-integrable and the following holds.

Theorem 1.3. *(i) (The Dominated Convergence Theorem) Suppose that (f_n) is a sequence of μ-integrable functions from X^Ω and (f_n) converges pointwise to a function $f : \Omega \to X$. Assume that, for every $p \in \circledast$, there exists a μ-integrable function $F_p : \Omega \to [0,\infty)$ such that $p(f_n) \leq F_p$, $n \in \mathbb{N}$. Then $f(\cdot)$ is a μ-integrable function and $\lim_{n\to\infty} \int_\Omega f_n\, d\mu = \int_\Omega f\, d\mu$.*

(ii) Let Y be a SCLCS, and let $T : X \to Y$ be a continuous linear mapping. If $f : \Omega \to X$ is μ-integrable, then $Tf : \Omega \to Y$ is likewise μ-integrable and

$$T \int_\Omega f\, d\mu = \int_\Omega Tf\, d\mu. \tag{1.8}$$

(iii) Let Y be a SCLCS, and let $T : D(T) \subseteq X \to Y$ be a closed linear mapping. If $f : \Omega \to D(T)$ is μ-integrable and $Tf : \Omega \to Y$ is likewise μ-integrable, then $\int_\Omega f\, d\mu \in D(T)$ and (1.8) holds.

Recent decades have witnessed a fast growing applications of fractional calculus and fractional differential equations to diverse scientific and engineering fields (cf. [5, 14, 30, 67, 70] and references cited therein for further information). In this paper, we mainly use the Caputo fractional derivatives. Let $\zeta > 0$. Then the Caputo fractional derivative $\mathbf{D}_t^\zeta u$ [5, 36] is defined for those functions $u \in C^{\lceil \zeta \rceil - 1}([0,\infty) : X)$ for which $g_{\lceil \zeta \rceil - \zeta} * (u - \sum_{j=0}^{\lceil \zeta \rceil - 1} u^{(j)}(0) g_{j+1}) \in C^{\lceil \zeta \rceil}([0,\infty) : X)$, by

$$\mathbf{D}_t^\zeta u(t) := \frac{d^{\lceil \zeta \rceil}}{dt^{\lceil \zeta \rceil}} \Big[g_{\lceil \zeta \rceil - \zeta} * \Big(u - \sum_{j=0}^{\lceil \zeta \rceil - 1} u^{(j)}(0) g_{j+1} \Big) \Big].$$

Define $C^r([0,T] : X)$ to be the vector space consisting of Hölder continuous functions $f : [0,T] \to X$ with the exponent r; if $r' \in (0,\infty) \setminus \mathbb{N}$, then we define $C^{r'}([0,T] : X)$ as the vector space consisting of those functions $f : [0,T] \to X$ for which $f \in C^{\lfloor r' \rfloor}([0,T] : X)$ and $f^{(\lfloor r' \rfloor)} \in C^{r' - \lfloor r' \rfloor}([0,T] : X)$. Without going into further details, we will only observe here that the existence of Caputo fractional derivative $\mathbf{D}_t^\zeta u$ implies $u \in C^{\lceil \zeta \rceil}((0,\infty) : X) \cap C^\zeta([0,T] : E)$, for each finite number $T > 0$. A proof is left to the interested reader.

We refer the reader to [5] for the notion of a Riemann-Liouville fractional derivative $D_t^\alpha u(t)$ of order $\alpha > 0$. The Mittag-Leffler function $E_{\beta,\gamma}(z)$ ($\beta > 0$, $\gamma \in \mathbb{R}$) is defined by

$$E_{\beta,\gamma}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\beta k + \gamma)}, \quad z \in \mathbb{C}.$$

Set, for short, $E_\beta(z) := E_{\beta,1}(z)$, $z \in \mathbb{C}$. If $\beta \in (0,1)$, then we define the Wright function $\Phi_\beta(\cdot)$ by

$$\Phi_\beta(t) := \mathcal{L}^{-1}\big(E_\beta(-\lambda)\big)(t), \quad t \geq 0,$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. For further information about the Mittag-Leffler and Wright functions, we refer the reader to [5], [36] and references cited there.

2. Multivalued linear operators in locally convex spaces

A multivalued map (multimap) $\mathcal{A} : X \to P(Y)$ is said to be a multivalued linear operator (MLO) if and only if the following holds:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of X;

(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x,\ y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

If $X = Y$, then we say that $\mathcal{A}$ is an MLO in X. An almost immediate consequence of definition is that $\mathcal{A}x + \mathcal{A}y = \mathcal{A}(x+y)$ for all $x,\ y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x = \mathcal{A}(\lambda x)$ for all $x \in D(\mathcal{A})$, $\lambda \neq 0$. Furthermore, for any $x,\ y \in D(\mathcal{A})$ and $\lambda,\ \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, we have $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. If $\mathcal{A}$ is an MLO, then $\mathcal{A}0$ is a linear manifold in Y and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Set $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. The set $\mathcal{A}^{-1}0 = \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ is called the kernel of $\mathcal{A}$ and it is denoted henceforth by $N(\mathcal{A})$ or $\mathrm{Kern}(\mathcal{A})$. The inverse $\mathcal{A}^{-1}$ of an MLO is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It is checked at once that $\mathcal{A}^{-1}$ is an MLO in X, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. If $N(\mathcal{A}) = \{0\}$, i.e., if $\mathcal{A}^{-1}$ is single-valued, then $\mathcal{A}$ is said to be injective. It is worth noting that $\mathcal{A}x = \mathcal{A}y$ for some two elements x and $y \in D(\mathcal{A})$, if and only if $\mathcal{A}x \cap \mathcal{A}y \neq \emptyset$; moreover, if $\mathcal{A}$ is injective, then the equality $\mathcal{A}x = \mathcal{A}y$ holds if and only if $x = y$.

For any mapping $\mathcal{A} : X \to P(Y)$ we define $\check{\mathcal{A}} := \{(x, y) : x \in D(\mathcal{A}),\ y \in \mathcal{A}x\}$. Then $\mathcal{A}$ is an MLO if and only if $\check{\mathcal{A}}$ is a linear relation in $X \times Y$, i.e., if and only if $\check{\mathcal{A}}$ is a linear subspace of $X \times Y$.

If $\mathcal{A},\ \mathcal{B} : X \to P(Y)$ are two MLOs, then we define its sum $\mathcal{A}+\mathcal{B}$ by $D(\mathcal{A}+\mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A}+\mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A}+\mathcal{B})$. It can be simply verified that $\mathcal{A}+\mathcal{B}$ is likewise an MLO.

Let $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : Y \to P(Z)$ be two MLOs, where Z is an SCLCS. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. Then $\mathcal{B}\mathcal{A} : X \to P(Z)$ is an MLO and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The scalar multiplication of an MLO $\mathcal{A} : X \to P(Y)$ with the number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$. It is clear that $z\mathcal{A} : X \to P(Y)$ is an MLO and $(\omega z)\mathcal{A} = \omega(z\mathcal{A}) = z(\omega\mathcal{A})$, $z,\ \omega \in \mathbb{C}$.

Suppose that X' is a linear subspace of X, and $\mathcal{A} : X \to P(Y)$ is an MLO. Then we define the restriction of operator $\mathcal{A}$ to the subspace X', $\mathcal{A}_{|X'}$ for short, by $D(\mathcal{A}_{|X'}) := D(\mathcal{A}) \cap X'$ and $\mathcal{A}_{|X'}x := \mathcal{A}x$, $x \in D(\mathcal{A}_{|X'})$. Clearly, $\mathcal{A}_{|X'} : X' \to P(Y)$ is an MLO. It is well known that an MLO $\mathcal{A} : X \to P(Y)$ is injective (resp., single-valued) if and only if $\mathcal{A}^{-1}\mathcal{A} = I_{|D(\mathcal{A})}$ (resp., $\mathcal{A}\mathcal{A}^{-1} = I^{Y}_{|R(\mathcal{A})}$).

The integer powers of an MLO $\mathcal{A} : X \to P(X)$ are defined recursively as follows: $\mathcal{A}^0 =: I$; if $\mathcal{A}^{n-1}$ is defined, set

$$D(\mathcal{A}^n) := \big\{x \in D(\mathcal{A}^{n-1}) : D(\mathcal{A}) \cap \mathcal{A}^{n-1}x \neq \emptyset\big\},$$

and

$$\mathcal{A}^n x := \big(\mathcal{A}\mathcal{A}^{n-1}\big)x = \bigcup_{y \in D(\mathcal{A}) \cap \mathcal{A}^{n-1}x} \mathcal{A}y, \quad x \in D(\mathcal{A}^n).$$

We can prove inductively that $(\mathcal{A}^n)^{-1} = (\mathcal{A}^{n-1})^{-1}\mathcal{A}^{-1} = (\mathcal{A}^{-1})^n =: \mathcal{A}^{-n}$, $n \in \mathbb{N}$ and $D((\lambda - \mathcal{A})^n) = D(\mathcal{A}^n)$, $n \in \mathbb{N}_0$. Moreover, if $\mathcal{A}$ is single-valued, then the above definitions are consistent with the usual definition of powers of $\mathcal{A}$.

If $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : X \to P(Y)$ are two MLOs, then we write $\mathcal{A} \subseteq \mathcal{B}$ if and only if $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$. Assume now that a linear single-valued operator $S : D(S) \subseteq X \to Y$ has domain $D(S) = D(\mathcal{A})$ and $S \subseteq \mathcal{A}$, where $\mathcal{A} : X \to P(Y)$ is an MLO. Then S is called a section of $\mathcal{A}$; if this is the case, we have $\mathcal{A}x = Sx + \mathcal{A}0$, $x \in D(\mathcal{A})$ and $R(\mathcal{A}) = R(S) + \mathcal{A}0$.

We say that an MLO operator $\mathcal{A} : X \to P(Y)$ is closed if for any nets (x_τ) in $D(\mathcal{A})$ and (y_τ) in Y such that $y_\tau \in \mathcal{A}x_\tau$ for all $\tau \in I$ we have that the suppositions $\lim_{\tau\to\infty} x_\tau = x$ and $\lim_{\tau\to\infty} y_\tau = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

We introduce the notion of a relatively closed MLO as follows [32]. We say that an MLO $\mathcal{A} : X \to P(Y)$ is relatively closed if and only if there exist auxiliary SCLCSs $X_\mathcal{A}$ and $Y_\mathcal{A}$ such that $D(\mathcal{A}) \subseteq X_\mathcal{A} \hookrightarrow X$, $R(\mathcal{A}) \subseteq Y_\mathcal{A} \hookrightarrow Y$ and $\mathcal{A}$ is closed in $X_\mathcal{A} \times Y_\mathcal{A}$; i.e., the assumptions $D(\mathcal{A}) \ni x_\tau \to x$ as $\tau \to \infty$ in $X_\mathcal{A}$ and $\mathcal{A}x_\tau \ni y_\tau \to y$ as $\tau \to \infty$ in $Y_\mathcal{A}$ implies that $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$. A relatively closed operator will also be called $X_\mathcal{A} \times Y_\mathcal{A}$-closed. For example, let $A,\ B : D \subseteq X \to Y$ be closed linear operators with the same domain D. Then the operator $A + B$ is not necessarily closed but it is always $[D(A)] \times Y$-closed (cf. [31, p. 170]). Examples presented in [32] can be simply reformulated for operators acting on locally convex spaces, as well:

Example 2.1. (i) If $\mathcal{A} : X \to P(Y)$ is an MLO, then $\overline{\mathcal{A}} : X \to P(Y)$ is likewise an MLO. This shows that any MLO has a closed linear extension, in contrast to the usually considered single-valued linear operators.

(ii) Let $A : D(A) \subseteq X \to Y$ be a single-valued linear operator that is $X_A \times Y_A$-closed, let $\mathcal{B} : X \to P(Y)$ be an MLO that is $X_\mathcal{B} \times Y_\mathcal{B}$-closed, and let $Y_A \hookrightarrow Y_\mathcal{B}$. Then the MLO $S = A + \mathcal{B}$ is $X_S \times Y_\mathcal{B}$-closed, where $X_S := D(A) \cap X_\mathcal{B}$ and the topology on X_S is induced by the system $(s_{p,q,r})$ of fundamental seminorms, defined as follows: $s_{p,q,r}(x) =: p(x)+p(Ax)+q(x)+r(Ax)$, $x \in X_S$ ($p \in \circledast_X,\ q \in \circledast_{X_\mathcal{B}},\ r \in \circledast_{Y_A}$).

(iii) Let $A : D(A) \subseteq X \to Y$ be a single-valued linear operator that is $X_A \times Y_A$-closed, let $\mathcal{B} : Y \to P(Z)$ be an MLO that is $Y_\mathcal{B} \times Z_\mathcal{B}$-closed, and let $Y_\mathcal{B} \hookrightarrow Y_A$. Then the MLO $C = \mathcal{B}A : X \to P(Z)$ is $X_C \times Z_\mathcal{B}$-closed, where $X_C := \{x \in D(A) : Ax \in Y_\mathcal{B}\}$ and the topology on X_C is induced by the system $(s_{p,q})$ of fundamental seminorms, defined as follows: $s_{p,q}(x) =: p(x)+p(Ax)+q(Ax)$, $x \in X_C$ ($p \in \circledast_X,\ q \in \circledast_{Y_\mathcal{B}}$).

(iv) Let $A : D(A) \subseteq X \to Y$ and $B : D(B) \subseteq X \to Y$ be two single-valued linear operators. Set

$$\mathcal{A} := B^{-1}A = \big\{(x, y) : x \in D(A),\ y \in D(B) \text{ and } Ax = By\big\}.$$

Then $\mathcal{A}$ is an MLO in X, and the following holds:

(a) If one of the operators $A,\ B$ is bounded and the other closed, then $\mathcal{A}$ is closed.
(b) If A is closed and B is $X_B \times Y$-closed, then $\mathcal{A}$ is $[D(A)] \times X_B$-closed.
(c) If B is closed and A is $X_A \times Y$-closed, then $\mathcal{A}$ is $X_A \times [D(B)]$-closed.
(d) If A is $X_A \times Y_A$-closed and B is $X_B \times Y_B$-closed, where $Y_B \hookrightarrow Y_A$, then $\mathcal{A}$ is $X_C \times X_B$-closed, where X_C is defined as in (iii).

If $\mathcal{A} : X \to P(Y)$ is an MLO, then we define the adjoint $\mathcal{A}^* : Y^* \to P(X^*)$ of $\mathcal{A}$ by its graph

$$\mathcal{A}^* := \big\{\big(y^*, x^*\big) \in Y^* \times X^* : \big\langle y^*, y\big\rangle = \big\langle x^*, x\big\rangle \text{ for all pairs } (x, y) \in \mathcal{A}\big\}.$$

It is simply verified that $\mathcal{A}^*$ is a closed MLO, and that $\langle y^*, y\rangle = 0$ whenever $y^* \in D(\mathcal{A}^*)$ and $y \in \mathcal{A}0$. Furthermore, $\mathcal{A}^*$ is single-valued provided that $\mathcal{A}$ is densely defined, $\mathcal{A}^* = \overline{\mathcal{A}}^*$ and the equations [17, (1.2)-(1.6)] continue to hold for adjoints of MLOs acting on locally convex spaces.

The following important lemma can be proved by using the Hahn-Banach theorem and the argumentation from [3].

Lemma 2.2. *Suppose that $\mathcal{A} : X \to P(Y)$ is an MLO and $\mathcal{A}$ is $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed. Assume, further, that $x_0 \in X$, $y_0 \in Y$ and $\langle x^*, x_0\rangle = \langle y^*, y_0\rangle$ for all pairs $(x^*, y^*) \in X^*_{\mathcal{A}} \times Y^*_{\mathcal{A}}$ satisfying that $\langle x^*, x\rangle = \langle y^*, y\rangle$ whenever $y \in \mathcal{A}x$. Then $y_0 \in \mathcal{A}x_0$.*

With Lemma 2.2 in view, we can simply prove the following extension of Theorem 1.3(iii) for relatively closed MLOs in locally convex spaces.

Theorem 2.3. *Suppose that $\mathcal{A} : X \to P(Y)$ is an MLO and $\mathcal{A}$ is $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed. Let $f : \Omega \to X_{\mathcal{A}}$ and $g : \Omega \to Y_{\mathcal{A}}$ be μ-integrable, and let $g(x) \in \mathcal{A}f(x)$, $x \in \Omega$. Then $\int_\Omega f\,d\mu \in D(\mathcal{A})$ and $\int_\Omega g\,d\mu \in \mathcal{A}\int_\Omega f\,d\mu$.*

In the remaining part of this section, we will analyze the C-resolvent sets of multivalued linear operators in locally convex spaces. Our standing assumptions will be that $\mathcal{A}$ is an MLO in X, as well as that $C \in L(X)$ is injective (the only exception will be Subsection 5.2, where C can be possibly non-injective) and $C\mathcal{A} \subseteq \mathcal{A}C$ (this is equivalent to say that, for any $(x, y) \in X \times X$, we have the implication $(x, y) \in \mathcal{A} \Rightarrow (Cx, Cy) \in \mathcal{A}$; by induction, we immediately get that $C\mathcal{A}^k \subseteq \mathcal{A}^kC$ for all $k \in \mathbb{N}$). Then the C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathcal{A})$;
(ii) $(\lambda - \mathcal{A})^{-1}C$ is a single-valued bounded operator on X.

The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$); the resolvent set of $\mathcal{A}$ is defined by $\rho(\mathcal{A}) := \rho_I(\mathcal{A})$, $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ ($\lambda \in \rho(\mathcal{A})$). We can almost trivially construct examples of MLOs for which $\rho(\mathcal{A}) = \emptyset$ and $\rho_C(\mathcal{A}) \neq \emptyset$; for example, let Y be a proper closed linear subspace of X, let $\mathcal{A}$ be an MLO in Y, and let $\lambda \in \mathbb{C}$ so that $(\lambda - \mathcal{A})^{-1} \in L(Y)$. Taking any injective operator $C \in L(X)$ with $R(C) \subseteq Y$, and looking $\mathcal{A} = \mathcal{A}_X$ as an MLO in X, it is clear that $\lambda \in \rho_C(\mathcal{A}_X)$ and $\rho(\mathcal{A}_X) = \emptyset$. In general case, if $\rho_C(\mathcal{A}) \neq \emptyset$, then for any $\lambda \in \rho_C(\mathcal{A})$ we have $\mathcal{A}0 = N((\lambda I - \mathcal{A})^{-1}C)$, as well as $\lambda \in \rho_C(\overline{\mathcal{A}})$, $\overline{\mathcal{A}} \subseteq C^{-1}\mathcal{A}C$ and $((\lambda - \mathcal{A})^{-1}C)^k(D(\mathcal{A}^l)) \subseteq D(\mathcal{A}^{k+l})$, $k,\ l \in \mathbb{N}_0$; here it is worth noting that the equality $\mathcal{A} = C^{-1}\mathcal{A}C$ holds provided, in addition, that $\rho(\mathcal{A}) \neq \emptyset$ (see the proofs of [12, Proposition 2.1, Lemma 2.3]). The basic properties of C-resolvent sets of single-valued linear operators [35, 36] continue to hold in our framework (observe, however, that there exist certain differences that we will not discuss here). For example, if $\rho(\mathcal{A}) \neq \emptyset$, then $\mathcal{A}$ is closed; it is well known that this statement does not hold if $\rho_C(\mathcal{A}) \neq \emptyset$ for some $C \neq I$ (cf. [12, Example 2.2]). Arguing as in the proofs of [17, Theorem 1.7-Theorem 1.9], we can deduce the validity of the following important theorem, which will be frequently used in the sequel.

Theorem 2.4. (i) *We have*

$$\big(\lambda - \mathcal{A}\big)^{-1}C\mathcal{A} \subseteq \lambda\big(\lambda - \mathcal{A}\big)^{-1}C - C \subseteq \mathcal{A}\big(\lambda - \mathcal{A}\big)^{-1}C, \quad \lambda \in \rho_C(\mathcal{A}).$$

The operator $(\lambda - \mathcal{A})^{-1}C\mathcal{A}$ is single-valued on $D(\mathcal{A})$ and $(\lambda - \mathcal{A})^{-1}C\mathcal{A}x = (\lambda - \mathcal{A})^{-1}Cy$, whenever $y \in \mathcal{A}x$ and $\lambda \in \rho_C(\mathcal{A})$.

(ii) *Suppose that $\lambda,\ \mu \in \rho_C(\mathcal{A})$. Then the resolvent equation*

$$\big(\lambda - \mathcal{A}\big)^{-1}C^2x - \big(\mu - \mathcal{A}\big)^{-1}C^2x = (\mu - \lambda)\big(\lambda - \mathcal{A}\big)^{-1}C\big(\mu - \mathcal{A}\big)^{-1}Cx, \quad x \in X$$

holds good. In particular, $(\lambda-\mathcal{A})^{-1}C(\mu-\mathcal{A})^{-1}C = (\mu-\mathcal{A})^{-1}C(\lambda-\mathcal{A})^{-1}C$.

By Theorem 2.4(i), it readily follows that the operator $\lambda(\lambda-\mathcal{A})^{-1}C-C \in L(X)$ is a bounded linear section of the MLO $\mathcal{A}(\lambda-\mathcal{A})^{-1}C$ ($\lambda \in \rho_C(\mathcal{A})$). Inductively, we can prove that, for every $x \in X$, $n \in \mathbb{N}_0$ and $\lambda \in \rho_C(\mathcal{A})$, we have $\mathrm{card}((\lambda-\overline{\mathcal{A}})^{-n}Cx) \leq 1$. Having in mind this fact, as well as the argumentation already seen many times in our previous research studies of C-resolvents of single-valued linear operators, we can prove the following extension of [36, Proposition 2.1.14] for MLOs in locally convex spaces.

Proposition 2.5. *Let* $\emptyset \neq \Omega \subseteq \rho_C(\mathcal{A})$ *be open, and let* $x \in X$.

(i) The local boundedness of the mapping $\lambda \mapsto (\lambda-\mathcal{A})^{-1}Cx$, $\lambda \in \Omega$, *resp. the assumption that* X *is barreled and the local boundedness of the mapping* $\lambda \mapsto (\lambda-\mathcal{A})^{-1}C$, $\lambda \in \Omega$, *implies the analyticity of the mapping* $\lambda \mapsto (\lambda-\mathcal{A})^{-1}C^3x$, $\lambda \in \Omega$, *resp.* $\lambda \mapsto (\lambda-\mathcal{A})^{-1}C^3$, $\lambda \in \Omega$. *Furthermore, if* $R(C)$ *is dense in* X, *resp. if* $R(C)$ *is dense in* X *and* X *is barreled, then the mapping* $\lambda \mapsto (\lambda-\mathcal{A})^{-1}Cx$, $\lambda \in \Omega$ *is analytic, resp. the mapping* $\lambda \mapsto (\lambda-\mathcal{A})^{-1}C$, $\lambda \in \Omega$ *is analytic.*

(ii) Suppose that $R(C)$ *is dense in* X. *Then the local boundedness of the mapping* $\lambda \mapsto (\lambda-\mathcal{A})^{-1}Cx$, $\lambda \in \Omega$ *implies its analyticity as well as* $Cx \in R((\lambda-\overline{\mathcal{A}})^n)$, $n \in \mathbb{N}$ *and*

$$\frac{d^{n-1}}{d\lambda^{n-1}}(\lambda-\mathcal{A})^{-1}Cx = (-1)^{n-1}(n-1)!(\lambda-\overline{\mathcal{A}})^{-n}Cx, \ n \in \mathbb{N}. \tag{2.1}$$

Furthermore, if X *is barreled, then the local boundedness of the mapping* $\lambda \mapsto (\lambda-\mathcal{A})^{-1}C$, $\lambda \in \Omega$ *implies its analyticity as well as* $R(C) \subseteq R((\lambda-\overline{\mathcal{A}})^n)$, $n \in \mathbb{N}$ *and*

$$\frac{d^{n-1}}{d\lambda^{n-1}}(\lambda-\mathcal{A})^{-1}C = (-1)^{n-1}(n-1)!(\lambda-\overline{\mathcal{A}})^{-n}C \in L(X), \ n \in \mathbb{N}. \tag{2.2}$$

(iii) The continuity of mapping $\lambda \mapsto (\lambda-\mathcal{A})^{-1}Cx$, $\lambda \in \Omega$ *implies its analyticity and* (2.1). *Furthermore, if* X *is barreled, then the continuity of mapping* $\lambda \mapsto (\lambda-\mathcal{A})^{-1}C$, $\lambda \in \Omega$ *implies its analyticity and* (2.2).

It is well known that $\rho_C(A)$ need not be an open subset of $\mathbb{C}$ if $C \neq I$ and A is a single-valued linear operator (cf. [12, Example 2.5]) and that $\rho(\mathcal{A})$ is an open subset of $\mathbb{C}$, provided that X is a Banach space and $\mathcal{A}$ is an MLO in X (cf. [17, Theorem 1.6]). The regular C-resolvent set of $\mathcal{A}$, $\rho_C^r(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \rho_C(\mathcal{A})$ for which $(\lambda-\mathcal{A})^{-1}C \in R(X)$, where $R(X)$ denotes the set of all regular bounded linear operators $A \in L(X)$, i.e., the operators $A \in L(X)$ for which there exists a positive constant $c > 0$ such that for each seminorm $p \in ⊛$ there exists another seminorm $q \in ⊛$ such that $p(A^nx) \leq c^n q(x)$, $x \in X$, $n \in \mathbb{N}$; the regular resolvent set of $\mathcal{A}$, $\rho^r(\mathcal{A})$ for short, is then defined by $\rho^r(\mathcal{A}) := \rho_I^r(\mathcal{A})$. By the argumentation contained in the proof of [17, Theorem 1.6], it readily follows that $\rho^r(\mathcal{A})$ is always an open subset of $\mathbb{C}$.

The generalized resolvent equations hold for C-resolvents of multivalued linear operators; more precisely, we have the following theorem which can be proved by induction.

Theorem 2.6. (i) *Let* $x \in X$, $k \in \mathbb{N}_0$ *and* $\lambda, z \in \rho_C(\mathcal{A})$ *with* $z \neq \lambda$. *Then the following holds:*

$$(z-\mathcal{A})^{-1}C\Big((\lambda-\mathcal{A})^{-1}C\Big)^k x$$

$$= \frac{(-1)^k}{(z-\lambda)^k}\big(z-\mathcal{A}\big)^{-1}C^{k+1}x + \sum_{i=1}^{k} \frac{(-1)^{k-i}\big((\lambda-\mathcal{A})^{-1}C\big)^i C^{k+1-i}x}{\big(z-\lambda\big)^{k+1-i}}.$$

(ii) *Let $k \in \mathbb{N}_0$, $x,\ y \in X$, $y \in (\lambda_0-\mathcal{A})^k x$ and $\lambda_0,\ z \in \rho_C(\mathcal{A})$ with $z \neq \lambda_0$. Then the following holds:*

$$\begin{aligned}\big(z-\mathcal{A}\big)^{-1}C^{k+1}x &= \frac{(-1)^k}{\big(z-\lambda_0\big)^k}\big(z-\mathcal{A}\big)^{-1}C^{k+1}y \\ &\quad + \sum_{i=1}^{k} \frac{(-1)^{k-i}\big((\lambda_0-\mathcal{A})^{-1}C\big)^i C^{k+1-i}y}{\big(z-\lambda_0\big)^{k+1-i}}.\end{aligned}$$

We close this subsection with the observation that the notion of C-resolvent set of a given MLO can be also introduced in the case that C is not injective. If this is the case, Theorem 2.4, Theorem 2.6 and an analogue of Proposition 2.5 continue to hold ([38]).

3. Laplace transform of functions with values in sequentially complete locally convex spaces

In this section, we assume that $\mu = dt$ is the Lebesgue measure on $\Omega = [0,\infty)$ and $f : [0,\infty) \to X$ is a locally Lebesgue integrable function (in the sense of Definition 1.2(i)). As in the Banach space case, we will denote the space consisting of such functions by $L^1_{\rm loc}([0,\infty) : X)$; similarly we define the space $L^1([0,\tau] : X)$ for $0 < \tau < \infty$. It is clear that (1.7) implies $\langle x^*, f(\cdot)\rangle \in L^1_{\rm loc}([0,\infty))$ for $x^* \in X^*$. The first normalized antiderivative $t \mapsto f^{[1]}(t) := F(t) := \int_0^t f(s)\,ds$, $t \geq 0$ of $f(\cdot)$ is continuous for $t \geq 0$, and we have that $\int_0^t p(f)\,d\mu < \infty$ for any $p \in \circledast$ and $t \geq 0$. Set $f^{[n]}(t) := \int_0^t g_n(t-s)f(s)\,ds$, $t \geq 0$.

A few auxiliary results on integration in sequentially complete locally convex spaces is collected in the following theorem, which seems to be new and not considered elsewhere in the existing literature:

Theorem 3.1. (i) *Suppose that $g \in C([0,\infty))$ and $f \in L^1_{\rm loc}([0,\infty) : X)$. Then $gf \in L^1_{\rm loc}([0,\infty) : X)$.*

(ii) *If $g \in L^1_{\rm loc}([0,\infty))$ and $f \in C([0,\infty) : X)$, then $gf \in L^1_{\rm loc}([0,\infty) : X)$.*

(iii) *(The partial integration) Suppose that $g \in AC_{\rm loc}([0,\infty))$. Then, for every $\tau \geq 0$, we have*

$$\int_0^\tau g(t)f(t)\,dt = g(\tau)F(\tau) - \int_0^\tau g'(t)F(t)\,dt. \tag{3.1}$$

Proof. Fix a number $\tau \in (0,\infty)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of simple functions in $X^{[0,\tau]}$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ a.e. $t \in K = [0,\tau]$ and for all $\epsilon > 0$ and each $p \in \circledast$ there is a number $n_0 = n_0(\epsilon,p)$ such that (1.4) holds. Then $\int_0^\tau f(t)\,dt = \lim_{n\to\infty} \int_0^\tau f_n(t)\,dt$ and the sequence $(p(f_n))_{n\in\mathbb{N}}$ is convergent in $L^1[0,\tau]$. By the proof of [61, Proposition 4.4.1], there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of simple functions in $\mathbb{C}^{[0,\tau]}$ such that $\lim_{n\to\infty} \|s_n - g\|_{L^\infty[0,\tau]} = 0$, $\sup_{n\in\mathbb{N}} \|s_n\|_{L^\infty[0,\tau]} \leq \|g\|_{L^\infty[0,\tau]}$ and that for all $\epsilon > 0$ and $p = |\cdot|$ there is a number $n_0 = n_0(\epsilon,p)$ such that (1.4) holds with the functions $f_n(\cdot)$ and $f_m(\cdot)$ replaced respectively with $s_n(\cdot)$ and $s_m(\cdot)$. Clearly, $(s_nf_n)_{n\in\mathbb{N}}$ is a sequence of simple functions in $X^{[0,\tau]}$ such that

$\lim_{n\to\infty} s_n(t)f_n(t) = g(t)f(t)$ a.e. $t \in [0,\tau]$. Furthermore, it can be easily seen that

$$\begin{aligned}
&\int_0^t p\big(s_n(t)f_n(t) - s_m(t)f_m(t)\big)\,dt \\
&\le \|s_n\|_{L^\infty[0,\tau]} \int_0^t p\big(f_n(t) - f_m(t)\big)\,dt + \|s_n - s_m\|_{L^\infty[0,\tau]} \int_0^t p\big(f_m(t)\big)\,dt \\
&\le \|g\|_{L^\infty[0,\tau]} \int_0^t p\big(f_n(t) - f_m(t)\big)\,dt \\
&\quad + \Big(\|s_n - g\|_{L^\infty[0,\tau]} + \|s_m - g\|_{L^\infty[0,\tau]}\Big) \int_0^t p\big(f_m(t)\big)\,dt, \quad m,\ n \in \mathbb{N}.
\end{aligned}$$

This proves (i). To prove (ii), observe first that using Definition 1.1 we can directly prove that a function $g_1 f_1(\cdot)$ belongs to the space $L^1([0,\tau] : X)$, provided that $f_1 : [0,\tau] \to X$ is a simple function and $g_1 \in L^1[0,\tau]$. Using the arguments contained in the proof of [61, Proposition 4.4.1] once more, we can find a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions in $X^{[0,\tau]}$ such that, for every $p \in \circledast$, $\lim_{n\to\infty} p(f_n - f)_{L^\infty[0,\tau]} = 0$, $\sup_{n\in\mathbb{N}} p(f_n)_{L^\infty[0,\tau]} \le p(f)_{L^\infty[0,\tau]}$ and that for all $\epsilon > 0$ there is a number $n_0 = n_0(\epsilon, p)$ such that (1.4) holds. Therefore, $(gf_n)_{n\in\mathbb{N}}$ is a sequence in $L^1([0,\tau] : X)$ and $\lim_{n\to\infty} g(t)f_n(t) = g(t)f(t)$ a.e. $t \in [0,\tau]$. Making use of the dominated convergence theorem (Theorem 1.3(i)), we obtain that $gf \in L^1_{\rm loc}([0,\infty) : X)$, as claimed. By (i) and (ii), the both integral in (3.1) are well-defined. Let $x^* \in X^*$. Using the partial integration in the Lebesgue integral and (1.7), we obtain that

$$\int_0^\tau g(t)\big\langle x^*, f(t)\big\rangle\,dt = g(\tau)\big\langle x^*, F(\tau)\big\rangle - \int_0^\tau g'(t)\big\langle x^*, F(t)\big\rangle\,dt.$$

Since x^* was arbitrary, it readily follows from (1.7) that (3.1) holds. The proof of the theorem is thereby complete. □

In the remaining part of this section, we are concerned with the existence of Laplace integral

$$(\mathcal{L}f)(\lambda) := \tilde{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\,dt := \lim_{\tau\to\infty} \int_0^\tau e^{-\lambda t} f(t)\,dt,$$

for $\lambda \in \mathbb{C}$. If $\tilde{f}(\lambda_0)$ exists for some $\lambda_0 \in \mathbb{C}$, then we define the abscissa of convergence of $\tilde{f}(\cdot)$ by

$$\mathrm{abs}_X(f) := \inf\big\{\Re\lambda : \tilde{f}(\lambda) \text{ exists}\big\};$$

otherwise, $\mathrm{abs}_X(f) := +\infty$. It is said that $f(\cdot)$ is Laplace transformable, or equivalently, that $f(\cdot)$ belongs to the class (P1)-X, if and only if $\mathrm{abs}_X(f) < \infty$. Assuming that there exists a number $\omega \in \mathbb{R}$ such that for each seminorm $p \in \circledast$ there exists a number $M_p > 0$ satisfying that $p(f(t)) \le M_p e^{\omega t}$, $t \ge 0$, we define $\omega_X(f) \in [-\infty, \infty)$ as the infimum of all numbers $\omega \in \mathbb{R}$ with the above property; if there is no such a number $\omega \in \mathbb{R}$, then we define $\omega_X(f) := +\infty$. Further on, we abbreviate $\omega_X(f)$ ($\mathrm{abs}_X(f)$) to $\omega(f)$ ($\mathrm{abs}(f)$), if there is no risk for confusion. Define

$${}_w\,\mathrm{abs}(f) := \inf\Big\{\lambda \in \mathbb{R} : \sup_{t>0}\Big|\int_0^t e^{-\lambda s}\big\langle x^*, f(s)\big\rangle\,ds\Big| < \infty \text{ for all } x^* \in X^*\Big\},$$

$F_\infty := \lim_{\tau\to\infty} F(\tau)$, if the limit exists in X, and $F_\infty := 0$, otherwise.

Keeping in mind Theorem 3.1, we can repeat literally the argumentation from [1, Section 1.4, pp. 27-30] in order to see that the following theorem holds good (the only essential difference occurs on l. 6, p. 29, where we can use [62, Mackey's theorem 23.15] in place of the uniform boundedness principle):

Theorem 3.2. *Let $f \in L^1_{\text{loc}}([0,\infty) : X)$. Then the following holds:*

(i) *The Laplace integral $\tilde{f}(\lambda)$ converges if $\Re\lambda > \operatorname{abs}(f)$ and diverges if $\Re\lambda < \operatorname{abs}(f)$. If $\Re\lambda = \operatorname{abs}(f)$, then the Laplace integral may or may not be convergent.*

(ii) $_w \operatorname{abs}(f) = \operatorname{abs}(f)$.

(iii) *Suppose that $\lambda \in \mathbb{C}$ and the limit $\lim_{\tau\to\infty} \int_0^t e^{-\lambda s} p(f(s))\,ds$ exists for any $p \in \circledast$. Then $\tilde{f}(\lambda)$ exists, as well.*

(iv) *We have*

$$abs(f) \le abs(p(f)) \le \omega(f), \quad p \in \circledast.$$

In general, any of these two inequalities can be strict.

(v) *We have*

$$\operatorname{abs}(f) = \omega\big(F - F_\infty\big), \tag{3.2}$$

$$\tilde{f}(\lambda) = F_\infty + \lambda \int_0^\infty e^{-\lambda t}\big(F(t) - F_\infty\big)\,dt,\ \Re\lambda > \omega\big(F - F_\infty\big), \tag{3.3}$$

$$\tilde{f}(\lambda) = \lambda \tilde{F}(\lambda), \quad \Re\lambda > \max(\operatorname{abs}(f), 0) \tag{3.4}$$

and

$$\operatorname{abs}(f) \le \omega \Leftrightarrow \omega(F) \le \omega \quad (\textit{if } \omega \ge 0).$$

In particular, $f(\cdot)$ is Laplace transformable if and only if $\omega(F) < \infty$.

Recall [79], a function $h(\cdot)$ belongs to the class $LT - X$ if and only if there exist a function $g \in C([0,\infty) : X)$ and a number $\omega \in \mathbb{R}$ such that $\omega(g) \le \omega < \infty$ and $h(\lambda) = (\mathcal{L}g)(\lambda)$ for $\lambda > \omega$; as observed in [36, Section 1.2], the assumption $h \in LT - X$ immediately implies that the function $\lambda \mapsto h(\lambda)$, $\lambda > \omega$ can be analytically extended to the right half plane $\{\lambda \in \mathbb{C} : \Re\lambda > \omega\}$. In the sequel, the set of all originals $g(\cdot)$ whose Laplace transform belongs to the class $LT - X$ will be abbreviated to $LT_{or} - X$. Keeping this observation and the equations (3.2)-(3.3) in mind, we can simply prove that the mapping $\lambda \mapsto \tilde{f}(\lambda)$, $\Re\lambda > \operatorname{abs}(f)$ is analytic, provided that $f \in (\text{P1}){-}X$. If this is the case, the following formula holds:

$$\frac{d^n}{d\lambda^n}\tilde{f}(\lambda) = (-1)^n \int_0^\infty e^{-\lambda t} t^n f(t)\,dt, \quad n \in \mathbb{N},\ \lambda \in \mathbb{C},\ \Re\lambda > \operatorname{abs}(f). \tag{3.5}$$

In the following theorem, we will collect various operational properties of vector-valued Laplace transform.

Theorem 3.3. *Let $f \in$ (P1)-X, $z \in \mathbb{C}$ and $s \ge 0$.*

(i) Put $g(t) := e^{-zt} f(t)$, $t \ge 0$. Then $g(\cdot)$ is Laplace transformable, $\operatorname{abs}(g) = \operatorname{abs}(f) - \Re z$ *and $\tilde{g}(\lambda) = \tilde{f}(\lambda + z)$, $\lambda \in \mathbb{C}$, $\Re\lambda > \operatorname{abs}(f) - \Re z$.*

(ii) Put $f_s(t) := f(t+s)$, $t \ge 0$, $h_s(t) := f(t-s)$, $t \ge s$ and $h_s(t) := 0$, $s \in [0,t]$. Then $\operatorname{abs}(f_s) = \operatorname{abs}(h_s) = \operatorname{abs}(f)$, $\tilde{f_s}(\lambda) = e^{\lambda s}(\tilde{f}(\lambda) - \int_0^s e^{-\lambda t} f(t)\,dt)$ and $\widetilde{h_s}(\lambda) = e^{-\lambda s}\tilde{f}(\lambda)$ ($\lambda \in \mathbb{C}$, $\Re\lambda > a$).

(iii) Let $T \in L(X,Y)$. Then $T \circ f \in$ (P1)-Y and $T\tilde{f}(\lambda) = (\widetilde{T \circ f})(\lambda)$ for $\lambda \in \mathbb{C}$, $\Re\lambda > \operatorname{abs}(f)$.

(iv) Suppose that $\mathcal{A} : X \to P(Y)$ is an MLO and $\mathcal{A}$ is $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed, as well as $f \in (P1) - X_{\mathcal{A}}$, $l \in (P1) - Y_{\mathcal{A}}$ and $(f(t), l(t)) \in \mathcal{A}$ for a.e. $t \ge 0$. Then $(\tilde{f}(\lambda), \tilde{l}(\lambda)) \in \mathcal{A}$, $\lambda \in \mathbb{C}$ for $\Re\lambda > \max(\operatorname{abs}(f), \operatorname{abs}(l))$.

(v) Suppose, in addition, $\omega(f) < \infty$. Put

$$j(t) := \int_0^\infty \frac{e^{-s^2/4t}}{\sqrt{\pi t}} f(s)\,ds := \lim_{\tau\to\infty} \int_0^\tau \frac{e^{-s^2/4t}}{\sqrt{\pi t}} f(s)\,ds, \quad t > 0,$$
$$k(t) := \int_0^\infty \frac{se^{-s^2/4t}}{2\sqrt{\pi}t^{\frac{3}{2}}} f(s)\,ds := \lim_{\tau\to\infty} \int_0^\tau \frac{se^{-s^2/4t}}{2\sqrt{\pi}t^{\frac{3}{2}}} f(s)\,ds, \quad t > 0.$$

Then $j(\cdot)$ and $k(\cdot)$ are Laplace transformable,

$$\max(\operatorname{abs}(j), \operatorname{abs}(k)) \le \big(\max(\omega(f), 0)\big)^2, \quad \tilde{j}(\lambda) = \frac{\tilde{f}\big(\sqrt{\lambda}\big)}{\sqrt{\lambda}}, \quad \tilde{k}(\lambda) = \tilde{f}\big(\sqrt{\lambda}\big)$$

for all $\lambda \in \mathbb{C}$ with $\Re\lambda > (\max(\omega(f), 0))^2$.

(vi) *Let $f \in$ (P1)-X, $h \in L^1_{\rm loc}([0,\infty))$ and $\operatorname{abs}(|h|) < \infty$. Suppose, in addition, that $f \in C([0,\infty) : X)$. Put*

$$(h * f)(t) := \int_0^t h(t-s) f(s)\,ds, \quad t \ge 0.$$

Then the mapping $t \mapsto (h * f)(t)$, $t \ge 0$ is continuous, $h * f \in$ (P1)-X, and

$$\widetilde{h * f}(\lambda) = \tilde{h}(\lambda)\tilde{f}(\lambda),\ \lambda \in \mathbb{C},\ \Re\lambda > \max\big(\operatorname{abs}(|h|), \operatorname{abs}(f)\big).$$

Proof. Keeping in mind Theorem 3.1, Theorem 1.3(ii) and Theorem 2.3, the assertions (i)-(iv) can be proved as in the Banach space case (cf. [1, Proposition 1.6.1-Proposition 1.6.3] for more details). Consider now the part (v). Let $\lambda \in \mathbb{C}$ with $\Re\lambda > (\max(\omega(f), 0))^2$ be fixed. Then $\Re(\sqrt{\lambda}) > \max(\omega(f), 0) \ge \max(\omega(F), 0)$ so that [36, Theorem 1.2.1(v)] implies in combination with (3.4) that $\tilde{f}(\sqrt{\lambda})$ exists, as well as that

$$\tilde{f}\big(\sqrt{\lambda}\big) = \frac{\tilde{F}\big(\sqrt{\lambda}\big)}{\sqrt{\lambda}} = \int_0^\infty e^{-\lambda t} \int_0^\infty \frac{e^{-s^2/4t}}{\sqrt{\pi t}} f(s)\,ds\,dt.$$

On the other hand, we can use the dominated convergence theorem and an elementary argumentation to prove that the mapping $t \mapsto k(t)$, $t > 0$ is continuous as well as that for each seminorm $p \in \circledast$ there exists a finite number $m_p > 0$ such that $p(k(t)) \le m_p t^{(-1)/2}$, $t \in (0,1]$. This simply implies $k \in L^1_{\rm loc}([0,\infty) : X)$. Since

$$\int_0^\infty e^{-\lambda t}\big\langle x^*, k(t)\big\rangle\,dt = \int_0^\infty e^{-\sqrt{\lambda}t}\big\langle x^*, f(t)\big\rangle\,dt, \quad x^* \in X^*, \tag{3.6}$$

we obtain

$$\lim_{\tau\to\infty}\big\langle x^*, \int_0^\tau e^{-\lambda t} k(t)\,dt\big\rangle = \langle x^*, \tilde{f}\big(\sqrt{\lambda}\big)\rangle, \quad x^* \in X^*.$$

By Theorem 3.1(i), we obtain that the mapping $\tau \mapsto \int_0^\tau e^{-\lambda t} k(t)\,dt$, $\tau \ge 0$ is continuous so that the previous equality implies $\sup_{\tau\ge 0} |\langle x^*, \int_0^\tau e^{-\lambda t} k(t)\,dt\rangle| < \infty$ for all $x^* \in X^*$. Therefore, Theorem 3.2(ii) shows that $\lambda >_w \operatorname{abs}(k) = \operatorname{abs}(k)$ and $\tilde{k}(\lambda)$ exists. Using again (3.6), it readily follows that $\tilde{k}(\lambda) = f(\sqrt{\lambda})$, as claimed. Similarly we can prove that $\tilde{j}(\lambda) = f(\sqrt{\lambda})/\sqrt{\lambda}$. Suppose, finally, that the requirements of (vi) hold. Then it is very simple to prove that the mapping $t \mapsto (h*f)(t)$, $t \ge 0$ is continuous as well as that $\omega(1 * h * f) = \omega(h * (1 * f)) < \infty$. An application of Theorem 3.2(v) yields that $h * f \in$ (P1)-X. Fix now a number $\lambda \in \mathbb{C}$ with $\Re\lambda > \max(\operatorname{abs}(|h|), \operatorname{abs}(f))$. Since $\operatorname{abs}(\langle x^*, f(\cdot)\rangle) \le \operatorname{abs}(f)$ for all $x^* \in X^*$, [1, Proposition 1.6.4] implies that $(\mathcal{L}(h * \langle x^*, f(\cdot)\rangle))(\lambda)$ exists. Using this fact, it readily follows that

$$\sup_{t>0}\Big|\int_0^t e^{-s\Re\lambda}\Big(h * \langle x^*, f(\cdot)\rangle\Big)(s)\,ds\Big| < \infty, \quad x^* \in X^*.$$

By Theorem 3.2(ii), we obtain that $\widetilde{h * f}(\lambda)$ exists. The equality $\widetilde{h * f}(\lambda) = \tilde{h}(\lambda)\tilde{f}(\lambda)$ can be proved in a routine manner. □

For the sequel, we need the notion of a Lebesgue point of a function $f \in L^1_{\rm loc}([0,\infty) : X)$. A point $t \geq 0$ is said to be a Lebesgue point of $f(\cdot)$ if and only if for each seminorm $p \in ⊛$, we have

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} p\big(f(s) - f(t)\big)\,ds = 0. \tag{3.7}$$

It is clear that any point of continuity of function $f(\cdot)$ is one of Lebesgue's points of $f(\cdot)$, as well as that the mapping $t \mapsto F(t)$, $t \geq 0$ is differentiable at any Lebesgue's point of $f(\cdot)$. Furthermore, a slight modification of the proof of [1, Proposition 1.2.2; a)/b)] shows that the following holds:

(Q1) For each seminorm $p \in ⊛$ there exists a set $N_p \subseteq [0,\infty)$ of Lebesgue's measure zero such that

$$\lim_{h\to 0} p\Big(\frac{1}{h}\int_t^{t+h} f(s)\,ds - f(t)\Big) = 0, \quad t \in [0,\infty) \setminus N_p$$

and that (3.7) holds for $t \in [0,\infty) \setminus N_p$.

In the case that X is a Fréchet space, (Q1) immediately implies that almost every point $t > 0$ is a Lebesgue point of $f(\cdot)$.

Using the proof of [1, Theorem 1.7.7], Theorem 3.1(iii), as well as the equations (1.6) and (3.5), we can simply prove that the Post-Widder inversion formula holds in our framework:

Theorem 3.4 (Post-Widder)**.** *Suppose $f \in (P1) - X$ and $t > 0$ is a Lebesgue point of $f(\cdot)$. Then*

$$f(t) = \lim_{n\to\infty}(-1)^n\frac{1}{n!}\Big(\frac{n}{t}\Big)^{n+1}\tilde{f}^{(n)}\Big(\frac{n}{t}\Big).$$

The situation is much more complicated if we consider the Phragmén-Doetsch inversion formula for the Laplace transform of functions with values in SCLCSs. The following result of this type will be sufficiently general for our purposes:

Theorem 3.5. *Let $f \in$ (P1)-X and $t \geq 0$. Then the following holds:*

$$f^{[2]}(t) = \lim_{\lambda\to\infty}\sum_{n=1}^{\infty}(-1)^{n-1}n!^{-1}e^{n\lambda t}\frac{\tilde{f}(n\lambda)}{n\lambda}.$$

Proof. Due to Theorem 3.2(v), we have $F \in C([0,\infty) : X)$ and $\omega(F) < \infty$. The result now follows easily from [36, Theorem 1.2.1(ix)]. □

Now we will state and prove the following uniqueness type theorem for the Laplace transform.

Theorem 3.6 (Uniqueness theorem for the Laplace transform)**.** *Suppose that* $f \in (P1)-X$, $\lambda_0 > \operatorname{abs}(f)$ *and* $\tilde{f}(\lambda) = 0$ *for all* $\lambda > \lambda_0$*. Then* $F(t) = 0$, $t \geq 0$, $f(t) = 0$ *if* $t > 0$ *is a Lebesgue point of* $f(\cdot)$*, and for each seminorm* $p \in \circledast$ *there exists a set* $N_p \subseteq [0, \infty)$ *of Lebesgue's measure zero such that* $p(f(t)) = 0$, $t \in [0, \infty) \setminus N_p$*. In particular, if* X *is a Fréchet space, then* $f(t) = 0$ *for a.e.* $t \geq 0$*.*

Proof. The function $t \mapsto F(t)$, $t \geq 0$ is continuous and by Theorem 3.2(v) we obtain that $\omega(F) < \infty$ and $\tilde{F}(\lambda) = 0$, $\lambda > \max(\lambda_0, 0)$. Now we can apply Theorem 3.4 in order to see that $F(t) = 0$, $t \geq 0$. The remaining part of proof is simple and therefore omitted. □

Remark 3.7. *Suppose that* $f \in L^1_{\text{loc}}([0, \infty) : X)$ *and for each seminorm* $p \in \circledast$ *there exists a set* $N_p \subseteq [0, \infty)$ *of Lebesgue's measure zero such that* $p(f(t)) = 0$, $t \in [0, \infty) \setminus N_p$*. Then* $\operatorname{abs}(f) = \operatorname{abs}(p(f)) = -\infty$ *(*$p \in \circledast$*) and* $\tilde{f}(\lambda) = 0$ *for all* $\lambda \in \mathbb{C}$*.*

The following converse of Theorem 3.3(iv) simply follows from an application of Theorem 3.5.

Proposition 3.8. *Suppose that* $\mathcal{A} : X \to P(Y)$ *is an MLO and* $\mathcal{A}$ *is* $X_{\mathcal{A}} \times Y_{\mathcal{A}}$*-closed, as well as* $f \in (P1) - X_{\mathcal{A}}$, $l \in (P1) - Y_{\mathcal{A}}$ *and* $(\tilde{f}(\lambda), \tilde{l}(\lambda)) \in \mathcal{A}$, $\lambda \in \mathbb{C}$ *for* $\Re\lambda > \max(\operatorname{abs}(f), \operatorname{abs}(l))$*. Then* $\mathcal{A}f^{[1]}(t) = l^{[1]}(t)$, $t \geq 0$ *and* $\mathcal{A}f(t) = l(t)$ *for any* $t > 0$ *which is a Lebesgue point of both functions* $f(t)$ *and* $l(t)$*.*

The method proposed by Xiao and Liang in [80] provides a sufficiently enough framework for the theoretical study of real and complex inversion methods for the Laplace transform of functions with values in SCLSCs, as well as for the studies of analytical properties and approximation of Laplace transform (see e.g. [79, Section 1.1.1] and [36, Section 1.2] for more details); this method can be successfully applied in the analysis of subordination principles for abstract time-fractional inclusions, as well (cf. Theorem 4.8 below). It is also worth noting that there exists a great number of theoretical results from the monograph [1], not mentioned so far, which can be reconsidered for the Laplace transformable functions with values in SCLSCs; for example, all structural results from [1, Section 4.1] continue to hold in our framework. Due primarily to the space limitations, in this paper we will not be able to consider many other important questions concerning the vector-valued Laplace transform of functions with values in SCLCSs.

At the end of this section, we would like to briefly explain how we can extend the definition of Laplace transformable functions to the multivalued ones. Let $0 < \tau \leq \infty$ and $\mathcal{F} : [0, \tau) \to P(X)$. A single-valued function $f : [0, \tau) \to X$ is called a section of $\mathcal{F}$ if and only if $f(t) \in \mathcal{F}(t)$ for all $t \in [0, \tau)$. We denote the set of all sections, resp., all continuous sections, of $\mathcal{F}$ by $\sec(\mathcal{F})$, resp., $\sec_c(\mathcal{F})$. Suppose now that $\tau = \infty$ and any function $f \in \sec(\mathcal{F})$ belongs to the class (P1)-X. Then we define $\operatorname{abs}_X(\mathcal{F}) := \sup\{\operatorname{abs}_X(f) : f \in \sec(v)\}$; $\mathcal{F}(\cdot)$ is said to be Laplace transformable if and only if $\operatorname{abs}_X(\mathcal{F}) < \infty$.

4. Abstract degenerate Volterra integro-differential inclusions

In the following general definition, we introduce various types of solutions to the abstract degenerate inclusions (1.1), (1.2) and (1.3).

Definition 4.1. Let $0 < \tau \leq \infty$, $\alpha > 0$, $a \in L^1_{\rm loc}([0,\tau))$, $a \neq 0$, $\mathcal{F} : [0,\tau) \to P(Y)$, and let $\mathcal{A} : X \to P(Y)$, $\mathcal{B} : X \to P(Y)$ be two given mappings (possibly non-linear).

(i) A function $u \in C([0,\tau) : X)$ is said to be a pre-solution of (1.1) if and only if $(a*u)(t) \in D(\mathcal{A})$ and $u(t) \in D(\mathcal{B})$ for $t \in [0,\tau)$, as well as (1.1) holds. By a solution of (1.1), we mean any pre-solution $u(\cdot)$ of (1.1) satisfying additionally that there exist functions $u_{\mathcal{B}} \in C([0,\tau) : Y)$ and $u_{a,\mathcal{A}} \in C([0,\tau) : Y)$ such that $u_{\mathcal{B}}(t) \in \mathcal{B}u(t)$ and $u_{a,\mathcal{A}}(t) \in \mathcal{A}\int_0^t a(t-s)u(s)\,ds$ for $t \in [0,\tau)$, as well as

$$u_{\mathcal{B}}(t) \in u_{a,\mathcal{A}}(t) + \mathcal{F}(t),\ t \in [0,\tau).$$

Strong solution of (1.1) is any function $u \in C([0,\tau) : X)$ satisfying that there exist two continuous functions $u_{\mathcal{B}} \in C([0,\tau) : Y)$ and $u_{\mathcal{A}} \in C([0,\tau) : Y)$ such that $u_{\mathcal{B}}(t) \in \mathcal{B}u(t)$, $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$ for all $t \in [0,\tau)$, and

$$u_{\mathcal{B}}(t) \in (a * u_{\mathcal{A}})(t) + \mathcal{F}(t), \quad t \in [0,\tau).$$

(ii) Let $B = \mathcal{B}$ be single-valued. By a p-solution of (1.2), we mean any X-valued function $t \mapsto u(t)$, $t \geq 0$ such that the term $t \mapsto \mathbf{D}_t^\alpha Bu(t)$, $t \geq 0$ is well-defined, $u(t) \in D(\mathcal{A})$ for $t \geq 0$, and the requirements of (1.2) hold; a pre-solution of (1.2) is any p-solution of (1.2) that is continuous for $t \geq 0$. Finally, a solution of (1.2) is any pre-solution $u(\cdot)$ of (1.2) satisfying additionally that there exists a function $u_{\mathcal{A}} \in C([0,\infty) : Y)$ such that $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$ for $t \geq 0$, and $\mathbf{D}_t^\alpha Bu(t) \in u_{\mathcal{A}}(t) + \mathcal{F}(t)$, $t \geq 0$.

(iii) By a pre-solution of (1.3), we mean any continuous X-valued function $t \mapsto u(t)$, $t \geq 0$ such that the term $t \mapsto \mathbf{D}_t^\alpha u(t)$, $t \geq 0$ is well defined and continuous, as well as that $\mathbf{D}_t^\alpha u(t) \in D(\mathcal{B})$ and $u(t) \in D(\mathcal{A})$ for $t \geq 0$, and that the requirements of (1.3) hold; a solution of (1.3) is any pre-solution $u(\cdot)$ of (1.3) satisfying additionally that there exist functions $u_{\alpha,\mathcal{B}} \in C([0,\infty) : Y)$ and $u_{\mathcal{A}} \in C([0,\infty) : Y)$ such that $u_{\alpha,\mathcal{B}}(t) \in \mathcal{B}\mathbf{D}_t^\alpha u(t)$ and $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$ for $t \geq 0$, as well as that $u_{\alpha,\mathcal{B}}(t) \in u_{\mathcal{A}}(t) + \mathcal{F}(t)$, $t \geq 0$.

Before proceeding further, we want to observe that the existence of solutions to (1.1), (1.2) or (1.3) immediately implies that $\sec_c(\mathcal{F}) \neq \emptyset$, as well as that any strong solution of (1.1) is already a solution of (1.1), provided that $\mathcal{A}$ and $\mathcal{B}$ are MLOs with $\mathcal{A}$ being closed; this can be simply verified with the help of Theorem 2.3. The notion of a (pre-)solution of problems (1.2) and (1.3) can be similarly defined on any finite interval $[0,\tau)$ or $[0,\tau]$, where $0 < \tau < \infty$, and extends so the notion of a strict solution of [17, problem (E) pp. 33-34] ($\mathcal{B} = I$, $\alpha = 1$, $\mathcal{F}(t) = f(t)$ is continuous single-valued). We refer the reader to [23]-[24] and [45] for related results about the wellposedness of problem (1.2), as well as to the monograph [15] by Dragoni, Macki, Nistri and Zecca for some other concepts of solutions to the abstract differential inclusions in abstract spaces.

In our further work, it will be assumed that $\mathcal{A}$ and $\mathcal{B}$ are multivalued linear operators. Observe that we cannot consider the qualitative properties of solutions of problems (1.1), (1.2) or (1.3) in full generality by a simple passing to the multivalued linear operators $\mathcal{B}^{-1}\mathcal{A}$ or $\mathcal{A}\mathcal{B}^{-1}$ (see the definition of a solution of (1.1)). Concerning this question, we have the following remark.

Remark 4.2. Suppose that $0 < \tau \leq \infty$, $\alpha > 0$, as well as that $A : D(A) \subseteq X \to Y$ and $B : D(B) \subseteq X \to Y$ are two single-valued linear operators. Then $B^{-1}A$ is an MLO in X, and AB^{-1} is an MLO in Y.

(i) Suppose that $u(\cdot)$ is a pre-solution (or, equivalently, solution) of problem (1.1) with $\mathcal{B} = I_X$, $\mathcal{A} = B^{-1}A$ and $\mathcal{F} = f : [0,\tau) \to D(B)$ being single-valued. Then $u \in C([0,\tau) : X)$ and $Bu(t) = A(a * u)(t) + Bf(t)$, $t \in [0,\tau)$. If, in addition to this, $B \in L(X,Y)$ and $u(\cdot)$ is a strong solution of problem (1.1) with the above requirements being satisfied, then the mappings $t \mapsto Au(t)$, $t \in [0,\tau)$ and $t \mapsto Bu(t)$, $t \in [0,\tau)$ are continuous, and $(a * Au)(t) = Bu(t) - Bf(t)$, $t \in [0,\tau)$.

(ii) Suppose that $v(\cdot)$ is a pre-solution (solution) of (1.2) with $\mathcal{B} = I_Y$, $\mathcal{A} = AB^{-1}$, $\mathcal{F} = f : [0,\tau) \to Y$ being single-valued, and $v_j = Bx_j$ $(0 \le j \le \lceil\alpha\rceil - 1)$. Let $B^{-1} \in L(Y,X)$. Then the function $u(t) := B^{-1}v(t)$, $t \ge 0$ is a pre-solution (solution) of (1.2) with $\mathcal{B} = B$ and $\mathcal{A} = A$.

(iii) Suppose that $\mathcal{F} = f : [0,\tau) \to D(B)$ is single-valued and $u(\cdot)$ is a pre-solution of problem $(\mathrm{DFP})_L$ with $\mathcal{B} = I_X$ and $\mathcal{A} = B^{-1}A$. Then $u(\cdot)$ is a pre-solution of problem $(\mathrm{DFP})_L$ with $\mathcal{B} = B$, $\mathcal{A} = A$ and $\mathcal{F}(t) = Bf(t)$, $t \in [0,\tau)$. If, in addition to this, $B \in L(X,Y)$ and $u(\cdot)$ is a solution of problem (1.3) with the above requirements being satisfied, then $u(\cdot)$ is a solution of problem $(\mathrm{DFP})_L$ with $\mathcal{B} = B$, $\mathcal{A} = A$ and $\mathcal{F}(t) = Bf(t)$, $t \in [0,\tau)$.

(iv) Suppose that $u : [0,\infty) \to D(A) \cap D(B)$. Then $u(\cdot)$ is a p-solution of problem $(\mathrm{DFP})_R$ with $\mathcal{B} = B$ and $\mathcal{A} = A$ if and only if $v = Bu(\cdot)$ is a pre-solution of problem

$$\begin{gathered} \mathbf{D}_t^\alpha v(t) \in AB^{-1}v(t) + \mathcal{F}(t), \quad t \ge 0, \\ v^{(j)}(0) = Bx_j, \quad 0 \le j \le \lceil\alpha\rceil - 1. \end{gathered}$$

(v) Suppose that $C_Y \in L(Y)$ is injective and the closed graph theorem holds for the mappings from Y into Y. Then we define the set $\rho^B_{C_Y}(A) := \{\lambda \in \mathbb{C} : \lambda B - A$ is injective and $(\lambda B - A)^{-1}C_Y \in L(Y)\}$. It can be simply checked that $\rho^B_{C_Y}(A) \subseteq \rho_{C_Y}(AB^{-1})$, as well as that

$$\big(\lambda - AB^{-1}\big)^{-1}C_Y = B\big(\lambda B - A\big)^{-1}C_Y, \quad \lambda \in \rho^B_{C_Y}(A). \tag{4.1}$$

This is an extension of [17, Theorem 1.14] and holds even in the case that the operator C_Y does not commute with AB^{-1}, when we define the C_Y-resolvent set of the operator $\lambda - AB^{-1}$ in the same way as before. Observe also that the assumption $D(A) \subseteq D(B)$, which has been used in [17, Section 1.6], is not necessary for the validity of (4.1).

(vi) Suppose that $X = Y$, $C \in L(X)$ is injective, $B \in L(X)$, $CA \subseteq AC$ and $CB \subseteq BC$. Define the set $\rho^B_C(A)$ as above. Then we have $\rho^B_C(A) \subseteq \rho_C(B^{-1}A)$ and

$$\big(\lambda - B^{-1}A\big)^{-1}Cx = \big(\lambda B - A\big)^{-1}CBx, \quad x \in X.$$

Furthermore, if $C = I$, $X \ne Y$ and $B \in L(X,Y)$, then $\rho^B(A) \subseteq \rho(B^{-1}A)$ and the previous equality holds.

Consider now the case in which the operator $\mathcal{A}$ is closed, the operator $\mathcal{B} = B$ is single-valued and the function $\mathcal{F}(t) = f(t)$ is Y-continuous at each point $t \ge 0$. Then any pre-solution $u(\cdot)$ of problem (1.2) is already a solution of this problem, and Theorem 2.3 in combination with the identity [5, (1.21)] implies that

$$Bu(t) - \sum_{k=0}^{\lceil\alpha\rceil-1} g_{k+1}(t)Bx_j - \big(g_\alpha * f\big)(t) \in \mathcal{A}\big(g_\alpha * u\big)(t), \quad t \geq 0.$$

Suppose, conversely, that there exists a function $u_{\mathcal{A}} \in C([0,\infty) : Y)$ such that $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$, $t \geq 0$ and

$$Bu(t) - \sum_{k=0}^{\lceil\alpha\rceil-1} g_{k+1}(t)Bx_j - \big(g_\alpha * f\big)(t) = \big(g_\alpha * u_{\mathcal{A}}\big)(t), \quad t \geq 0.$$

Then it can be simply verified that $u(\cdot)$ is a solution of problem (1.2); it is noteworthy that we do not need the assumption on closedness of $\mathcal{A}$ in this direction. Even in the case that $\mathcal{A} = A$ is a closed single-valued linear operator, a corresponding statement for the problem (1.3) cannot be proved. Suppose, finally, that the operators $\mathcal{A}$ and $\mathcal{B}$ are closed, $u(\cdot)$ is a solution of problem (1.2), the function $\mathcal{F}(t) = f(t)$ is Y-continuous at each point $t \geq 0$, as well as the functions $u_{\alpha,\mathcal{B}} \in C([0,\infty) : Y)$ and $u_{\mathcal{A}} \in C([0,\infty) : Y)$ satisfy the requirements stated in Definition 4.1(iii). Using again Theorem 2.3 and the identity [5, (1.21)], it readily follows that

$$\begin{aligned} &\mathcal{B}\Big[u(t) - \sum_{k=0}^{\lceil\alpha\rceil-1} g_{k+1}(t)x_j\Big] \ni \big(g_\alpha * u_{\alpha,\mathcal{B}}\big)(t) \\ &= \big(g_\alpha * u_{\mathcal{A}}\big)(t) + \big(g_\alpha * f\big)(t) \in \mathcal{A}\big(g_\alpha * u\big)(t) + \big(g_\alpha * f\big)(t), \quad t \geq 0. \end{aligned}$$

The proof of following important theorem can be deduced by using Theorem 2.3, Theorem 3.3[(iv),(vi)], Theorem 3.5 and the argumentation already seen in the proof of [31, Theorem 3.1] (cf. also [32, Fundamental Lemma 3.1]); observe that we do not use the assumption on the exponential boundedness of function $u(t)$ here. After formulation, we will only include the most relevant details needed for the proof of implication (iii) $\Rightarrow$ (iv).

Theorem 4.3. *Suppose that $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : X \to P(Y)$ are MLOs, as well as that $\mathcal{A}$ is $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed. Assume, further, that $a \in L^1_{\mathrm{loc}}([0,\infty))$, $a \neq 0$, $\mathrm{abs}(|a|) < \infty$, $u \in C([0,\infty) : X)$, $u \in (P1) - X$, as well as that $u(t) \in D(\mathcal{B})$, $t \geq 0$, $a * u \in C([0,\infty) : X_{\mathcal{A}})$, $a * u \in$ (P1)$- X_{\mathcal{A}}$, $\mathrm{abs}_{Y_{\mathcal{A}}}(\mathcal{B}u) < \infty$, $\mathrm{abs}_{Y_{\mathcal{A}}}(\mathcal{F}) < \infty$, and $\omega > \max(0, \omega_X(u), \mathrm{abs}_{Y_{\mathcal{A}}}(\mathcal{B}u), \mathrm{abs}_{Y_{\mathcal{A}}}(\mathcal{F}), \mathrm{abs}_{X_{\mathcal{A}}}(a*u))$. Consider the following assertions:*

- (i) *$u(\cdot)$ is a solution of (1.1) with $\tau = \infty$.*
- (ii) *$u(\cdot)$ is a pre-solution of (1.1) with $\tau = \infty$.*
- (iii) *For any section $u_{\mathcal{B}} \in sec(\mathcal{B}u)$ there is a section $f \in sec(\mathcal{F})$ such that*
$$\widetilde{u_{\mathcal{B}}}(\lambda) - \tilde{f}(\lambda) \in \tilde{a}(\lambda)\mathcal{A}\tilde{u}(\lambda), \quad \Re\lambda > \omega,\ \tilde{a}(\lambda) \neq 0.$$
- (iv) *For any section $u_{\mathcal{B}} \in sec(\mathcal{B}u)$ there is a section $f \in sec(\mathcal{F})$ such that*
$$\widetilde{u_{\mathcal{B}}}(\lambda) - \tilde{f}(\lambda) \in \tilde{a}(\lambda)\mathcal{A}\tilde{u}(\lambda), \quad \lambda \in \mathbb{N},\ \lambda > \omega,\ \tilde{a}(\lambda) \neq 0. \tag{4.2}$$
- (v) *For any section $u_{\mathcal{B}} \in sec(\mathcal{B}u)$ there is a section $f \in sec(\mathcal{F})$ such that*
$$\big(1 * u_{\mathcal{B}}\big)(t) - (1 * f)(t) \in \mathcal{A}(1 * a * u)(t), \quad t \geq 0. \tag{4.3}$$

Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). Furthermore, if $\mathcal{B} = B$ is single-valued, $Bu \in C([0,\infty) : Y_{\mathcal{A}})$ and $\mathcal{F} = f \in C([0,\infty) : Y_{\mathcal{A}})$ is single-valued, then the above is equivalent.

Sketch of proof for (iv) ⇒ (v). Suppose that for any section $u_{\mathcal{B}} \in \sec(\mathcal{B}u)$ there is a section $f \in \sec(\mathcal{F})$ such that (4.2) holds. Let a number $\lambda \in \mathbb{N}$ with $\lambda > \omega$ and $\tilde{a}(\lambda) = 0$ be temorarily fixed. Then there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in (λ, ∞) such that $\tilde{a}(\lambda_n) \neq 0$ and $\lim_{n\to+\infty} \lambda_n = \lambda$. Since $(\tilde{a}(\lambda_n)\tilde{u}(\lambda_n), \widetilde{u_{\mathcal{B}}}(\lambda_n) - \tilde{f}(\lambda_n)) \in \mathcal{A}$, $n \in \mathbb{N}$, i.e., $(\widetilde{a * u}(\lambda_n), \widetilde{u_{\mathcal{B}}}(\lambda_n) - \tilde{f}(\lambda_n)) \in \mathcal{A}$, $n \in \mathbb{N}$, and $\mathcal{A}$ is $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed, it readily folows that $(\widetilde{a * u}(\lambda), \widetilde{u_{\mathcal{B}}}(\lambda) - \tilde{f}(\lambda)) \in \mathcal{A}$; in other words, $(0, \widetilde{u_{\mathcal{B}}}(\lambda) - \tilde{f}(\lambda)) \in \mathcal{A}$. By the foregoing, we have that $(\widetilde{a * u}(\lambda), \widetilde{u_{\mathcal{B}}}(\lambda) - \tilde{f}(\lambda)) \in \mathcal{A}$ for all $\lambda \in \mathbb{N}$ with $\lambda > \omega$. Using Theorem 2.3, we obtain that $\int_0^\infty e^{-\lambda t}(u_{\mathcal{B}} - f)^{[2]}(t)\,dt \in \mathcal{A}\int_0^\infty e^{-\lambda t}(a * u)^{[2]}(t)\,dt$ ($\lambda \in \mathbb{N}$, $\lambda > \omega$) and now we can apply Theorem 3.5, along with the $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closedness of $\mathcal{A}$, in order to see that $u_{\mathcal{B}}^{[2]}(t) - f^{[2]}(t) \in \mathcal{A}(a*u)^{[2]}(t)$, $t \geq 0$. This simply implies (4.3).

Remark 4.4. *Observe that we do not require any type of closedness of the operator $\mathcal{B}$ in the formulation of Theorem 4.3. Even in the case that $X = Y$ and $\mathcal{B} = B = I$, we cannot differentiate the equation* (4.3) *once more without making an additional assumption that $\mathcal{F} = f \in C([0,\infty) : Y_{\mathcal{A}})$ is single-valued (cf.* [31, l. -1, p. 173; l. 1-3, p. 174]*, where the author has made a small mistake in the consideration; in actual fact, the equation* [31, (3.1)] *has to be valid for some $f \in sec_c(\mathcal{F})$ in order for the proof of implication (iii) ⇒ (i) of* [31, Theorem 3.1] *to work).*

If Ω is a non-empty open subset of $\mathbb{C}$ and $G : \Omega \to X$ is an analytic mapping that it is not identically equal to the zero function, then we can simply prove that for each zero λ_0 of $G(\cdot)$ there exists a uniquely determined natural number $n \in \mathbb{N}$ such that $G^{(j)}(\lambda_0) = 0$ for $0 \leq j \leq n-1$ and $G^{(n)}(\lambda_0) \neq 0$. Owing to this fact, we can repeat almost literally the arguments given in the proof of [31, Theorem 3.2] to verify the validity of the following Ljubich uniqueness type theorem:

Theorem 4.5. *Suppose $\mathcal{A} : X \to P(Y)$ is an MLO, $\mathcal{B} = B : D(B) \subseteq X \to Y$ is a single-valued linear operator, $\mathcal{A}$ is $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed and B is $X_B \times Y_B$-closed, where $Y_{\mathcal{A}} \hookrightarrow Y_B$. Assume, further, that $a \in L^1_{\mathrm{loc}}([0,\infty))$, $a \neq 0$, $\mathrm{abs}(|a|) < \infty$, $\mathcal{F} = f \in C([0,\infty) : Y_{\mathcal{A}})$ is single-valued, $\mathrm{abs}_{Y_{\mathcal{A}}}(f) < \infty$, and there exist a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of complex numbers and a number $\omega > \mathrm{abs}(|a|)$ such that $\lim_{k\to\infty} \Re\lambda_k = +\infty$, $\tilde{a}(\lambda_k) \neq 0$, $k \in \mathbb{N}$, and*

$$\frac{1}{\tilde{a}(\lambda_k)} Bx \notin \mathcal{A}x, \quad k \in \mathbb{N},\ 0 \neq x \in D(\mathcal{A}) \cap D(B).$$

Then there exists a unique pre-solution of (1.1)*, with $\tau = \infty$, satisfying that $u \in (P1) - X_B$, $u(t) \in D(B)$, $t \geq 0$, $Bu \in C([0,\infty) : Y_{\mathcal{A}})$, $a * u \in C([0,\infty) : X_{\mathcal{A}})$, $a * u \in (P1) - X_{\mathcal{A}}$ and $\mathrm{abs}_{Y_{\mathcal{A}}}(Bu) < \infty$.*

In the following extension of [36, Theorem 2.1.34], we will prove one more Ljubich's uniqueness criterium for abstract Cauchy problems with multivalued linear operators (cf. also [32, Theorem 3.5] and [36, Theorem 2.10.44]).

Theorem 4.6. *Suppose $\alpha > 0$, $\lambda > 0$, $\mathcal{A}$ is an MLO in X, $\{(n\lambda)^\alpha : n \in \mathbb{N}\} \subseteq \rho_C(\mathcal{A})$ and, for every $\sigma > 0$ and $x \in X$,*

$$\lim_{n\to\infty} \frac{\big((n\lambda)^\alpha - \mathcal{A}\big)^{-1}Cx}{e^{n\lambda\sigma}} = 0.$$

Then, for every $x_0, \cdots, x_{\lceil\alpha\rceil-1} \in X$, there exists at most one pre-solution of the initial value problem (1.2) *with $\mathcal{B} = I$.*

Proof. It suffices to show that the zero function is the only pre-solution of problem (1.2) with $\mathcal{B} = I$ and the initial values $x_0, \cdots, x_{\lceil\alpha\rceil-1}$ chosen to be zeroes. Let $u(\cdot)$ be a pre-solution of such a problem. Set $z_n(t) := ((n\lambda)^\alpha - \mathcal{A})^{-1}Cu(t)$, $t \geq 0$, $n \in \mathbb{N}$. Then it can be easily checked with the help of Theorem 2.4(i) that $z_n(\cdot)$ is a solution of the initial value problem:

$$\begin{gathered} z_n \in C^{\lceil\alpha\rceil}((0,\infty) : X) \cap C^{\lceil\alpha\rceil-1}([0,\infty) : X), \\ \mathbf{D}_t^\alpha z_n(t) = (n\lambda)^\alpha z_n(t) - Cu(t),\ t > 0, \\ z_n^{(j)}(0) = 0,\ 0 \leq j \leq \lceil\alpha\rceil - 1. \end{gathered}$$

This implies $z_n(t) = -(u * \cdot^{\alpha-1}E_{\alpha,\alpha}((n\lambda)^\alpha \cdot^{\alpha-1}))(t)$, $t \geq 0$, $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} e^{-n\lambda\sigma}\int_0^t s^{\alpha-1}E_{\alpha,\alpha}\big((n\lambda)^\alpha s^\alpha\big)Cu(t-s)\,ds = 0 \quad (t > 0,\ \sigma > 0).$$

Now we can argue as in the second part of proof of [36, Theorem 2.1.34] so as to conclude that $u(t) = 0$, $t \geq 0$ (in the case that $\alpha \in \mathbb{N}$, the assertion can be proved by a trustworthy passing to the theory of abstract Cauchy problems of first order since [36, Lemma 2.1.33(i)] admits an extension to multivalued linear operators). □

Remark 4.7. Observe that, in the formulation of Theorem 4.6, we do not require any type of closedness of the operator $\mathcal{A}$.

The following theorem is very similar to [5, Theorem 3.1, Theorem 3.3] and [36, Theorem 2.4.2]. Because of its importance, we will include the most relevant details of proof.

Theorem 4.8 (Subordination principle for abstract time-fractional inclusions)**.** *Suppose that $0 < \alpha < \beta$, $\gamma = \alpha/\beta$, $\mathcal{A} : X \to P(Y)$ is an MLO, $\mathcal{B} = B : D(B) \subseteq X \to Y$ is a single-valued linear operator, $\mathcal{A}$ is $X_{\mathcal{A}} \times Y_{\mathcal{A}}$-closed and B is $X_B \times Y_B$-closed, where $X_{\mathcal{A}} \hookrightarrow X_B$ and $Y_{\mathcal{A}} \hookrightarrow Y_B$.*

Assume, further, that $f_\beta \in LT_{or} - Y_{\mathcal{A}}$ is single-valued and there exists a pre-solution (or, equivalently, solution) $u(t) := u_\beta(t)$ of (1.1), with $\tau = \infty$, $a(t) = g_\beta(t)$ and $\mathcal{F} = f_\beta$, satisfying that $u_\beta \in LT_{or} - X_B$, $Bu_\beta \in LT_{or} - Y_{\mathcal{A}}$, $g_\beta * u_\beta \in LT_{or} - X_{\mathcal{A}}$ and that for each seminorm $p \in \circledast_{X_B}$ there exists $\omega_p \geq 0$ such that $p(u_\beta(t)) = O(e^{\omega_p t})$, $t \geq 0$, $p \in \circledast_{X_B}$.

We define

$$\begin{gathered} u_\alpha(t) := \int_0^\infty t^{-\gamma}\Phi_\gamma\big(st^{-\gamma}\big)u_\beta(s)\,ds, \quad t > 0 \text{ and } u_\alpha(0) := u_\beta(0); \\ f_\alpha(t)x := \int_0^\infty t^{-\gamma}\Phi_\gamma\big(st^{-\gamma}\big)f_\beta(s)\,ds, \quad t > 0 \text{ and } f_\alpha(0) := f_\beta(0). \end{gathered}$$

Then $u_\alpha(t)$ is a solution of (1.1), with $\tau = \infty$, $a(t) = g_\alpha(t)$ and $\mathcal{F}(t) = f_\alpha(t) \in LT_{or} - Y_{\mathcal{A}}$, satisfying additionally that $u_\alpha \in LT_{or} - X_B$, $Bu_\alpha \in LT_{or} - Y_{\mathcal{A}}$, $g_\alpha * u_\alpha \in LT_{or} - X_{\mathcal{A}}$ and

$$p\big(u_\alpha(t)\big) = O\Big(\exp\big(\omega_p^{1/\gamma}t\big)\Big),\ p \in \circledast_{X_B},\ t \geq 0. \tag{4.4}$$

Let $p \in \circledast_{X_B}$ be fixed. Then the condition

$$p\big(u_\beta(t)\big) = O\Big(\big(1 + t^{\xi_p}\big)e^{\omega_p t}\Big) \text{ for some } \xi_p \geq 0, \tag{4.5}$$

resp.,

$$p\big(u_\beta(t)\big) = O\Big(t^{\xi_p} e^{\omega_p t}\Big),\ t \geq 0 \tag{4.6}$$

implies that

$$p\big(u_\alpha(t)\big) = O\Big(\big(1+t^{\xi_p\gamma}\big)\big(1+\omega_p t^{\xi_p(1-\gamma)}\big)\exp\big(\omega_p^{1/\gamma}t\big)\Big),\ t \geq 0, \tag{4.7}$$

resp.,

$$p\big(u_\alpha(t)\big) = O\Big(t^{\xi_p\gamma}\big(1+\omega_p t^{\xi_p(1-\gamma)}\big)\exp\big(\omega_p^{1/\gamma}t\big)\Big),\ t \geq 0. \tag{4.8}$$

Furthermore, the following holds:

- (i) *The mapping $t \mapsto u_\alpha(t)$, $t > 0$ admits an extension to $\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$ and the mapping $z \mapsto u_\alpha(z)$, $z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)}$ is analytic.*
- (ii) *Let $\varepsilon \in (0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi))$. If, for every $p \in \circledast$, one has $\omega_p = 0$, then for each $\theta \in (0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi))$ the following holds: $\lim_{z\to 0, z\in\overline{\Sigma_\theta}} u_\alpha(z) = u_\alpha(0)$.*
- (iii) *If $\omega_p > 0$ for some $p \in \circledast$, then for each $\theta \in (0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2}))$ the following holds: $\lim_{z\to 0, z\in\overline{\Sigma_\theta}} u_\alpha(z) = u_\alpha(0)$.*

Proof. The proofs of (i)-(iii) follows similarly as in that of [5, Theorem 3.3], while the proof that the condition (4.5), resp. (4.6), implies (4.7), resp. (4.8), follows similarly as in that of [36, Theorem 2.4.2]. Furthermore, it can be easily seen that the estimate (4.4) holds for solution $u_\alpha(\cdot)$. By Theorem 4.3, we should only show that $u_\alpha \in LT_{or} - X_B$, $Bu_\alpha \in LT_{or} - Y_{\mathcal{A}}$, $f_\alpha \in LT_{or} - Y_{\mathcal{A}}$, $g_\alpha * u_\alpha \in LT_{or} - X_{\mathcal{A}}$ and

$$\widetilde{Bu_\alpha}(\lambda) - \widetilde{f_\alpha}(\lambda) \in \lambda^{-\alpha}\mathcal{A}\widetilde{u_\alpha}(\lambda), \quad \lambda > \omega \text{ suff. large.} \tag{4.9}$$

Since $u_\beta \in LT_{or} - X_B$, the proof of [5, Theorem 3.1] immediately implies that $u_\alpha \in LT_{or} - X_B$, as well as that $\widetilde{u_\alpha}(\lambda) = \lambda^{\gamma-1}\widetilde{u_\beta}(\lambda^\beta)$, $\lambda > \omega$ suff. large. Similarly, we have that $f_\alpha \in LT_{or} - Y_{\mathcal{A}}$ and $\tilde{f}_\alpha(\lambda) = \lambda^{\gamma-1}\tilde{f}_\beta(\lambda^\beta)$, $\lambda > \omega$ suff. large. Keeping in mind that $X_{\mathcal{A}} \hookrightarrow X_B$ and $g_\beta * u_\beta \in LT_{or} - X_{\mathcal{A}}$, we can prove that

$$\big(g_\alpha * u_\alpha\big)(t) = \int_0^\infty t^{-\gamma}\Phi_\gamma\big(st^{-\gamma}\big)\big(g_\beta * u_\beta\big)(s)\,ds, \quad t > 0$$

by performing the Laplace transform (the convergence of last integral is taken for the topology of $X_{\mathcal{A}}$). This simply implies that $g_\alpha * u_\alpha \in LT_{or} - X_{\mathcal{A}}$ and $(\mathcal{L}(g_\alpha * u_\alpha))(\lambda) = \lambda^{\gamma-1}(\mathcal{L}(g_\beta * u_\beta))(\lambda^\gamma)$, $\lambda > \omega$ suff. large. Since $Y_{\mathcal{A}} \hookrightarrow Y_B$, a similar line of reasoning shows that

$$Bu_\alpha(t) = \int_0^\infty t^{-\gamma}\Phi_\gamma\big(st^{-\gamma}\big)\big(Bu_\beta\big)(s)\,ds, \quad t > 0$$

(the convergence of this integral is taken for the topology of $Y_{\mathcal{A}}$) and $\widetilde{Bu_\alpha}(\lambda) = B\widetilde{u_\alpha}(\lambda)$ for all sufficiently large values of $\lambda > \omega$. The proof of (4.9) now follows from a simple computation. □

We can similarly prove the following subordination principles for abstract degenerate Volterra inclusions in locally convex spaces (cf. [68, Section 4] and [36, Theorem 2.1.8, Theorem 2.8.7] for more details concerning non-degenerate case and, especially, the case in which $b(t) = g_1(t)$ or $b(t) = g_2(t)$).

Theorem 4.9. *Let $b(t)$ and $c(t)$ satisfy (P1)-$\mathbb{C}$, let $\int_0^\infty e^{-\beta t}|b(t)|\,dt < \infty$ for some $\beta \geq 0$, and let*

$$\alpha = \tilde{c}^{-1}\Big(\frac{1}{\beta}\Big) \text{ if } \int_0^\infty c(t)\,dt > \frac{1}{\beta},\ \alpha = 0 \text{ otherwise.}$$

Suppose that $\operatorname{abs}(|a|) < \infty$, $\tilde{a}(\lambda) = \tilde{b}(\frac{1}{\tilde{c}(\lambda)})$, $\lambda > \alpha$, $\mathcal{A} : X \to P(Y)$ *is an MLO,* $\mathcal{B} = B : D(B) \subseteq X \to Y$ *is a single-valued linear operator,* $\mathcal{A}$ *is* $X_{\mathcal{A}} \times Y_{\mathcal{A}}$*-closed and* B *is* $X_B \times Y_B$*-closed, where* $X_{\mathcal{A}} \hookrightarrow X_B$ *and* $Y_{\mathcal{A}} \hookrightarrow Y_B$.

Assume, further, that $f_\beta \in LT_{or} - Y_{\mathcal{A}}$ *is single-valued and there exists a pre-solution (or, equivalently, solution)* $u(t) := u_b(t)$ *of* (1.1), *with* $\tau = \infty$, $a(t)$ *replaced with* $b(t)$ *therein, and* $\mathcal{F} = f_b$, *satisfying that* $u_b \in LT_{or} - X_B$, $Bu_b \in LT_{or} - Y_{\mathcal{A}}$, $b * u_b \in LT_{or} - X_{\mathcal{A}}$ *and the family* $\{e^{-\omega_b t}u_b(t) : t \geq 0\}$ *is bounded in* X_B $(\omega_b \geq 0)$.

Let $c(t)$ *be completely positive and let there exist a function* $f_a \in LT_{or} - Y_{\mathcal{A}}$ *such that*

$$\widetilde{f_a}(\lambda) = \frac{1}{\lambda\tilde{c}(\lambda)}\widetilde{f_b}\Big(\frac{1}{\tilde{c}(\lambda)}\Big),\ \lambda > \omega_0,\ \widetilde{f_b}\Big(\frac{1}{\tilde{c}(\lambda)}\Big) \neq 0, \text{ for some } \omega_0 > 0.$$

Let

$$\omega_a = \tilde{c}^{-1}\Big(\frac{1}{\omega_b}\Big) \text{ if } \int_0^\infty c(t)\,dt > \frac{1}{\omega_b},\ \omega_a = 0 \text{ otherwise.}$$

Then, for every $r \in (0,1]$, *there exists a solution* $u(t) := u_{a,r}(t)$ *of* (1.1), *with* $\tau = \infty$, $a(t)$ *and* $\mathcal{F} = f_r := g_r * f_a$, *satisfying that* $u_{a,r} \in LT_{or} - X_B$, $Bu_{a,r} \in LT_{or} - Y_{\mathcal{A}}$, $a * u_{a,r} \in LT_{or} - X_{\mathcal{A}}$ *and the set* $\{e^{-\omega_a t}u_{a,r}(t) : t \geq 0\}$ *is bounded in* X_B, *if* $\omega_b = 0$ *or* $\omega_b\tilde{c}(0) \neq 1$, *resp., the set* $\{e^{-\varepsilon t}u_{a,r}(t) : t \geq 0\}$ *is bounded in* X_B *for any* $\varepsilon > 0$, *if* $\omega_b > 0$ *and* $\omega_b\tilde{c}(0) = 1$.

Furthermore, the function $t \mapsto u_{a,r}(t) \in X_B$, $t \geq 0$ *is locally Hölder continuous with the exponent* $r \in (0,1]$.

Remark 4.10. (i) In Theorem 4.8 and Theorem 4.9, we have only proved the existence of a solution of the subordinated inclusion. The uniqueness of solutions can be proved, for example, by using Theorem 4.5, Theorem 4.6 or [36, Theorem 2.2.6].

(ii) In Theorem 4.9, we have faced ourselves with a loss of regularity for solutions of the subordinated problem. Even in the case that $X = Y$ and $B = I$, it is not so simple to prove the existence of a solution of problem (1.1), with $\tau = \infty$, $a(t)$ and $\mathcal{F} = f_a$, without imposing some additional unpleasant conditions. In the next section, we will introduce various types of solution operator families for the abstract Volterra inclusion (1.1) and there we will reconsider the problem of loss of regularity for solutions of the subordinated problem once more (cf. Theorem 5.7).

5. Multivalued linear operators as subgenerators of (a,k)-regularized C-resolvent solution operator families

In [36, Section 2.8], the class of (a,k)-regularized (C_1,C_2)-existence and uniqueness families has been introduced and analyzed within the theory of abstract non-degenerate Volterra equations. The main aim of this section is to consider multivalued linear operators in locally convex spaces as subgenerators of (a,k)-regularized (C_1,C_2)-existence and uniqueness families, as well as to consider in more detail the class of (a,k)-regularized C-resolvent families. Unless specified otherwise, we assume that $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{\mathrm{loc}}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$

is an MLO, $C_1 \in L(Y,X)$, $C_2 \in L(X)$ is injective, $C \in L(X)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$.

The following definition is an extension of [36, Definition 2.8.2] ($X = Y$, A is a closed single-valued linear operator on X) and [72, Definition 3.5] ($X = Y$, $C = C_1$, $a(t) = k(t) = 1$).

Definition 5.1. Suppose $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{\rm loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ is an MLO, $C_1 \in L(Y,X)$, and $C_2 \in L(X)$ is injective.

(i) Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)} \subseteq L(Y,X)\times L(X)$ if and only if the mappings $t \mapsto R_1(t)y$, $t \geq 0$ and $t \mapsto R_2(t)x$, $t \in [0,\tau)$ are continuous for every fixed $x \in X$ and $y \in Y$, as well as the following conditions hold:

$$\Big(\int_0^t a(t-s)R_1(s)y\,ds, R_1(t)y - k(t)C_1y\Big) \in \mathcal{A}, \quad t \in [0,\tau),\ y \in Y; \tag{5.1}$$

$$\int_0^t a(t-s)R_2(s)y\,ds = R_2(t)x - k(t)C_2x,\ \text{whenever } t \in [0,\tau) \text{ and } (x,y) \in \mathcal{A}. \tag{5.2}$$

(ii) Let $(R_1(t))_{t\in[0,\tau)} \subseteq L(Y,X)$ be strongly continuous. Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)}$ if and only if (5.1) holds.

(iii) Let $(R_2(t))_{t\in[0,\tau)} \subseteq L(X)$ be strongly continuous. Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ if and only if (5.2) holds.

As an immediate consequence of definition, we have that $R(R_1(0)-k(0)C_1) \subseteq \mathcal{A}0$ as well as that $R_2(t)\mathcal{A}$ is single-valued for any $t \geq 0$, and $R_2(t)y = 0$ for any $y \in \mathcal{A}0$ and $t \geq 0$.

Now we will extend the definition of an (a,k)-regularized C-resolvent family subgenerated by a single-valued linear operator (cf. [36, Definition 2.1.1]).

Definition 5.2. Suppose that $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{\rm loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ is an MLO, $C \in L(X)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Then it is said that a strongly continuous operator family $(R(t))_{t\in[0,\tau)} \subseteq L(X)$ is an (a,k)-regularized C-resolvent family with a subgenerator $\mathcal{A}$ if and only if $(R(t))_{t\in[0,\tau)}$ is a mild (a,k)-regularized C-uniqueness family having $\mathcal{A}$ as subgenerator, $R(t)C = CR(t)$ and $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ $(t \geq 0)$.

An (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ is said to be locally equicontinuous if and only if, for every $t \in (0,\tau)$, the family $\{R(s) : s \in [0,t]\}$ is equicontinuous. In the case $\tau = \infty$, $(R(t))_{t\geq 0}$ is said to be exponentially equicontinuous (equicontinuous) if there exists $\omega \in \mathbb{R}$ ($\omega = 0$) such that the family $\{e^{-\omega t}R(t) : t \geq 0\}$ is equicontinuous; the infimum of such numbers is said to be the exponential type of $(R(t))_{t\geq 0}$. The above notion can be simply understood for the classes of mild (a,k)-regularized C_1-existence families and mild (a,k)-regularized C_2-uniqueness families; a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)} \subseteq L(Y,X)\times L(X)$ is said to be locally equicontinuous (exponentially equicontinuous, provided that $\tau = \infty$) if and only if both operator families $(R_1(t))_{t\geq 0}$ and $(R_2(t))_{t\geq 0}$ are.

It would take too long to consider the notion of q-exponential equicontinuity for the classes of mild (a,k)-regularized C_1-existence families and mild (a,k)-regularized C_2-uniqueness families (cf. [36, Section 2.4] for more details about non-degenerate case). If $k(t)=g_{\alpha+1}(t)$, where $\alpha\geq 0$, then it is also said that $(R(t))_{t\in[0,\tau)}$ is an α-times integrated (a,C)-resolvent family; 0-times integrated (a,C)-resolvent family is further abbreviated to (a,C)-resolvent family. We will accept a similar terminology for the classes of mild (a,k)-regularized C_1-existence families and mild (a,k)-regularized C_2-uniqueness families; in the case of consideration of convoluted C-semigroups, it will be always assumed that the condition (5.1) holds with $a(t)=1$ and the operator C_1 replaced by C. Let us mention in passing that the operator semigroups generated by multivalued linear operators have been analyzed by A. G. Baskakov in [4].

The following proposition can be proved with the help of Theorems 2.3 and 3.1(ii).

Proposition 5.3. *Suppose that $(R_1(t),R_2(t))_{t\in[0,\tau)}\subseteq L(Y,X)\times L(X)$ is a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$ and $(R(t))_{t\in[0,\tau)}\subseteq L(X)$ is an (a,k)-regularized C-resolvent family with a subgenerator $\mathcal{A}$. Let $b\in L^1_{\rm loc}([0,\tau))$ be such that $a*b\neq 0$ in $L^1([0,\tau))$ and $k*b\neq 0$ in $C([0,\tau))$. Then $((b*R_2)(t))_{t\geq 0}$ is a mild (a,k)-regularized C_2-uniqueness family with a subgenerator $\mathcal{A}$. Furthermore, the following holds:*

- (i) *Let $\mathcal{A}$ be $X^1_{\mathcal{A}}\times X^2_{\mathcal{A}}$-closed. Suppose that, for every $y\in Y$, the mapping $t\mapsto (a*R_1)(t)y$, $t\in[0,\tau)$ is continuous in $X^1_{\mathcal{A}}$ and the mapping $t\mapsto R_1(t)y$, $t\in[0,\tau)$ is continuous in $X^2_{\mathcal{A}}$. Then $((b*R_1)(t))_{t\geq 0}$ is a mild (a,k)-regularized C_1-existence family with a subgenerator $\mathcal{A}$.*
- (ii) *Let $\mathcal{A}$ be $X^1_{\mathcal{A}}\times X^2_{\mathcal{A}}$-closed. Suppose that, for every $x\in D(\mathcal{A})$ and $y\in R(\mathcal{A})$, the mapping $t\mapsto R(t)x$, $t\in[0,\tau)$ is continuous in $X^1_{\mathcal{A}}$ and the mapping $t\mapsto R(t)y$, $t\in[0,\tau)$ is continuous in $X^2_{\mathcal{A}}$. Then $((b*R)(t))_{t\geq 0}$ is a (a,k)-regularized C-regularized family with a subgenerator $\mathcal{A}$.*

Although the parts (i) and (ii) of the above proposition have been stated for $X^1_{\mathcal{A}}\times X^2_{\mathcal{A}}$-closed subgenerators, the most important case in our further study will be that in which $X^1_{\mathcal{A}}=X^2_{\mathcal{A}}=X$. This is primarily caused by the following fact: Let $\mathcal{A}$ be a subgenerator of a mild (a,k)-regularized C_1-existence family (mild (a,k)-regularized C_2-uniqueness family; mild (a,k)-regularized C-resolvent family) $(R_1(t))_{t\in[0,\tau)}$ $((R_2(t))_{t\in[0,\tau)}$; $(R(t))_{t\in[0,\tau)})$. Then $\overline{\mathcal{A}}$ is likewise a subgenerator of $(R_1(t))_{t\in[0,\tau)}$ $((R_2(t))_{t\in[0,\tau)}$; $(R(t))_{t\in[0,\tau)}$, provided in addition that $(R_2(t))_{t\in[0,\tau)}$; $(R(t))_{t\in[0,\tau)}$ is locally equicontinuous).

Suppose that $(R_1(t),R_2(t))_{t\in[0,\tau)}$ is a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$. Arguing as in non-degenerate case (cf. the paragraph directly preceding [36, Definition 2.8.3]), we may conclude that

$$\begin{aligned}&\big(a*R_2\big)(s)R_1(t)y-R_2(s)\big(a*R_1\big)(t)y\\&=k(t)\big(a*R_2\big)(s)C_1y-k(s)C_2\big(a*R_1\big)(t)y,\quad t\in[0,\tau),\ y\in Y.\end{aligned}\tag{5.3}$$

The integral generator of mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ (mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)}$) is defined by

$$\mathcal{A}_{\rm int}:=\Big\{(x,y)\in X\times X: R_2(t)x-k(t)C_2x=\int_0^t a(t-s)R_2(s)y\,ds,\ t\in[0,\tau)\Big\};$$

we define the integral generator of an (a,k)-regularized C-regularized family $(R(t))_{t\in[0,\tau)}$ in the same way as above. The integral generator $\mathcal{A}_{\rm int}$ is an MLO in X which is, in fact, the maximal subgenerator of $(R_2(t))_{t\in[0,\tau)}$ $((R(t))_{t\in[0,\tau)})$ with respect to the set inclusion; furthermore, the assumption $R_2(t)C_2 = C_2R_2(t)$, $t \in [0,\tau)$ implies that $C_2^{-1}\mathcal{A}_{\rm int}C_2 = \mathcal{A}_{\rm int}$ so that $C^{-1}\mathcal{A}_{\rm int}C = \mathcal{A}_{\rm int}$ for resolvent families. The local equicontinuity of $(R_2(t))_{t\in[0,\tau)}$ $((R(t))_{t\in[0,\tau)})$ immediately implies that $\mathcal{A}_{\rm int}$ is closed. Observe that, in the above definition of integral generator, we do not require that the function $a(t)$ is a kernel on $[0,\tau)$, as in non-degenerate case. In the case of resolvent families, the following holds:

(i) Suppose that $(R(t))_{t\in[0,\tau)}$ is locally equicontinuous and $\mathcal{A}$ is a closed subgenerator of $(R(t))_{t\in[0,\tau)}$. Then
$$\Big(\int_0^t a(t-s)R(s)x\,ds, R(t)x - k(t)Cx\Big) \in \mathcal{A}, \tag{5.4}$$
for $t \in [0,\tau)$, $x \in \overline{D(\mathcal{A})}$.

(ii) If $\mathcal{A}$ is a subgenerator of $(R(t))_{t\in[0,\tau)}$, then $C^{-1}\mathcal{A}C$ is a subgenerator of $(R(t))_{t\in[0,\tau)}$, too.

(iii) Suppose that $a(t)$ is a kernel on $[0,\tau)$, $\mathcal{A}$ and $\mathcal{B}$ are two subgenerators of $(R(t))_{t\in[0,\tau)}$, and $x \in D(\mathcal{A}) \cap D(\mathcal{B})$. Then $R(t)(y-z) = 0$, $t \in [0,\tau)$ for each $y \in \mathcal{A}x$ and $z \in \mathcal{B}x$.

(iv) Let $\mathcal{A}$ be a subgenerator of $(R(t))_{t\in[0,\tau)}$, and let $\lambda \in \rho_C(\mathcal{A})$ $(\lambda \in \rho(\mathcal{A}))$. Suppose that $x \in X$, $y = (\lambda-\mathcal{A})^{-1}Cx$ $(y = (\lambda-\mathcal{A})^{-1}x)$ and $z \in \mathcal{A}y$. Then Theorem 2.4(i) implies that $\lambda(\lambda-\mathcal{A})^{-1}Cx - Cx \in \mathcal{A}(\lambda-\mathcal{A})^{-1}Cx = \mathcal{A}y$ $(\lambda(\lambda-\mathcal{A})^{-1}x - x \in \mathcal{A}(\lambda-\mathcal{A})^{-1}x = \mathcal{A}y)$, so that $R(t)y - k(t)Cy \in \mathcal{A}\int_0^t a(t-s)R(s)[\lambda(\lambda-\mathcal{A})^{-1}Cx - Cx]\,ds = \mathcal{A}\{\lambda(\lambda-\mathcal{A})^{-1}C\int_0^t a(t-s)R(s)x\,ds - \int_0^t a(t-s)R(s)Cx\,ds\}$, $t \in [0,\tau)$ and $\int_0^t a(t-s)R(s)Cx\,ds \in D(\mathcal{A})$, $t \in [0,\tau)$; from this, we may conclude that $R(t)Cx - k(t)C^2x \in (\lambda-\mathcal{A})\mathcal{A}(\lambda-\mathcal{A})^{-1}C\int_0^t a(t-s)R(s)x\,ds$, $t \in [0,\tau)$; similarly, we have that $\int_0^t a(t-s)R(s)x\,ds \in D(\mathcal{A})$ and $R(t)x - k(t)Cx \in (\lambda-\mathcal{A})\mathcal{A}(\lambda-\mathcal{A})^{-1}\int_0^t a(t-s)R(s)x\,ds$, $t \in [0,\tau)$, provided that $\lambda \in \rho(\mathcal{A})$.

The following extensions of [36, Theorems 2.8.5 and 2.1.5] are stated without proofs.

Theorem 5.4. *Suppose $\mathcal{A}$ is a closed MLO in X, $C_1 \in L(Y,X)$, $C_2 \in L(X)$, C_2 is injective, $\omega_0 \geq 0$ and $\omega \geq \max(\omega_0, \operatorname{abs}(|a|), \operatorname{abs}(k))$.*

(i) Let $(R_1(t), R_2(t))_{t\geq 0} \subseteq L(Y,X) \times L(X)$ be strongly continuous, and let the family $\{e^{-\omega t}R_i(t) : t \geq 0\}$ be equicontinuous for $i = 1,2$.

(a) *Suppose $(R_1(t), R_2(t))_{t\geq 0}$ is a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$. Then, for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, the operator $I - \tilde{a}(\lambda)\mathcal{A}$ is injective, $R(C_1) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$,*
$$\tilde{k}(\lambda)\big(I - \tilde{a}(\lambda)\mathcal{A}\big)^{-1}C_1y = \int_0^\infty e^{-\lambda t}R_1(t)y\,dt, \quad y \in Y, \tag{5.5}$$
$$\Big\{\frac{1}{\tilde{a}(z)} : \Re z > \omega, \tilde{k}(z)\tilde{a}(z) \neq 0\Big\} \subseteq \rho_{C_1}(\mathcal{A}), \tag{5.6}$$
$$\tilde{k}(\lambda)C_2x = \int_0^\infty e^{-\lambda t}\big[R_2(t)x - \big(a * R_2\big)(t)y\big]\,dt, \tag{5.7}$$

whenever $(x,y) \in \mathcal{A}$. Here, $\rho_{C_1}(\mathcal{A})$ is defined in the obvious way.

(b) *Let (5.6) hold, and let (5.5) and (5.7) hold for any $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$. Then $(R_1(t), R_2(t))_{t\geq 0}$ is a mild (a,k)-regularized (C_1, C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$.*

(ii) Let $(R_1(t))_{t\geq 0}$ be strongly continuous, and let the family $\{e^{-\omega t}R_1(t) : t \geq 0\}$ be equicontinuous. Then $(R_1(t))_{t\geq 0}$ is a mild (a,k)-regularized C_1-existence family with a subgenerator $\mathcal{A}$ if and only if for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, one has $R(C_1) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$ and

$$\tilde{k}(\lambda)C_1 y \in \big(I - \tilde{a}(\lambda)\mathcal{A}\big)\int_0^\infty e^{-\lambda t}R_1(t)y\,dt, \quad y \in Y.$$

(iii) Let $(R_2(t))_{t\geq 0}$ be strongly continuous, and let the family $\{e^{-\omega t}R_2(t) : t \geq 0\}$ be equicontinuous. Then $(R_2(t))_{t\geq 0}$ is a mild (a,k)-regularized C_2-uniqueness family with a subgenerator $\mathcal{A}$ if and only if for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, the operator $I - \tilde{a}(\lambda)\mathcal{A}$ is injective and (5.7) holds.

Theorem 5.5. *Let $(R(t))_{t\geq 0} \subseteq L(X)$ be a strongly continuous operator family such that there exists $\omega \geq 0$ satisfying that the family $\{e^{-\omega t}R(t) : t \geq 0\}$ is equicontinuous, and let $\omega_0 > \max(\omega, \operatorname{abs}(|a|), \operatorname{abs}(k))$. Suppose that $\mathcal{A}$ is a closed MLO in X and $C\mathcal{A} \subseteq \mathcal{A}C$.*

(i) Assume that $\mathcal{A}$ is a subgenerator of the global (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ satisfying (5.1) for all $x = y \in X$. Then, for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, the operator $I - \tilde{a}(\lambda)\mathcal{A}$ is injective, $R(C) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$,

$$\tilde{k}(\lambda)\big(I - \tilde{a}(\lambda)\mathcal{A}\big)^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \tag{5.8}$$

for $x \in X$, $\Re\lambda > \omega_0$, $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$,

$$\Big\{\frac{1}{\tilde{a}(\lambda)} : \Re\lambda > \omega_0,\ \tilde{k}(\lambda)\tilde{a}(\lambda) \neq 0\Big\} \subseteq \rho_C(\mathcal{A}) \tag{5.9}$$

and $R(s)R(t) = R(t)R(s)$ for $t, s \geq 0$.

(ii) Assume (5.8) and (5.9). Then $\mathcal{A}$ is a subgenerator of the global (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ satisfying (5.1) for all $x = y \in X$ and $R(s)R(t) = R(t)R(s)$, $t,\ s \geq 0$.

Remark 5.6. (i) Suppose that $(R(t))_{t\geq 0}$ is a degenerate exponentially equicontinuous (a,k)-regularized C-resolvent family in the sense of [45, Definition 2.2], and $B \in L(X)$. Using Remark 2.1(iv)/(a), Remark 4.2(iv) and Theorem 5.5(ii), it can be easily seen that $(R(t))_{t\geq 0}$ is an exponentially equicontinuous (a,k)-regularized C-resolvent family with a closed subgenerator $B^{-1}A$.

(ii) Suppose that $n \in \mathbb{N}$, X and Y are Banach spaces, $A : D(A) \subseteq X \to Y$ is closed, $B \in L(X,Y)$ and $(V(t))_{t\geq 0} \subseteq L(X)$ is a degenerate exponentially bounded n-times integrated semigroup generated by linear operators A, B, in the sense of [63, Definition 1.5.3]. Then the arguments mentioned above show that $(V(t))_{t\geq 0}$ is an exponentially bounded n-times integrated (g_1, I)-regularized family (semigroup) with a closed subgenerator $B^{-1}A$.

(iii) Let $n \in \mathbb{N}_0$. Due to Theorem 5.5(ii), the notion of an exponentially bounded (a,k)-regularized C-resolvent family extends the notion of a degenerate exponentially bounded n-times integrated semigroup generated by an MLO, introduced in [63, Definition 1.6.6, Definition 1.6.8].

(iv) Suppose that $(R(t))_{t\geq 0} \subseteq L(X,[D(B)])$ is an exponentially equicontinuous (a,k)-regularized C-resolvent family generated by A, B, in the sense of [46, Definition 2.5]. Then [46, Theorem 2.3(i)] in combination with Remark 4.2(v) and Theorem 5.5(ii) implies that $(BR(t))_{t\geq 0}$ is an exponentially equicontinuous (a,k)-regularized C-resolvent family generated by $\overline{B^{-1}A}$ (recall that $B^{-1}A$ is closed provided that $C=I$).

The proof of following extension of [36, Theorems 2.1.8(i) and 2.8.7(i)] is standard and therefore omitted; we can similarly reformulate Theorem 4.8 and [36, Proposition 2.1.16] for the class of mild (a,k)-regularized (C_1,C_2)-existence and uniqueness families ((a,k)-regularized C-resolvent families). Here it is only worth noting that the existence of a mild (a,k_1)-regularized C_1-existence family $(R_{0,1}(t))_{t\geq 0}$ in the second part of theorem is not automatically guaranteed by the denseness of $\mathcal{A}$ (even in the case that the operator $\mathcal{A}=A$ is single-valued, it seems that the condition $C_1A \subseteq AC_1$ is necessary for such a mild existence family to exist).

Theorem 5.7. *Suppose $C_1 \in L(Y,X)$, $C_2 \in L(X)$ is injective, $\mathcal{A}$ is a closed MLO in X, $C \in L(X)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Let $b(t)$ and $c(t)$ satisfy (P1)-$\mathbb{C}$, let $\int_0^\infty e^{-\beta t}|b(t)|\,dt < \infty$ for some $\beta \geq 0$, and let*

$$\alpha = \tilde{c}^{-1}\Big(\frac{1}{\beta}\Big) \text{ if } \int_0^\infty c(t)\,dt > \frac{1}{\beta},\ \alpha = 0 \text{ otherwise.}$$

Suppose, further, that $\operatorname{abs}(|a|) < \infty$ and $\tilde{a}(\lambda) = \tilde{b}(\frac{1}{\tilde{c}(\lambda)})$, $\lambda \geq \alpha$. Let $\mathcal{A}$ be a subgenerator of a (b,k)-regularized C_1-existence family $(R_1(t))_{t\geq 0}$ ((b,k)-regularized C_2-uniqueness family $(R_2(t))_{t\geq 0}$; (b,k)-regularized C-resolvent family $(R_0(t))_{t\geq 0}$ with the property that (5.1) holds for $R_1(\cdot)$ replaced with $R_0(\cdot)$ and each $x=y\in X$) satisfying that the family $\{e^{-\omega_b t}R_1(t) : t \geq 0\}$ ($\{e^{-\omega_b t}R_2(t) : t \geq 0\}$; $\{e^{-\omega_b t}R(t) : t \geq 0\}$) is equicontinuous for some $\omega_b \geq 0$.

Assume, further, that $c(t)$ is completely positive and that there exists a scalar-valued continuous kernel $k_1(t)$ satisfying (P1)-$\mathbb{C}$ and

$$\tilde{k_1}(\lambda) = \frac{1}{\lambda\tilde{c}(\lambda)}\tilde{k}\Big(\frac{1}{\tilde{c}(\lambda)}\Big),\ \lambda > \omega_0,\ \tilde{k}\Big(\frac{1}{\tilde{c}(\lambda)}\Big) \neq 0,\ \text{for some } \omega_0 > 0.$$

Let

$$\omega_a = \tilde{c}^{-1}\Big(\frac{1}{\omega_b}\Big) \text{ if } \int_0^\infty c(t)\,dt > \frac{1}{\omega_b},\ \omega_a = 0 \text{ otherwise.}$$

Then, for every $r \in (0,1]$, $\mathcal{A}$ is a subgenerator of a global $(a,k_1 * g_r)$-regularized C_1-existence family $(R_{r,1}(t))_{t\geq 0}$ ($(a,k_1 * g_r)$-regularized C_2-uniqueness family $(R_{r,2}(t))_{t\geq 0}$; $(a,k_1 * g_r)$-regularized C-resolvent family $(R_{r,0}(t))_{t\geq 0}$ with the property that (5.1) holds for $R_1(\cdot)$ replaced with $R_{r,0}(\cdot)$ and each $x=y\in X$) such that the family $\{e^{-\omega_a t}R_{r,i}(t) : t \geq 0\}$ is equicontinuous and that the mapping $t \mapsto R_{r,i}(t)$, $t \geq 0$ is locally Hölder continuous with exponent r, if $\omega_b = 0$ or $\omega_b\tilde{c}(0) \neq 1$ ($i=0,1,2$), resp., for every $\varepsilon > 0$, there exists $M_\varepsilon \geq 1$ such that the family $\{e^{-\varepsilon t}R_{r,i}(t) : t \geq 0\}$ is equicontinuous and that the mapping $t \mapsto R_{r,i}(t)$, $t \geq 0$ is locally Hölder continuous with exponent r, if $\omega_b > 0$ and $\omega_b\tilde{c}(0) = 1$ ($i=0,1,2$).

Furthermore, if $\mathcal{A}$ is densely defined, then $\mathcal{A}$ is a subgenerator of a global (a,k_1)-regularized C_2-uniqueness family $(R_{0,2}(t))_{t\geq 0}$ ((a,k_1)-regularized C-resolvent family $(R_{0,0}(t))_{t\geq 0}$ with the property that (5.1) holds for $R_1(\cdot)$ replaced with $R_{0,0}(\cdot)$ and *each $x = y \in X$) such that the family $\{e^{-\omega_a t}R_i(t) : t \geq 0\}$ is equicontinuous, resp., for every $\varepsilon > 0$, the family $\{e^{-\varepsilon t}R_i(t) : t \geq 0\}$ is equicontinuous ($i = 1, 2$).*

Let $(R_1(t), R_2(t))_{t\in[0,\tau)}$ be a mild (a, k)-regularized (C_1, C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$. Then it is straightforward to see that the function $t \mapsto R_1(t)y$, $t \in [0, \tau)$ $(y \in Y)$, resp. $t \mapsto R_2(t)x$, $t \in [0, \tau)$ $(x \in D(\mathcal{A}))$, is a solution of problem (1.1) with $\mathcal{B} = I$ and $f(t) = k(t)C_1y$, $t \in [0, \tau)$, resp. a strong solution of (1.1) with $\mathcal{B} = I$ and $f(t) = k(t)C_2x$, $t \in [0, \tau)$, provided additionally in the last case that $R_2(t)x \in D(\mathcal{A})$, $t \in [0, \tau)$ and $R_2(t)\mathcal{A}x \subseteq \mathcal{A}R_2(t)x$, $t \in [0, \tau)$. Furthermore, it is very simple to transmit the assertions of [36, Proposition 2.8.8, Proposition 2.8.9] to mild (a, k)-regularized (C_1, C_2)-existence and uniqueness families subgenerated by multivalued linear operators:

Proposition 5.8. *(i) Suppose that $(R_1(t), R_2(t))_{t\in[0,\tau)}$ is a mild (a, k)-regularized (C_1, C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$, as well as that $(R_2(t))_{t\in[0,\tau)}$ is locally equicontinuous and the functions $a(t)$ and $k(t)$ are kernels on $[0, \tau)$. Then $C_2R_1(t) = R_2(t)C_1$, $t \in [0, \tau)$.*

(ii) Suppose that $(R_2(t))_{t\in[0,\tau)}$ is a locally equicontinuous mild (a, k)-regularized C_2-uniqueness family with a subgenerator $\mathcal{A}$. Then every strong solution $u(t)$ of (1.1) with $\mathcal{B} = I$ and $\mathcal{F} = f \in C([0, \tau) : X)$ satisfies

$$\big(R_2 * f\big)(t) = \big(kC_2 * u\big)(t), \quad 0 \leq t < \tau. \tag{5.10}$$

Furthermore, the problem (1.1) has at most one pre-solution provided, in addition, that the functions $a(t)$ and $k(t)$ are kernels on on $[0, \tau)$ and the function $\mathcal{F}(t)$ is single-valued.

The first part of the following theorem is an extension of [36, Theorem 2.1.28(ii)] and its validity can be verified with the help of proof of [59, Theorem 2.7], Lemma 2.2 and Theorem 2.3; the second part of theorem is an extension of [36, Proposition 2.1.31] and can be shown by the arguments contained in the proof of [66, Theorem 2.5], along with Lemma 2.2.

Theorem 5.9. *(i) Suppose that $(R(t))_{t\in[0,\tau)}$ is a locally equicontinuous (a, k)-regularized C-resolvent family generated by $\mathcal{A}$, the equation (5.1) holds for each $y = x \in X$, with $R_1(\cdot)$ and C_1 replaced therein with $R(\cdot)$ and C, respectively, $k(t)$ is a kernel on $[0, \tau)$, u, $f \in C([0, \tau) : X)$, and (5.10) holds with $R_2(\cdot)$ and C_2 replaced therein with $R(\cdot)$ and C, respectively. Then $u(t)$ is a solution of the abstract Volterra inclusion (1.1) with $\mathcal{B} = I$ and $\mathcal{F} = f$.*

(ii) Suppose that the functions $a(t)$ and $k(t)$ are kernels on $[0, \tau)$, and $\mathcal{A}$ is a closed MLO in X. Consider the following assertions:

- (a) *$\mathcal{A}$ is a subgenerator of a locally equicontinuous (a, k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ satisfying the equation (5.1) for each $y = x \in X$, with $R_1(\cdot)$ and C_1 replaced therein by $R(\cdot)$ and C, respectively.*
- (b) *For every $x \in X$, there exists a unique solution of (1.1) with $\mathcal{B} = I$ and $\mathcal{F}(t) = f(t) = k(t)Cx$, $t \in [0, \tau)$.*

Then (a) ⇒ (b). If, in addition, X is a Fréchet space, then the above are equivalent.

Before proceeding further, it should be noticed that some additional conditions ensure the validity of implication (b) ⇒ (a) in complete locally convex spaces. We will explain this fact for the problem (1.3), where after integration we have $a(t) = g_\alpha(t)$. Assume that there exists a unique solution of problem (1.3) with

$\mathcal{B} = I$, $\mathcal{F}(t) \equiv 0$, $x_0 \in C(D(\mathcal{A}))$ and $x_j = 0$, $1 \le j \le \lceil\alpha\rceil - 1$. If, in addition to this, X is complete, $\mathcal{A}$ is closed, $C\mathcal{A} \subseteq \mathcal{A}C$ and for each seminorm $p \in \circledast$ and $T > 0$ there exist $q \in \circledast$ and $c > 0$ such that $p(u(t; Cx)) \le cq(x)$, $x \in D(\mathcal{A})$, $t \in [0, T]$, then the arguments used in non-degenerate case (see e.g. [40, p. 304]) show that $\mathcal{A}$ is a subgenerator of a locally equicontinuous (g_α, C)-resolvent family $(R_\alpha(t))_{t\ge0}$.

The proof of following complex characterization theorem for (a, k)-regularized C-resolvent families is left to the reader as an easy exercise.

Theorem 5.10. *Let $\omega_0 > \max(0, \operatorname{abs}(|a|), \operatorname{abs}(k))$, and let $\mathcal{A}$ be a closed MLO in X. Assume that, for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0$ and $\tilde{a}(\lambda)\tilde{k}(\lambda) \ne 0$, the operator $I - \tilde{a}(\lambda)\mathcal{A}$ is injective and $R(C) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$. If there exists a function $\Upsilon : \{\lambda \in \mathbb{C} : \Re\lambda > \omega_0\} \to L(X)$ which satisfies:*

- (i) $\Upsilon(\lambda) = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}C$, $\Re\lambda > \omega_0$, $\tilde{a}(\lambda)\tilde{k}(\lambda) \ne 0$,
- (ii) *the mapping $\lambda \mapsto \Upsilon(\lambda)x$, $\Re\lambda > \omega_0$ is analytic for every fixed $x \in X$,*
- (iii) *there exists $r \ge -1$ such that the family $\{\lambda^{-r}\Upsilon(\lambda) : \Re\lambda > \omega_0\} \subseteq L(X)$ is equicontinuous,*

then, for every $\alpha > 1$, $\mathcal{A}$ is a subgenerator of a global $(a, k * g_{\alpha+r})$-regularized C-resolvent family $(R_\alpha(t))_{t\ge0}$ which satisfies that the family $\{e^{-\omega_0 t}R_\alpha(t) : t \ge 0\} \subseteq L(X)$ is equicontinuous. Furthermore, $(R_\alpha(t))_{t\ge0}$ is a mild $(a, k * g_{\alpha+r})$-regularized C-existence family having $\mathcal{A}$ as subgenerator.

In the first part of following example, we will briefly explain how one can use multiplication operators for construction of local integrated semigroups generated by multivalued operators; in the second part of example, we will apply the complex characterization theorem for proving the existence of a very specific exponentially equicontinuous, convoluted fractional resolvent family (cf. [42, Example 2.5] for an example of a locally defined solution of an abstract degenerate multi-term fractional problem).

Example 5.11. (i) (cf. also [2, Example 4.4(c)]) Suppose that $1 \le p \le \infty$, $X := L^p(1, \infty)$, $1 < a < b < \infty$, $J := [a, b]$, $m_b(x) := \chi_J(x)$ and $m_a(x) := x + ie^x$ $(x > 1)$. Consider the multiplication operators $A : D(A) \to X$ and $B \in L(X)$, where $D(A) := \{f(x) \in X : (x + ie^x)f(x) \in X\}$, $Af(x) := (x + ie^x)f(x)$ and $Bf(x) := m_b(x)f(x)$ $(x > 1,\ f \in X)$. Then it is very simple to prove that, for every $\alpha \in (0, 1)$, the resolvent set of the multivalued linear operator $\mathcal{A} := B^{-1}A$ contains the exponential region $E(\alpha, 1) := \{x + iy : x \ge 1,\ |y| \le e^{\alpha x}\}$, as well as that $(\lambda - \mathcal{A})^{-1}f(x) = (\lambda B - A)^{-1}Bf(x) = m_b(x)f(x)/\lambda m_b(x) - m_a(x)$ for $x > 1$, $f \in X$. Furthermore, the operator $\mathcal{A}$ generates a local once integrated semigroup $(S_1(t))_{t\in[0,1]}$, given by

$$(S_1(t)f)(x) = \begin{cases} \big(x + ie^x\big)^{-1}\big[e^{t(x+ie^x)} - 1\big]f(x), & t \in [0, 1],\ x \notin J,\ f \in X, \\ 0, & t \in [0, 1],\ x \in J,\ f \in X. \end{cases}$$

(ii) Put $X := \{f \in C^\infty([0, \infty)) : \lim_{x\to+\infty} f^{(k)}(x) = 0 \text{ for all } k \in \mathbb{N}_0\}$ and $||f||_k := \sum_{j=0}^{k} \sup_{x\ge0} |f^{(j)}(x)|$, $f \in X$, $k \in \mathbb{N}_0$. Then the topology induced by these norms turns X into a Fréchet space (cf. also [36, Example 2.4.6(ii)]). Let $\alpha \in (0, 1)$ and $J = [a, b] \subseteq [0, \infty)$ be such that $\overline{\Sigma_{\alpha\pi/2}} \cap \{x + ie^x : x \in J\} = \emptyset$, and let $m_b \in C^\infty([0, \infty))$ satisfy $0 \le m_b(x) \le 1$, $x \ge 0$, $m_b(x) = 1$, $x \notin J$ and $m_b(x) = 0$, $x \in [a + \epsilon, b - \epsilon]$ for some $\epsilon > 0$. As in the first part of this example, we consider the multiplication operators $A : D(A) \to X$ and $B \in L(X)$, where $D(A) = \{f(x) \in E :$

$(x+ie^x)f(x) \in X\}$, $Af(x) := (x+ie^x)f(x)$ and $Bf(x) := m_b(x)f(x)$ ($x \geq 0$, $f \in X$). In a recent research study with Pilipović and Velinov [53], we have shown that A cannot be the generator of any local integrated semigroup in X, as well as that A generates an ultradistribution semigroup of Beurling class. Set $\mathcal{A} := B^{-1}A$. We will prove that there exists a sufficiently large number $\omega > 0$ such that for each $s > 1$ and $d > 0$ the operator family $\{e^{-d|\lambda|^{1/s}}(\lambda - \mathcal{A})^{-1} : \Re\lambda > \omega,\ \lambda \in \Sigma_{\alpha\pi/2}\} \subseteq L(X)$ is equicontinuous, which immediately implies by Theorem 5.10 that $\mathcal{A}$ generates an exponentially equicontinuous $(g_\alpha, \mathcal{L}^{-1}(e^{-d|\lambda|^{\alpha/s}}))$-regularized resolvent family. It is clear that the resolvent of $\mathcal{A}$ will be given by $(\lambda-\mathcal{A})^{-1}f(x) = (\lambda B - A)^{-1}Bf(x) = m_b(x)f(x)/\lambda m_b(x) - m_a(x)$ for $x \geq 0$, $f \in X$. Since $m_b(x)f(x)/\lambda m_b(x) - m_a(x) = 1/\lambda - (x+ie^x)$ for $x \notin J$, our first task will be to estimate the derivatives of function $1/\lambda - (\cdot + ie^{\cdot})$ outside the interval J. In order to do that, observe first that any complex number $\lambda \in \mathbb{C} \setminus S$, where $S := \{x + ie^x : x \geq 0\}$, belongs to the resolvent set of A and

$$\big(\lambda - A\big)^{-1} f(x) = \frac{f(x)}{\lambda - \big(x + ie^x\big)}, \quad \lambda \in \mathbb{C} \setminus S,\ x \geq 0.$$

Fix, after that, numbers $s > 1$, $d > 0$, $a > 0$, $b > 1$ satisfying that $x - \ln(((x-b)/a)^s + 1) \geq 1$, $x \geq b$. Set $\Omega := \{\lambda \in \mathbb{C} : \Re\lambda \geq a|\Im\lambda|^{1/s} + b\}$ and denote by Γ the upwards oriented boundary of the region Ω. Inductively, we can prove that for each number $n \in \mathbb{N}$ there exist complex polynomials $P_j(z) = \sum_{l=0}^{j} a_{j,l} z^l$ ($1 \leq j \leq n$) such that $\deg(P_j) = j$, $|a_{j,l}| \leq (n+1)!$ ($1 \leq j \leq n$, $0 \leq l \leq j$) and

$$\frac{d^n}{dx^n}\Big(\lambda - \big(x + ie^x\big)\Big)^{-1} = \sum_{j=1}^{n+1}\Big(\lambda - \big(x + ie^x\big)\Big)^{-j-1} P_j\big(e^x\big), \quad x \geq 0,\ \lambda \in \mathbb{C} \setminus S. \tag{5.11}$$

Suppose $\lambda \in \Omega$ and $x \geq 0$. If $|\Im\lambda - e^x| \geq 1$, then we have the estimate

$$\begin{aligned} \frac{e^{2jx}}{\big(\Re\lambda - x\big)^{2k} + \big(\Im\lambda - e^x\big)^{2k}} &\leq \frac{e^{2jx}}{\big(\Im\lambda - e^x\big)^{2k}} \\ &\leq 2^{2j}\big(1 + |\Im\lambda|\big)^{2j}, \quad k \in \mathbb{N}_0,\ 0 \leq j < k. \end{aligned} \tag{5.12}$$

If $|\Im\lambda - e^x| < 1$, then $\Im\lambda > 0$, $0 \leq x < \ln(\Im\lambda + 1)$, and

$$\begin{aligned} \frac{e^{2jx}}{\big(\Re\lambda - x\big)^{2k} + \big(\Im\lambda - e^x\big)^{2k}} &\leq \frac{e^{2jx}}{\big(\Re\lambda - x\big)^{2k}} \\ &\leq \frac{\big(\Im\lambda + 1\big)^j}{\Re\lambda - \ln\big(((\Re\lambda - b)/a)^s + 1\big)} \\ &\leq \big(\Im\lambda + 1\big)^j, \quad k \in \mathbb{N}_0,\ 0 \leq j < k. \end{aligned} \tag{5.13}$$

Let $\omega' > 0$ be such that $\{\lambda \in \Sigma_{\alpha\pi/2} : \Re\lambda > \omega'\} \subseteq \Omega$. Combining (5.11)-(5.13), it can be simply proved that for each number $n \in \mathbb{N}$ there exists a finite constant $c_n > 0$ such that

$$\sum_{k=0}^{n} \sup_{x \geq 0, x \notin J} \Big|\frac{d^n}{dx^n}\Big(\lambda - \big(x + ie^x\big)\Big)^{-1}\Big| \leq c_n e^{d|\lambda|^{1/s}}, \tag{5.14}$$

for $\lambda \in \Sigma_{\alpha\pi/2}$ and $\Re\lambda > \omega'$.

We can similarly prove an estimate of type (5.14) for the derivatives of function $(\lambda m_b(x) - (x+ie^x))^{-1}$ on the interval J, which is well-defined for $\lambda \in \Sigma_{\alpha\pi/2}$ because of assumption $0 \leq m_b(x) \leq 1$, $x \geq 0$ and the condition $\overline{\Sigma_{\alpha\pi/2}} \cap \{x+ie^x : x \in J\} = \emptyset$. In actual fact, an induction argument shows that for each number $n \in \mathbb{N}$ there exist numbers $a_{j,l_1,\cdots,l_s}$ such that $|a_{j,l_1,\cdots,l_s}| \leq (n+1)!$ $(1 \leq j \leq n,\ 0 \leq l \leq j)$ and that, for every $x \in J$ and $\lambda \in \Sigma_{\alpha\pi/2}$,

$$\begin{aligned} &\frac{d^n}{dx^n}\Big(\lambda m_b(x) - \big(x+ie^x\big)\Big)^{-1} \\ &= \sum_{j=1}^{n+1}\Big(\lambda m_b(x) - \big(x+ie^x\big)\Big)^{-j-1} \\ &\quad \times \sum_{l=0}^{j} a_{j,l_1,\cdots,l_s} \prod_{l_1m_1+\cdots+l_sm_s=n} \Big(\lambda m_b^{(l_j)}(x) - m_a^{(l_j)}(x)\Big)^{m_j}. \end{aligned} \tag{5.15}$$

Since $d :=\mathrm{dist}(\overline{\Sigma_{\alpha\pi/2}}, \{x+ie^x : x \in J\})$ is a positive real number and $|(\lambda m_b^{(l_j)}(x) - m_a^{(l_j)}(x))^{m_j}| \leq c^{m_j}|\lambda|^{m_j}$ for all $\lambda \in \Sigma_{\alpha\pi/2}$ with $\Re\lambda > \omega$, where the number $\omega > \omega'$ is sufficiently large, (5.15) shows that for each number $n \in \mathbb{N}$ there exists a finite number $c_n' > 0$ such that

$$\sum_{k=0}^{n} \sup_{x\geq 0, x\in J}\Big|\frac{d^n}{dx^n}\Big(\lambda m_b(x) - \big(x+ie^x\big)\Big)^{-1}\Big| \leq c_n' e^{d|\lambda|^{1/s}}, \tag{5.16}$$

for $\lambda \in \Sigma_{\alpha\pi/2}$ and $\Re\lambda > \omega$.

By (5.14) and (5.16), we have that the operator family $\{e^{-d|\lambda|^{1/s}}(\lambda - \mathcal{A})^{-1} : \lambda \in \Sigma_{\alpha\pi/2},\ \Re\lambda > \omega\} \subseteq L(X)$ is equicontinuous, as claimed.

Now we would like to tell something more about the importance of condition $k(0) \neq 0$ in the part (ii) of subsequent theorem. If all the necessary requirements hold, the arguments contained in the proof of [32, Theorem 3.6] imply the existence of a global $(a, k * g_1)$-regularized C-resolvent family $(R_1(t))_{t\geq 0}$ subgenerated by $\mathcal{A}$, which additionally satisfies that for each $t \geq 0$ the operator $R_1(t)\mathcal{A}$ is single-valued on $D(\mathcal{A})$. Then it is necessary to differentiate the equality $R_1(t)x - (k*g_1)(t)Cx = \int_0^t a(t-s)R_1(s)\mathcal{A}x\,ds$, $t \geq 0$, $x \in D(\mathcal{A})$ and to employ the fact that $(\frac{d}{dt}R_1(t)x)_{t=0} = k(0)Cx$ $(x \in \overline{D(\mathcal{A})})$ (cf. the proof of [32, Theorem 3.6], as well as the proofs of [17, Proposition 2.1] and [36, Proposition 2.1.7]) in order to see that the function $R : D(R) \equiv \{\tilde{a}(\lambda)^{-1} : \lambda > b,\ \tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0\} \to L(\overline{D(\mathcal{A})})$, given by $R(\tilde{a}(\lambda)^{-1}) := (\tilde{a}(\lambda)^{-1} - \mathcal{A})^{-1}C$, $\lambda \in D(R)$, is a C-pseudoresolvent in the sense of [57, Definition 3.1], satisfying additionally that $N(R(\lambda)) = \{0\}$, $\lambda \in D(R)$. Only after that, we can use [57, Theorem 3.4] with a view to prove the existence of a single-valued linear operator A, with domain and range contained in $\overline{D(\mathcal{A})}$, which satisfies the properties required in (ii): this consideration shows the full importance of concepts introduced in Definition 5.1 and Definition 5.2 in integrated and convoluted case $k(0) = 0$. Keeping in mind Theorem 2.4(i) and the argumentation contained in the proofs of [32, Theorem 3.6] and [36, Theorem 1.2.6], the remaining parts of following theorem can be deduced, more or less, as in non-degenerate case.

Theorem 5.12. *Suppose* $\omega \in \mathbb{R}$, $\mathrm{abs}(k) < \infty$, $\mathrm{abs}(|a|) < \infty$, $\mathcal{A}$ *is a closed MLO in* X, $\lambda_0 \in \rho_C(\mathcal{A})$, $b \geq \max(0, \omega, \mathrm{abs}(|a|), \mathrm{abs}(k))$,

$$\Big\{\frac{1}{\tilde{a}(\lambda)} : \lambda > b,\ \tilde{k}(\lambda)\tilde{a}(\lambda) \neq 0\Big\} \subseteq \rho_C(\mathcal{A}),$$

the function $H : D(H) \equiv \{\lambda > b : \tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0\} \to L(X)$*, given by* $H(\lambda)x = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}Cx$*,* $x \in X$*,* $\lambda \in D(H)$*, satisfies that the mapping* $\lambda \mapsto H(\lambda)x$*,* $\lambda \in D(H)$ *is infinitely differentiable for every fixed* $x \in X$ *and, for every* $p \in \circledast$*, there exist* $c_p > 0$ *and* $r_p \in \circledast$ *such that*

$$p\Big(l!^{-1}(\lambda-\omega)^{l+1}\frac{d^l}{d\lambda^l}H(\lambda)x\Big) \le c_p r_p(x),\ x \in X,\ \lambda \in D(H),\ l \in \mathbb{N}_0. \tag{5.17}$$

Then, for every $r \in (0,1]$*, the operator* $\mathcal{A}$ *is a subgenerator of a global* $(a, k*g_r)$*-regularized* C*-resolvent family* $(R_r(t))_{t\ge 0}$ *satisfying that, for every* $p \in \circledast$*,*

$$p\big(R_r(t+h)x - R_r(t)x\big) \le \frac{2c_p r_p(x)}{r\Gamma(r)}\max\big(e^{\omega(t+h)},1\big)h^r,\ t \ge 0,\ h > 0,\ x \in X,$$

and that, for every $p \in \circledast$ *and* $B \in \mathcal{B}$*, the mapping* $t \mapsto p_B(R_r(t))$*,* $t \ge 0$ *is locally Hölder continuous with exponent* r*; furthermore,* $(R_r(t))_{t\ge 0}$ *is a mild* $(a, k*g_r)$*-regularized* C*-existence family having* $\mathcal{A}$ *as subgenerator, and the following holds:*

(i) *Suppose that* $\mathcal{A}$ *is densely defined. Then* $\mathcal{A}$ *is a subgenerator of a global* (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\ge 0} \subseteq L(X)$ *satisfying that the family* $\{e^{-\omega t}R(t) : t \ge 0\} \subseteq L(X)$ *is equicontinuous. Furthermore,* $(R(t))_{t\ge 0}$ *is a mild* (a,k)*-regularized* C*-existence family having* $\mathcal{A}$ *as subgenerator.*

(ii) *Suppose that* $k(0) \neq 0$*. Then the operator* $C' := C_{|\overline{D(\mathcal{A})}} \in L(\overline{D(\mathcal{A})})$ *is injective,* $\mathcal{A}0$ *is a closed subspace of* X*,* $\overline{D(\mathcal{A})} \cap \mathcal{A}0 = \{0\}$*, and we have the following: Define the operator* $A : D(A) \subseteq \overline{D(\mathcal{A})} \to \overline{D(\mathcal{A})}$ *by*

$$D(A) := \big\{x \in \overline{D(\mathcal{A})} : Cx = \big(\lambda_0 - \mathcal{A}\big)^{-1}Cy \text{ for some } y \in \overline{D(\mathcal{A})}\big\}$$

and

$$Ax := C^{-1}\mathcal{A}Cx, \quad x \in D(A).$$

Then A *is a well-defined single-valued closed linear operator in* $\overline{D(\mathcal{A})}$*, and moreover,* A *is the integral generator of a global* (a,k)*-regularized* C'*-resolvent family* $(S(t))_{t\ge 0} \subseteq L(\overline{D(\mathcal{A})})$ *satisfying that the family* $\{e^{-\omega t}S(t) : t \ge 0\} \subseteq L(\overline{D(\mathcal{A})})$ *is equicontinuous,* $A\int_0^t a(t-s)S(s)x\,ds = S(t)x - k(t)Cx$*,* $t \in [0,\tau)$*,* $x \in \overline{D(\mathcal{A})}$ *and* $R_1(t)x = \int_0^t S(s)x\,ds$*,* $t \ge 0$*,* $x \in \overline{D(\mathcal{A})}$*.*

In the following proposition, which extends the assertions of [59, Proposition 2.5] and [36, Proposition 2.1.4(ii)], we will reconsider the condition $k(0) \neq 0$ from Theorem 5.12 once more. A straightforward proof is omitted.

Proposition 5.13. *Let* $\mathcal{A}$ *be a closed subgenerator of a mild* (a,k)*-regularized* C_1*-resolvent family* $(R_1(t))_{t\in[0,\tau)}$ *(mild* (a,k)*-regularized* C_2*-uniqueness family* $(R_2(t))_{t\in[0,\tau)}$*;* (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\in[0,\tau)}$*). If* $k(t)$ *is absolutely continuous and* $k(0) \neq 0$*, then* $\mathcal{A}$ *is a subgenerator of a mild* (a,g_1)*-regularized* C_1*-resolvent family* $(R_1(t))_{t\in[0,\tau)}$ *(mild* (a,g_1)*-regularized* C_2*-uniqueness family* $(R_2(t))_{t\in[0,\tau)}$*;* (a,g_1)*-regularized* C*-resolvent family* $(R(t))_{t\in[0,\tau)}$*).*

Now we would like to present some illustrative applications of results obtained so far.

Example 5.14. Let $\alpha \in (0,1)$.

(i) ([17]) Consider the following time-fractional analogue of homogeneous counterpart of problem [17, Example 2.1, (2.18)]:

$$
\begin{gathered}
\mathbf{D}_t^{\alpha}[m(x)v_{\alpha}(t,x)] = -\frac{\partial}{\partial x} v_{\alpha}(t,x), \quad t \geq 0,\ x \in \mathbb{R}; \\
m(x)v_{\alpha}(0,x) = u_0(x), \quad x \in \mathbb{R}.
\end{gathered} \tag{5.18}
$$

Let $X = Y := L^2(\mathbb{R})$, and let the operator $A := -d/dx$ act on X with its maximal distributional domain $H^1(\mathbb{R})$.

(a) Suppose first that $(Bf)(x) := \chi_{(-\infty,a)\cap(b,\infty)}(x)f(x)$, $x \in \mathbb{R}$ ($f \in X$), where $-\infty < a < b < \infty$. Then $B \in L(X)$, $B = B^*$, $B^2 = B$ and (5.18) is formulated in X in the abstract form

$$
\begin{gathered}
B^*\mathbf{D}_t^{\alpha} Bv_{\alpha}(t) = \mathbf{D}_t^{\alpha} Bv_{\alpha}(t) = Av_{\alpha}(t), \quad t \geq 0; \\
Bv_{\alpha}(0) = u_0.
\end{gathered} \tag{5.19}
$$

Further on, the multivalued linear operator $\mathcal{A} := (B^*)^{-1}AB^{-1}$ is maximal dissipative in the sense of [17, Definition, p. 35] and $\|(\lambda - \mathcal{A})^{-1}\| \leq \lambda^{-1}$, $\lambda > 0$. By the foregoing, we know that the operator $\mathcal{A}$ is single-valued on $\overline{D(\mathcal{A})}$; with a little abuse of notation, we will denote by $T \subseteq \mathcal{A}$ the single-valued linear operator which generates a bounded strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\overline{D(\mathcal{A})}$ (cf. Theorem 5.12(ii), where we have denoted this operator by A).

Using [32, Theorem 3.6(a)] and the consideration from the paragraph directly preceding the formulation of [17, Theorem 2.8], it readily follows that $D(T) = D(\mathcal{A})$. Suppose now that $u_0 = Bv_0$, where $v_0 \in D(A)$ and $Av_0 \in R(B^*)$, i.e., that $u_0 \in D(\mathcal{A}) = D(T)$ (cf. the proof of [17, Theorem 2.10]). From [17, Theorem 2.8, Theorem 2.10], the problem (5.19), with $\alpha = 1$, has a unique solution $v_1(t)$ satisfying $Bv_1(t) = T(t)u_0$; moreover,

$$
(d/dt)Bv_1(t) = B^*(d/dt)Bv_1(t) = Av_1(t) = (d/dt)T(t)u_0 = T(t)Tu_0, \tag{5.20}
$$

for $t \geq 0$. Since condition [17, (2.14)] holds, we obtain that there exists $\lambda_0 > 0$ such that $(\lambda_0 B - A)^{-1} \in L(X)$; hence, $v_1(\cdot) = (\lambda_0 B - A)^{-1}(\lambda_0 B - A)v_1(\cdot) \in C([0,\infty) : X)$ is bounded, as well as $(d/dt)Bv_1(t)$, $Bv_1(t)$ and $Av_1(t)$ are continuous and bounded for $t \geq 0$. Define $v_{\alpha}(t) := \int_0^{\infty} t^{-\alpha}\Phi_{\alpha}(st^{-\alpha})v_1(s)\,ds$, $t > 0$ and $v_{\alpha}(0) := v_1(0)$. Using Theorem 4.8 and the arguments contained in its proof, it readily follows that the function $v_{\alpha}(\cdot)$ is a bounded solution of problem (5.19), satisfying in addition that the functions $t \mapsto v_{\alpha}(\cdot)$, $t > 0$ and $t \mapsto Av_{\alpha}(\cdot)$, $t > 0$ can be analytically extended to the sector $\Sigma_{\min((\frac{1}{\alpha}-1)\frac{\pi}{2},\pi)}$. The uniqueness of solutions of problem (5.19) can be proved with the help of Theorem 4.6.

(b) Suppose now that $(Bf)(x) := \chi_{(a,\infty,a)}(x)f(x)$, $x \in \mathbb{R}$ ($f \in X$), where $-\infty < a < \infty$. Then $B \in L(X)$, $B = B^*$, $B^2 = B$ and the conclusions established in the part (a) of this example, ending with the equation (5.20), continue to hold. In our concrete situation, we have the validity of condition [17, (2.11)] but not the condition [17, (2.14)], in general. Define $f_{\alpha}(t) := \int_0^{\infty} t^{-\alpha}\Phi_{\alpha}(st^{-\alpha})Bv_1(s)\,ds$, $t > 0$, $f_{\alpha}(0) := Bv_1(0) = u_0$, $h_{\alpha}(t) := \int_0^{\infty} t^{-\alpha}\Phi_{\alpha}(st^{-\alpha})Av_1(s)\,ds$, $t > 0$ and $h_{\alpha}(0) := Av_1(0)$.

By the foregoing, we have that $f_{\alpha},\ h_{\alpha} \in C([0,\infty) : X)$ are bounded and $\mathbf{D}_t^{\alpha} f_{\alpha}(t) = h_{\alpha}(t)$, $t \geq 0$, which simply implies $Bh_{\alpha}(t) = h_{\alpha}(t)$, $t \geq 0$.

By (5.20), we have that $Av_1(t) = T(t)Tu_0 \in B^{-1}[Av_1(t)]$ and $BAv_1(t) = Av_1(t)$ $(t \geq 0)$, whence we may conclude that $Av_1(t) \in \mathcal{A}[Bv_1(t)]$ $(t \geq 0)$. Since $\mathcal{A}$ is closed, an application of Theorem 2.3 yields that $h_\alpha(t) = Bh_\alpha(t) \in AB^{-1}f_\alpha(t)$ $(t \geq 0)$; consequently, the function $t \mapsto f_\alpha(t)$, $t \geq 0$ is a pre-solution of problem (1.2) with $B \equiv I$, $\mathcal{F}(t) \equiv 0$ and, by Remark 4.2(iv), the problem (5.19) has a bounded p-solution $v_\alpha(\cdot)$ satisfying, in addition, that the functions $t \mapsto Bv_\alpha(\cdot)$, $t > 0$ and $t \mapsto Av_\alpha(\cdot)$, $t > 0$ can be analytically extended to the sector $\Sigma_{\min((\frac{1}{\alpha}-1)\frac{\pi}{2},\pi)}$. The uniqueness follows again from an essential application of Theorem 4.6.

(ii) ([31]-[32]) Here we would like to observe, without going into full details, that we can similarly prove some results on the existence and uniqueness of analytical solutions of the abstract Volterra equation

$$\frac{\partial}{\partial r} v_\alpha(t,r) = a(r) \int_0^t g_\alpha(t-s) v_\alpha(s,r)\, ds + f(t,r), \quad t \geq 0,\ r \in [0,1],$$

on the sector $\Sigma_{\min((\frac{1}{\alpha}-1)\frac{\pi}{2},\pi)}$, where $a \in C^1[0,1]$ and the mapping $t \mapsto f(t,\cdot)$, $t \geq 0$ is continuous and exponentially bounded with the values in the Banach space $C[0,1]$ (cf. [31, Example 1] and Theorem 4.8); using Theorem 4.9 instead of Theorem 4.8, we can consider the well-posedness in $C[0,1]$ for the equation

$$\frac{\partial}{\partial r} v_c(t,r) = a(r) \int_0^t c(t-s) v_c(s,r)\, ds + f(t,r), \quad t \geq 0,\ r \in [0,1],$$

where $c(\cdot)$ is a completely positive function.

Fractional Maxwell's equations have gained much attention in recent years (see e.g. [10], [27], [60], [74], [83] and references cited therein for more details on the subject). Here we want to briefly explain how we can use the analysis of Favini and Yagi [17, Exampe 2.2] for proving the existence and uniqueness of analytical solutions of certain classes of inhomogeneous abstract time-fractional Maxwell's equations in $\mathbb{R}^3$; the time-fractional analogues of Poisson-wave equations (see e.g. [17, Example 2.3, Example 6.23]) will be considered somewhere else.

Consider the following abstract time-fractional Maxwell's equations:

$$\operatorname{rot} E = -\mathbf{D}_t^\alpha B, \quad \operatorname{rot} H = \mathbf{D}_t^\alpha D + J \tag{5.21}$$

in $\mathbb{R}^3$, where E (resp. H) denotes the electric (resp. magnetic) field intensity, B (resp. D) denotes the electric (resp. magnetic) flux density, and where J is the current density. It is assumed that the medium which fills the space $\mathbb{R}^3$ is linear but possibly anisotropic and nonhomogeneous, which means that $D = \epsilon E$, $B = \mu H$ and $J = \sigma E + J'$ with some 3×3 real matrices $\epsilon(x)$, $\mu(x)$, $\sigma(x)$ $(x \in \mathbb{R}^3)$ and J' being a given forced current density. Let any component of $\epsilon(x)$, $\mu(x)$, $\sigma(x)$ be a bounded, measurable function in $\mathbb{R}^3$, let the conditions [17, (2.23)-(2.25)] hold, and let $f(t) = -(J'(\cdot,t)\ \ 0)^T$. Then we can formulate the problem (5.21) in the abstract form

$$\begin{gathered} B^*\mathbf{D}_t^\alpha Bv_1(t) = Av_1(t) + f(t), \quad t \geq 0; \\ Bv_1(0) = u_0, \end{gathered} \tag{5.22}$$

in the space $X := \{L^2(\mathbb{R}^3)\}^6$, using the bounded self-adjoint operator B of multiplication by $\sqrt{c(x)}$ acting in X, and A being the closed linear operator in X given by [17, (2.27)].

In our concrete situation, the conditions [17, (2.10) and (2.14)] hold, so that the assumptions $f \in C^2([0,\infty) : X)$ and $u_0 = Bv_0$ for some $v_0 \in D(A)$ satisfying $Av_0 + f(0) \in R(B^*)$ ensure by [17, Corollary 2.11] that the problem (5.22) has a unique strict solution $v_1(\cdot)$ in the sense of equation [17, (2.13)]. Suppose, additionally, that the function $f''(t)$ is exponentially bounded. Then we can use [17, Theorem 2.5], the proof of [17, Corollary 2.11] and the arguments from the part (i)/(a) of this example in order to see that the solution $v_1 \in C([0,\infty) : X)$ is exponentially bounded, as well as that $H(t) := (d/dt)Bv_1(t)$, $Bv_1(t)$ and $Av_1(t)$ are continuous and exponentially bounded for $t \geq 0$. Define $v_\alpha(t)$ and $f_\alpha(t)$ as before, $H_\alpha(t) := \int_0^\infty t^{-\alpha}\Phi_\alpha(st^{-\alpha})H(s)\,ds,\ t > 0$ and $H_\alpha(0) := H(0)$. Performing the Laplace transform, it can be simply verifed that $(g_{1-\alpha} * (Bv_\alpha - u_0))(t) = \int_0^t H_\alpha(s)\,ds,\ t \geq 0$, so that $\mathbf{D}_t^\alpha Bv_\alpha(t)$ exists and equals to $H_\alpha(t)$. On the other hand, we have

$$B^*Bv_1(t) = A\big(g_1 * v_1\big)(t) + B^*u_0 + \int_0^t f(s)\,ds, \quad t \geq 0,$$

so that

$$B^*Bv_\alpha(t) = A\big(g_\alpha * v_\alpha\big)(t) + B^*u_0 + \int_0^t f_\alpha(s)\,ds, \quad t \geq 0$$

by Theorem 4.8. This implies $\mathbf{D}_t^\alpha B^*Bv_\alpha(t) = Av_\alpha(t) + f_\alpha(t)$ and, since $\mathbf{D}_t^\alpha Bv_\alpha(t)$ exists, $B^*\mathbf{D}_t^\alpha Bv_\alpha(t) = Av_\alpha(t) + f_\alpha(t)$, $t \geq 0$. Clearly, $Bv_\alpha(0) = u_0$ so that $v_\alpha \in C([0,\infty) : X)$ is an exponentially bounded solution of problem

$$\begin{gathered} B^*\mathbf{D}_t^\alpha Bv_\alpha(t) = Av_\alpha(t) + f_\alpha(t), \quad t \geq 0; \\ Bv_\alpha(0) = u_0, \end{gathered} \tag{5.23}$$

that is analytically extensible on the sector $\Sigma_{\min((\frac{1}{\alpha}-1)\frac{\pi}{2},\pi)}$ and satisfies, in addition, that the mapping $Av_\alpha \in C([0,\infty) : X)$ is exponentially bounded and analytically extensible on the same sector, as well. The uniqueness of solutions of problem (P_α) follows from Theorem 4.6.

We end this example with the observation that Theorem 4.8 and Theorem 4.9 can be successfully applied in the analysis of a large class of abstract degenerate Volterra integro-differential equations that are subordinated, in a certain sense, to degenerate differential equations of first and second order for which we know that are well posed [17, 22, 25, 71, 73, 75, 76].

Concerning the adjoint type theorems, it should be noticed that the assertions of [36, Theorem 2.1.12(i)/(ii); Theorem 2.1.13] continue to hold for (a,k)-regularized C-regularized families subgenerated by closed multivalued linear operators. Furthermore, it is not necessary to assume that the operator $\mathcal{A}$ is densely defined in the case of consideration of [36, Theorem 2.1.12(i)].

Suppose now that $\mathcal{A}$ is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, $n \in \mathbb{N}$ and $x_j \in \mathcal{A}x_{j-1}$ for $1 \leq j \leq n$. Then we can prove inductively that, for every $t \in [0,\tau)$,

$$R(t)x = k(t)Cx_0 + \sum_{j=1}^{n-1}\big(a^{*,j} * k\big)(t)Cx_j + \big(a^{*,n} * R(\cdot)x_n\big)(t). \tag{5.24}$$

Keeping in mind the identity (5.24), Theorem 2.3 and Proposition 5.8(ii), it is almost straightforward to transfer the assertion of [36, Proposition 2.1.32] to degenerate case.

Proposition 5.15. *(i) Suppose $\alpha \in (0,\infty) \setminus \mathbb{N}$, $x \in D(\mathcal{A})$ as well as $C^{-1}f$, $f_{\mathcal{A}} \in C([0,\tau) : X)$, $f_{\mathcal{A}}(t) \in \mathcal{A}C^{-1}f(t)$, $t \in [0,\tau)$ and $\mathcal{A}$ is a closed subgenerator of a (g_α, C)-regularized resolvent family $(R(t))_{t\in[0,\tau)}$. Set $v(t) := (g_{\lceil\alpha\rceil-\alpha} * f)(t)$, $t \in [0,\tau)$. If $v \in C^{\lceil\alpha\rceil-1}([0,\tau) : X)$ and $v^{(k)}(0) = 0$ for $1 \le k \le \lceil\alpha\rceil - 2$, then the function $u(t) := R(t)x + (R * C^{-1}f)(t)$, $t \in [0,\tau)$ is a unique solution of the following abstract time-fractional inclusion:*

$$\begin{gathered} u \in C^{\lceil\alpha\rceil}((0,\tau) : X) \cap C^{\lceil\alpha\rceil-1}([0,\tau) : X), \\ \mathbf{D}_t^\alpha u(t) \in \mathcal{A}u(t) + \frac{d^{\lceil\alpha\rceil-1}}{dt^{\lceil\alpha\rceil-1}}\big(g_{\lceil\alpha\rceil-\alpha} * f\big)(t),\ t \in [0,\tau), \\ u(0) = Cx,\ u^{(k)}(0) = 0,\ 1 \le k \le \lceil\alpha\rceil - 1. \end{gathered}$$

(ii) Suppose $r \ge 0$, $n \in \mathbb{N} \setminus \{1\}$, $x_j \in \mathcal{A}x_{j-1}$ for $1 \le j \le n$, $f_j(t) \in \mathcal{A}f_{j-1}(t)$ for $t \in [0,\tau)$ and $1 \le j \le n$, $f_j \in C([0,\tau) : X)$ for $0 \le j \le n$, and $\mathcal{A}$ is a closed subgenerator of a $(g_{1/n}, g_{r+1})$-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$. Then the function $v(t) := R(t)x + (R * C^{-1}f)(t)x$, $t \in [0,\tau)$ is a unique solution of the following abstract time-fractional inclusion

$$\begin{gathered} v \in C^1((0,\tau) : X) \cap C([0,\tau) : X), \\ \begin{aligned} v'(t) \in \mathcal{A}v(t) &+ \sum_{j=1}^{n-1} g_{(j/n)+r}(t)Cx_j + \sum_{j=0}^{n-1}\big(g_{(j/n)+r} * f_j\big)(t) \\ &+ \frac{d}{dt}g_{r+1}(t)Cx,\ t \in (0,\tau), \end{aligned} \\ v(0) = g_{r+1}(0)Cx. \end{gathered}$$

Furthermore, $v \in C^1([0,\tau) : X)$ provided that $r \ge 1$ or $x = 0$ and $r \ge 0$.

5.1. **Differential and analytical properties of (a,k)-regularized C-resolvent families.** The main structural characterizations of differential and analytical (a,k)-regularized C-resolvent families generated by single-valued linear operators continue to hold in our framework (cf. [17, Chapter III], [6, 7, 19] for some references on infinitely differentiable semigroups generated by MLOs). We will use the following definition.

Definition 5.16. (see [36, Definition 2.2.1] for non-degenerate case)

(i) Suppose that $\mathcal{A}$ is an MLO in X. Let $\alpha \in (0,\pi]$, and let $(R(t))_{t\ge0}$ be an (a,k)-regularized C-resolvent family which do have $\mathcal{A}$ as a subgenerator. Then it is said that $(R(t))_{t\ge0}$ is an analytic (a,k)-regularized C-resolvent family of angle α, if there exists a function $\mathbf{R} : \Sigma_\alpha \to L(X)$ which satisfies that, for every $x \in X$, the mapping $z \mapsto \mathbf{R}(z)x$, $z \in \Sigma_\alpha$ is analytic as well as that:

(a) $\mathbf{R}(t) = R(t),\ t > 0$ and
(b) $\lim_{z\to0, z\in\Sigma_\gamma} \mathbf{R}(z)x = k(0)Cx$ for all $\gamma \in (0,\alpha)$ and $x \in X$.

(ii) Let $(R(t))_{t\ge0}$ be an analytic (a,k)-regularized C-resolvent family of angle $\alpha \in (0,\pi]$. Then it is said that $(R(t))_{t\ge0}$ is an exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family of angle α, resp. equicontinuous analytic (a,k)-regularized C-resolvent family of angle α, if for every $\gamma \in (0,\alpha)$, there exists $\omega_\gamma \ge 0$, resp. $\omega_\gamma = 0$, such that the family $\{e^{-\omega_\gamma \Re z}\mathbf{R}(z) : z \in \Sigma_\gamma\} \subseteq L(X)$ is equicontinuous. Since there is no risk for confusion, we will identify in the sequel $R(\cdot)$ and $\mathbf{R}(\cdot)$.

In the following example, we consider a time-fractional analogue of the linearized Benney-Luke equation in L^2-spaces and there we will meet some interesting examples of exponentially bounded, analytic fractional resolvent families of bounded operators whose angle of analyticity can be strictly greater than $\pi/2$; in our approach, we do not use neither multivalued linear operators nor relatively p-radial operators ([17], [73]). The method employed by G. A. Sviridyuk and V. E. Fedorov [73] for the usually considered Benney-Luke equation of first order can be very hepful for achieving the final conclusions stated in (i)-(ii), as well as for the concrete choice of the state space X_0 below (cf. also [38, Example 2.2.49, Example 2.2.53] for our recent study of fractional analogues of the abstract Barenblatt-Zheltov-Kochina equation in finite domains, where we have used the pure Laplace transform techniques from [42]).

Example 5.17. Suppose that $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary, and Δ is the Dirichlet Laplacian in $X := L^2(\Omega)$, acting with domain $H^2(\Omega) \cap H_0^1(\Omega)$. By $\{\lambda_k\}[= \sigma(\Delta)]$ we denote the eigenvalues of Δ in $L^2(\Omega)$ (recall that $0 < -\lambda_1 \leq -\lambda_2 \cdots \leq -\lambda_k \leq \cdots \to +\infty$ as $k \to \infty$; cf. [77, Section 5.6], [1, Section 6] and [73, Section 1.3] for more details) numbered in nonascending order with regard to multiplicities. By $\{\phi_k\} \subseteq C^\infty(\Omega)$ we denote the corresponding set of mutually orthogonal eigenfunctions. Then, for every $\zeta > 0$, we define the spectral fractional power $C_\zeta \in L(X)$ of $-\Delta$ by $C_\zeta \cdot := (-\Delta)^{-(\zeta)/2} \cdot := \sum_{k\geq 1} \langle \cdot, \phi_k\rangle (-\lambda_k)^{-(\zeta/2)}\phi_k$ (cf. [69] for more details). Then C_ζ is injective and $R(C) =: D((-\Delta)^{\zeta/2}) = \{f \in L^2(\Omega) : \sum_{k\geq 1} |\langle f, \phi_k\rangle|^2 (-\lambda_k)^\zeta < \infty\}$.

Let $\lambda \in \sigma(\Delta)$, let $0 < \eta \leq 2$, and let $\alpha,\ \beta > 0$. Consider the time-fractional analogue of the linearized Benney-Luke equation

$$
\begin{gathered}
(\lambda - \Delta)\mathbf{D}_t^\eta u(t,x) = \big(\alpha\Delta - \beta\Delta^2\big)u(t,x) + f(t,x), \quad t \geq 0,\ x \in \Omega, \\
\Big(\frac{\partial^k}{\partial t^k} u(t,x)\Big)_{t=0} = u_k(x), \quad x \in \Omega,\ 0 \leq k \leq \lceil \eta \rceil - 1, \\
u(t,x) = \Delta u(t,x) = 0, \quad t \geq 0,\ x \in \partial\Omega,
\end{gathered} \tag{5.25}
$$

for which is known that plays an important role in evolution modeling of some problems appearing in the theory of liquid filtration. Denote by X_0 the vector space of those functions from X that are orthogonal to the eigenfunctions $\phi_k(\cdot)$ for $\lambda_k = \lambda$. Then X_0 is a closed subspace of X, and therefore, becomes the Banach space equipped with the topology inherited by the X-norm (cf. [73, Example 5.3.1, Theorem 5.3.2] for the case $\eta = 1$). On the other hand, the operators $A := \alpha\Delta - \beta\Delta^2$ and $B := \lambda - \Delta$, acting with maximal domains, are closed in $L^2(\Omega)$. Set $\theta := \min((\pi/\eta) - (\pi/2), \pi)$. Using the Parseval equality, the asymptotic expansion formula [5, (1.28)] and an elementary argumentation, we can simply prove that the operator family $(T_\eta(z))_{z\in\Sigma_\theta\cup\{0\}} \subseteq L(X_0)$, given by

$$
t \mapsto T_\eta(z)\cdot := \sum_{k|\lambda_k\neq\lambda} E_\eta\Big(\frac{\alpha\lambda_k - \beta\lambda_k^2}{\lambda - \lambda_k} z^\eta\Big)\langle \cdot, \phi_k\rangle\phi_k, \quad z \in \Sigma_\theta \cup \{0\},
$$

is well-defined, provided $\eta \in (0,2)$. If $\eta = 2$, then we define $(T_2(t))_{t\geq 0} \subseteq L(X_0)$ in the same way as above; since $E_2(z^2) = \cosh(z)$, we have that, for every $t \geq 0$,

$$
T_2(t)\cdot = \frac{1}{2}\sum_{k|\lambda_k\neq\lambda}\Big[e^{it\big((\beta\lambda^2-\alpha\lambda_k)/(\lambda-\lambda_k)\big)^{1/2}} - e^{-it\big((\beta\lambda^2-\alpha\lambda_k)/(\lambda-\lambda_k)\big)^{1/2}}\Big]\langle \cdot, \phi_k\rangle\phi_k,
$$

and that $(T_2(t))_{t\geq 0}$ is bounded in the uniform operator norm. Differentiating $T_2(t)$ term by term, it can be easily seen that the mapping $t\mapsto T_2(t)f$, $t\geq 0$ is continuously differentiable for any $f\in D((-\Delta)^{1/2})\cap X_0$, and therefore, continuous. Since $D((-\Delta)^{1/2})\cap X_0$ is dense in X_0 and $(T_2(t))_{t\geq 0}$ is bounded, the usual arguments shows that $(T_2(t))_{t\geq 0}$ is strongly continuous. Now we can proceed as in the proof of Theorem 4.8 in order to see that, for every $\eta\in(0,2)$, $(T_\eta(t))_{t\geq 0}$ is an exponentially bounded, analytic (g_η,I)-regularized resolvent family of angle θ. A straightforward computation shows that, for every $\eta\in(0,2]$, the integral generator $\mathcal{A}$ of $(T_\eta(t))_{t\geq 0}$ is a closed single-valued operator in X_0, given by $\mathcal{A}=\{(f,g)\in X_0\times X_0:(\lambda-\lambda_k)\langle g,\phi_k\rangle=(\alpha\lambda_k-\beta\lambda_k^2)\langle f,\phi_k\rangle$ for all $k\in\mathbb{N}$ with $\lambda_k\neq\lambda\}$; in particular, $\mathcal{A}$ is an extension of the operator $B^{-1}A_{|X_0}$. It is also clear that $(T_\eta(t))_{t\geq 0}$ is a mild (g_η,I)-existence family generated by $\mathcal{A}$. Keeping in mind the identity [5, (1.25)], we can carry out a direct computation showing that the homogeneous counterpart of problem (5.25) $\equiv$ (5.25) with $x_j=0$ for $1\leq j\leq\lceil\zeta\rceil-1$, has an exponentially bounded pre-solution $u_{h,0}(t)=T_\eta(t)x_0$, $t\geq 0$ for any $x_k\in D(A)\cap X_0$ $(0\leq k\leq\lceil\eta\rceil-1)$, which seems to be an optimal result in the case that $\eta\leq 1$. Concerning the homogeneous counterpart of problem (5.25) with $x_0=0$, its solution $u_{h,1}(t)$ has to be find in the form $u_{h,1}(t)=\int_0^t T_\eta(s)x_1\,ds$, $t\geq 0$.

Consider first the case $\eta\in(1,2)$. Then for each $k\in\mathbb{N}$ with $\lambda_k\neq\lambda$, we have

$$\begin{aligned}
&\frac{d^2}{dt^2}\Big[g_{2-\eta}*\Big(E_\eta\Big(\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}\cdot^\eta\Big)-1\Big)\Big](t)\\
&=\mathbf{D}_t^\eta E_\eta\Big(\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}t^\eta\Big)\\
&=\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}E_\eta\Big(\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}t^\eta\Big),\quad t\geq 0.
\end{aligned}$$

On the other hand, expanding the function $E_\eta(\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}\cdot^\eta)-1$ in a power series we obtain that

$$\frac{d}{dt}\Big[g_{2-\eta}*\Big(E_\eta\Big(\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}\cdot^\eta\Big)-1\Big)\Big](t)=t\sum_{n=0}^\infty\frac{\big(\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}t^\eta\big)^{n+1}t^{n\eta}}{\Gamma(n\eta+2)},\ t\geq 0.$$

The previous two equalities together imply that $\frac{d}{dt}[g_{2-\eta}*(E_\eta(\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}\cdot^\eta)-1)](t)=\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}\int_0^t E_\eta(\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}s^\eta)\,ds$, $t\geq 0$ and

$$\begin{aligned}
\mathbf{D}_t^\eta\big[g_1*T_\eta(\cdot)x_1\big](t)&=\sum_{k|\lambda_k\neq\lambda}\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}\int_0^t E_\eta\Big(\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}s^\eta\Big)\,ds\Big\langle x_1,\phi_k\Big\rangle\phi_k\\
&=\sum_{k|\lambda_k\neq\lambda}\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}tE_{\eta,2}\Big(\frac{\alpha\lambda_k-\beta\lambda_k^2}{\lambda-\lambda_k}t^\eta\Big)\Big\langle x_1,\phi_k\Big\rangle\phi_k,\ t\geq 0.
\end{aligned}$$

Using again the asymptotic expansion formula [5, (1.28)], we obtain that the above series converges for any $x_1\in X_0$ and belongs to $D(B)$ provided, in addition, that $x_1\in D(B)\cap X_0$. In this case, the equality $B\mathbf{D}_t^\eta u_{h,1}(t)=Au_{h,1}(t)$, $t\geq 0$ readily follows, so that the function $u_h(t):=u_{h,0}(t)+u_{h,1}(t)$, $t\geq 0$ is a pre-solution of problem (1.3) provided that $x_0\in D(A)\cap X_0$ and $x_1\in D(B)\cap X_0$ (with $X=Y=L^2(\Omega)$ in Definition 4.1(iii)); furthermore, the mappings $t\mapsto u_h(t)\in L^2(\Omega)$, $t>0$ and $t\mapsto Bu_h(t)\in L^2(\Omega)$, $t>0$ can be analytically extended to the sector Σ_θ. The situation is slightly different in the case that $\eta = 2$ since we cannot use the formula [5, (1.28)]; then a simple computation shows that, formally, for every $t \geq 0$,

$$\begin{aligned} Bu''_{h,1}(t) &= Au_{h,1}(t) \\ &= \frac{1}{2} \sum_{k|\lambda_k \neq \lambda} \Big[i\big((\beta\lambda^2 - \alpha\lambda_k)/(\lambda - \lambda_k)\big)^{1/2} e^{it((\beta\lambda^2-\alpha\lambda_k)/(\lambda-\lambda_k))^{1/2}} \\ &\quad - i\big((\beta\lambda^2 - \alpha\lambda_k)/(\lambda - \lambda_k)\big)^{1/2} e^{-it((\beta\lambda^2-\alpha\lambda_k)/(\lambda-\lambda_k))^{1/2}} \Big] \langle x_1, \phi_k\rangle \phi_k. \end{aligned}$$

Hence, the function $u_h(t) := u_{h,0}(t)+u_{h,1}(t)$, $t \geq 0$ is a pre-solution of problem (1.3) with $x_0 \in D(A) \cap X_0$ and $x_1 \in D((-\Delta)^{3/2}) \cap X_0$. The range of any pre-solution of problem (5.25) with $f = 0$ must be contained in X_0, so that the uniqueness of solutions of problem (5.25) follows from its linearity and Proposition 5.8(ii).

Before considering the inhomogeneous problem (5.25), we would like to observe that the assumptions $(x, y) \in \mathcal{A}$ and $x \in D(A)$ imply $(x, y) \in B^{-1}A_{|X_0}$. Keeping in mind this remark, Theorem 2.3, as well as the fact that the assertion of [68, Proposition 2.1(iii)] admits a reformulation in our framework, we can simply prove that for any function $h \in W^{1,1}_{\rm loc}([0,\infty) : X_0)$ satisfying that

$$t \mapsto \sum_{k|\lambda_k \neq \lambda} \big(\alpha\lambda_k - \beta\lambda_k^2\big)\Big\langle \frac{d}{dt}(g_\eta * h)(t), \phi_k\Big\rangle \phi_k \in L^1_{\rm loc}([0,\infty) : X_0), \tag{5.26}$$

the function $u_{Bh}(t) := \int_0^t T_\eta(t-s)\frac{d}{ds}(g_\eta * h)\, ds$, $t \geq 0$ is a solution of problem (5.25) with $f = Bh$. On the other hand, the operator B annihilates any function from $\mathrm{span}\{\phi_k : k|\lambda = \lambda_k\}$ so that the function $t \mapsto \sum_{k|\lambda_k=\lambda} \frac{\langle f(t),\phi_k\rangle}{\beta\lambda_k^2-\alpha\lambda_k}\phi_k$, $t \geq 0$ is a pre-solution of problem (5.25) with $f = \sum_{k|\lambda_k=\lambda}\langle f(\cdot), \phi_k\rangle\phi_k$, provided that the following condition holds

(A1) : $\mathbf{D}_t^\eta\langle f(t), \phi_k\rangle$ exists in $L^2(\Omega)$ for $k|\lambda = \lambda_k$, $\langle x_0, \phi_k\rangle = 0$ for $k|\lambda \neq \lambda_k$, $\langle x_1, \phi_k\rangle = 0$ for $k|\lambda \neq \lambda_k$, $1 < \eta \leq 2$, $\langle x_0, \phi_k\rangle = \frac{\langle f(0),\phi_k\rangle}{\beta\lambda_k^2-\alpha\lambda_k}$ for $k|\lambda = \lambda_k$, and $\langle x_1, \phi_k\rangle = \frac{\langle f'(0),\phi_k\rangle}{\beta\lambda_k^2-\alpha\lambda_k}$ for $k|\lambda = \lambda_k$, $1 < \eta \leq 2$.

Summa summarum, we have the following:

(i) $0 < \eta < 2$: Suppose that $x_0 \in D(A) \cap X_0$, $x_1 \in D(B) \cap X_0$, if $\eta > 1$, $\sum_{k|\lambda_k\neq\lambda} \frac{\langle f(\cdot),\phi_k\rangle}{\lambda-\lambda_k}\phi_k = h \in W^{1,1}_{\rm loc}([0,\infty) : X_0)$ satisfies (5.26), and the condition (A1) holds. Then there exists a unique pre-solution of problem (5.25).

(ii) $\eta = 2$: Suppose $x_1 \in D((-\Delta)^{3/2}) \cap X_0$ and the remaining assumptions from (i) hold. Then there exists a unique pre-solution of problem (5.25).

Observe also that our results on the well-posedness of fractional analogue of the Benney-Luke equation, based on a very simple approach, are completely new provided that $\eta > 1$, as well as that we have obtained some new results on the well-posedness of the inhomogeneous Cauchy problem $P_{\eta,f}$ in the case that $\eta < 1$ (cf. [22, Theorem 4.2] for the first result in this direction).

The following theorem can be deduced by making use of the argumentation contained in the proof of [39, Theorem 2.16]. Here we would like to observe that the equality $R_{\lambda,\mu} = 0$, stated on [39, p. 12, l. 4], can be proved by taking the Laplace transform of term appearing on [39, p. 12, l. 1-2] in variable μ, and by using the strong analyticity of mapping $\lambda \mapsto F(\lambda) \in L(X)$, $\lambda \in N$, along with the equality $R_{\lambda,\mu} = 0$ for $\Re\lambda > \omega$, $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$ (the repeated use of identity [39, (2.30)] on [39, p. 12, l.4] is wrong and makes a circulus vitiosus):

Theorem 5.18. *(see* [36, Theorem 2.2.4] *for non-degenerate case) Suppose that* $\alpha \in (0, \pi/2]$, $\operatorname{abs}(k) < \infty$, $\operatorname{abs}(|a|) < \infty$, *and* $\tilde{k}(\lambda)$ *can be analytically continued to a function* $g : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to \mathbb{C}$, *where* $\omega \geq \max(0, \operatorname{abs}(k), \operatorname{abs}(|a|))$. *Suppose, further, that* $\mathcal{A}$ *is a closed subgenerator of an analytic* (a, k)*-regularized* C*-resolvent family* $(R(t))_{t\geq 0}$ *of angle* α *satisfying that the family* $\{e^{-\omega z}R(z) : z \in \Sigma_\gamma\} \subseteq L(X)$ *is equicontinuous for all angles* $\gamma \in (0, \alpha)$, *as well as that equation* (5.1) *holds for each* $y = x \in X$, *with* $R_1(\cdot)$ *and* C_1 *replaced therein by* $R(\cdot)$ *and* C, *respectively. Set*

$$N := \big\{\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\alpha} : g(\lambda) \neq 0\big\}.$$

Then N *is an open connected subset of* $\mathbb{C}$. *Furthermore, the existence of an analytic function* $\hat{a} : N \to \mathbb{C}$ *such that* $\hat{a}(\lambda) = \tilde{a}(\lambda)$, $\Re\lambda > \omega$ *implies that the operator* $I - \hat{a}(\lambda)\mathcal{A}$ *is injective for every* $\lambda \in N$, $R(C) \subseteq R(I - \hat{a}(\lambda)C^{-1}\mathcal{A}C)$ *for every* $\lambda \in N_1 := \{\lambda \in N : \hat{a}(\lambda) \neq 0\}$, *the operator* $(I - \hat{a}(\lambda)C^{-1}\mathcal{A}C)^{-1}C \in L(X)$ *is single-valued* ($\lambda \in N_1$), *the family*

$$\Big\{(\lambda - \omega)g(\lambda)\big(I - \hat{a}(\lambda)C^{-1}\mathcal{A}C\big)^{-1}C : \lambda \in\ N_1 \cap\ (\omega + \Sigma_{\frac{\pi}{2}+\gamma_1})\Big\} \subseteq L(X)$$

is equicontinuous for every angle $\gamma_1 \in (0, \alpha)$, *the mapping*

$$\lambda \mapsto \big(I - \hat{a}(\lambda)C^{-1}\mathcal{A}C\big)^{-1}Cx,\ \lambda \in N_1 \text{ is analytic for every } x \in X,$$

and

$$\lim_{\lambda\to+\infty, \tilde{a}(\lambda)\tilde{k}(\lambda)\neq 0} \lambda\tilde{k}(\lambda)\big(I - \tilde{a}(\lambda)\mathcal{A}\big)^{-1}Cx = R(0)x,\ x \in X.$$

Keeping in mind Lemma 2.2, Theorem 2.3 and Theorem 5.5, we can repeat almost literally the proof of [36, Theorem 2.2.5] in order to see that the following result holds.

Theorem 5.19. *Assume that* $\mathcal{A}$ *is a closed MLO in* X, $C\mathcal{A} \subseteq \mathcal{A}C$, $\alpha \in (0, \pi/2]$, $\operatorname{abs}(k) < \infty$, $\operatorname{abs}(|a|) < \infty$ *and* $\omega \geq \max(0, \operatorname{abs}(k), \operatorname{abs}(|a|))$. *Assume, further, that for every* $\lambda \in \mathbb{C}$ *with* $\Re\lambda > \omega$ *and* $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, *the operator* $I - \tilde{a}(\lambda)\mathcal{A}$ *is injective with* $R(C) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$.

If there exist a function $q : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X)$ *and an operator* $D \in L(X)$ *such that, for every* $x \in X$, *the mapping* $\lambda \mapsto q(\lambda)x$, $\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\alpha}$ *is analytic as well as that*

$$q(\lambda)x = \tilde{k}(\lambda)\big(I - \tilde{a}(\lambda)\mathcal{A}\big)^{-1}Cx,\ \Re\lambda > \omega,\ \tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0,\ x \in X,$$

for every $\gamma \in (0, \alpha)$, *the family* $\{(\lambda - \omega)q(\lambda) : \lambda \in \omega + \Sigma_{\frac{\pi}{2}+\gamma}\} \subseteq L(X)$ *is equicontinuous and*

$$\lim_{\lambda\to+\infty} \lambda q(\lambda)x = Dx,\ x \in X,\ \text{if } \overline{D(\mathcal{A})} \neq X,$$

then $\mathcal{A}$ *is a subgenerator of an exponentially equicontinuous, analytic* (a, k)*-regularized* C*-resolvent family* $(R(t))_{t\geq 0}$ *of angle* α *satisfying that* $R(z)\mathcal{A} \subseteq \mathcal{A}R(z)$, $z \in \Sigma_\alpha$, *the family* $\{e^{-\omega z}R(z) : z \in \Sigma_\gamma\} \subseteq L(X)$ *is equicontinuous for all angles* $\gamma \in (0, \alpha)$, *as well as that equation* (5.1) *holds for each* $y = x \in X$, *with* $R_1(\cdot)$ *and* C_1 *replaced therein by* $R(\cdot)$ *and* C, *respectively.*

Suppose that $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary. As explained by Falaleev and Orlov in [16], the equation

$$\begin{gathered}(\alpha-\Delta)u_{tt}=\beta\Delta u_t+\Delta u+\int_0^t g(t-s)\Delta u(s,x)\,ds, \quad t>0,\ x\in\Omega;\\ u(0,x)=\phi(x), \quad u_t(0,x)=\psi(x),\end{gathered} \tag{5.27}$$

where $g \in L^1_{\rm loc}([0,\infty))$, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}\setminus\{0\}$, appears in some models of nonlinear viscoelasticity provided that $n = 3$. In the following illustrative example, we will consider the well-posedness of equation (5.27) following the approaches from [42] and this article.

Example 5.20. Let Δ be the Dirichlet Laplacian in $X := L^2(\Omega)$, acting with domain $H^2(\Omega)\cap H^1_0(\Omega)$. As in Example 5.17, we will denote by $\{\lambda_k\}$ $[= \sigma(\Delta)]$ the eigenvalues of Δ in $L^2(\Omega)$ numbered in nonascending order with regard to multiplicities; $\{\phi_k\} \subseteq C^\infty(\Omega)$ denotes the corresponding set of mutually orthogonal eigenfunctions. Integrating (5.27) twice with the respect to the time-variable t, we obtain the associated integral equation

$$\begin{aligned}(\alpha-\Delta)u(t)&=(\alpha+(\beta-1)\Delta)\phi(x)+t(\alpha-\Delta)\psi+\beta\Delta\big(g_1*u\big)(t)\\ &\quad+\Delta\big(g_2*u\big)(t)+\Delta\big(g_2*g*u\big)(t), \quad t\geq 0.\end{aligned} \tag{5.28}$$

Set $B := \alpha - \Delta$, $A_2 := \beta\Delta$, $A_1 = A_0 := \Delta$ (acting with the Dirichlet boundary conditions), $a_2(t) := g_1(t)$, $a_1(t) := g_2(t)$, $a_0(t) := (g_2 * g)(t)$, and

$$\mathcal{P}_\lambda := \frac{\lambda^2+\beta\lambda+\tilde{g}(\lambda)+1}{\lambda^2}\Big[\frac{\alpha\lambda^2}{\lambda^2+\beta\lambda+\tilde{g}(\lambda)+1}-\Delta\Big].$$

Suppose that $\alpha = \lambda_{k_0} \in \sigma(\Delta)$ for some $k_0 \in \mathbb{N}$ and the function $g(t)$ is Laplace transformable (in [42, Example 3.15], we have considered the case $\alpha > 0$, with the state space being $L^p(\Omega)$ for some $1 \leq p < \infty$). Then there exist constants $M \geq 1$ and $\omega \geq 0$ such that $|\int_0^t g(s)\,ds| \leq Me^{\omega t}$, $t \geq 0$ and

$$\lambda\int_0^\infty e^{-\lambda t}\int_0^t g(s)\,ds\,dt=\int_0^\infty e^{-\lambda t}g(t)\,dt,$$

and $\lambda > \omega$, which simply implies that the set $\{\tilde{g}(\lambda) : \lambda > \omega + 1\}$ is bounded.

Define $D : L^2(\Omega) \to L^2(\Omega)$ by $Df := (-1)\beta^{-1}\sum_{k=1}^\infty\langle\phi_k, f\rangle\phi_k$, $f \in L^2(\Omega)$. Using Parseval's equality, it can be simply verified that $D, BD \in L(L^2(\Omega))$; furthermore, $\|R(\lambda : \Delta)\| = O\big(|\alpha-\lambda|^{-1}\big)$ as $\lambda \to \alpha$ (see [44, Example, pp. 57-58]). Using the resolvent equation and these facts, we obtain the existence of a sufficiently large real number $R > 0$ such that $\mathcal{P}_\lambda^{-1} \in L(L^2(\Omega))$ for $|\lambda| \geq R$, as well as that

$$\begin{gathered}|\lambda|^{-2}\Big[\|\mathcal{P}_\lambda^{-1}\|+\|B\mathcal{P}_\lambda^{-1}\|+\sum_{j=0}^{2}\|\tilde{a_j}(\lambda)A_j\mathcal{P}_\lambda^{-1}\|\Big]\leq M, \quad |\lambda|\geq R,\\ \lim_{|\lambda|\to\infty}\lambda^{-1}\mathcal{P}_\lambda^{-1}f=Df, \quad \lim_{|\lambda|\to\infty}\lambda^{-1}B\mathcal{P}_\lambda^{-1}f=BDf,\\ \lim_{|\lambda|\to\infty}\lambda^{-1}\tilde{a_j}(\lambda)\mathcal{P}_\lambda^{-1}f=0, \quad 0\leq j\leq 2 \quad (f\in L^2(\Omega)).\end{gathered} \tag{5.29}$$

Using [42, Theorem 3.9], we obtain that there exists an exponentially bounded once integrated I-existence family $(E_1(t))_{t\geq 0}$ for (5.28), in the sense of [42, Definition 3.8(i)], satisfying additionally that for each $f \in L^2(\Omega)$ the mappings $t \mapsto E_1(t)f$, $t > 0$, $t \mapsto BE_1(t)f$, $t > 0$ and $t \mapsto A_j(a_j * E_1)(t)f$, $t > 0$ can be analytically extended to the sector $\Sigma_{\pi/2}$; furthermore, $(E_1(t))_{t\geq 0}$ is an exponentially bounded once integrated I-uniqueness family for (5.28), in the sense of [42, Definition 3.8(ii)]. Therefore, for every $\phi,\ \psi \in H^2(\Omega)\cap H_0^1(\Omega)$, there exists a unique strong solution of the associated once integrated problem (5.28)

$$\begin{aligned}(\alpha-\Delta)u(t) &= t(\alpha+(\beta-1)\Delta)\phi(x)+\frac{t^2}{2}(\alpha-\Delta)\psi+\beta\Delta\big(g_1*u\big)(t)\\ &\quad +\Delta\big(g_2*u\big)(t)+\Delta\big(g_2*g*u\big)(t),\quad t\geq 0,\end{aligned}\tag{5.30}$$

given by $u(t)=E_1(t)(\alpha+(\beta-1)\Delta)\phi+\int_0^t E_1(s)(\alpha-\Delta)\psi\,ds$, $t\geq 0$. On the other hand, equation (5.29) taken together with the equality $\lim_{|\lambda|\to\infty}\lambda^{-1}B\mathcal{P}_\lambda^{-1}f = BDf$, Theorem 5.19 and Remark 4.2(v) implies that for each $\theta\in(-\pi,\pi]$ the MLO $e^{i\theta}AB^{-1}$ generates an exponentially bounded, analytic once integrated $(b+g_2(t)+(g_2*g)(t),I)$-regularized resolvent family $(E_{1,B}(t)\equiv BE_1(t))_{t\geq 0}$ of angle $\pi/2$. Since [36, Theorem 2.1.29(ii)] holds in our framework, this immediately yields some results on the existence and uniqueness of analytical (possible, entire, cf. [44, Theorem 2.2]) solutions of the problem (5.30) with the term $t(\alpha+(\beta-1)\Delta)\phi+\frac{t^2}{2}(\alpha-\Delta)\psi$ replaced by a general inhomogeneity $f(t)$.

The classes of exponentially equicontinuous, analytic (a,k)-regularized C_1-existence families and (a,k)-regularized C_2-uniqueness families can be introduced and analyzed, as well. For the sequel, we need the following notion.

Definition 5.21. Let $X=Y$, and let $\mathcal{A}$ be a subgenerator of a C_1-existence family $(R_1(t))_{t\geq 0}$ (cf. Definition 5.1(i) with $a(t)\equiv 1$ and $k(t)\equiv 1$). Then $(R_1(t))_{t\geq 0}$ is said to be entire if, for every $x\in X$, the mapping $t\mapsto R_1(t)x$, $t\geq 0$ can be analytically extended to the whole complex plane.

Using the arguments in the proof of [41, Theorem 3.15], we can deduce the following result.

Theorem 5.22. *Suppose $r\geq 0$, $\theta\in(0,\pi/2)$, $\mathcal{A}$ is a closed MLO and $-\mathcal{A}$ is a subgenerator of an exponentially equicontinuous, analytic r-times integrated C-semigroup $(S_r(t))_{t\geq 0}$ of angle θ. Then there exists an operator $C_1\in L(X)$ such that $\mathcal{A}$ is a subgenerator of an entire C_1-existence family in X.*

Remark 5.23. (i) It ought to be observed that we do not require the injectivity of operator C_1 here. The operators $T_\alpha(z)$ and $S_{\alpha,z_0}(z)$, appearing in the proof of [41, Theorem 3.15], annulate on the subspace $\mathcal{A}0$.

(ii) Theorem 5.22 is closely linked with the assertions of [40, Theorem 2.1, Theorem 2.2]. These results can be extended to abstract degenerate fractional differential inclusions, as well.

Example 5.24. In a great number of research papers, many authors have considered infinitely differentiable semigroups generated by multivalued linear operators of form AB^{-1} or $B^{-1}A$, where the operators A and B satisfy the condition [17, (3.14)], or its slight modification, with certain real constants $0<\beta\leq\alpha\leq 1$, $\gamma\in\mathbb{R}$ and $c,\ C>0$ (in our notation, we have $A=L$ and $B=M$). The validity of this condition with $\alpha=1$ (see e.g. [17, Example 3.3, 3.6]) immediately implies by Theorem 5.19 and Remark 4.2(v) that the operator AB^{-1} generates an exponentially bounded, analytic σ-times integrated semigroup of angle $\Sigma_{\operatorname{arcctan}(1/c)}$, provided that $\sigma>1-\beta$; in the concrete situation of [17, Example 3.4, 3.5], the above holds with the operator AB^{-1} replaced by $B^{-1}A$.

Unfortunately, this fact is not sufficiently enough for taking up a fairly complete study of the abstract degenerate Cauchy problems that are subordinated to those appearing in the above-mentioned examples and, concerning this question, we will only want to mention that the subordination fractional operator families can be constructed since the semigroups considered in [17, Chapter III] have a removable singularity at zero (cf. the proof of [5, Theorem 3.1], [47] and the forthcoming monograph [38] for more details). On the other hand, from the point of view of possible applications of Theorem 5.22, it is very important to know that the operators AB^{-1} or $B^{-1}A$ generate exponentially bounded, analytic integrated semigroups. This enables us to consider the abstract degenerate Cauchy problems that are backward to those appearing in [17, Examples 3.3–3.6]. For example, we can consider the following modification of the backward Poisson heat equation in the space $L^p(\Omega)$:

$$\begin{gathered}\frac{\partial}{\partial t}[m(x)v(t,x)] = -\Delta v + bv, \quad t \geq 0,\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega,\end{gathered} \tag{5.31}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$ and $1 < p < \infty$. Let B be the multiplication in $L^p(\Omega)$ with $m(x)$, and let $A = \Delta - b$ act with the Dirichlet boundary conditions. Then Theorem 5.22 implies that there exists an operator $C_1 \in L(L^p(\Omega))$ such that $\mathcal{A} = -AB^{-1}$ is a subgenerator of an entire C_1-existence family; hence, for every $u_0 \in R(C_1)$, the problem (5.31) has a unique solution $t \mapsto u(t)$, $t \geq 0$ which can be extended entirely to the whole complex plane. Furthermore, it can be proved that the set of all initial values u_0 for which there exists a unique solution of problem (5.31) is dense in $L^p(\Omega)$ provided that there exists a constant $d > 0$ such that $|m(x)| \geq d$ a.e. $x \in \Omega$.

In the following example, we consider the existence and uniqueness of solutions of abstract degenerate relaxation Cauchy problems that are not subordinated to those of first order.

Example 5.25. It is clear that the examples presented in [17, Chapter III] can serve one for consideration of a wide class of abstract degenerate relaxation equations that are not subordinated to the problems of first order (a fairly complete analysis of such equations is quite non-trivial and we shall skip all related details for convenience): Suppose that the condition [17, (3.1)] holds with certain real constants $0 < \beta \leq \alpha \leq 1$, $c,\ M > 0$, as well as that $\theta \in (\pi/2, 0)$, $\zeta \in (0,1)$ and $\frac{\pi}{2} > \pi - \arctan\frac{1}{c} + \theta > \frac{1}{2}\pi\zeta$. Then $\Sigma_{\pi - \arctan\frac{1}{c} + \theta} \subseteq \rho(e^{i\theta}\mathcal{A})$ and, in general, $\rho(e^{i\theta}\mathcal{A})$ does not contain any right half-plane. An application of Theorem 5.19 shows that the operator $e^{i\theta}\mathcal{A}$ generates an exponentially bounded, analytic (g_ζ, g_{r+1})-regularized resolvent family of angle $\theta' := \min((\pi - \arctan(1/c) + \theta - (\pi\zeta/2))/\zeta, \pi/2)$, where $r > \zeta(1-\beta)$, if $\mathcal{A}$ is not densely defined, and $r = \zeta(1-\beta)$, otherwise.

Suppose now that $x \in E$, $1 - \zeta > \eta > 1 - \zeta\beta$, $\delta > 0$, $0 < \gamma < \theta'$, $t > 0$ is fixed temporarily, $\Gamma_1 := \{re^{i((\pi/2)+\gamma)} : r \geq t^{-1}\} \cup \{t^{-1}e^{i\theta} : \theta \in [0, (\pi/2) + \gamma]\}$, $\Gamma_2 := \{re^{-i((\pi/2)+\gamma)} : r \geq t^{-1}\} \cup \{t^{-1}e^{i\theta} : \theta \in [-(\pi/2) - \gamma, 0]\}$ and $\Gamma := \Gamma_1 \cup \Gamma_2$ is oriented counterclockwise. Define $u(0) := 0$ and

$$u(t) := \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{-\eta}\Big(\lambda^\zeta - e^{i\theta}\mathcal{A}\Big)^{-1} x\, d\lambda.$$

Arguing as in [1, Theorem 2.6.1, Theorem 2.6.4], it readily follows that $u \in C([0,\infty) : E)$, $\|u(t)\| = O(t^{\eta+\zeta\beta-1})$, $t \geq 0$ and that the mapping $t \mapsto u(t)$, $t > 0$ can be analytically extended to the sector $\Sigma_{\theta'}$. Keeping in mind Theorem 2.3 and Theorem 2.4(i), we obtain that there exists a continuous section $t \mapsto u_{\mathcal{A},\theta,\zeta}(t)$, $t > 0$ of the multivalued mapping $t \mapsto e^{i\theta}\mathcal{A}(g_\zeta * u)(t)$, $t > 0$, with the meaning clear, such that

$$u(t) = u_{\mathcal{A},\theta,\zeta}(t) + g_{\eta+\zeta}(t)x, \quad t > 0.$$

Observe, finally, that the Riemann-Liouville fractional derivative $D_t^\zeta u(t)$ need not be defined here.

In the sequel, we need the following notion. Suppose that a sequence $(M_n)_{n\in\mathbb{N}_0}$ of positive real numbers satisfies $M_0 = 1$, as well as the following conditions:

(A2) $M_p^2 \leq M_{p+1}M_{p-1},\ p \in \mathbb{N}$,

(A2) $M_p \leq AH^p \min_{p_1,p_2\in\mathbb{N},p_1+p_2=p} M_{p_1}M_{p_2}$, $n \in \mathbb{N}$, for some $A > 1$ and $H > 1$,

(A2)' $\sum_{p=1}^\infty \frac{M_{p-1}}{M_p} < \infty$.

Set

$$\omega_L(t) := \sum_{n=0}^\infty \frac{t^n}{M_n},\ t \geq 0.$$

The most important results concerning differential properties of non-degenerate (a,k)-regularized C-resolvent families remain true, with almost minimal reformulations, in our new setting. The proofs of following extensions of [36, Theorem 2.2.15, Theorem 2.2.17] are omitted.

Theorem 5.26. *Suppose that $\mathcal{A}$ is a closed MLO in X,* $\operatorname{abs}(k) < \infty$, $\operatorname{abs}(|a|) < \infty$, *$r \geq -1$ and there exists $\omega \geq \max(0, \operatorname{abs}(k), \operatorname{abs}(|a|))$ such that, for every $z \in \{\lambda \in \mathbb{C} : \Re\lambda > \omega,\ \tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0\}$, we have that the operator $I - \tilde{a}(z)\mathcal{A}$ is injective and $R(C) \subseteq R(I - \tilde{a}(z)\mathcal{A})$. If, additionally, for every $\sigma > 0$, there exist $C_\sigma > 0$ and an open neighborhood $\Omega_{\sigma,\omega}$ of the region*

$$\Lambda_{\sigma,\omega} := \big\{\lambda \in \mathbb{C} : \Re\lambda \leq \omega,\ \Re\lambda \geq -\sigma \ln|\Im\lambda| + C_\sigma\big\} \cup \{\lambda \in \mathbb{C} : \Re\lambda \geq \omega\},$$

and a function $h_\sigma : \Omega_{\sigma,\omega} \to L(X)$ such that, for every $x \in X$, the mapping $\lambda \mapsto h_\sigma(\lambda)x$, $\lambda \in \Omega_{\sigma,\omega}$ is analytic as well as that $h_\sigma(\lambda) = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}C$, $\Re\lambda > \omega$, $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, and that the family $\{|\lambda|^{-r}h_\sigma(\lambda) : \lambda \in \Lambda_{\sigma,\omega}\}$ is equicontinuous, then, for every $\zeta > 1$, $\mathcal{A}$ is a subgenerator of an exponentially equicontinuous $(a, k * g_{\zeta+r})$-regularized C-resolvent family $(R_\zeta(t))_{t\geq0}$ satisfying that the mapping $t \mapsto R_\zeta(t)$, $t > 0$ is infinitely differentiable in $L(X)$.

Theorem 5.27. *Let $(M_n)_{n\in\mathbb{N}_0}$ satisfy (M.1), (M.2) and (M.3)'.*

(i) Suppose that $\operatorname{abs}(k) < \infty$, $\operatorname{abs}(|a|) < \infty$, *$\mathcal{A}$ is a closed subgenerator of a (local) (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$,*
$\omega > \max(0, \operatorname{abs}(k), \operatorname{abs}(|a|))$ and $m \in \mathbb{N}$. Denote, for every $\varepsilon \in (0,1)$ and a corresponding $K_\varepsilon > 0$,

$$F_{\varepsilon,\omega} := \big\{\lambda \in \mathbb{C} : \Re\lambda \geq -\ln \omega_L\big(K_\varepsilon|\Im\lambda|\big) + \omega\big\}.$$

Assume that, for every $\varepsilon \in (0,1)$, there exist $K_\varepsilon > 0$, an open neighborhood $O_{\varepsilon,\omega}$ of the region $G_{\varepsilon,\omega} := \{\lambda \in \mathbb{C} : \Re\lambda \geq \omega,\ \tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0\} \cup \{\lambda \in F_{\varepsilon,\omega} : \Re\lambda \leq \omega\}$, a mapping $h_\varepsilon : O_{\varepsilon,\omega} \to L(E)$ and analytic mappings $f_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$, $g_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$ such that:

(a) *$f_\varepsilon(\lambda) = \tilde{k}(\lambda)$, $\Re\lambda > \omega$; $g_\varepsilon(\lambda) = \tilde{a}(\lambda)$, $\Re\lambda > \omega$,*

(b) *for every* $\lambda \in F_{\varepsilon,\omega}$, *the operator* $I - g_\varepsilon(\lambda)\mathcal{A}$ *is injective and* $R(C) \subseteq R(I - g_\varepsilon(\lambda)\mathcal{A})$,

(c) *for every* $x \in X$, *the mapping* $\lambda \mapsto h_\varepsilon(\lambda)x$, $\lambda \in G_{\varepsilon,\omega}$ *is analytic,* $h_\varepsilon(\lambda) = f_\varepsilon(\lambda)(I - g_\varepsilon(\lambda)\mathcal{A})^{-1}C$, $\lambda \in G_{\varepsilon,\omega}$,

(d) *the family* $\{(1 + |\lambda|)^{-m}e^{-\varepsilon|\Re\lambda|}h_\varepsilon(\lambda) : \lambda \in F_{\varepsilon,\omega},\ \Re\lambda \leq \omega\} \subseteq L(X)$ *is equicontinuous and the family* $\{(1+|\lambda|)^{-m}h_\varepsilon(\lambda) : \lambda \in \mathbb{C},\ \Re\lambda \geq \omega\} \subseteq L(X)$ *is equicontinuous.*

Then the mapping $t \mapsto R(t)$, $t \in (0, \tau)$ *is infinitely differentiable in* $L(X)$ *and, for every compact set* $K \subseteq (0, \tau)$, *there exists* $h_K > 0$ *such that the set* $\{\frac{h_K^n \frac{d^n}{dt^n} R(t)}{M_n} : t \in K,\ n \in \mathbb{N}_0\}$ *is equicontinuous.*

(ii) Suppose that $\operatorname{abs}(k) < \infty$, $\operatorname{abs}(|a|) < \infty$, $\mathcal{A}$ *is a closed subgenerator of a (local)* (a, k)*-regularized* C*-resolvent family* $(R(t))_{t\in[0,\tau)}$,
$\omega > \max(0, \operatorname{abs}(k), \operatorname{abs}(|a|))$ *and* $m \in \mathbb{N}$. *Denote, for every* $\varepsilon \in (0, 1)$, $\rho \in [1,\infty)$ *and a corresponding* $K_\varepsilon > 0$,

$$F_{\varepsilon,\omega,\rho} := \{\lambda \in \mathbb{C} : \Re\lambda \geq -K_\varepsilon|\Im\lambda|^{1/\rho} + \omega\}.$$

Assume that, for every $\varepsilon \in (0, 1)$, *there exist* $K_\varepsilon > 0$, *an open neighborhood* $O_{\varepsilon,\omega}$ *of the region* $G_{\varepsilon,\omega,\rho} := \{\lambda \in \mathbb{C} : \Re\lambda \geq \omega,\ \tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0\} \cup \{\lambda \in F_{\varepsilon,\omega,\rho} : \Re\lambda \leq \omega\}$, *a mapping* $h_\varepsilon : O_{\varepsilon,\omega} \to L(X)$ *and analytic mappings* $f_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$ *and* $g_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$ *such that the conditions (i)(a)-(d) of this theorem hold with* $F_{\varepsilon,\omega}$, *resp.* $G_{\varepsilon,\omega}$, *replaced by* $F_{\varepsilon,\omega,\rho}$, *resp.* $G_{\varepsilon,\omega,\rho}$. *Then the mapping* $t \mapsto R(t)$, $t \in (0, \tau)$ *is infinitely differentiable in* $L(X)$ *and, for every compact set* $K \subseteq (0, \tau)$, *there exists* $h_K > 0$ *such that the set* $\{\frac{h_K^n \frac{d^n}{dt^n} R(t)}{n!^\rho} : t \in K,\ n \in \mathbb{N}_0\}$ *is equicontinuous.*

Let us recall that the case $\rho = 1$ in Theorem 5.27 is very important because it gives a sufficient condition for an (a, k)-regularized C-resolvent family to be real analytic.

Suppose now that $n \in \mathbb{N}$, $|a|(t)$ satisfies (P1)-$\mathbb{C}$ and $\operatorname{abs}(a) = 0$. Following [68, Definition 3.3, p. 69], we say that $a(t)$ is n-regular if and only if there exists $c > 0$ such that $|\lambda^m \hat{a}^{(m)}(\lambda)| \leq c|\hat{a}(\lambda)|$, $\lambda \in \mathbb{C}_+$, $1 \leq m \leq n$. Set $a^{(-1)}(t) := \int_0^t a(s)\,ds$, $t \geq 0$ and suppose that $a(t)$ and $b(t)$ are n-regular for some $n \in \mathbb{N}$. Then we know that $\hat{a}(\lambda) \neq 0$, $\lambda \in \mathbb{C}_+$, as well as that $(a * b)(t)$ and $a^{(-1)}(t)$ are n-regular, and that $a'(t)$ is n-regular provided that $\operatorname{abs}(a') = 0$.

Following [68, Definition 3.1, p. 68] and [36, Definition 2.1.23], it will be said that the abstract Volterra inclusion (1.1) with $\mathcal{B} = I$ (denoted henceforth by the same symbol) is (kC)-parabolic if and only if the following holds:

(i) $|a|(t)$ and $k(t)$ satisfy (P1)-$\mathbb{C}$ and there exist meromorphic extensions of the functions $\tilde{a}(\lambda)$ and $\tilde{k}(\lambda)$ on $\mathbb{C}_+$, denoted by $\hat{a}(\lambda)$ and $\hat{k}(\lambda)$. Let N be the subset of $\mathbb{C}_+$ which consists of all zeros and possible poles of $\hat{a}(\lambda)$ and $\hat{k}(\lambda)$.

(ii) There exists $M \geq 1$ such that, for every $\lambda \in \mathbb{C}_+ \setminus N$, $1/\hat{a}(\lambda) \in \rho_C(\mathcal{A})$ and $||\hat{k}(\lambda)(I - \hat{a}(\lambda)\mathcal{A})^{-1}C|| \leq M/|\lambda|$.

If $k(t) \equiv 1$, resp. $C = I$, then it is also said that (1.1) is C-parabolic, resp. k-parabolic.

Now we are ready to formulate the following extension of [36, Theorem 2.1.24].

Theorem 5.28. *Assume* $n \in \mathbb{N}$, $a(t)$ *is* n*-regular,* $(X, \|\cdot\|)$ *is a Banach space,* $\mathcal{A}$ *is a closed MLO in* X, *the abstract Volterra inclusion* (1.1) *is* C*-parabolic, and the* *mapping* $\lambda \mapsto (I-\tilde{a}(\lambda)\mathcal{A})^{-1}C$, $\lambda \in \mathbb{C}_+$ *is continuous. Then, for every* $\alpha \in (0,1]$, $\mathcal{A}$ *is a subgenerator of an* $(a, g_{\alpha+1})$*-regularized* C^2*-resolvent family* $(S_\alpha(t))_{t\geq 0}$ *which satisfies* $\sup_{h>0,t\geq 0} h^{-\alpha}\|S_\alpha(t+h)-S_\alpha(t)\| < \infty$, $D_t^\alpha S_\alpha(t)C^{k-1} \in C^{k-1}((0,\infty) : L(X))$, $1 \leq k \leq n$ *as well as:*

$$\|t^j D_t^j D_t^\alpha S_\alpha(t)C^{k-1}\| \leq M, \quad t \geq 0,\ 1 \leq k \leq n,\ 0 \leq j \leq k-1, \tag{5.32}$$

$$\begin{aligned} &\|t^k D_t^{k-1} D_t^\alpha S_\alpha(t)C^{k-1} - s^k D_s^{k-1} D_s^\alpha S_\alpha(s)C^{k-1}\| \\ &\leq M|t-s|\Big(1+\ln\frac{t}{t-s}\Big), \quad 0 \leq s < t < \infty,\ 1 \leq k \leq n, \end{aligned} \tag{5.33}$$

and, for every $T > 0$, $\varepsilon > 0$ *and* $k \in \mathbb{N}_n$, *there exists* $M_{T,k}^\varepsilon > 0$ *such that*

$$\begin{aligned} &\|t^k D_t^{k-1} D_t^\alpha S_\alpha(t)C^{k-1} - s^k D_s^{k-1} D_s^\alpha S_\alpha(s)C^{k-1}\| \\ &\leq M_{T,k}^\varepsilon (t-s)^{1-\varepsilon},\ 0 \leq s < t \leq T,\ 1 \leq k \leq n. \end{aligned} \tag{5.34}$$

Furthermore, if $\mathcal{A}$ *is densely defined, then* $\mathcal{A}$ *is a subgenerator of a bounded* (a, C^2)*-regularized resolvent family* $(S(t))_{t\geq 0}$, *satisfying additionally that the mapping* $t \mapsto S(t)C^{k-1}$, $t > 0$ *is in class* $C^{k-1}((0,\infty) : L(X))$, $1 \leq k \leq n$ *and that* (5.32)-(5.34) *hold with* $D_t^\alpha S_\alpha(t)C^{k-1}$ *replaced by* $S(t)C^{k-1}$ $(1 \leq k \leq n)$ *therein.*

The representation formula [68, (3.41), p. 81] and the assertions of [68, Corollary 3.2-Corollary 3.3, pp. 74-75] can be extended to exponentially bounded (a, C)-regularized resolvent families subgenerated by multivalued linear operators, as well. For more details about parabolicity of abstract non-degenerate Volterra equations, we refer the reader to [68, Chapter I, Section 3].

5.2. **Non-injectivity of regularizing operators C_2 and C.** In this subsection, we consider multivalued linear operators as subgenerators of mild (a, k)-regularized (C_1, C_2)-resolvent operator families and (a, k)-regularized C-resolvent operator families. We use the same notion and notation as before but now we allow that the operators C_2 and C are possibly non-injective (see Definition 5.1-Definition 5.2). Without any doubt, this choice has some obvious displeasing consequences on the uniqueness of corresponding abstract Volterra integro-differential inclusions (see Proposition 5.8(ii) and Theorem 5.9(ii)).

As before, we assume that X and Y are two SCLCSs, $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{\mathrm{loc}}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ is an MLO, $C_1 \in L(Y, X)$, C, $C_2 \in L(X)$ and $C\mathcal{A} \subseteq \mathcal{A}C$. We define the integral generator $\mathcal{A}_{\mathrm{int}}$ of a mild (a, k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ (mild (a, k)-regularized (C_1, C_2)-existence and uniqueness family $(R_1(t), R_2(t))_{t\in[0,\tau)}$; (a, k)-regularized C-regularized family $(R(t))_{t\in[0,\tau)}$) in the same way as for injective operators C and C_2. Then we have that $\mathcal{A}_{\mathrm{int}} \subseteq C_2^{-1}\mathcal{A}_{\mathrm{int}}C_2$ ($\mathcal{A}_{\mathrm{int}} \subseteq C^{-1}\mathcal{A}_{\mathrm{int}}C$) is still the maximal subgenerator of $(R_2(t))_{t\in[0,\tau)}$ ($(R(t))_{t\in[0,\tau)}$) with respect to the set inclusion and the local equicontinuity of $(R_2(t))_{t\in[0,\tau)}$ ($(R(t))_{t\in[0,\tau)}$) implies that $\mathcal{A}_{\mathrm{int}}$ is closed; as the next illustrative example shows, $C^{-1}\mathcal{A}_{\mathrm{int}}C$ need not be a subgenerator of $(R(t))_{t\in[0,\tau)}$ and the inclusion $C^{-1}\mathcal{A}_{\mathrm{int}}C \subseteq \mathcal{A}_{\mathrm{int}}$ is not true for resolvent operator families, in general [50].

Suppose that $a(t)$ is a kernel on $[0,\tau)$, $\mathcal{A}$ and $\mathcal{B}$ are two subgenerators of an (a, k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, and $x \in D(\mathcal{A}) \cap D(\mathcal{B})$. Then $R(t)(y - z) = 0$, $t \in [0,\tau)$ for each $y \in \mathcal{A}x$ and $z \in \mathcal{B}x$. Furthermore, the local equicontinuity of $(R(t))_{t\in[0,\tau)}$ and the closedness of $\mathcal{A}$ imply that the inclusion (5.4) continues to hold without injectivity of C being assumed.

In the following definition, we introduce the notion of an (a,k,C)-subgenerator of any strongly continuous operator family $(Z(t))_{t\in[0,\tau)} \subseteq L(X)$. This definition extends the corresponding ones introduced by Kuo [54, 55, Definition 2.4] in the setting of Banach spaces, where it has also been assumed that the operator $\mathcal{A}=A$ is linear and single-valued.

Definition 5.29. Let $0<\tau\leq\infty$, $C\in L(X)$, $a\in L^1_{\rm loc}([0,\tau))$, $a\neq 0$, $k\in C([0,\tau))$ and $k\neq 0$. Suppose that $(Z(t))_{t\in[0,\tau)}\subseteq L(X)$ is a strongly continuous operator family. By an (a,k,C)-subgenerator of $(Z(t))_{t\in[0,\tau)}$ we mean any MLO $\mathcal{A}$ in X satisfying the following two conditions:

(i) $Z(t)x-k(t)Cx=\int_0^t a(t-s)Z(s)y\,ds$, whenever $t\in[0,\tau)$ and $y\in\mathcal{A}x$.

(ii) For all $x\in X$ and $t\in[0,\tau)$, we have $\int_0^t a(t-s)Z(s)x\,ds\in D(\mathcal{A})$ and $Z(t)x-k(t)Cx\in\mathcal{A}\int_0^t a(t-s)Z(s)x\,ds$.

The (a,k,C)-integral generator $\mathcal{A}_{\rm int}$ of $(Z(t))_{t\in[0,\tau)}$ (integral generator, if there is no risk for confusion) is defined by

$$\mathcal{A}_{\rm int}:=\Big\{(x,y)\in X\times X: Z(t)x-k(t)Cx=\int_0^t a(t-s)Z(s)y\,ds \text{ for all } t\in[0,\tau)\Big\}.$$

If $\mathcal{A}$ is a subgenerator of $(Z(t))_{t\in[0,\tau)}$, then it is clear that $(Z(t))_{t\in[0,\tau)}$ is a mild (a,k)-regularized (C,C)-existence and uniqueness family which do have $\mathcal{A}$ as subgenerator. Since we have not assumed that $\mathcal{A}$ commutes with C or $(Z(t))_{t\in[0,\tau)}$, it does not follow automatically from Definition 5.29 that $(Z(t))_{t\in[0,\tau)}$ is an (a,k)-regularized C-resolvent family with subgenerator $\mathcal{A}$.

By $\chi(Z)$ we denote the set consisting of all subgenerators of $(Z(t))_{t\in[0,\tau)}$. The local equicontinuity of $(Z(t))_{t\in[0,\tau)}$ yields that for each subgenerator $\mathcal{A}\in\chi(Z)$ we have $\overline{\mathcal{A}}\in\chi(Z)$. The set $\chi(Z)$ can have infinitely many elements; if $\mathcal{A}\in\chi(Z)$, then $\mathcal{A}\subseteq\mathcal{A}_{\rm int}$ (cf. [58, Example 4.10, 4.11]; in these examples, the partially ordered set $(\chi_{sv}(Z),\subseteq)$, where $\chi_{sv}(Z)$ denotes the set consisting of all single-valued linear subgenerators of $(Z(t))_{t\in[0,\tau)}$, does not have the greatest element) and, if $\chi(Z)$ is finite, then it need not be a singleton [35]. In general, the set $\chi(Z)$ can be empty and the integral generator of $(Z(t))_{t\in[0,\tau)}$ need not be a subgenerator of $(Z(t))_{t\in[0,\tau)}$ in the case that $\tau<\infty$; see [50] for a counterexample given for local C-regularized semigroups.

If $\mathcal{A}$ and $\mathcal{B}$ are subgenerators of $(Z(t))_{t\in[0,\tau)}$, then for any complex numbers α, β such that $\alpha+\beta=1$ we have that $\alpha\mathcal{A}+\beta\mathcal{B}$ is a subgenerator of $(Z(t))_{t\in[0,\tau)}$. Set $\mathcal{A}\wedge B:=(1/2)\mathcal{A}+(1/2)\mathcal{B}$. We define the operator $\mathcal{A}\vee_0\mathcal{B}$ by $D(\mathcal{A}\vee_0\mathcal{B}):=$span$[D(\mathcal{A})\cup D(\mathcal{B})]$ and

$$\mathcal{A}\vee_0\mathcal{B}(ax+by):=aAx+bBy,\ x\in D(\mathcal{A}),\ y\in D(\mathcal{B}),\ a,\ b\in\mathbb{C};$$

$\mathcal{A}\vee\mathcal{B}:=\overline{\mathcal{A}\vee_0\mathcal{B}}$. Then $\mathcal{A}\vee_0\mathcal{B}$ is a subgenerator of $(Z(t))_{t\in[0,\tau)}$, and $\mathcal{A}\vee\mathcal{B}$ is a subgenerator of $(Z(t))_{t\in[0,\tau)}$, provided that $(Z(t))_{t\in[0,\tau)}$ is locally equicontinuous. In the case of non-degenerate K-convoluted C-semigroups, C injective, it is well known that the set $\chi(Z)$, equipped with the operations $\wedge$ and $\vee$, forms a complete Boolean lattice ([78], [35, Remark 2.1.8(ii)-(iii)]). We will not discuss the properties of $(\chi(Z),\wedge,\vee)$ in general case.

If $\mathcal{A}$ is a closed, $\mathcal{A} \in \chi(Z)$, $0 \in \operatorname{supp}(a)$ and $y \in \mathcal{A}x$, then we have $(\int_0^t a(t-s)Z(s)x\,ds, Z(t)x - k(t)Cx) \in \mathcal{A}$, $t \in [0,\tau)$, i.e.,

$$\Big(\int_0^t a(t-s)Z(s)x\,ds, \int_0^t a(t-s)Z(s)y\,ds\Big) \in \mathcal{A}, \quad t \in [0,\tau). \tag{5.35}$$

Suppose now that $\tau_0 \in (0,\tau)$. By [35, Theorem 3.4.40], there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1[0,\tau_0]$ such that $(a * f_n)(t) \to g_1(t)$ in $L^1[0,\tau_0]$. Then the closedness of $\mathcal{A}$ along with (5.35) shows that $(\int_0^t g_1(t-s)Z(s)x\,ds, \int_0^t g_1(t-s)Z(s)y\,ds) \in \mathcal{A}$, $t \in [0,\tau_0]$. After differentiation, we obtain that $(Z(t)x, Z(t)y) \in \mathcal{A}$, $t \in [0,\tau_0]$ and since τ_0 was arbitrary, we have that $Z(t)\mathcal{A} \subseteq \mathcal{A}Z(t)$, $t \in [0,\tau)$ for any closed subgenerator $\mathcal{A}$ of $(Z(t))_{t\in[0,\tau)}$. If this is the case and $Z(t)C = CZ(t)$, $t \in [0,\tau)$, then $C^{-1}\mathcal{A}C$ also commutes with $Z(t)$: Suppose that $(x,y) \in C^{-1}\mathcal{A}C$. Then $Cy \in \mathcal{A}Cx$, $CZ(t)x = Z(t)Cx \in D(\mathcal{A})$, $t \in [0,\tau)$ and $CZ(t)y = Z(t)Cy \in Z(t)\mathcal{A}Cx \subseteq \mathcal{A}CZ(t)x = \mathcal{A}Z(t)Cx$, $t \in [0,\tau)$ so that $Z(t)y \in C^{-1}\mathcal{A}CZ(t)x$, $t \in [0,\tau)$ and $Z(t)[C^{-1}\mathcal{A}C] \subseteq [C^{-1}\mathcal{A}C]Z(t)$, $t \in [0,\tau)$.

Suppose again that $\mathcal{A}$ is a closed subgenerator of $(Z(t))_{t\in[0,\tau)}$, $0 \in \operatorname{supp}(a)$ and $y \in \mathcal{A}x$. Then $(\int_0^t a(t-s)Z(s)y\,ds, Z(t)y - k(t)Cy) = (Z(t)x - k(t)Cx, Z(t)y - k(t)Cy) \in \mathcal{A}$, $t \in [0,\tau)$. Since $(Z(t)x, Z(t)y) \in \mathcal{A}$, $t \in [0,\tau)$, the above easily implies that $(Cx, Cy) \in \mathcal{A}$ so that $C\mathcal{A} \subseteq \mathcal{A}C$, i.e., $\mathcal{A} \subseteq C^{-1}\mathcal{A}C$. Now we proceed by repeating some parts of the proof of [35, Proposition 2.1.6(i)]. Let $(x,y) \in \mathcal{A}_{\text{int}}$. As above, we have $(\int_0^t a(t-s)Z(s)x\,ds, \int_0^t a(t-s)Z(s)y\,ds) \in \mathcal{A}$, $t \in [0,\tau)$ and $(Z(t)x, Z(t)y) \in \overline{\mathcal{A}} = \mathcal{A}$, $t \in [0,\tau)$. This implies $Z(t)y \in \mathcal{A}Z(t)x = \mathcal{A}[\Theta(t)Cx + \int_0^t a(t-s)Z(s)y\,ds]$, $t \in [0,\tau)$ and, since $\int_0^t a(t-s)Z(s)y\,ds \in D(\mathcal{A})$ for $t \in [0,\tau)$, $Cx \in D(\mathcal{A})$ as well as $0 \in \mathcal{A}[\Theta(t)Cx + \int_0^t a(t-s)Z(s)y\,ds - \int_0^t a(t-s)Z(s)y\,ds] - \Theta(t)Cy$, $t \in [0,\tau)$. Hence, $Cy \in \mathcal{A}Cx$ and $\mathcal{A}_{\text{int}} \subseteq C^{-1}\mathcal{A}C$. If, additionally, the operator C is injective and $Z(t)C = CZ(t)$, $t \in [0,\tau)$, then we can simply verify that $C^{-1}\mathcal{A}C$ is likewise a closed subgenerator of $(W(t))_{t\in[0,\tau)}$, so that $\mathcal{A}_{\text{int}} = C^{-1}\mathcal{A}C$ by previously proved inclusion $\hat{\mathcal{A}} \subseteq C^{-1}\mathcal{A}C$ and the fact that $\mathcal{A}_{\text{int}}$ extends any subgenerator from $\chi(W)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two subgenerators of $(Z(t))_{t\in[0,\tau)}$, let $\mathcal{B}$ be closed, and let $a(t)$ kernel on $[0,\tau)$. Suppose that $y \in \mathcal{A}x$. Then $(\int_0^t a(t-s)Z(s)y\,ds, Z(t)y - k(t)Cy) = (Z(t)x - k(t)Cx, Z(t)y - k(t)Cy) \in \mathcal{B}$, $t \in [0,\tau)$, which implies by Theorem 2.3 that $((a*Z)(t)x - (a*k)(t)Cx, (a*Z)(t)y - (a*k)(t)Cy) \in \mathcal{B}$, $t \in [0,\tau)$. Since $(a*Z)(t)x \in D(\mathcal{B})$, $t \in [0,\tau)$, the above implies that $Cx \in D(\mathcal{B})$. Hence, $C(D(\mathcal{A})) \subseteq D(\mathcal{B})$.

We continue by observing that Proposition 5.3, Proposition 5.8, Proposition 5.13, the equation (5.3) and assertions clarified in the paragraph directly after Theorem 5.7 continue to hold without any terminological changes. If $(R_1(t), R_2(t))_{t\in[0,\tau)}$ is strongly continuous and (5.3) holds, then it can be easily seen that the integral generator $\mathcal{A}_{\text{int}}$ of $(R_2(t))_{t\in[0,\tau)}$ is a subgenerator of a mild (a,k)-regularized (C_1, C_2)-existence and uniqueness family $(R_1(t), R_2(t))_{t\in[0,\tau)}$. This is no longer true if (5.3) holds only for $0 \le t,\ s,\ t+s < \tau$; see [50] for more details.

Proposition 5.30. *Suppose that $\mathcal{A}$ is a closed MLO, $0 < \tau \le \infty$, $a \in L^1_{\text{loc}}([0,\tau))$, $a * a \neq 0$ in $L^1_{\text{loc}}([0,\tau))$, $k \in C([0,\tau))$ and $k \neq 0$. If $\pm\mathcal{A}$ are subgenerators of mild (a,k)-regularized C_1-existence families $(R_{1,\pm}(t))_{t\in[0,\tau)}$ (mild (a,k)-regularized C_2-uniqueness families $(R_{2,\pm}(t))_{t\in[0,\tau)}$; (a,k)-regularized C-resolvent families $(R_{\pm}(t))_{t\in[0,\tau)}$), then $\mathcal{A}^2$ is a subgenerator of a mild $(a*a,k)$-regularized C_1-existence* *family* $(R_1(t) \equiv (1/2)R_1(t) + (1/2)R_{1,-}(t))_{t\in[0,\tau)}$ *(mild* $(a*a,k)$*-regularized* C_2*-uniqueness family* $(R_2(t) \equiv (1/2)R_2(t) + (1/2)R_{2,-}(t))_{t\in[0,\tau)}$*; mild* $(a*a,k)$*-regularized* C*-resolvent family* $(R(t) \equiv (1/2)R_+(t) + (1/2)R_-(t))_{t\in[0,\tau)}$*).*

Proof. We prove the proposition only for mild (a,k)-regularized C_1-existence families. Let $x \in E$ and $t \in [0,\tau)$ be fixed. Then $\frac{1}{2}[R_{1,+}(t)x - R_{1,-}(t)x] = \frac{1}{2}[R_{1,+}(t)x - k(t)C_1x] - [R_{1,-}(t)x - k(t)C_1x] \in \frac{1}{2}\mathcal{A}(a*R_{1,+}(\cdot)x)(t) + \frac{1}{2}\mathcal{A}(a*R_{1,-}(\cdot)x)(t) = \mathcal{A}(a*R_1(\cdot)x)(t)$. Applying Theorem 2.3, we obtain that $\frac{1}{2}(a*[R_{1,+}(\cdot)x - R_{1,-}(\cdot)x])(t) \in \mathcal{A}(a*a*R_1(\cdot)x)(t)$. Since $\pm\mathcal{A}$ are subgenerators of mild (a,k)-regularized C_1-existence families $(R_{1,\pm}(t))_{t\in[0,\tau)}$, the above inclusion implies $(a*a*R_1(\cdot)x)(t) \in D(\mathcal{A}^2)$ and $\frac{1}{2}([R_{1,+}(t)x - k(t)C_1x] + [R_{1,-}(t)x - k(t)C_1x]) = R_1(t)x - k(t)C_1x \in \mathcal{A}^2(a*a*R_1(\cdot)x)(t)$, as required. □

The following analogues of Theorems 5.4[(i),(iii)] and 5.5 hold.

Theorem 5.31. *Suppose that* $\mathcal{A}$ *is a closed MLO in* X, $C_1 \in L(Y,X)$, $C_2 \in L(X)$, $|a(t)|$ *and* $k(t)$ *satisfy* (P1)*, as well as that* $(R_1(t),R_2(t))_{t\geq 0} \subseteq L(Y,X)\times L(X)$ *is strongly continuous. Let* $\omega \geq \max(0,\mathrm{abs}(|a|),\mathrm{abs}(k))$ *be such that the operator family* $\{e^{-\omega t}R_i(t) : t \geq 0\}$ *is equicontinuous for* $i = 1,2$*. Then the following holds:*

(i) $(R_1(t),R_2(t))_{t\geq 0}$ *is a mild* (a,k)*-regularized* (C_1,C_2)*-existence and uniqueness family with a subgenerator* $\mathcal{A}$ *if and only if for every* $\lambda \in \mathbb{C}$ *with* $\Re\lambda > \omega$ *and* $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$*, we have* $R(C_1) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$*,*
$$\int_0^\infty e^{-\lambda t}R_1(t)y\,dt \in \tilde{k}(\lambda)\big(I - \tilde{a}(\lambda)\mathcal{A}\big)^{-1}C_1y,\ \ y \in Y, \tag{5.36}$$
$$\tilde{k}(\lambda)C_2x = \int_0^\infty e^{-\lambda t}\big[R_2(t)x - \big(a*R_2\big)(t)y\big]\,dt, \quad \text{whenever } (x,y) \in \mathcal{A}. \tag{5.37}$$

(ii) $(R_2(t))_{t\geq 0}$ *is a mild* (a,k)*-regularized* C_2*-uniqueness family with a subgenerator* $\mathcal{A}$ *if and only if* (5.37) *holds for* $\Re\lambda > \omega$*.*

Theorem 5.32. *Suppose that* $\mathcal{A}$ *is a closed MLO in* X, $C \in L(X)$, $C\mathcal{A} \subseteq \mathcal{A}C$, $|a(t)|$ *and* $k(t)$ *satisfy* (P1)*, as well as that* $(R(t))_{t\geq 0} \subseteq L(X)$ *is strongly continuous and commutes with* C *on* X*. Let* $\omega \geq \max(0,\mathrm{abs}(|a|),\mathrm{abs}(k))$ *be such that the operator family* $\{e^{-\omega t}R(t) : t \geq 0\}$ *is equicontinuous. Then* $(R(t))_{t\geq 0}$ *is an* (a,k)*-regularized* C*-resolvent family with a subgenerator* $\mathcal{A}$ *if and only if for every* $\lambda \in \mathbb{C}$ *with* $\Re\lambda > \omega$ *and* $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$*, we have* $R(C) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$*,* (5.36) *holds with* $R_1(\cdot)$*,* C_1 *and* Y*,* y *replaced with* $R(\cdot)$*,* C *and* X*,* x *therein, as well as* (5.37) *holds with* $R_2(\cdot)$ *and* C_2 *replaced with* $R(\cdot)$ *and* C *therein.*

Keeping in mind Theorem 5.32 and [36, Theorem 1.2.2], it is very simple to prove the following complex characterization theorem (cf. Theorem 5.10):

Theorem 5.33. *Suppose that* $\mathcal{A}$ *is a closed MLO in* X, $C \in L(X)$, $C\mathcal{A} \subseteq \mathcal{A}C$, $|a(t)|$ *and* $k(t)$ *satisfy* (P1)*,* $\omega_0 > \max(0,\mathrm{abs}(|a|),\mathrm{abs}(k))$ *and, for every* $\lambda \in \mathbb{C}$ *with* $\Re\lambda > \omega_0$ *and* $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$*, we have* $R(C) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$*. If there exists a function* $\Upsilon : \{\lambda \in \mathbb{C} : \Re\lambda > \omega_0\} \to L(X)$ *which satisfies:*

(a) $\Upsilon(\lambda)x \in \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}Cx$ *for* $\Re\lambda > \omega_0$*,* $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$*,* $x \in X$*,*
(b) *the mapping* $\lambda \mapsto \Upsilon(\lambda)x$*,* $\Re\lambda > \omega_0$ *is analytic for every fixed* $x \in X$*,*
(c) *there exists* $r \geq -1$ *such that the family* $\{\lambda^{-r}\Upsilon(\lambda) : \Re\lambda > \omega_0\} \subseteq L(X)$ *is equicontinuous,*
(d) $\Upsilon(\lambda)x - \tilde{a}(\lambda)\Upsilon(\lambda)y = \tilde{k}(\lambda)Cx$ *for* $\Re\lambda > \omega_0$*,* $(x,y) \in \mathcal{A}$*, and*

(e) $\Upsilon(\lambda)Cx = C\Upsilon(\lambda)x$ *for* $\Re\lambda > \omega_0$, $x \in X$,

then, for every $\alpha > 1$, $\mathcal{A}$ *is a subgenerator of a global* $(a, k * g_{\alpha+r})$*-regularized* C*-resolvent family* $(R_\alpha(t))_{t\geq 0}$ *which satisfies that the family* $\{e^{-\omega_0 t}R_\alpha(t) : t \geq 0\} \subseteq L(X)$ *is equicontinuous.*

The real representation theorem for generation of degenerate (a, k)-regularized C-resolvent families can be also formulated but the assertion of Theorem 5.12(ii) is not attainable in the case that the operator C is not injective. The assertion of Theorem 5.7 continues to hold with minimal terminological changes. Since the identity (5.24) holds for degenerate (a, k)-regularized C-resolvent families, with C being not injective, Proposition 5.15 can be reformulated without substantial difficulties, as well, but we cannot prove the uniqueness of solutions of corresponding abstract time-fractional inclusions.

As already mentioned, the adjoint type theorems [36, Theorem 2.1.12(i)/(ii); Theorem 2.1.13] continue to hold for (a, k)-regularized C-regularized families subgenerated by closed multivalued linear operators and it is not necessary to assume that the operator $\mathcal{A}$ is densely defined in the case of consideration of [36, Theorem 2.1.12(i)]. All this remains true if the operator C is not injective, when we also do not need to assume that $R(C)$ is dense in X.

If C is not injective, then we introduce the notion of (exponential equicontinuous) analyticity of degenerate (a, k)-regularized C-resolvent families in the same way as in Definition 5.16. Then Theorem 5.18 does not admit a satisfactory reformulation in our new frame. On the other hand, the assertion of Theorem 5.19 can be rephrased by taking into consideration the conditions (d)-(e) from Theorem 5.33. Differential properties of degenerate (a, k)-regularized C-resolvent families clarified in Theorem 5.26-Theorem 5.27 continue to hold after a reformulation of the same type.

During the peer-review process, the author has published several research papers about degenerate (a, k)-regularized C-resolvent families and their applications. Various subclasses of degenerate convoluted C-semigroups and degenerate convoluted C-cosine functions in locally convex spaces have been investigated in [50]. Perturbation results for abstract degenerate Volterra integro-differential equations have been examined in [51], while the approximation and convergence of degenerate (a, k)-regularized C-resolvent families have been examined in [52].

6. Conclusions and final remarks

In this research article, we have analyzed the abstract degenerate Volterra integro-differential equations in sequentially complete locally convex spaces. We have systematically investigated the class of degenerate (a, k)-regularized C-resolvent families subgenerated by multivalued linear operators and examined many interesting topics including the generation of (a, k)-regularized C-resolvent families, smoothing properties of (a, k)-regularized C-resolvent families and subordination principles. We have also examined the class of mild (a, k)-regularized C_1-existence families, the class of mild (a, k)-regularized C_2-uniqueness families and provided a new theoretical concept of vector-valued Laplace transform. In addition to the above, we have presented many useful comments, open problems, examples and illustrative applications of our theoretical results.

The material of this paper has recently been published as a part of the research monograph [38]; the almost periodic type solutions of the abstract degenerate

Volterra integro-differential equations have recently been analyzed in the research monograph [37]. We close the paper with the observation that we have obeyed the multivalued linear operators approach here; this approach, although very dominant when compared with the other existing methods and theoretical strategies in this theory, is not sufficiently adequate to cover all related problems regarding the abstract degenerate Volterra integro-differential equations. For some other concepts of solution operator families, we may refer to [42, 43, 45, 46].

Finally, we would like to emphasize that almost anything relevant has been said about the existence and uniqueness of the positive solutions to the abstract degenerate Volterra integro-differential equations in ordered Banach spaces.

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