

**LOCAL WELL-POSEDNESS AND STANDING WAVES WITH
PRESCRIBED MASS FOR SCHRÖDINGER-POISSON SYSTEMS
WITH A LOGARITHMIC POTENTIAL IN \mathbb{R}^2**

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ABSTRACT. In this article, we consider planar Schrödinger-Poisson systems with a logarithmic external potential $W(x) = \ln(1 + |x|^2)$ and a general nonlinear term f . We obtain conditions for the local well-posedness of the Cauchy problem in the energy space. By introducing some suitable assumptions on f , we prove the existence of the global minimizer. In addition, with the help of the local well-posedness, we show that the set of ground state standing waves is orbitally stable.

1. INTRODUCTION

We consider the planar Schrödinger-Poisson system

$$\begin{aligned}i\psi_t - \Delta\psi + W(x)\psi + \gamma\omega\psi &= f(\psi), \quad \forall(t, x) \in \mathbb{R}^{1+2}, \\ \Delta\omega &= |\psi|^2, \\ \psi(0, x) &= \psi_0(x),\end{aligned}\tag{1.1}$$

where $\psi : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{C}$ is the (time-dependent) wave function, $x \mapsto W(x)$ is a real external potential and $\gamma \in \mathbb{R}$. The function ω represents an internal potential for a nonlocal self-interaction of the wave function ψ , and the nonlinear term f is used to model the interaction among particles. Such a system arises from quantum mechanics [3, 5, 16] and in semiconductor theory [18, 19]. We refer the reader to [2, 13] for more details on its physical aspects.

An important topic is to establish conditions for the well-posedness of Cauchy problem (1.1). From a mathematical point of view, the second equation in the system determines $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}$ up to harmonic functions, it is natural to choose ω as the Newton potential of $|\psi|^2$, i.e. the convolution of $|\psi|^2$ with the fundamental solution $\Phi(x) = \frac{1}{2\pi} \ln|x|$ of the Laplacian. Thus the Newtonian potential ω is given by

$$\omega = \frac{1}{2\pi} (\ln|x| * |\psi|^2).$$

We note that the Newtonian potential ω diverges at the spatial infinity no matter how fast ψ decays. In view of this, Masaki [20, 21] proposed a new approach

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to deal with such a nonlocal term, which can be decomposed into a sum of the linear logarithmic potential and a good remainder. By using the perturbation method, the global well-posedness for the Cauchy problem (1.1) with $W(x) \equiv 0$ and $f(\psi) = |\psi|^{p-2}\psi$ ($p > 2$) is established in the space \mathcal{B} given by

$$\mathcal{B} := \left\{ \psi \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \ln(\sqrt{1+|x|^2}) |\psi(x)|^2 dx < \infty \right\}.$$

Another interesting topic on (1.1) is to study the standing wave solution of the form

$$\psi(x, t) = e^{i\lambda t} u(x),$$

where $\lambda \in \mathbb{R}$ and $u : \mathbb{R}^2 \rightarrow \mathbb{R}$. Then (1.1) is reduced to the system

$$\begin{aligned} -\Delta u + (W(x) - \lambda)u + \gamma\omega u &= f(u) \quad \text{in } \mathbb{R}^2, \\ -\Delta \omega &= u^2 \quad \text{in } \mathbb{R}^2, \end{aligned} \tag{1.2}$$

which can be further written as the integro-differential equation

$$-\Delta u + (W(x) - \lambda)u + \gamma(\Phi * |u^2|)u = f(u), \quad \forall x \in \mathbb{R}^2. \tag{1.3}$$

At least formally, the energy functional associated with (1.3) is

$$\begin{aligned} E(u) &= \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + (W(x) - \lambda)u^2) dx \\ &\quad + \frac{\gamma}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x-y|) |u(x)|^2 |u(y)|^2 dx dy - \int_{\mathbb{R}^2} F(u) dx, \end{aligned}$$

where $F(t) = \int_0^t f(s) ds$. Obviously, if u is a critical point of E , then the pair $(u, \Phi * |u|^2)$ is a weak solution of (1.2). However, the energy functional E is not well-defined on the natural Sobolev space $H^1(\mathbb{R}^2)$, since the logarithm term changes sign and is neither bounded from above nor from below. Inspired by [24], Cingolani and Weth [11] developed a variational framework of ((1.3) with $W(x) \equiv 0$ in the smaller Hilbert space

$$X := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \ln(1+|x|) u^2 dx < \infty \right\},$$

endowed with the norm

$$\|u\|_X^2 := \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2(1 + \ln(1+|x|^2))) dx.$$

If the frequency λ is a fixed and assigned parameter, then solutions of (1.3) can be obtained as critical points of the functional E in X . Under various types of potentials W and nonlinearities f , there has been much study on this case in recent years, see, for example [1, 8, 9, 14]. For other nonlocal problems, we refer the reader to [22, 25, 26, 27, 29].

If we would like to find solutions of (1.3) with the frequency λ unknown, then λ appears as a Lagrange multiplier, and L^2 -norms of solutions are prescribed, i.e.

$$\int_{\mathbb{R}^2} |u|^2 dx = c \quad \text{for a given } c > 0,$$

which are usually called normalized solutions. This study seems particularly meaningful from the physical point of view, since solutions of (1.1) conserve their mass along time. When $W(x) \equiv 0$, Cingolani and Jeanjean [10] proved the existence and

multiplicity of normalized solutions for 1.3 with $f(u) = |u|^{p-2}u$ ($p > 2$). When the logarithmic external potential

$$W(x) = \ln(1 + |x|^2) \tag{1.4}$$

is considered in (1.3), Dolbeault, Frank and Jeanjean [12] studied the existence of normalized solutions for (1.3) with $f(u) = \ln |u|^2 u$, and recently Guo, Liang and Li [15] proved the existence and uniqueness of L^2 -critical constraint minimization problem, i.e. $f(u) = |u|^{p-2}u$ with $p = 4$.

Inspired by the analysis mentioned above, in this paper we are concerned with a class of planar Schrödinger-Poisson systems with a logarithmic external potential (1.4) and a general nonlinearity f . First of all, we shall establish conditions of the local well-posedness for the Cauchy problem (1.1). Secondly, we shall focus on the existence of global minimizer when f satisfies some suitable assumptions. In addition, with the help of the local well-posedness of the Cauchy problem (1.1), the orbital stability of the set of ground states is explored as well.

To find normalized solutions of (1.3), we consider the associated energy functional

$$\begin{aligned} J(u) := & \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + \ln(1 + |x|^2)u^2) dx \\ & + \frac{\gamma}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x - y|)u^2(x)u^2(y) dx dy - \int_{\mathbb{R}^2} F(u) dx. \end{aligned} \tag{1.5}$$

under the constraint

$$S(c) := \left\{ u \in \mathcal{H} : \int_{\mathbb{R}^2} u^2 dx = c \right\},$$

where

$$\mathcal{H} := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \ln(1 + |x|^2)u^2 dx < \infty \right\},$$

endowed with the norm $\|u\|_{\mathcal{H}} := \|u\|_{H^1} + \|u\|_*$, here

$$\|u\|_*^2 = \int_{\mathbb{R}^2} \ln(1 + |x|^2)u^2 dx.$$

We now summarize our main results.

Theorem 1.1. *Assume that f satisfies*

- (A1) $f \in C(\mathbb{R}, \mathbb{R})$ and $f(0) = 0$,
- (A2) $f(e^{i\theta}z) = e^{i\theta}f(z)$,
- (A3) there exist z_1, z_2 and a constant $L > 0$ such that

$$|f(z_1) - f(z_2)| \leq L|z_1 - z_2|(1 + |z_1| + |z_2|)^2.$$

Then the Cauchy problem (1.1) is local well-posed in \mathcal{H} . That is, for any $\psi_0 \in \mathcal{H}$, there exists an existence time $T = T(\|\psi_0\|_{\mathcal{H}})$ and a unique solution $\psi \in C((-T, T); \mathcal{H}) \cap L^{q_0}((-T, T); L^{r_0}) \cap C^1((-T, T); \mathcal{H}')$ of (1.1), where (q_0, r_0) be an admissible pair with $r_0 > 2$.

Theorem 1.2. *Assume that condition (A1) holds. In addition, we assume that f satisfies*

- (A4) $\lim_{t \rightarrow 0} \frac{f(t)}{t} = 0$;
- (A5) $\limsup_{t \rightarrow \infty} \frac{f(t)t}{|t|^4} = 0$.

Then there exists a constant $c_* > 0$ such that for $0 < c < c_*$, the infimum

$$\mathbf{J}_c := \inf_{u \in S(c)} J(u)$$

is achieved by some $u_c \in S(c)$, i.e. $J(u_c) = \mathbf{J}_c$.

It is easy to find some examples on the nonlinearity f satisfying conditions (A1), (A4), and (A5), such as

$$f(t) = |t|^{p-2}t + |t|^{q-2}t \quad \text{with } 2 < q < p < 4.$$

By Theorem 1.2, we know that the set of ground states

$$\mathcal{M}_c := \{e^{i\lambda t}u(x) : u \in S(c) \text{ and } J(u) = \mathbf{J}_c\}$$

is not empty. Then we have the following stability result.

Theorem 1.3. *Under the assumptions of Theorems 1.1 and 1.2, the set of ground states \mathcal{M}_c is orbitally stable. That is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\psi_0 = \psi(0, x) \in \mathcal{H}$ satisfying $\inf_{u \in \mathcal{M}_c} \|\psi_0 - u\|_{\mathcal{H}} < \delta$, the solution $\psi(t, x)$ of system (1.1) satisfies*

$$\sup_{t \in [0, T)} \inf_{u \in \mathcal{M}_c} \|\psi(t, x) - u\|_{\mathcal{H}} < \varepsilon$$

where T is the maximal existence time for $\psi(t, x)$.

2. PRELIMINARY RESULTS

For sake of convenience, we set

$$A(u) := \int_{\mathbb{R}^2} |\nabla u|^2 dx \quad \text{and} \quad V(u) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x - y|) u^2(x) u^2(y) dx dy.$$

Then the energy functional J defined in (1.5) can be rewritten as

$$J(u) := \frac{1}{2}A(u) + \frac{1}{2} \int_{\mathbb{R}^2} \ln(1 + |x|^2) u^2(x) dx + \frac{\gamma}{8\pi} V(u) - \int_{\mathbb{R}^2} F(u) dx.$$

Definition 2.1. We say that a pair (q, r) is Strichartz admissible if $2 \leq r < \infty$ and $\frac{2}{q} = 1 - \frac{2}{r}$.

Lemma 2.2 (Strichartz estimates [6]). *For any $T > 0$, the following properties hold:*

(i) *let $\varphi \in L^2(\mathbb{R}^2)$. For any admissible pair (q, r) , we have*

$$\|e^{it\Delta}\varphi\|_{L^q((-T, T); L^r)} \lesssim \|\varphi\|_{L^2};$$

(ii) *let $I \subset (-T, T)$ be an interval and $t_0 \in \bar{I}$. For any admissible pairs (q, r) and (γ, ρ) , we have*

$$\left\| \int_{t_0}^t e^{i(t-s)\Delta} F(s) ds \right\|_{L^q(I; L^r)} \lesssim \|F\|_{L^{\gamma'}(I; L^{\rho'})}$$

for every $F \in L^{\gamma'}(I; L^{\rho'})$.

Lemma 2.3 ([21, Lemma 2.2]). *Let P be an arbitrary weight function satisfying $\nabla P, \Delta P \in L^\infty(\mathbb{R}^2)$. Then for all $T > 0$ and admissible pair (q, r) , we have*

$$\begin{aligned} \|\nabla[e^{it\mathbf{A}}]\varphi\|_{L^q((-T, T); L^r)} &\lesssim |T| \|\varphi\|_{L^2}, \\ \|[P, e^{it\mathbf{A}}]\varphi\|_{L^q((-T, T); L^r)} &\lesssim |T| \|(1 + \nabla)\varphi\|_{L^2}, \end{aligned}$$

where

$$\mathbf{A} := -\Delta + m \ln(1 + |x|^2) \tag{2.1}$$

with $m := \frac{\gamma}{2\pi} \|\psi_0\|_{L^2}^2 + 1$.

Lemma 2.4 ([21, Lemma 2.3]). *Let*

$$K(x, y) = \frac{\ln\left(\frac{|x-y|}{\langle x \rangle}\right)}{1 + \ln\langle y \rangle} \quad \text{for } x, y \in \mathbb{R}^2.$$

For any $p \in [1, \infty)$ and $\epsilon > 0$, there exist a function $H(x, y) \geq 0$ with $\|H\|_{L^\infty L^p_x} \leq \epsilon$ and a constant $C_0 > 0$ such that

$$|K(x, y)| \leq C_0 + H(x, y)$$

for all $(x, y) \in \mathbb{R}^{2+2}$.

Lemma 2.5 (Gagliardo-Nirenberg Inequality [17]). (i). *Let $r > 2$. Then there exists a sharp constant $K_{GN} > 0$ such that*

$$\|u\|_r \leq K_{GN}^{1/r} \|\nabla u\|_2^{\frac{r-2}{r}} \|u\|_2^{2/r}.$$

(ii) (Hardy-Littlewood-Sobolev inequality [30]). *Let $t, r > 1$ and $0 < \alpha < N$ with $\frac{1}{t} + \frac{N-\alpha}{N} + \frac{1}{r} = 2$. For $\bar{f} \in L^t(\mathbb{R}^N)$ and $\bar{h} \in L^r(\mathbb{R}^N)$, there exists a sharp constant $C(t, N, \alpha, r)$, independent of u and v , such that*

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\bar{f}(x)\bar{h}(x)}{|x-y|^{N-\alpha}} dx dy \leq C(t, N, \alpha, r) \|\bar{f}\|_t \|\bar{h}\|_r.$$

As in [15], we introduce the symmetric bilinear forms

$$B_1(u, v) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(1 + |x-y|^2) u(x)v(y) dx dy,$$

$$B_2(u, v) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln\left(1 + \frac{1}{|x-y|^2}\right) u(x)v(y) dx dy,$$

$$B_0(u, v) = \frac{1}{2} [B_1(u, v) - B_2(u, v)] = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x-y|) u(x)v(y) dx dy.$$

Clearly, $V(u) = B_0(u^2, u^2) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x-y|) u^2(x)u^2(y) dx dy$. By the continuous embedding from \mathcal{H} into $L^s(\mathbb{R}^2)$ for $s \in [2, \infty)$, the functionals $B_i(u^2, v^2)$ are well-defined on $\mathcal{H} \times \mathcal{H}$ for $i = 0, 1, 2$. Moreover, we define the associated functionals on \mathcal{H} as follows

$$V_1(u) = B_1(u^2, u^2) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(1 + |x-y|^2) u^2(x)u^2(y) dx dy,$$

$$V_2(u) = B_2(u^2, u^2) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln\left(1 + \frac{1}{|x-y|^2}\right) u^2(x)u^2(y) dx dy.$$

Lemma 2.6 ([15, Lemma 2.1]). *The following statements are valid:*

- (i) *the space \mathcal{H} is compactly embedded in $L^s(\mathbb{R}^2)$ for all $s \in [2, \infty)$;*
- (ii) *the functionals V, V_1, V_2 and J are of class C^1 on \mathcal{H} . Moreover, $V'_i(u)v = 4B_i(u^2, uv)$ for $u, v \in \mathcal{H}$ and $i = 1, 2$;*
- (iii) *V_2 is continuous (in fact continuously differentiable) on $L^{8/3}(\mathbb{R}^2)$;*
- (iv) *V_1 is weakly lower semi-continuous on $H^1(\mathbb{R}^2)$;*
- (v) *V is weakly lower semi-continuous on \mathcal{H} .*

3. LOCAL WELL-POSEDNESS OF THE CAUCHY PROBLEM

Following the ideas in [21], the following decomposition holds

$$\gamma\omega\psi = \frac{\gamma}{2\pi}\|\psi\|_{L^2}^2(\ln\langle x\rangle)\psi + \frac{\gamma}{2\pi}\psi \int_{\mathbb{R}^2} \ln\left(\frac{|x-y|}{\langle x\rangle}\right)|\psi(y)|^2 dy,$$

where $\langle x \rangle := (1 + |x|^2)$. According to the conservation of mass $\|\psi\|_{L^2} = \|\psi_0\|_{L^2}$, we have

$$m = \frac{\gamma}{2\pi}\|\psi_0\|_{L^2}^2 + 1 > 0.$$

Then, (1.1) is rewritten as

$$\begin{aligned} i\partial_t\psi + (-\Delta + m\ln\langle x\rangle)\psi &= -\frac{\gamma}{2\pi}\psi \int_{\mathbb{R}^2} \ln\left(\frac{|x-y|}{\langle x\rangle}\right)|\psi(y)|^2 dy + f(\psi), \quad \forall(t, x) \in \mathbb{R}^{1+2}, \\ \psi(0, x) &= \psi_0(x). \end{aligned}$$

We note that \mathbf{A} defined as (2.1) is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ (see [23]). Since $|\partial^\alpha(\ln\langle x\rangle)| \rightarrow 0$ as $|x| \rightarrow \infty$ for $|\alpha| = 2$ and $\partial^\alpha(\ln\langle x\rangle) \in L^\infty(\mathbb{R}^2)$ for $|\alpha| \geq 3$, the potential is subquadratic. Then for any $t \in [-T, T]$, we have

$$\|e^{it\mathbf{A}}\varphi\|_{L^\infty} \lesssim |t|^{-1}\|\varphi\|_{L^1},$$

(see [28]). Once we know this type of estimate, the Strichartz estimates follow by interpolation.

We are ready to prove Theorem 1.1. We write $L^p((-T, T); \mathcal{H}) = L_T^p\mathcal{H}$ for short. We define the Banach space

$$\mathcal{H}_{T, M} := \{\psi \in L_T^\infty\mathcal{H} \mid \|\psi\|_{\mathcal{H}_T} \leq M\}$$

with the norm

$$\|\psi\|_{\mathcal{H}_T} := \|\psi\|_{L_T^\infty\mathcal{H}} + \|\psi\|_{L_T^{q_0}W^{1, r_0}} + \|\sqrt{\ln\langle x\rangle}\psi\|_{L_T^{q_0}L^{r_0}}.$$

Now we show that if $r_0 > 2$, then there exist $M = M(\|\psi_0\|_{\mathcal{H}})$ and $T = T(\|\psi_0\|_{\mathcal{H}})$ such that

$$Q[\psi] := e^{it\mathbf{A}}\psi_0 + i \int_0^t e^{i(t-s)\mathbf{A}} \left(\frac{\gamma\psi}{2\pi} \int_{\mathbb{R}^2} \ln\left(\frac{|x-y|}{\langle x\rangle}\right)|\psi(y)|^2 dy - f(\psi) \right) ds$$

becomes a contraction map from $\mathcal{H}_{T, M}$ to itself. Set

$$K(x, y) := \frac{\ln\left(\frac{|x-y|}{\langle x\rangle}\right)}{1 + \ln\langle y\rangle}.$$

By Lemma 2.4, there exist a nonnegative function $H \in L_y^\infty L_y^{r_0'}$ and a constant $C_0 > 0$ such that

$$|K(x, y)| \leq C_0 + H(x, y).$$

Recall that $r_0 \in (2, \infty)$ and so $r_0' := r_0/(r_0 - 1) \in (1, 2)$. We hence see that

$$\omega\psi = \int K(x, y)(1 + \ln\langle y\rangle)|\psi(y)|^2\psi(x) dy$$

satisfying

$$\|\omega\psi\|_{L^2} \lesssim (\|\psi\|_{L^2} + \|\psi\|_{L^{r_0}})\|\sqrt{1 + \ln\langle x\rangle}\psi\|_{L^2}^2.$$

Taking the L_T^1 -norm one has

$$\|\omega\psi\|_{L_T^1 L^2} \lesssim \left(T\|\psi\|_{L_T^\infty L^2} + T^{\frac{r_0+2}{2r_0}}\|\psi\|_{L_T^{q_0} L^{r_0}} \right) \|\sqrt{1 + \ln\langle x\rangle}\psi\|_{L_T^\infty L^2}^2. \quad (3.1)$$

Note that $|f(\psi)| \leq L(1 + |\psi|)^2|\psi|$. Then we have

$$\begin{aligned} \|f\|_{L^2} &\lesssim \|\psi\|_{L^2} + \|\psi\|_{L^{2r_0/(r_0-2)}} \|\psi\|_{L^{r_0}} + \|\psi^2\|_{L^{2r_0/(r_0-2)}} \|\psi\|_{L^{r_0}} \\ &\lesssim \|\psi\|_{L^2} + (\|\psi\|_{L^2} + \|\nabla\psi\|_{L^2} + \|\psi\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2) \|\psi\|_{L^{r_0}}. \end{aligned}$$

Similarly, taking the L^1_T -norm yields

$$\begin{aligned} \|f\|_{L^1_T L^2} &\lesssim T^{\frac{r_0+2}{2r_0}} \left(\|\psi\|_{L^\infty_T L^2} + \|\nabla\psi\|_{L^\infty_T L^2} + \|\psi\|_{L^\infty_T L^2}^2 \right. \\ &\quad \left. + \|\nabla\psi\|_{L^\infty_T L^2}^2 \right) \|\psi\|_{L^{q_0}_T L^{r_0}} + T \|\psi\|_{L^\infty_T L^2}. \end{aligned} \tag{3.2}$$

By Strichartz estimates, we have

$$\begin{aligned} \|Q[\psi]\|_{L^\infty_T L^2} + \|Q[\psi]\|_{L^{q_0}_T L^{r_0}} \\ \lesssim \|\psi_0\|_{L^2} + T \|\psi\|_{\mathcal{H}_T} + T^{\frac{r_0+2}{2r_0}} \|\psi\|_{\mathcal{H}_T}^2 + (T + T^{\frac{r_0+2}{2r_0}}) \|\psi\|_{\mathcal{H}_T}^3. \end{aligned} \tag{3.3}$$

Next, we estimate $\nabla Q[\psi]$. It is easy to see that

$$\begin{aligned} \nabla Q[\psi] &= e^{it\mathbf{A}} \nabla \psi_0 + [\nabla, e^{it\mathbf{A}}] \psi_0 + \frac{i\gamma}{2\pi} \int_0^t e^{i(t-s)\mathbf{A}} (\nabla(\omega\psi) - \nabla f)(s) ds \\ &\quad + \frac{i\gamma}{2\pi} \int_0^t [\nabla, e^{i(t-s)\mathbf{A}}] (\omega\psi - f)(s) ds. \end{aligned}$$

From Lemma 2.3 with $(q, r) = (\infty, 2)$, we deduce that

$$\int_0^t \|[\nabla, e^{i(t-s)\mathbf{A}}] (\omega\psi)(s)\|_{L^2} ds \leq \int_0^t (t-s) \|\omega\psi(s)\|_{L^2} ds \leq |t| \|\omega\psi\|_{L^1_T L^2}.$$

Similarly, we have

$$\begin{aligned} \int_0^t \|[\nabla, e^{i(t-s)\mathbf{A}}] f(s)\|_{L^2} ds &\leq \int_0^t (t-s) \|f(s)\|_{L^2} ds \leq |t| \|f\|_{L^1_T L^2}, \\ \|[\nabla, e^{it\mathbf{A}}] \psi_0\|_{L^\infty_T L^2} &\lesssim T \|\psi_0\|_{L^2}. \end{aligned}$$

Similar to (3.1), we infer that

$$\|\omega \nabla \psi\|_{L^1_T L^2} \lesssim \left(T \|\nabla \psi\|_{L^\infty_T L^2} + T^{\frac{r_0+2}{2r_0}} \|\nabla \psi\|_{L^{q_0}_T L^{r_0}} \right) \|\sqrt{1 + \ln \langle x \rangle} \psi\|_{L^\infty_T L^2}^2. \tag{3.4}$$

Now, let us estimate $(\nabla \omega)\psi$. It can be written as

$$(\nabla \omega(x))\psi(x) = \left[\int_{\mathbb{R}^2} \left(\frac{x-y}{|x-y|^2} - \frac{2x}{1+x^2} \right) |\psi(y)|^2 dy \right] \psi(x).$$

It follows from the Hardy-Littlewood-Sobolev and the Sobolev inequalities that

$$\begin{aligned} \|(\nabla \omega)\psi\|_{L^2} &\lesssim \|(|x|^{-1} * |\psi|^2) + \langle \cdot \rangle^{-1} \|\psi\|_{L^2}^2\|_{L^{2r_0/(r_0-2)}} \|\psi\|_{L^{r_0}} \\ &\lesssim (\|\psi\|_{L^{2r_0/(r_0-1)}}^2 + \|\psi\|_{L^2}^2) \|\psi\|_{L^{r_0}} \\ &\lesssim (\|\psi\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2) \|\psi\|_{L^{r_0}}, \end{aligned}$$

which implies that

$$\|(\nabla \omega)\psi\|_{L^1_T L^2} \lesssim T^{\frac{r_0+2}{2r_0}} \left(\|\psi\|_{L^\infty_T L^2}^2 + \|\nabla\psi\|_{L^\infty_T L^2}^2 \right) \|\psi\|_{L^{q_0}_T L^{r_0}}. \tag{3.5}$$

By condition (A3) one has

$$\begin{aligned} \|\nabla f\|_{L^1_T L^2} &\lesssim T \|\nabla \psi\|_{L^\infty_T L^2} + T^{\frac{r_0+2}{2r_0}} (\|\psi\|_{L^\infty_T L^2} + \|\nabla\psi\|_{L^\infty_T L^2}) \|\nabla\psi\|_{L^{q_0}_T L^{r_0}} \\ &\quad + T^{\frac{r_0+2}{2r_0}} \left(\|\psi\|_{L^\infty_T L^2}^2 + \|\nabla\psi\|_{L^\infty_T L^2}^2 \right) \|\nabla\psi\|_{L^{q_0}_T L^{r_0}}. \end{aligned}$$

We deduce from the Strichartz estimates that

$$\begin{aligned} & \|\nabla Q[\psi]\|_{L_T^\infty L^2} + \|\nabla Q[\psi]\|_{L_T^{q_0} L^{r_0}} \\ & \lesssim T\|\nabla\psi_0\|_{\mathcal{H}} + T\|\psi\|_{\mathcal{H}_T} + T^{\frac{r_0+2}{2r_0}}\|\psi\|_{\mathcal{H}_T}^2 + \left(T + T^{\frac{r_0+2}{2r_0}}\right)\|\psi\|_{\mathcal{H}_T}^3. \end{aligned} \quad (3.6)$$

Let us proceed to the estimate

$$\sqrt{1 + \ln\langle x \rangle} Q[\psi] = e^{it\mathbf{A}} \sqrt{1 + \ln\langle x \rangle} \psi_0 + \frac{i\gamma}{2\pi} \int_0^t e^{i(t-s)\mathbf{A}} \sqrt{1 + \ln\langle x \rangle} (\omega\psi - f)(s) ds + R,$$

where

$$R := [\sqrt{1 + \ln\langle x \rangle}, e^{it\mathbf{A}}] \psi_0 + \frac{i\gamma}{2\pi} \int_0^t [\sqrt{1 + \ln\langle x \rangle}, e^{i(t-s)\mathbf{A}}] (\omega\psi - f)(s) ds. \quad (3.7)$$

Let $G = \sqrt{1 + \ln\langle x \rangle}$. It follows from Lemma 2.3 and (3.1)–(3.5) that

$$\begin{aligned} \|R\|_{L_T^\infty L^2} + \|R\|_{L_T^{q_0} L^{r_0}} & \lesssim T\|\psi_0\|_{\mathcal{H}} + T\|(1 + \nabla)(\omega\psi)\|_{L_T^1 L^2} + T\|(1 + \nabla)f\|_{L_T^1 L^2} \\ & \lesssim T\|\psi_0\|_{\mathcal{H}} + T\left(T + T^{\frac{r_0+2}{2r_0}}\right)\|\psi\|_{\mathcal{H}_T}^3 \\ & \quad + T\left(T\|\psi\|_{\mathcal{H}_T} + T^{\frac{r_0+2}{2r_0}}\|\psi\|_{\mathcal{H}_T}^2 + T^{\frac{r_0+2}{2r_0}}\|\psi\|_{\mathcal{H}_T}^3\right). \end{aligned}$$

As in (3.1), we have

$$\begin{aligned} \|\omega(G\psi)\|_{L_T^1 L^2} & \lesssim \left(T\|G\psi\|_{L_T^\infty L^2} + T^{\frac{r_0+2}{2r_0}}\|G\psi\|_{L_T^{q_0} L^{r_0}}\right)\|G\psi\|_{L_T^\infty L^2}^2 \\ & \lesssim \left(T + T^{\frac{r_0+2}{2r_0}}\right)\|\psi\|_{\mathcal{H}_T}^3, \end{aligned}$$

and

$$\begin{aligned} \|fG\|_{L_T^1 L^2} & \lesssim T\|G\psi\|_{L_T^\infty L^2} + T^{\frac{r_0+2}{2r_0}}\left(\|\psi\|_{L_T^\infty L^2} + \|\nabla\psi\|_{L_T^\infty L^2}\right)\|G\psi\|_{L_T^{q_0} L^{r_0}} \\ & \quad + T^{\frac{r_0+2}{2r_0}}\left(\|\psi\|_{L_T^\infty L^2}^2 + \|\nabla\psi\|_{L_T^\infty L^2}^2\right)\|\omega\psi\|_{L_T^{q_0} L^{r_0}} \\ & \lesssim T\|\psi\|_{\mathcal{H}_T} + T^{\frac{r_0+2}{2r_0}}\|\psi\|_{\mathcal{H}_T}^2 + T^{\frac{r_0+2}{2r_0}}\|\psi\|_{\mathcal{H}_T}^3. \end{aligned}$$

From the Strichartz estimates we have

$$\begin{aligned} & \|\sqrt{\ln\langle x \rangle} Q[\psi]\|_{L_T^\infty L^2} + \|\sqrt{\ln\langle x \rangle} Q[\psi]\|_{L_T^{q_0} L^{r_0}} \\ & \lesssim T\left[\|\psi_0\|_{\mathcal{H}} + (1 + T)\|\psi\|_{\mathcal{H}_T} + T^{\frac{r_0+2}{2r_0}}\|\psi\|_{\mathcal{H}_T}^2 + (T + T^{\frac{r_0+2}{2r_0}})\|\psi\|_{\mathcal{H}_T}^3\right]. \end{aligned} \quad (3.8)$$

Thus, it follows from (3.3), (3.6), and (3.8) that

$$\|Q[\psi]\|_{\mathcal{H}_T} \lesssim (1 + T)\left[\|\psi_0\|_{\mathcal{H}} + T\|\psi\|_{\mathcal{H}_T} + T^{\frac{r_0+2}{2r_0}}\|\psi\|_{\mathcal{H}_T}^2 + (T + T^{\frac{r_0+2}{2r_0}})\|\psi\|_{\mathcal{H}_T}^3\right].$$

A similar argument shows that

$$\begin{aligned} \|Q[\psi_1] - Q[\psi_2]\|_{\mathcal{H}_T} & \lesssim (1 + T)\left(T + T^{\frac{r_0+2}{2r_0}}\right)\left(\|\psi_1\|_{\mathcal{H}_T} + \|\psi_2\|_{\mathcal{H}_T}\right)^2 \\ & \quad + \|\psi_1\|_{\mathcal{H}_T} + \|\psi_2\|_{\mathcal{H}_T}\|\psi_1 - \psi_2\|_{\mathcal{H}_T}. \end{aligned}$$

Hence if we take $M \geq 2\|\psi_0\|_{\mathcal{H}}$, then there exists $T = T(M)$ such that Q is a contraction map from $\mathcal{H}_{T,M}$ to itself. A similar argument shows that Q has a unique fixed point in this space.

4. EXISTENCE OF A GLOBAL MINIMIZER

Lemma 4.1. *Assume that (A1), (A4), (A5) hold. Then there exists $c_* > 0$ such that the energy functional J is bounded from below on $S(c)$ for $0 < c < c_*$.*

Proof. Let $\varepsilon > 0$ be arbitrary. By conditions (A4) and (A5), there exists $C_\varepsilon > 0$ such that

$$|F(t)| \leq \varepsilon|t|^2 + C_\varepsilon|t|^4 \quad \text{for all } t \in \mathbb{R}.$$

For $u \in S(c)$, it follows from Lemma 2.5 that

$$\begin{aligned} \int_{\mathbb{R}^2} |F(u)|dx &\leq \varepsilon \int_{\mathbb{R}^2} |u|^2 dx + C_\varepsilon \int_{\mathbb{R}^2} |u|^4 dx \\ &\leq \varepsilon \int_{\mathbb{R}^2} |u|^2 dx + C_\varepsilon K_{GN} \int_{\mathbb{R}^2} |u|^2 dx \int_{\mathbb{R}^2} |\nabla u|^2 dx \\ &= c\varepsilon + cK_{GN}C_\varepsilon A(u). \end{aligned} \tag{4.1}$$

Since $0 < \ln(1+r) < r$ holds for all $r > 0$, by the Hardy-Littlewood-Sobolev inequality, there exists a constant $C > 0$ such that

$$\begin{aligned} \frac{1}{2}|V_2(u)| &= \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln\left(1 + \frac{1}{|x-y|^2}\right) u^2(x)u^2(y) dx dy \\ &\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln\left(1 + \frac{1}{|x-y|}\right) u^2(x)u^2(y) dx dy \\ &\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|} u^2(x)u^2(y) dx dy \\ &\leq Cc^{3/2}A(u)^{1/2}. \end{aligned} \tag{4.2}$$

From this, (4.1) and (4.2), we obtain

$$\begin{aligned} J(u) &\geq \frac{1}{2}A(u) + \int_{\mathbb{R}^2} \ln(1+|x|^2)u^2(x) dx + \frac{\gamma}{16\pi}(V_1(u) - V_2(u)) - \int_{\mathbb{R}^2} F(u) dx \\ &\geq \frac{1}{2}A(u) + \frac{1}{2} \int_{\mathbb{R}^2} \ln(1+|x|^2)u^2(x) dx - \frac{\gamma}{16\pi}V_2(u) - c\varepsilon - cK_{GN}C_\varepsilon A(u) \\ &\geq \frac{1}{2}A(u) + \frac{1}{2} \int_{\mathbb{R}^2} \ln(1+|x|^2)u^2(x) dx - \frac{\gamma}{8\pi}Cc^{3/2}A(u)^{1/2} \\ &\quad - c\varepsilon - cK_{GN}C_\varepsilon A(u) \\ &\geq \left(\frac{1}{2} - cC_\varepsilon K_{GN}\right)A(u) + \frac{1}{2} \int_{\mathbb{R}^2} \ln(1+|x|^2)u^2(x) dx \\ &\quad - \frac{\gamma}{8\pi}Cc^{3/2}A(u)^{1/2} - c\varepsilon, \end{aligned} \tag{4.3}$$

which implies that $J(u)$ is bounded from below on $S(c)$ when $c < c_* := \frac{1}{2K_{GN}C_\varepsilon}$. The proof is complete. \square

Proof of Theorem 1.2. By Lemma 4.1, we know that

$$\mathbf{J}_c = \inf_{u \in S(c)} J(u) > -\infty.$$

Then there exists a minimizing sequence $\{u_n\} \subset S(c)$ such that $\lim_{n \rightarrow \infty} J(u_n) = \mathbf{J}_c$. From (4.3) it follows that $A(u_n)$ and $\int_{\mathbb{R}^2} \ln(1+|x|^2)u_n^2 dx$ are bounded uniformly

in n . Since $\{u_n\} \in S(c)$, we can deduce that $\{u_n\}$ is bounded uniformly in \mathcal{H} . According to Lemma 2.6(i), it follows from that $u_c \in S(c)$, and

$$\int_{\mathbb{R}^2} F(u_n) dx \rightarrow \int_{\mathbb{R}^2} F(u_c) dx \quad \text{as } n \rightarrow \infty, \quad (4.4)$$

where we have used the Brezis-Lieb lemma [4]. Moreover, by Lemma 2.6(v), we have

$$\begin{aligned} & \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x-y|) u^2(x) u^2(y) dx dy \\ & \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x-y|) u_n^2(x) u_n^2(y) dx dy. \end{aligned} \quad (4.5)$$

Thus, by (4.4), (4.5) and the weakly lower semi-continuity, we obtain

$$\mathbf{J}_c \leq J(u_c) \leq \liminf_{n \rightarrow \infty} J(u_n) = \mathbf{J}_c,$$

which indicates that $J(u_c) = \mathbf{J}_c$, that is, u_c is a minimizer of \mathbf{J}_c for $c < c_*$.

Since $J(u_n) \rightarrow J(u_c)$ and $V_2(u_n) \rightarrow V_2(u_c)$ as $n \rightarrow \infty$, together with (4.4) again, we obtain

$$\begin{aligned} & \frac{1}{2} [A(u_n) - A(u_c)] + \frac{1}{2} \int_{\mathbb{R}^2} \ln(1+|x|^2) (u_n^2(x) - u_c^2(x)) dx \\ & + \frac{\gamma}{16\pi} [V_1(u_n) - V_1(u_c)] = o(1). \end{aligned}$$

Note that

$$\begin{aligned} & A(u_c) \leq \liminf_{n \rightarrow \infty} A(u_n), \quad V_1(u_c) \leq \liminf_{n \rightarrow \infty} V_1(u_n), \\ & \int_{\mathbb{R}^2} \ln(1+|x|^2) u_c^2 dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} \ln(1+|x|^2) u_n^2 dx. \end{aligned}$$

Then we have

$$\begin{aligned} & A(u_n) \rightarrow A(u_c) \quad \text{and} \quad V_1(u_n) \rightarrow V_1(u_c) \quad \text{as } n \rightarrow \infty, \\ & \int_{\mathbb{R}^2} \ln(1+|x|^2) u_n^2 dx \rightarrow \int_{\mathbb{R}^2} \ln(1+|x|^2) u_c^2 dx \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence, we deduce that $u_n \rightarrow u_c$ in \mathcal{H} . The proof is complete. \square

Proof of Theorem 1.3. Following the classical arguments of Cazenave and Lions [7], we assume that there exist an $\varepsilon_0 > 0$, $\{\delta_n\} \subset \mathbb{R}^+$ a decreasing sequence converging to 0, and $\{\psi_n\} \subset \mathcal{H}$ satisfying $\inf_{u \in \mathcal{M}_c} \|\psi_n(0, x) - u\|_{\mathcal{H}} < \delta_n$ such that

$$\inf_{u \in \mathcal{M}_c} \|\psi_n(t_n, x) - u\|_{\mathcal{H}} \geq \varepsilon_0,$$

where $\psi(t_n, x)$ is the unique solution of (1.1) with the initial value $\psi_n(0, x)$. We observe that $\|\psi_n(0, x)\|_{L^2}^2 \rightarrow c$ as $n \rightarrow \infty$ and that $J(\psi_n(0, x)) \rightarrow \mathbf{J}_c$ by the continuity of J . According to the conservation laws of the energy and mass, we have

$$\|\psi_n(t, x)\|_{L^2}^2 = \|\psi_n(0, x)\|_{L^2}^2 \rightarrow c \quad \text{as } n \rightarrow \infty, \quad (4.6)$$

$$J(\psi_n(t, x)) = J(\psi_n(0, x)) \rightarrow \mathbf{J}_c \quad \text{as } n \rightarrow \infty. \quad (4.7)$$

Now, let $\phi_n(t_n, x) = \frac{\sqrt{c} \psi_n(t, x)}{\|\psi_n(t, x)\|_{L^2}}$. Then by (4.6) one has $\|\phi_n(t_n, x)\|_{L^2}^2 = c$. Moreover, it follows from (4.7) that

$$J(\phi_n(t_n, x))$$

$$\begin{aligned}
 &= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \phi_n(t_n, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \ln(1 + |x|^2) \phi_n^2(t_n, x) dx \\
 &\quad + \frac{\gamma}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x - y|) |\phi_n(t_n, x)|^2 |\phi_n(t_n, y)|^2 dx dy - \int_{\mathbb{R}^2} F(\phi(t_n, x)) dx \\
 &= \frac{c}{2\|\psi_n(t, x)\|_{L^2}^2} \int_{\mathbb{R}^2} |\nabla \psi_n(t, x)|^2 dx \\
 &\quad + \frac{c}{2\|\psi_n(t, x)\|_{L^2}^2} \int_{\mathbb{R}^2} \ln(1 + |x|^2) \psi_n^2(t, x) dx \\
 &\quad + \frac{\gamma c^2}{8\pi\|\psi_n(t, x)\|_{L^2}^4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x - y|) |\psi_n(t, x)|^2 |\psi_n(t, y)|^2 dx dy \\
 &\quad - \int_{\mathbb{R}^2} F\left(\frac{\sqrt{c}\psi_n(t, x)}{\|\psi_n(t, x)\|_{L^2}}\right) dx \\
 &\rightarrow \mathbf{J}_c \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

So, $\{\phi_n(t_n, x)\}$ is a minimizing sequence to \mathbf{J}_c . Thus, there exists $\tilde{u} \in S(c)$ such that

$$\|\phi_n(t_n, x) - \tilde{u}\|_{\mathcal{H}} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{4.8}$$

Since

$$\begin{aligned}
 &\|\phi_n(t_n, x) - \tilde{u}\|_{\mathcal{H}} \\
 &= \|\phi_n(t_n, x) - \tilde{u}\|_{H^1} + \left(\int_{\mathbb{R}^2} \ln(1 + |x|^2) |\phi_n(t_n, x) - \tilde{u}|^2 dx\right)^{1/2} \\
 &= \left\| \frac{\sqrt{c}\psi_n(t, x)}{\|\psi_n(t, x)\|_{L^2}} - \tilde{u} \right\|_{H^1} \\
 &\quad + \left(\int_{\mathbb{R}^2} \ln(1 + |x|^2) \left| \frac{\sqrt{c}\psi_n(t, x)}{\|\psi_n(t, x)\|_{L^2}} - \tilde{u} \right|^2 dx\right)^{1/2} \\
 &\geq \inf_{u \in \mathcal{M}_c} \|\psi_n(t_n, x) - u\|_{\mathcal{H}} \geq \varepsilon_0,
 \end{aligned}$$

which contradicts with (4.8). The proof is complete. □

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