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PARAMETER-DEPENDENT PERIODIC PROBLEMS FOR NON-AUTONOMOUS DUFFING EQUATIONS WITH SIGN-CHANGING FORCING TERM

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Abstract. We study the existence, exact multiplicity, and structure of the set of positive solutions to the periodic problem

$$u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u + \mu f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega),$$

where $\mu \in \mathbb{R}$ is a parameter. We assume that $p, h, f \in L([0, \omega])$, $\lambda > 1$, and the function h is non-negative. The results obtained extend the results known in the existing literature. We do not require that the Green's function of the corresponding linear problem be positive and we allow the forcing term f to change its sign.

1. Statement of the problem

We consider the periodic problem

$$u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u + \mu f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{1.1}$$

where $p, h, f \in L([0, \omega])$, $h \geq 0$ a.e. on $[0, \omega]$, $\lambda > 1$, and $\mu \in \mathbb{R}$ is a parameter. By a solution to problem (1.1), as usual, we understand a function $u\colon [0, \omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies the given equation almost everywhere, and meets the periodic conditions.

In [11], we considered problem (1.1) with $\mu = 0$ and we showed, among other things, that for the existence of a positive solution it is necessary that $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. Using a technique developed in [11], we provided in [15] effective conditions for the existence and exact multiplicity of positive solutions to the periodic problem for a non-autonomous Duffing equation with a sign-changing forcing term, i.e., problem (1.1) with $\mu = 1$. In the present paper, we conclude our studies and show, in the case of $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, the existence/non-existence as well as the exact multiplicity of sign-constant solutions to problem (1.1) depending on the choice of the parameter μ. The results obtained are compared with the results known for the autonomous case and the results available in the existing literature.

For the results covering the multiplicity and local/global bifurcations of periodic solutions to super-linear equations (and their systems), we refer the readers, for instance, to [1, 2, 3, 4, 6, 8, 12, 13] (see also the references therein). We studied

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a bifurcation of positive solutions to problem (1.1), with the non-positive function h, in [14].

In [2], the authors study the parameter-dependent problem

$$x'' + cx' + a(t)x - b(t)x^3 = \lambda d(t); \quad x(0) = x(T),\ x'(0) = x'(T), \tag{1.2}$$

where $c > 0$, $\lambda \in \mathbb{R}$ is a parameter, and $a, b, d\colon [0,T] \to \mathbb{R}$ are continuous functions such that

$$a(t) \le \frac{\pi^2}{T^2} + \frac{c^2}{4} \quad \text{for } t \in [0,T], \quad \int_0^T a(s)\,\mathrm{d}s > 0, \tag{1.3}$$

and

$$b(t) > 0, \quad d(t) > 0 \quad \text{for } t \in [0,T]. \tag{1.4}$$

Theorem 1.1 ([2, Theorem 1.1])**.** *Assume that* (1.3) *and* (1.4) *hold. Then, all solutions to* (1.2) *are of one sign and there is* $\lambda_0 > 0$ *such that*

(1) *problem* (1.2) *has a unique solution which is negative (positive) and unstable for* $\lambda > \lambda_0$ *(*$\lambda < -\lambda_0$*),*
(2) *problem* (1.2) *has exactly three ordered solutions for* $|\lambda| < |\lambda_0|$*. Moreover, the middle solution is asymptotically stable and the remaining two are unstable. When* $-\lambda_0 < \lambda < 0$*, the maximal solution is positive and the other two are negative. When* $\lambda = 0$*, problem* (1.2) *has one positive, one* 0*, and one negative solution. When* $0 < \lambda < \lambda_0$*, the minimal solution is negative and the other two are positive.*
(3) *problem* (1.2) *has exactly two one-signed solutions for* $\lambda = \pm\lambda_0$*; both of them are unstable.*

Recently, Liang [8] proved the conclusion of Theorem 1.1 under the positivity of a, b, d and the hypothesis $\|a\|_p \le (1 + c^2)K(2p^*)$ with some $p \ge 1$. It seems from the proof of Theorem 1.1 that its conclusions, which concern the existence and multiplicity of solutions, remain true even in the case of $c = 0$.

In Section 3, we extend the conclusions of Theorem 1.1 for the case of undamped Duffing equation (i.e., for $c = 0$). Moreover, we weaken hypotheses (1.3) and (1.4). In particular, (1.3) is replaced by a weaker assumption $-a \in \mathcal{V}^+(T)$ (see Definition 2.1), b may be equal to zero on a set of positive measure, and d may change its sign so that $(-a, d) \in \mathcal{U}(T)$ (see Definition 2.7). Furthermore, we prove the existence/non-existence of solutions to problem (1.2), with $c = 0$, depending on the choice of the parameter λ in the case of $a(t) > \frac{\pi^2}{T^2}$ on a set of positive measure.

At the end of this section, we show, as a motivation, what happens in the autonomous case of (1.1). If $p(t) := -a$, then $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ if and only if $a > 0$ (see Remark 2.4). Therefore, we consider the equation

$$x'' = -ax + b|x|^\lambda \operatorname{sgn} x + \mu, \tag{1.5}$$

where $a > 0$ and $b, \mu \in \mathbb{R}$. In this paper, we are interested in the equation in (1.1) with a non-negative h and, thus, we assume that $b > 0$ in (1.5). By direct calculation, the phase portraits of this equation can be elaborated depending on the choice of the parameter μ and, thus, one can prove the following proposition concerning periodic solutions to equation (1.5).

Proposition 1.2. *Let* $\lambda > 1$ *and* $a, b > 0$*. Then, the following conclusions hold:*

(i) *If* $\mu > \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$*, then equation* (1.5) *has a unique negative equilibrium (saddle) and no other periodic solutions occur.*

(ii) *If* $\mu = \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, *then equation* (1.5) *has a unique positive equilibrium (cusp), a unique negative equilibrium (saddle), and no other periodic solutions occur.*

(iii) *If* $0 < \mu < \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, *then equation* (1.5) *possesses exactly two positive equilibria* $x_1 > x_2$ *(*x_1 *is a saddle and* x_2 *is a center), a unique negative equilibrium* x_3 *(saddle), and non-constant (both positive and sign-changing) periodic solutions with different periods. Moreover, all non-constant periodic solutions oscillate around* x_2 *between* x_3 *and* x_1.

(iv) *If* $\mu = 0$, *then equation* (1.5) *possesses a unique positive equilibrium* x_0 *(saddle), a trivial equilibrium (center), a unique negative equilibrium* $-x_0$, *and non-constant sign-changing periodic solutions with different periods. Moreover, all non-constant periodic solutions oscillate around* 0 *between* $-x_0$ *and* x_0.

(v) *If* $-\frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}} < \mu < 0$, *then equation* (1.5) *possesses exactly two negative equilibria* $x_1 < x_2$ *(*x_1 *is a saddle and* x_2 *is a center), a unique positive equilibrium* x_3 *(saddle), and non-constant (both positive and sign-changing) periodic solutions with different periods. Moreover, all non-constant periodic solutions oscillate around* x_2 *between* x_1 *and* x_3.

(vi) *If* $\mu = -\frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, *then equation* (1.5) *has a unique negative equilibrium (cusp), a unique positive equilibrium (saddle), and no other periodic solutions occur.*

(vii) *If* $\mu < -\frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, *then equation* (1.5) *has a unique positive equilibrium (saddle) and no other periodic solutions occur.*

2. Notation and definitions

The following notation is used throughout this article:

- $\mathbb{R}$ is the set of real numbers. For $x \in \mathbb{R}$, we put $[x]_+ = \frac{1}{2}(|x| + x)$ and $[x]_- = \frac{1}{2}(|x| - x)$.
- $C(I)$ denotes the set of continuous real functions defined on the interval $I \subseteq \mathbb{R}$. For $u \in C([a,b])$, we put $\|u\|_C = \max\{|u(t)| : t \in [a,b]\}$.
- $AC^1([a,b])$ is the set of functions $u\colon [a,b] \to \mathbb{R}$ which are absolutely continuous together with their first derivatives.
- $AC_\ell([a,b])$ (resp. $AC_u([a,b])$) is the set of absolutely continuous functions $u\colon [a,b] \to \mathbb{R}$ such that u' admits the representation $u'(t) = \gamma(t) + \sigma(t)$ for a. e. $t \in [a,b]$, where $\gamma\colon [a,b] \to \mathbb{R}$ is absolutely continuous and $\sigma\colon [a,b] \to \mathbb{R}$ is a non-decreasing (resp. non-increasing) function whose derivative is equal to zero almost everywhere on $[a,b]$.
- $L([a,b])$ is the Banach space of Lebesgue integrable functions $p\colon [a,b] \to \mathbb{R}$ equipped with the norm $\|p\|_L = \int_a^b |p(s)|\,\mathrm{d}s$. The symbol $\operatorname{Int} A$ stands for the interior of the set $A \subset L([a,b])$.

Definition 2.1 ([10, Definition 0.1])**.** We say that a function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}^+(\omega)$ (resp. $\mathcal{V}^-(\omega)$) if, for any function $u \in AC^1([0,\omega])$ satisfying

$$u''(t) \ge p(t)u(t) \quad \text{for a.e. } t \in [0,\omega], \quad u(0) = u(\omega), \quad u'(0) = u'(\omega),$$

the inequality

$$u(t) \ge 0 \quad \text{for } t \in [0,\omega] \quad \big(\text{resp. } u(t) \le 0 \quad \text{for } t \in [0,\omega]\big)$$

holds.

Remark 2.2. In an alternative terminology, $p \in \mathcal{V}^-(\omega)$ (resp. $p \in \mathcal{V}^+(\omega)$) means that the maximum principle (resp. the anti-maximum principle) holds for the linear periodic problem

$$u'' = p(t)u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega). \tag{2.1}$$

Definition 2.3 ([10, Definition 0.2])**.** *We say that a function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}_0(\omega)$ if problem* (2.1) *has a positive solution.*

Remark 2.4. *Let $\omega > 0$. If $p(t) := p_0$ for $t \in [0,\omega]$, then one can show by direct calculation that:*

- $\triangleright$ *$p \in \mathcal{V}^-(\omega)$ if and only if $p_0 > 0$,*
- $\triangleright$ *$p \in \mathcal{V}_0(\omega)$ if and only if $p_0 = 0$,*
- $\triangleright$ *$p \in \mathcal{V}^+(\omega)$ if and only if $p_0 \in \big[-\frac{\pi^2}{\omega^2}, 0\big[$,*
- $\triangleright$ *$p \in \operatorname{Int}\mathcal{V}^+(\omega)$ if and only if $p_0 \in \big]-\frac{\pi^2}{\omega^2}, 0\big[$.*

When the function $p \in L([0,\omega])$ is not constant, efficient conditions for p to belong to each of the sets $\mathcal{V}^+(\omega)$ and $\mathcal{V}^-(\omega)$ are provided in [10] *(see also* [1, 16]*).*

Remark 2.5. It is well known that, if the homogeneous problem (2.1) has only the trivial solution, then, for any $f \in L([0,\omega])$, the problem

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{2.2}$$

possesses a unique solution u and this solution satisfies

$$|u(t)| \le \Delta(p) \int_0^\omega |f(s)|\,\mathrm{d}s \quad \text{for } t \in [0,\omega],$$

where $\Delta(p)$, depending only on p, denotes a norm of the Green's operator of problem (2.1). Clearly, $\Delta(p) > 0$.

Assuming that $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, we extend the function p periodically to the whole real axis denoting it by the same symbol. It is proved in [10, Section 6] that, for any $a \in \mathbb{R}$, the problem

$$u'' = p(t)u; \quad u(a) = 1,\ u(a+\omega) = 1$$

has a unique solution u_a and $u_a(t) > 0$ for $t \in [0,\omega]$. We put

$$\Gamma(p) := \sup\big\{\|u_a\|_C : a \in [0,\omega]\big\} \mathrm{e}^{\int_0^\omega [p(s)]_+\,\mathrm{d}s}. \tag{2.3}$$

It is clear that $\Gamma(p) \ge 1$.

Remark 2.6. If $p \in \mathcal{V}^+(\omega)$, then the number $\Delta(p)$ defined in Remark 2.5 can be estimated, for example, by the a maximal value of the Green's function of problem (2.1) (see, e.g., [16]). On the other hand, assuming $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, some estimates of the number $\Gamma(p)$ are provided in [10, Section 6].

For instance, if $p(t) := p_0$ for $t \in [0,\omega]$ and $p_0 \in [-\frac{\pi^2}{\omega^2}, 0[$, resp. $p_0 \in \big]-\frac{\pi^2}{\omega^2}, 0\big[$, then

$$\Delta(p) \le \Big(2\sqrt{|p_0|}\sin\frac{\omega\sqrt{|p_0|}}{2}\Big)^{-1}, \quad \text{resp.} \quad \Gamma(p) = \Big(\cos\frac{\omega\sqrt{|p_0|}}{2}\Big)^{-1}.$$

Definition 2.7 ([10, Definition 16.1])**.** Let $p, f \in L([0,\omega])$. We say that a pair (p,f) belongs to the set $\mathcal{U}(\omega)$, if problem (2.1) has a unique solution which is positive.

3. Main results

This section contains formulations of all the main results of the paper. Their proofs are presented in detail in Section 5.

We start with the most general statement of the paper, which provides the existence/non-existence results in the case of $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. This condition is satisfied, for instance, if

$$\int_0^\omega p(s)\,\mathrm{d}s \le 0, \quad p(t) \not\equiv 0$$

(see Lemma 4.15). Note also that, for the Duffing equation with the constant coefficients

$$x'' + ax - bx^3 = \mu f(t),$$

the above-mentioned condition is satisfied if and only if $a > 0$.

Theorem 3.1. *Let $\lambda > 1$, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $f(t) \not\equiv 0$, and*

$$h(t) > 0 \quad \text{for a.e. } t \in [0, \omega]. \tag{3.1}$$

Then, there exist $-\infty \le \mu_* < 0$ and $0 < \mu^* \le +\infty$ such that the following conclusions hold:

(1) *For any $\mu \in]\mu_*, \mu^*[$, problem (1.1) has a positive solution u^* such that every solution u to problem (1.1) satisfies*

$$\text{either} \quad u(t) < u^*(t) \quad \text{for } t \in [0, \omega], \quad \text{or} \quad u(t) \equiv u^*(t). \tag{3.2}$$

Moreover, for any couple of distinct positive solutions u_1, u_2 to (1.1) satisfying

$$u_1(t) \not\equiv u^*(t), \quad u_2(t) \not\equiv u^*(t), \tag{3.3}$$

the conditions

$$\begin{aligned} &\min\{u_1(t) - u_2(t) : t \in [0, \omega]\} < 0, \\ &\max\{u_1(t) - u_2(t) : t \in [0, \omega]\} > 0 \end{aligned} \tag{3.4}$$

hold.

(2) *If $\mu^* < +\infty$, then*
  (a) *for $\mu > \mu^*$, problem (1.1) has no positive solution,*
  (b) *for $\mu = \mu^*$, problem (1.1) has a unique non-negative solution u^* and every solution u to (1.1) satisfies (3.2).*

(3) *If $\mu_* > -\infty$, then*
  (a) *for $\mu < \mu_*$, problem (1.1) has no positive solution,*
  (b) *for $\mu = \mu_*$, problem (1.1) has a unique non-negative solution u^* and every solution u to (1.1) satisfies (3.2).*

(4) *If $\int_0^\omega f(s)\,\mathrm{d}s > 0$ (resp. $\int_0^\omega f(s)\,\mathrm{d}s < 0$), then $\mu^* < +\infty$ (resp. $\mu_* > -\infty$).*

Corollary 3.2. *Let $\lambda > 1$, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $f(t) \not\equiv 0$, and condition (3.1) hold. Then, there exists $0 < \mu_0 < +\infty$ such that, for any $\mu \in] - \mu_0, \mu_0[$, problem (1.1) has a negative solution u_* and a positive solution u^* such that every solution u to problem (1.1) different from u_*, u^* satisfies*

$$u_*(t) < u(t) < u^*(t) \quad \text{for } t \in [0, \omega]. \tag{3.5}$$

Remark 3.3. The conclusions of Theorem 3.1 and Corollary 3.2 extend the conclusions of Proposition 1.2 for non-autonomous Duffing equations with a sign-changing forcing term. Indeed, let $\omega > 0$ and

$$p(t) := -a, \quad h(t) := b, \quad f(t) := 1 \quad \text{for } t \in [0, \omega],$$

where $a, b > 0$. Then, condition (3.1) holds and, by Remark 2.4, we obtain $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. We emphasize, in particular, the conclusion of Corollary 3.2 which claims that there exists $0 < \mu_0 < +\infty$ such that, for any $\mu \in]-\mu_0, \mu_0[$, equation (1.5) has a maximal (resp. a minimal) ω-periodic solution which is positive (resp. negative); compare it with conclusions (iii), (iv), (v) of Proposition 1.2.

We now provide a lower (resp. an upper) estimate of the number μ^* (resp. μ_*) appearing in the conclusion of Theorem 3.1.

Proposition 3.4. *Let $\lambda > 1$, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $f(t) \not\equiv 0$, h satisfy* (3.1)*, and μ_*, μ^* be the numbers appearing in the conclusion of Theorem 3.1. If $[f(t)]_+ \not\equiv 0$, then*

$$\mu^* \geq \frac{1}{\int_0^\omega [f(s)]_+ \,\mathrm{d}s} \sup \Big\{ \frac{r}{\Delta\big(p + r^{\lambda-1} h\big)} : r > 0, \; p + r^{\lambda-1} h \in \mathcal{V}^+(\omega) \Big\}, \tag{3.6}$$

and, if $[f(t)]_- \not\equiv 0$, then

$$\mu_* \leq -\frac{1}{\int_0^\omega [f(s)]_- \,\mathrm{d}s} \sup \Big\{ \frac{r}{\Delta\big(p + r^{\lambda-1} h\big)} : r > 0, \; p + r^{\lambda-1} h \in \mathcal{V}^+(\omega) \Big\}, \tag{3.7}$$

where Δ is defined in Remark 2.5.

Remark 3.5. Let $\lambda > 1$, $\omega > 0$, and

$$p(t) := -a, \quad h(t) := b \quad \text{for } t \in [0, \omega], \tag{3.8}$$

where $a, b > 0$, and

$$\Phi(a, b, \lambda, \omega) := \begin{cases} \frac{2\omega}{\pi} \, \frac{(\lambda-1)a}{\lambda} \big(\frac{a}{\lambda b}\big)^{\frac{1}{\lambda-1}} & \text{if } a < \frac{\lambda}{\lambda-1} \left(\frac{\pi}{\omega}\right)^2, \\ \frac{2\pi}{\omega} \big[\frac{1}{b}\big(a - \frac{\pi^2}{\omega^2}\big)\big]^{\frac{1}{\lambda-1}} & \text{if } a \geq \frac{\lambda}{\lambda-1} \big(\frac{\pi}{\omega}\big)^2. \end{cases}$$

It follows from the proof of [15, Corollary 3.19] that, if $[f(t)]_+ \not\equiv 0$ and $[f(t)]_- \not\equiv 0$, then

$$\mu^* \geq \frac{\Phi(a, b, \lambda, \omega)}{\int_0^\omega [f(s)]_+ \,\mathrm{d}s}, \quad \mu_* \leq -\frac{\Phi(a, b, \lambda, \omega)}{\int_0^\omega [f(s)]_- \,\mathrm{d}s}.$$

If

$$f(t) \geq 0 \quad \text{for } t \in [0, \omega], \quad f(t) \not\equiv 0, \tag{3.9}$$

then it follows from [15, Theorem 3.15(3)] that, for any $\mu > 0$, problem (1.1) has a unique negative solution. Therefore, the conclusions of Theorem 3.1 can be refined as follows.

Theorem 3.6. *Let $\lambda > 1$, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ and conditions* (3.1) *and* (3.9) *be fulfilled. Then, there exists $0 < \mu_0 < +\infty$ such that the following conclusions hold:*

1. *For any $\mu > \mu_0$, problem* (1.1) *has a unique negative solution u_* and no positive solution. Moreover, every solution u to* (1.1) *satisfies*

$$\text{either} \quad u(t) > u_*(t) \quad \text{for } t \in [0, \omega], \quad \text{or} \quad u(t) \equiv u_*(t). \tag{3.10}$$

2. *For $\mu = \mu_0$, problem* (1.1) *has a unique negative solution u_* and a unique non-negative solution u^*. Moreover, every solution u to problem* (1.1) *different from u_*, u^* satisfies* (3.5).

(3) *For* $\mu \in]0, \mu_0[$, *problem* (1.1) *has a unique negative solution* u_* *and a positive solution* u^* *such that every solution* u *to problem* (1.1) *different from* u_*, u^* *satisfies* (3.5).
(4) *For* $\mu = 0$, *problem* (1.1) *has a unique positive solution* u_0, *the trivial solution, and a unique negative solution* $-u_0$. *Moreover, every solution* u *to problem* (1.1) *different from* u_*, u^* *changes its sign and satisfies* (3.5).
(5) *For* $\mu \in]-\mu_0, 0[$, *problem* (1.1) *has a unique negative solution* u_* *and a positive solution* u^* *such that every solution* u *to problem* (1.1) *different from* u_*, u^* *satisfies* (3.5).
(6) *For* $\mu = -\mu_0$, *problem* (1.1) *has a unique non-positive solution* u_* *and a unique positive solution* u^*. *Moreover, every solution* u *to problem* (1.1) *different from* u_*, u^* *satisfies* (3.5).
(7) *For any* $\mu < -\mu_0$, *problem* (1.1) *has a unique positive solution* u^* *an no negative solution. Moreover, every solution* u *to* (1.1) *satisfies* (3.2).

Remark 3.7. It follows from Theorem 3.1(1) that, in Theorem 3.6(3,5), if u_1, u_2 are distinct positive (resp. negative) solutions to problem (1.1) different from u^* (resp. u_*), then conditions (3.4) hold.

Remark 3.8. Let $\omega > 0$ and

$$p(t) := -a, \quad h(t) := b, \quad f(t) := 1 \quad \text{for } t \in [0, \omega],$$

where $a, b > 0$. Then, conditions (3.1) and (3.9) hold, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ (see Remark 2.4), and all the conclusions of Theorem 3.6 are in compliance with those in Proposition 1.2.

We showed in [11, Example 2.8] that assuming $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, hypothesis (3.1) in Theorems 3.1 and 3.6 (i.e. the positivity of h a. e. on $[0, \omega]$) is essential for the existence of a positive solution to problem (1.1) with $\mu = 0$ and cannot be weakened to the non-negativity of h. However, under a stronger assumption on the coefficient p, namely, $p \in \mathcal{V}^+(\omega)$, hypothesis (3.1) of Theorems 3.1 and 3.6 can be relaxed to

$$h(t) \geq 0 \quad \text{for a.e. } t \in [0, \omega], \quad h(t) \not\equiv 0. \tag{3.11}$$

Theorem 3.9. *Let* $\lambda > 1$, $p \in \mathcal{V}^+(\omega)$, h *satisfy* (3.11), *and*

$$(p, f) \in \mathcal{U}(\omega), \quad \int_0^\omega f(s)\,\mathrm{d}s > 0. \tag{3.12}$$

Then, there exist $-\infty \leq \mu_* < 0$ *and* $0 < \mu^* < +\infty$ *such that the following conclusions hold:*

(1) *For any* $\mu > \mu^*$, *problem* (1.1) *has no positive solution.*
(2) *For* $\mu = \mu^*$, *problem* (1.1) *has a unique positive solution* u^* *and, moreover, every solution* u *to problem* (1.1) *satisfies* (3.2).
(3) *For* $\mu \in]0, \mu^*[$, *problem* (1.1) *has exactly two positive solutions* u_1, u_2 *and these solutions satisfy*
$$u_1(t) > u_2(t) > 0 \quad \text{for } t \in [0, \omega]. \tag{3.13}$$
Moreover, every solution u *to problem* (1.1) *different from* u_1 *is such that*
$$u(t) < u_1(t) \quad \text{for } t \in [0, \omega]. \tag{3.14}$$
(4) *For* $\mu = 0$, *problem* (1.1) *has exactly three solutions: a positive solution* u_0, *the trivial solution, a negative solution* $-u_0$.

(5) *For* $\mu \in]\mu_*, 0[$, *problem* (1.1) *has either one or two positive solutions. Moreover,* (1.1) *has a positive solution* u^* *such that every solution to problem* (1.1) *satisfies* (3.2).
(6) *If* $\mu_* > -\infty$, *then, for any* $\mu < \mu_*$, *problem* (1.1) *has no positive solution.*

Remark 3.10. Assume that hypotheses of Theorem 3.9 hold and $\mu_* > -\infty$. If, moreover, $h(t) > 0$ for a. e. $t \in [0, \omega]$, then it follows from Theorem 3.1(3b) that problem (1.1) with $\mu = \mu_*$ has a unique non-negative solution u^* and, moreover, every solution to (1.1) with $\mu = \mu_*$ satisfies (3.2).

Open questions. The following two questions remain open in Theorem 3.9:
(1) Does the inequality $\mu_* > -\infty$ hold without any additional assumption?
(2) What happens in the case of $\mu = \mu_*$, if $\mu_* > -\infty$ and $h(t) = 0$ on a set of positive measure?

Remark 3.11. It is proved in [10, Theorem 16.4] that, if $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, then the inclusion $(p, f) \in \mathcal{U}(\omega)$ holds for every function $f \in L([0, \omega])$ satisfying $f(t) \not\equiv 0$ and

$$\int_0^\omega [f(s)]_+ \,\mathrm{d}s \geq \Gamma(p) \int_0^\omega [f(s)]_- \,\mathrm{d}s,$$

where Γ is given by (2.3).

On the other hand, if $p \in \mathcal{V}^+(\omega)$ and f satisfies (3.9), then $(p, f) \in \mathcal{U}(\omega)$ as well (see [10, Remark 9.2]).

Remark 3.12. In [1], to show a possible use of the main results, the authors consider the parameter-dependent periodic problem for the forced Mathieu-Duffing equation

$$z'' = -(e + b\cos(t))z + \nu z^3 + c(t); \quad z(0) = z(2\pi),\ z'(0) = z'(2\pi), \tag{3.15}$$

where $e \geq 0$ and $b \in \mathbb{R}$ are such that $e + |b| > 0$ and

$$\|[e + b\cos(\cdot)]_+\|_{L^\alpha} \leq \max\big\{K(2\alpha^*, 2\pi) : \alpha \geq 1\big\},$$

K is the so-called best Sobolev constant, c satisfies $\big(-(e+b\cos(\cdot)), c\big) \in \mathcal{U}(2\pi)$, and $\nu \in \mathbb{R}$ is a parameter. It is proved in [1, Corollary 45] that *there exits* $\nu_0 > 0$ *such that problem* (3.15) *has at least two positive solutions provided that* $0 < \nu < \nu_0$. Putting $u(t) := \sqrt{\nu}\, z(t)$, problem (3.15) is equivalent, in some sense, with problem (1.1) in which $p(t) := -(e + b\cos(t))$, $h(t) := 1$, $f(t) := c(t)$, $\lambda := 3$, and $\mu := \sqrt{\nu}$.

Since $-(e + b\cos(\cdot)) \in \mathcal{V}^+(\omega)$ in the case considered, Theorem 3.9 complements the conclusion of [1, Corollary 45] as follows: *There exists* $\nu_0 > 0$ *such that problem* (3.15) *has exactly two positive solutions provided that* $0 < \nu < \nu_0$, *a unique positive solution provided that* $\nu = \nu_0$, *and no positive solution provided that* $\nu > \nu_0$.

Theorem 3.9 guarantees the existence of certain "critical" values μ_*, μ^* of the parameter μ such that crossing these values, a bifurcation of positive solutions to problem (1.1) occurs. From an application point of view, the estimates of these numbers are also needed.

Proposition 3.13. *Let* $\lambda > 1$, $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, h *satisfy* (3.11), *and*

$$\int_0^\omega [f(s)]_+ \,\mathrm{d}s > \Gamma(p) \int_0^\omega [f(s)]_- \,\mathrm{d}s > 0, \tag{3.16}$$

where Γ is given by (2.3)*. Then, the numbers μ_*, μ^* appearing in the conclusion of Theorem 3.9 satisfy*

$$\mu_* \leq -\frac{(\lambda-1)\,[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\lambda\left[\lambda\int_0^\omega h(s)\,\mathrm{d}s\right]^{\frac{1}{\lambda-1}}\int_0^\omega [f(s)]_-\,\mathrm{d}s}\,, \tag{3.17}$$

$$\mu^* \geq \frac{(\lambda-1)\,[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\lambda\left[\lambda\int_0^\omega h(s)\,\mathrm{d}s\right]^{\frac{1}{\lambda-1}}\int_0^\omega [f(s)]_+\,\mathrm{d}s}\,, \tag{3.18}$$

where Δ is defined in Remark 2.5, and

$$\mu^* < \frac{(\lambda-1)[\Gamma(p)\int_0^\omega [p(s)]_-\,\mathrm{d}s - \int_0^\omega [p(s)]_+\,\mathrm{d}s]^{\frac{\lambda}{\lambda-1}}}{\lambda[\lambda\int_0^\omega h(s)\,\mathrm{d}s]^{\frac{1}{\lambda-1}}\left[\int_0^\omega [f(s)]_+\,\mathrm{d}s - \Gamma(p)\int_0^\omega [f(s)]_-\,\mathrm{d}s\right]}\,. \tag{3.19}$$

If the forcing term f is non-negative, then, similarly as in Theorem 3.6, the conclusions of Theorem 3.9 can be extended as follows.

Theorem 3.14. *Let $\lambda > 1$, $p \in \mathcal{V}^+(\omega)$, and conditions* (3.9) *and* (3.11) *be fulfilled. Then, there exists $0 < \mu_0 < +\infty$ such that the following conclusions hold:*

(1) *For any $\mu > \mu_0$, problem* (1.1) *has a unique solution which is negative.*
(2) *For $\mu = \mu_0$, problem* (1.1) *has exactly two solutions: one positive and one negative.*
(3) *For $\mu \in]0, \mu_0[$, problem* (1.1) *has exactly three solutions u_1, u_2, u_3 and these solutions satisfy*
$$u_1(t) > u_2(t) > 0, \quad u_3(t) < 0 \quad \text{for } t \in [0,\omega].$$
(4) *For $\mu = 0$, problem* (1.1) *has exactly three solutions: a positive solution u_0, the trivial solution, a negative solution $-u_0$.*
(5) *For $\mu \in]-\mu_0, 0[$, problem* (1.1) *has exactly three solutions u_1, u_2, u_3 and these solutions satisfy*
$$u_1(t) < u_2(t) < 0, \quad u_3(t) > 0 \quad \text{for } t \in [0,\omega].$$
(6) *For $\mu = -\mu_0$, problem* (1.1) *has exactly two solutions: one positive and one negative.*
(7) *For any $\mu < -\mu_0$, problem* (1.1) *has a unique solution which is positive.*

Remark 3.15. Theorem 3.14 extends the conclusions of Theorem 1.1 for the case of $c = 0$ and confirms a conjecture formulated in [2, Remark 3, p. 2502] because, at least in case of $c = 0$, the conclusions of Theorem 1.1 (except for the asymptotic stability) are still true for d which changes its sign (and belongs to a certain class of functions).

We finally provide the upper and lower estimates of the number μ_0 appearing in Theorem 3.14, which follow immediately from Proposition 3.13.

Proposition 3.16. *Let $\lambda > 1$, $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, and conditions* (3.9) *and* (3.11) *hold. Then, the number μ_0 appearing in the conclusion of Theorem 3.14 satisfies*

$$\mu_0 \geq \frac{(\lambda-1)\,[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\lambda\left[\lambda\int_0^\omega h(s)\,\mathrm{d}s\right]^{\frac{1}{\lambda-1}}\int_0^\omega f(s)\,\mathrm{d}s}\,,$$

where Δ is defined in Remark 2.5, and

$$\mu_0 < \frac{(\lambda-1)\left[\Gamma(p)\int_0^\omega [p(s)]_- \,\mathrm{d}s - \int_0^\omega [p(s)]_+ \,\mathrm{d}s\right]^{\frac{\lambda}{\lambda-1}}}{\lambda\left[\lambda\int_0^\omega h(s)\,\mathrm{d}s\right]^{\frac{1}{\lambda-1}}\int_0^\omega f(s)\,\mathrm{d}s\,\mathrm{d}s},$$

where Γ is given by (2.3)*.*

Remark 3.17. Let $\lambda > 1$, $\omega > 0$, and

$$p(t) := -a, \quad h(t) := b, \quad f(t) := 1 \quad \text{for } t \in [0,\omega],$$

where $0 < a \le \frac{\pi^2}{\omega^2}$ and $b > 0$. Then, conditions (3.1), (3.9), and (3.11) hold, $p \in \mathcal{V}^+(\omega)$ (see Remark 2.4), and all the conclusions of Theorem 3.14 coincide with those in Proposition 1.2. Moreover, from Remark 2.6, Remark 3.5, and Proposition 3.16, we obtain

$$\frac{2}{\pi}\,\frac{(\lambda-1)a}{\lambda}\Big(\frac{a}{\lambda b}\Big)^{\frac{1}{\lambda-1}} \le \mu_0 < \Big(\frac{1}{\cos\frac{\omega\sqrt{a}}{2}}\Big)^{\frac{\lambda}{\lambda-1}}\frac{(\lambda-1)a}{\lambda}\Big(\frac{a}{\lambda b}\Big)^{\frac{1}{\lambda-1}};$$

compare it with the number appearing in Proposition 1.2.

4. Auxiliary statements

We first recall some results stated in [15].

Lemma 4.1 ([15, Theorem 3.15(2,3)])**.** *Let $\lambda > 1$, $\mu \in \mathbb{R}$, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, and h satisfy* (3.1)*. Then, the following conclusions hold:*

(1) *Assume that there exists a positive function $\alpha \in AC_\ell([0,\omega])$ such that*

$$\alpha''(t) \ge p(t)\alpha(t) + h(t)\alpha^\lambda(t) + \mu f(t) \quad \text{for a.e. } t \in [0,\omega], \tag{4.1}$$

$$\alpha(0) = \alpha(\omega), \quad \alpha'(0) = \alpha'(\omega). \tag{4.2}$$

Then, problem (1.1) *has a positive solution u^* satisfying*

$$u^*(t) \ge \alpha(t) \quad \text{for } t \in [0,\omega] \tag{4.3}$$

such that every solution u to problem (1.1) *satisfies* (3.2)*. Moreover, for any couple of distinct positive solutions u_1, u_2 to* (1.1) *satisfying* (3.3)*, conditions* (3.4) *hold.*

(2) *If $\mu f(t) \le 0$ holds for a. e. $t \in [0,\omega]$, then problem* (1.1) *has a unique positive solution.*

Lemma 4.2 ([15, Corollary 3.16])**.** *Let $\lambda > 1$, $\mu \in \mathbb{R}$, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, h satisfy* (3.1) *and*

$$\int_0^\omega [\mu f(s)]_+ \,\mathrm{d}s < \sup\Big\{\frac{r}{\Delta\big(p + r^{\lambda-1}h\big)} : r > 0,\ p + r^{\lambda-1}h \in \mathcal{V}^+(\omega)\Big\},$$

where Δ is defined in Remark 2.5. Then, there exists a positive function $\alpha \in AC^1([0,\omega])$ satisfying (4.1) *and* (4.2)*.*

Lemma 4.3 ([15, Theorem 3.25(1,3,4,5)])**.** *Let $\lambda > 1$, $\mu \in \mathbb{R}$, $p \in \mathcal{V}^+(\omega)$, and h satisfy* (3.11)*. Then, the following conclusions hold:*

(1) *Problem* (1.1) *has at most two positive solutions.*
(2) *Assume that there exists a positive function $\alpha \in AC_\ell([0,\omega])$ satisfying* (4.1) *and* (4.2)*. Then, problem* (1.1) *has a positive solution u^* satisfying* (4.3) *such that, for every solution u to problem* (1.1)*, condition* (3.2) *holds.*

(3) *Assume that* $(p, \mu f) \in \mathcal{U}(\omega)$ *and there exist functions* $\alpha_1 \in AC_\ell([0,\omega])$ *and* $\alpha_2 \in AC^1([0,\omega])$ *such that*

$$0 < \alpha_2(t) < \alpha_1(t) \quad \text{for } t \in [0,\omega], \tag{4.4}$$

$$\alpha_k''(t) \geq p(t)\alpha_k(t) + h(t)\alpha_k^\lambda(t) + \mu f(t) \quad \text{for a.e. } t \in [0,\omega],\ k = 1,2, \tag{4.5}$$

$$\alpha_k(0) = \alpha_k(\omega), \quad \alpha_k'(0) = \alpha_k'(\omega),\ k = 1,2. \tag{4.6}$$

Then, problem (1.1) *possesses exactly two positive solutions* u_1, u_2 *and these solutions satisfy* (3.13). *Moreover, for every solution* u *to problem* (1.1) *different from* u_1, *condition* (3.14) *holds.*

(4) *If* $\mu f(t) \leq 0$ *holds for a. e.* $t \in [0,\omega]$, *then problem* (1.1) *has a unique positive solution.*

Lemma 4.4 ([15, Corollary 3.29])**.** *Let* $\lambda > 1$, $\mu \in \mathbb{R}$, $p \in \mathcal{V}^+(\omega)$, *and* h *satisfy* (3.11). *Then, the following conclusions hold:*

(1) *If*

$$0 < \int_0^\omega [\mu f(s)]_+ \,\mathrm{d}s \leq \frac{\lambda - 1}{\lambda\,[\Delta(p)]^{\frac{\lambda}{\lambda-1}}\,[\lambda \int_0^\omega h(s)\,\mathrm{d}s]^{\frac{1}{\lambda-1}}}\,,$$

where Δ *is defined in Remark 2.5, then there exists a positive function* $\alpha \in AC^1([0,\omega])$ *satisfying* (4.1) *and* (4.2).

(2) *If* $(p, \mu f) \in \mathcal{U}(\omega)$ *and*

$$0 < \int_0^\omega [\mu f(s)]_+ \,\mathrm{d}s < \frac{\lambda - 1}{\lambda\,[\Delta(p)]^{\frac{\lambda}{\lambda-1}}\,\big[\lambda \int_0^\omega h(s)\,\mathrm{d}s\big]^{\frac{1}{\lambda-1}}}\,, \tag{4.7}$$

where Δ *is defined in Remark 2.5, then there exists functions* $\alpha_1, \alpha_2 \in AC^1([0,\omega])$ *satisfying* (4.4), (4.5), *and* (4.6).

Lemma 4.5 ([15, Theorem 3.32])**.** *Let* $\lambda > 1$, $\mu \in \mathbb{R} \setminus \{0\}$, $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, h *satisfy* (3.11), $f(t) \not\equiv 0$, *and*

$$\begin{aligned} &\int_0^\omega [\mu f(s)]_+ \,\mathrm{d}s - \Gamma(p) \int_0^\omega [\mu f(s)]_- \,\mathrm{d}s \\ &\qquad \geq \frac{\lambda - 1}{\lambda}\,\frac{\big|\Gamma(p)\int_0^\omega [p(s)]_- \,\mathrm{d}s - \int_0^\omega [p(s)]_+ \,\mathrm{d}s\big|^{\frac{\lambda}{\lambda-1}}}{\big[\lambda \int_0^\omega h(s)\,\mathrm{d}s\big]^{\frac{1}{\lambda-1}}}\,. \end{aligned}$$

where Γ *is given by* (2.3). *Then, problem* (1.1) *has no non-negative solution.*

We now provide several lemmas needed in the proofs of the main results.

Lemma 4.6. *Let* $\lambda > 1$, $\mu \geq 0$, f *satisfy* (3.9), *and either*

$$p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega), \quad h(t) > 0 \quad \text{for a.e. } t \in [0,\omega], \tag{4.8}$$

or

$$p \in \mathcal{V}^+(\omega), \quad h(t) \geq 0 \quad \text{for a.e. } t \in [0,\omega], \quad h(t) \not\equiv 0. \tag{4.9}$$

Then, problem (1.1) *has a unique negative solution* u_* *and, moreover, every solution* u *to problem* (1.1) *satisfies* (3.10).

Proof. It is clear that u is a solution to problem (1.1) if and only if $-u$ is a solution to the problem

$$z'' = p(t)z + h(t)|z|^\lambda \operatorname{sgn} z - \mu f(t); \quad z(0) = z(\omega),\ z'(0) = z'(\omega). \tag{4.10}$$

It follows from Lemmas 4.1(2) and 4.3(4) that, in both cases (4.8) and (4.9), problem (4.10) has a unique positive solution z^*. Moreover, Lemmas 4.1(1) and 4.3(2) (with $\alpha(t) := z^*(t)$) guarantee that, in both cases (4.8) and (4.9), every solution z to problem (4.10) satisfies

$$\text{either} \quad z(t) < z^*(t) \quad \text{for } t \in [0,\omega], \quad \text{or} \quad z(t) \equiv z^*(t).$$

Therefore, the conclusion of the lemma holds with $u_* := -z^*$. □

Lemma 4.7. *Let $\lambda > 1$, $\mu > 0$, $p \in \mathcal{V}^+(\omega)$, and conditions* (3.9) *and* (3.11) *hold. Then, every solution to problem* (1.1) *is either positive or negative.*

Proof. Putting $q(t,x) := h(t)|x|^{\lambda-1}\operatorname{sgn} x$ for a.e. $t \in [0,\omega]$ and all $x \in \mathbb{R}$, the conclusion of the lemma follows immediately from [15, Theorem 3.13(4)]. □

Lemma 4.8. *Let $\lambda > 1$, $p \in \mathcal{V}^+(\omega)$, conditions* (3.11) *and* (3.12) *hold, and $\{\mu_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive numbers. Let, for any $n \in \mathbb{N}$, u_n be a positive solution to problem* (1.1) *with $\mu = \mu_n$. Then, the sequence $\{\|u_n\|_C\}_{n=1}^{\infty}$ is bounded and $\lim_{n\to+\infty} \mu_n < +\infty$.*

Proof. For any $n \in \mathbb{N}$, Lemma 4.3(2) (with $\alpha(t) := u_n(t)$ and $\mu := \mu_n$) implies that problem (1.1) with $\mu = \mu_n$ has a positive solution u_n^* such that

$$\text{every solution } u \text{ to (1.1) with } \mu = \mu_n \text{ satisfies } u(t) \le u_n^*(t) \text{ for } t \in [0,\omega]. \tag{4.11}$$

Let $n \in \mathbb{N}$ be arbitrary. Put

$$\alpha(t) := \frac{\mu_n}{\mu_{n+1}}\, u_{n+1}^*(t) \quad \text{for } t \in [0,\omega].$$

Then, in view of (3.11) and the condition $\mu_n \le \mu_{n+1}$, (1.1) with $\mu = \mu_{n+1}$ yields (4.2) and

$$\begin{aligned}\alpha''(t) &= p(t)\alpha(t) + \Big(\frac{\mu_{n+1}}{\mu_n}\Big)^{\lambda-1} h(t)\alpha^{\lambda}(t) + \mu_n f(t)\\ &\ge p(t)\alpha(t) + h(t)\alpha^{\lambda}(t) + \mu_n f(t) \quad \text{for a.e. } t \in [0,\omega].\end{aligned}$$

Therefore, it follows from Lemma 4.3(2) (with $\mu := \mu_n$) and (4.11) that

$$\frac{\mu_n}{\mu_{n+1}}\, u_{n+1}^*(t) \le u_n^*(t) \quad \text{for } t \in [0,\omega].$$

Consequently,

$$\frac{\mu_n}{\|u_n^*\|_C} \le \frac{\mu_{n+1}}{\|u_{n+1}^*\|_C} \quad \text{for } t \in [0,\omega],\ n \in \mathbb{N}. \tag{4.12}$$

We now show that

$$\sup\big\{\|u_n\|_C : n \in \mathbb{N}\big\} < +\infty. \tag{4.13}$$

Suppose on the contrary that (4.13) does not hold. Then, in view of (4.11), we can assume without loss of generality that

$$\lim_{n\to+\infty} \|u_n^*\|_C = +\infty. \tag{4.14}$$

Put

$$v_n(t) := \frac{u_n^*(t)}{\|u_n^*\|_C} \quad \text{for } t \in [0,\omega],\ n \in \mathbb{N}. \tag{4.15}$$

Clearly,

$$\|v_n\|_C = 1, \quad v_n(t) > 0 \quad \text{for } t \in [0,\omega],\ n \in \mathbb{N}. \tag{4.16}$$

It follows from (1.1) with $\mu = \mu_n$ that, for any $n \in \mathbb{N}$,

$$v_n''(t) = p(t)v_n(t) + \|u_n^*\|_C^{\lambda-1} h(t)v_n^{\lambda}(t) + \frac{\mu_n}{\|u_n^*\|_C} f(t) \quad \text{for a.e. } t \in [0,\omega], \tag{4.17}$$

which yields

$$0 = \int_0^{\omega} p(s)v_n(s)\,\mathrm{d}s + \|u_n^*\|_C^{\lambda-1} \int_0^{\omega} h(s)v_n^{\lambda}(s)\,\mathrm{d}s + \frac{\mu_n}{\|u_n^*\|_C} \int_0^{\omega} f(s)\,\mathrm{d}s \tag{4.18}$$

for $n \in \mathbb{N}$. In view of (4.16), from the latter equality, we obtain

$$\|u_n^*\|_C^{\lambda-1} \int_0^{\omega} h(s)v_n^{\lambda}(s)\,\mathrm{d}s + \frac{\mu_n}{\|u_n^*\|_C} \int_0^{\omega} f(s)\,\mathrm{d}s \le \int_0^{\omega} |p(s)|\,\mathrm{d}s \quad \text{for } n \in \mathbb{N}. \tag{4.19}$$

Put

$$A := \sup\Big\{\|u_n^*\|_C^{\lambda-1} \int_0^{\omega} h(s)v_n^{\lambda}(s)\,\mathrm{d}s : n \in \mathbb{N}\Big\} \tag{4.20}$$

$$B := \sup\Big\{\frac{\mu_n}{\|u_n^*\|_C} : n \in \mathbb{N}\Big\}. \tag{4.21}$$

It follows from (3.11), (3.12), (4.16), and (4.19) that $A \in]0,+\infty[$ and $B \in]0,+\infty[$. For any $n \in \mathbb{N}$, we choose $t_n \in [0,\omega]$ such that $v_n'(t_n) = 0$. In view of (4.16), (4.20), and (4.21), integrating (4.17) from t_n to t, we obtain

$$\begin{aligned}
|v_n'(t)| &= \Big|\int_{t_n}^{t} \Big[p(s)v_n(s) + \|u_n^*\|_C^{\lambda-1} h(s)v_n^{\lambda}(s) + \frac{\mu_n}{\|u_n^*\|_C} f(s)\Big]\,\mathrm{d}s\Big| \\
&\le \int_0^{\omega} |p(s)|v_n(s)\,\mathrm{d}s + \|u_n^*\|_C^{\lambda-1} \int_0^{\omega} h(s)v_n^{\lambda}(s)\,\mathrm{d}s + \frac{\mu_n}{\|u_n^*\|_C} \int_0^{\omega} |f(s)|\,\mathrm{d}s \\
&\le \int_0^{\omega} |p(s)|\,\mathrm{d}s + A + B\int_0^{\omega} |f(s)|\,\mathrm{d}s \quad \text{for } t \in [0,\omega],\ n \in \mathbb{N}.
\end{aligned}$$

Therefore, the sequences $\{\|v_n\|_C\}_{n=1}^{\infty}$ and $\{\|v_n'\|_C\}_{n=1}^{\infty}$ are bounded and, thus, by the Arzelá-Ascoli theorem, we can assume without loss of generality that

$$\lim_{n\to+\infty} v_n(t) = v_0(t) \quad \text{uniformly on } [0,\omega], \tag{4.22}$$

where $v_0 \in C([0,\omega])$.

It follows from the hypothesis $(p,f) \in \mathcal{U}(\omega)$ that the problem

$$v'' = p(t)v + f(t); \quad v(0) = v(\omega),\ v'(0) = v'(\omega) \tag{4.23}$$

has a unique solution v which is positive. Putting $z(t) := v_n(t) - \frac{\mu_n}{\|u_n^*\|_C} v(t)$ for $t \in [0,\omega]$, $n \in \mathbb{N}$, and taking into account (3.11) and (4.16), from (4.17) and (4.23), we obtain

$$z''(t) \ge p(t)z(t) \quad \text{for a.e. } t \in [0,\omega], \quad z(0) = z(\omega), \quad z'(0) = z'(\omega)$$

and, thus, the hypothesis $p \in \mathcal{V}^+(\omega)$ yields $z_n(t) \ge 0$ for $t \in [0,\omega]$, i.e.,

$$v_n(t) \ge \frac{\mu_n}{\|u_n^*\|_C} v(t) \ge \frac{\mu_1}{\|u_1^*\|_C} v(t) \quad \text{for } t \in [0,\omega],\ n \in \mathbb{N},$$

because (4.12) holds and v is positive. Consequently, by (4.22), we obtain

$$v_0(t) \ge \frac{\mu_1}{\|u_1^*\|_C} v(t) > 0 \quad \text{for } t \in [0,\omega]. \tag{4.24}$$

On the other hand, in view of (3.12), (4.19) yields

$$\int_0^\omega h(s)v_n^\lambda(s)\,\mathrm{d}s \le \frac{1}{\|u_n^*\|_C^{\lambda-1}}\int_0^\omega |p(s)|\,\mathrm{d}s \quad \text{for } n\in\mathbb{N}$$

and, therefore, passing the limit for $n\to+\infty$ and taking into account (4.14) and (4.22), we conclude that

$$\int_0^\omega h(s)v_0^\lambda(s)\,\mathrm{d}s \le 0. \tag{4.25}$$

However, by (4.24), the latter inequality contradicts (3.11). The obtained contradiction proves that (4.13) holds.

Now we show that $\lim_{n\to+\infty}\mu_n < +\infty$. Suppose on the contrary that

$$\lim_{n\to+\infty}\mu_n = +\infty. \tag{4.26}$$

Integrating the equation in (1.1) with $\mu=\mu_n$ over the interval $[0,\omega]$, we obtain

$$0 = \int_0^\omega p(s)u_n(s)\,\mathrm{d}s + \int_0^\omega h(s)u_n^\lambda(s)\,\mathrm{d}s + \mu_n\int_0^\omega f(s)\,\mathrm{d}s \quad \text{for } n\in\mathbb{N}$$

which, in view of (3.11) and the positivity of u_n, yields

$$\int_0^\omega f(s)\,\mathrm{d}s \le \frac{\|u_n\|_C}{\mu_n}\int_0^\omega |p(s)|\,\mathrm{d}s \quad \text{for } n\in\mathbb{N}.$$

Taking into account (4.13), (4.26) and passing the limit for $n\to+\infty$, we obtain $\int_0^\omega f(s)\,\mathrm{d}s \le 0$, which contradicts (3.12). $\square$

Lemma 4.9. *Let $\lambda>1$, $p\notin\mathcal{V}^-(\omega)\cup\mathcal{V}_0(\omega)$, h satisfy* (3.1),

$$\int_0^\omega f(s)\,\mathrm{d}s > 0, \tag{4.27}$$

and $\{\mu_n\}_{n=1}^\infty$ be a sequence of positive numbers. Let, for any $n\in\mathbb{N}$, u_n^* be a positive solution to problem (1.1) *with $\mu=\mu_n$. Then, the sequences $\{\|u_n^*\|_C\}_{n=1}^\infty$ and $\{\mu_n\}_{n=1}^\infty$ are bounded.*

Proof. We first show that

$$\sup\big\{\|u_n^*\|_C : n\in\mathbb{N}\big\} < +\infty. \tag{4.28}$$

Suppose on the contrary that (4.28) does not hold. Then, we can assume without loss of generality that (4.14) is satisfied. Define the functions v_n by (4.15). It is clear that (4.16) holds and, in much the same way as in the proof of Lemma 4.8, we show that the sequences $\{\|v_n\|_C\}_{n=1}^\infty$ and $\{\|v_n'\|_C\}_{n=1}^\infty$ are bounded. Therefore, by the Arzelá-Ascoli theorem, we can assume without loss of generality that (4.22) is satisfied, where $v_0\in C([0,\omega])$. From (4.16) and (4.22), we obtain

$$v_0(t)\ge 0 \quad \text{for } t\in[0,\omega], \quad \|v_0\|_C = 1. \tag{4.29}$$

Moreover, in much that same way as in the proof of Lemma 4.8, we derive inequality (4.25) which, in view of (4.29), contradicts (3.1).

Now we show that the sequence $\{\mu_n\}_{n=1}^\infty$ is bounded. Suppose on the contrary that $\sup\{\mu_n : n\in\mathbb{N}\}=+\infty$. Then, we can assume without loss of generality that (4.26) holds. Integrating the equation in (1.1) with $\mu=\mu_n$ over the interval $[0,\omega]$, we obtain

$$0 = \int_0^\omega p(s)u_n^*(s)\,\mathrm{d}s + \int_0^\omega h(s)\big(u_n^*(s)\big)^\lambda\,\mathrm{d}s + \mu_n\int_0^\omega f(s)\,\mathrm{d}s \quad \text{for } n\in\mathbb{N}$$

which, in view of (3.1) and the positivity of u_n^*, yields

$$\int_0^\omega f(s)\,\mathrm{d}s \le \frac{\|u_n^*\|_C}{\mu_n}\int_0^\omega |p(s)|\,\mathrm{d}s \quad \text{for } n \in \mathbb{N}.$$

Taking into account (4.26), (4.28) and passing the limit for $n \to +\infty$, we obtain $\int_0^\omega f(s)\,\mathrm{d}s \le 0$, which contradicts (4.27). □

Lemma 4.10. *Let $\lambda > 1$, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, h satisfy (3.1), and $\{\mu_n\}_{n=1}^\infty$ be a bounded sequence of real numbers. Let, for any $n \in \mathbb{N}$, u_n^* be a positive solution to problem (1.1) with $\mu = \mu_n$. Then, the sequence $\{\|u_n^*\|_C\}_{n=1}^\infty$ is bounded.*

Proof. Suppose on the contrary that $\sup\{\|u_n^*\|_C : n \in \mathbb{N}\} = \infty$. Then, we can assume without loss of generality that (4.14) holds. Let the functions v_n be given by (4.15). It is clear that (4.16) and (4.19) are fulfilled. Define the numbers A, B by formulas (4.20) and (4.21), respectively. Since the sequence $\{\mu_n\}_{n=1}^\infty$ is bounded and $\{\|u_n^*\|_C\}_{n=1}^\infty$ satisfies (4.14), we conclude easily that $B \in]0,+\infty[$. Hence, in view of (3.1) and (4.16), condition (4.19) yields

$$\|u_n^*\|_C^{\lambda-1}\int_0^\omega h(s)v_n^\lambda(s)\,\mathrm{d}s \le \int_0^\omega |p(s)|\,\mathrm{d}s + B\int_0^\omega |f(s)|\,\mathrm{d}s \quad \text{for } n \in \mathbb{N}, \tag{4.30}$$

which guarantees that $A \in]0,+\infty[$. In much that same way as in the proof of Lemma 4.8, we show that that the sequences $\{\|v_n\|_C\}_{n=1}^\infty$ and $\{\|v_n'\|_C\}_{n=1}^\infty$ are bounded and, thus, by the Arzelá-Ascoli theorem, we can assume without loss of generality that (4.22) is satisfied, where $v_0 \in C([0,\omega])$. From (4.16) and (4.22), we obtain (4.29).

On the other hand, (4.30) yields

$$\int_0^\omega h(s)v_n^\lambda(s)\,\mathrm{d}s \le \frac{1}{\|u_n^*\|_C^{\lambda-1}}\Big(\int_0^\omega |p(s)|\,\mathrm{d}s + B\int_0^\omega |f(s)|\,\mathrm{d}s\Big) \quad \text{for } n \in \mathbb{N}$$

and, therefore, passing the limit for $n \to +\infty$ and taking into account (4.14) and (4.22), we obtain (4.25). However, in view of (3.1), condition (4.25) contradicts (4.29). □

The following lemma follows from the well-known Gronwall-Bellman's lemma for the systems of first-order ODEs (see, e. g., [7, §1.7] or [17] for the case of continuous ℓ).

Lemma 4.11. *Let $\ell \in L([a,b])$ be a non-negative function and $w \in AC^1([a,b])$ be such that*

$$\begin{gathered} w(t) \ge 0 \quad \text{for } t \in [a,b], \quad w(a) = 0, \quad w'(a) = 0, \\ w''(t) \le \ell(t)w(t) \quad \text{for a.e. } t \in [a,b]. \end{gathered} \tag{4.31}$$

Then, $w(t) \equiv 0$.

Lemma 4.12. *Let $\lambda > 1$, $\mu^* > 0$, $p \in \mathcal{V}^+(\omega)$, h satisfy (3.11), and there exist a positive function $\alpha \in AC^1([0,\omega])$ such that (4.1) with $\mu = \mu^*$ and (4.2) hold. Then, for any $\mu \in]0,\mu^*[$, there exist functions $\alpha_1, \alpha_2 \in AC^1([0,\omega])$ satisfying conditions (4.4), (4.5), and (4.6).*

Proof. Let $\mu \in]0,\mu^*[$ be arbitrary and $\alpha_2(t) := \frac{\mu}{\mu^*}\alpha(t)$ for $t \in [0,\omega]$. It follows from (4.1) with $\mu = \mu^*$ and (4.2) that

$$\alpha_2(t) > 0 \quad \text{for } t \in [0,\omega], \tag{4.32}$$

$$\alpha_2(0) = \alpha_2(\omega), \quad \alpha_2'(0) = \alpha_2'(\omega), \tag{4.33}$$

$$\begin{aligned} \alpha_2''(t) &= p(t)\alpha_2(t) + \Big(\frac{\mu^*}{\mu}\Big)^{\lambda-1} h(t)\alpha_2^\lambda(t) + \mu f(t) \\ &\geq p(t)\alpha_2(t) + h(t)\alpha_2^\lambda(t) + \mu f(t) \quad \text{for a.e. } t \in [0,\omega], \end{aligned} \tag{4.34}$$

$$\operatorname{meas}\big\{t \in [0,\omega] : \alpha_2''(t) > p(t)\alpha_2(t) + h(t)\alpha_2^\lambda(t) + \mu f(t)\big\} > 0, \tag{4.35}$$

because $0 < \mu < \mu^*$ and h satisfies (3.11).

Therefore, Lemma 4.3(2) (with $\alpha(t) := \alpha_2(t)$), problem (1.1) has a solution α_1 such that

$$\alpha_1(t) \geq \alpha_2(t) \quad \text{for } t \in [0,\omega]. \tag{4.36}$$

Consequently, the functions α_1, α_2 satisfy conditions (4.5) and (4.6). We finally show that (4.4) is fulfilled as well. Suppose on the contrary that (4.4) does not hold. Extend the functions p, h, f, α_1, α_2 periodically to the whole real axis denoting them by the same symbols. Then, in view of (4.32) and (4.36), there exists $a \in [0,\omega[$ such that

$$\alpha_1(a) = \alpha_2(a), \quad \alpha_1'(a) = \alpha_2'(a). \tag{4.37}$$

Put

$$\begin{gathered} w(t) := \alpha_1(t) - \alpha_2(t) \quad \text{for } t \in [a, a+\omega], \\ \varphi(t) := g\big(\alpha_1(t), \alpha_2(t)\big) \quad \text{for } t \in [a, a+\omega], \end{gathered}$$

where

$$g(x,y) := \begin{cases} \frac{x^\lambda - y^\lambda}{x-y} & \text{for } x, y \in \mathbb{R},\ x \neq y, \\ \lambda |x|^{\lambda-1} \operatorname{sgn} x & \text{for } x, y \in \mathbb{R},\ x = y. \end{cases}$$

It is not difficult to verify that $g\colon \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and, thus, the function φ is continuous and non-negative on $[a, a+\omega]$. By (4.36) and (4.37), w satisfies (4.31) with $b = a + \omega$. Since α_1 is a solution to problem (1.1) and α_2 satisfies (4.34), we have

$$\begin{aligned} w''(t) &\leq p(t)w(t) + h(t)\Big(\alpha_1^\lambda(t) - \alpha_2^\lambda(t)\Big) \\ &\leq \big(|p(t)| + h(t)\varphi(t)\big) w(t) \quad \text{for a.e. } t \in [a, a+\omega]. \end{aligned}$$

Therefore, Lemma 4.11 (with $\ell(t) := |p(t)| + h(t)\varphi(t)$ and $b := a+\omega$) yields $w(t) \equiv 0$, i., e., $\alpha_1(t) \equiv \alpha_2(t)$. However, this contradicts condition (4.35), because α_1 is a solution to problem (1.1). $\square$

Lemma 4.13. *Let $\lambda > 1$, $\mu^* > 0$, $p, h, f \in L([0,\omega])$, h satisfy (3.11), and there exist functions $\alpha_1, \alpha_2 \in AC^1([0,\omega])$ such that (4.5) with $\mu = \mu^*$ and (4.6) hold and*

$$0 \leq \alpha_2(t) < \alpha_1(t) \quad \text{for } t \in [0,\omega]. \tag{4.38}$$

Then, there exist $\mu > \mu^*$ and a positive function $\alpha \in AC^1([0,\omega])$ satisfying (4.1) with $\mu = \mu^*$ and (4.2).

The proof of the above lemma is similar to the proof of [14, Lemma 4.9] and thus, it is omitted.

Lemma 4.14. *Let $\lambda > 1$, $\mu^* \in \mathbb{R}$, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, and h satisfy (3.1). Then, for any $c > 0$, there exists a function $\beta \in AC^1([0,\omega])$ such that*

$$\beta''(t) \leq p(t)\beta(t) + h(t)\beta^\lambda(t) + \mu^* f(t) \quad \text{for a.e. } t \in [0,\omega], \tag{4.39}$$

$$\beta(0) = \beta(\omega), \quad \beta'(0) = \beta'(\omega), \tag{4.40}$$

$$\beta(t) \geq c \quad \textit{for } t \in [0, \omega]. \tag{4.41}$$

Proof. Put

$$q_0(t, x) := h(t)|x|^{\lambda - 1} \quad \text{for a.e. } t \in [0, \omega] \text{ and all } x \in \mathbb{R}.$$

Since $\lim_{x \to +\infty} x^{\lambda-1} \int_E h(s)\,\mathrm{d}s = +\infty$ for every $E \subseteq [0, \omega]$, $\operatorname{meas} E > 0$, it follows from [15, Lemma 4.15] that there exists $R > 0$ such that $p + q_0(\cdot, R) \in \mathcal{V}^-(\omega)$. Therefore, the conclusion of the lemma follows from [15, Proposition 4.21] (with $q(t, x) := q_0(t, x)$ and $x_0 := 0$). ☐

Lemma 4.15 ([10, Proposition 10.8, Remark 0.7]). *If $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, then either $\int_0^\omega p(s)\,\mathrm{d}s > 0$ or $p(t) \equiv 0$.*

5. Proofs of main results

Proof of Theorem 3.1. Put

$$\mathcal{A} := \big\{\mu \in \mathbb{R} : \text{problem (1.1) has a positive solution}\big\}. \tag{5.1}$$

In view of Lemmas 4.1(1) and 4.2, there exists $\varepsilon > 0$ such that $\,]-\varepsilon, \varepsilon[\, \cap \mathcal{A} \neq \emptyset$. Let

$$\mu_* := \inf \mathcal{A}, \quad \mu^* := \sup \mathcal{A}. \tag{5.2}$$

Then, $-\infty \leq \mu_* < 0$ and $0 < \mu^* \leq +\infty$.

Conclusion (1)*:* Let $\mu_0 \in \mathcal{A} \setminus \{0\}$ be arbitrary and $\mu \in \mathbb{R}$ be such that $0 < |\mu| \leq |\mu_0|$ and $\operatorname{sgn} \mu = \operatorname{sgn} \mu_0$. Let, moreover, u_0 be a positive solution to problem (1.1) with $\mu = \mu_0$. Put

$$\alpha(t) := \frac{\mu}{\mu_0}\, u_0(t) \quad \text{for } t \in [0, \omega]. \tag{5.3}$$

Clearly, $\alpha(t) > 0$ for $t \in [0, \omega]$. It follows from (1.1) with $\mu = \mu_0$ that α satisfies (4.2) and

$$\begin{aligned} \alpha''(t) &= p(t)\alpha(t) + \big(\frac{\mu_0}{\mu}\big)^{\lambda - 1} h(t)\alpha^\lambda(t) + \mu f(t) \\ &\geq p(t)\alpha(t) + h(t)\alpha^\lambda(t) + \mu f(t) \quad \text{for a.e. } t \in [0, \omega], \end{aligned} \tag{5.4}$$

because $|\mu_0| \geq |\mu| > 0$ and (3.1) holds. Therefore, Lemma 4.1(1) yields $\mu \in \mathcal{A}$. Consequently, $]\mu_*, \mu^*[\, \subseteq \mathcal{A}$ and, thus, conclusion (1) of the theorem follows from Lemma 4.1(1).

Conclusion (2)*:* Assume that $\mu^* < +\infty$. Then, it follows immediately from (5.1) and (5.2) that conclusion (2a) of the theorem holds.

Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of positive numbers such that

$$\mu_n \in \mathcal{A} \quad \text{for } n \in \mathbb{N}, \quad \lim_{n \to +\infty} \mu_n = \mu^* \tag{5.5}$$

and, for any $n \in \mathbb{N}$, let u_n be a solution to problem (1.1) with $\mu = \mu_n$. Lemma 4.10 yields (4.13). By the standard arguments using in the proof of a well-possedness of the periodic problems for second-order ODEs, one can show that there exists a subsequence $\{u_{n_k}\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ such that

$$\lim_{k \to +\infty} u_{n_k}^{(i)}(t) = (u^*)^{(i)}(t) \quad \text{uniformly on } [0, \omega],\ i = 0, 1, \tag{5.6}$$

where $u^* \in AC^1([0,\omega])$ is a solution to problem (1.1) with $\mu = \mu^*$. All the functions u_{n_k} are positive and, thus, it is clear that

$$u^*(t) \geq 0 \quad \text{for } t \in [0,\omega].$$

We now prove that u^* is a unique non-negative solution to problem (1.1) with $\mu = \mu^*$. Suppose on the contrary that u_* is a non-negative solution to (1.1) with $\mu = \mu^*$ such that

$$u_*(\xi) \neq u^*(\xi) \quad \text{for some } \xi \in [0,\omega]. \tag{5.7}$$

Put

$$\alpha(t) := \max\big\{u_*(t), u^*(t)\big\} \quad \text{for } t \in [0,\omega].$$

It is not difficult to verify that $\alpha \in AC_\ell([0,\omega])$, condition (4.2) with $\mu = \mu^*$ holds, and

$$\alpha(a) = \alpha(\omega), \quad \alpha'(a) \geq \alpha'(\omega). \tag{5.8}$$

Let us show that

$$\alpha(t) > 0 \quad \text{for } t \in [0,\omega]. \tag{5.9}$$

If this condition does not hold, then, in view of the non-negativity of u_*, u^*, there exists $t_0 \in [0,\omega]$ such that

$$u_*(t_0) = 0, \quad u^*(t_0) = 0. \tag{5.10}$$

Extend the functions p, h, f, u_*, u^* periodically to the whole real axis denoting them by the same symbols. Then, using (5.10) and the non-negativity of u_*, u^*, we obtain

$$u_*'(t_0) = 0, \quad (u^*)'(t_0) = 0. \tag{5.11}$$

Since the function $x \mapsto |x|^\lambda \operatorname{sgn} x$ is Lipschitz on every compact interval, for any $c_1, c_2 \in \mathbb{R}$, the Cauchy problem

$$u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u + \mu^* f(t); \quad u(t_0) = c_1,\ u'(t_0) = c_2 \tag{5.12}$$

is uniquely solvable. Therefore, (5.10) and (5.11) yield $u_*(t) \equiv u^*(t)$, which contradicts (5.7). Hence, (5.9) holds. Now, in view of (4.2) with $\mu = \mu^*$, (5.8), and (5.9), it follows from Lemma 4.1(1) that problem (1.1) with $\mu = \mu^*$ has a positive solution $\tilde{u}^*$ such that

$$0 \leq u_*(t) < \tilde{u}^*(t) \quad \text{for } t \in [0,\omega] \quad \text{or} \quad 0 \leq u^*(t) < \tilde{u}^*(t) \quad \text{for } t \in [0,\omega].$$

Therefore, Lemma 4.13 guarantees that there exist $\tilde{\mu} > \mu^*$ and a positive function $\tilde{\alpha} \in AC^1([0,\omega])$ satisfying

$$\tilde{\alpha}''(t) \geq p(t)\tilde{\alpha}(t) + h(t)\tilde{\alpha}^\lambda(t) + \tilde{\mu} f(t) \quad \text{for a.e. } t \in [0,\omega], \tag{5.13}$$

$$\tilde{\alpha}(0) = \tilde{\alpha}(\omega), \quad \tilde{\alpha}'(0) = \tilde{\alpha}'(\omega). \tag{5.14}$$

Consequently, it follows from Lemma 4.1 (with $\alpha(t) := \tilde{\alpha}(t)$ and $\mu := \tilde{\mu}$) that problem (1.1) with $\mu = \tilde{\mu}$ has at least one positive solution, which contradicts the above-proved conclusion (2a). The contradiction obtained proves that u^* is a unique non-negative solution to problem (1.1) with $\mu = \mu^*$.

It remains to show that every solution u to problem (1.1) with $\mu = \mu^*$ satisfies (3.2). Indeed, suppose on the contrary that u is a solution to problem (1.1) with $\mu = \mu^*$ such that (3.2) does not hold. We have mentioned above that, for any $t_0 \in [0,\omega]$ and $c_1, c_2 \in \mathbb{R}$, the Cauchy problem (5.12) is uniquely solvable and, thus, the solution u satisfies

$$\max\big\{u(t) - u^*(t) : t \in [0,\omega]\big\} > 0. \tag{5.15}$$

Put

$$\alpha(t) := \max\big\{u(t), u^*(t)\big\} \quad \text{for } t \in [0,\omega]. \tag{5.16}$$

It is not difficult to verify that $\alpha \in AC_\ell([0,\omega])$ and conditions (4.2) with $\mu = \mu^*$ and (5.8) hold. Moreover, it follows from Lemma 4.14 that there exists $\beta \in AC^1([0,\omega])$ satisfying (4.39), (4.40), and

$$\beta(t) \geq \alpha(t) \quad \text{for } t \in [0,\omega]. \tag{5.17}$$

Therefore, by (4.2) with $\mu = \mu^*$, (4.39), (4.40), (5.8), and (5.17), we conclude that α and β form a well-ordered pair of lower and upper functions and, thus, problem (1.1) with $\mu = \mu^*$ has a solution $\hat{u}$ such that

$$\alpha(t) \leq \hat{u}(t) \leq \beta(t) \quad \text{for } t \in [0,\omega].$$

However, this condition, together with (5.15) and (5.16), implies that $\hat{u}$ is a non-negative solution to problem (1.1) with $\mu = \mu^*$ different from u^*, which contradicts the above-proved fact concerning the uniqueness of the non-negative solution u^*.

Conclusion (3)*:* It can be proved in much the same way as conclusion (2) considering $-\mu$ and $-f$ instead of μ and f.

Conclusion (4)*:* It follows immediately from Lemma 4.9. □

Proof of Corollary 3.2. It is clear that u is a solution to problem (1.1) if and only if $-u$ is a solution to problem (4.10). Therefore, the conclusion of the corollary follows from Theorem 3.1(1). □

Proof of Proposition 3.4. Let μ_*, μ^* be the numbers appearing in the conclusion of Theorem 3.1.

Assume that $[f(t)]_+ \not\equiv 0$ and suppose on the contrary that (3.6) does not hold, i.e.,

$$\mu^* < \frac{1}{\int_0^\omega [f(s)]_+ \,\mathrm{d}s} \sup\Big\{\frac{r}{\Delta\big(p + r^{\lambda-1}h\big)} : r > 0,\ p + r^{\lambda-1}h \in \mathcal{V}^+(\omega)\Big\},$$

where Δ is defined by Remark 2.5. Then, $\mu^* \in]0,+\infty[$ and there exists $\varepsilon > 1$ such that

$$\int_0^\omega [\varepsilon\mu^* f(s)]_+ \,\mathrm{d}s < \sup\Big\{\frac{r}{\Delta\big(p + r^{\lambda-1}h\big)} : r > 0,\ p + r^{\lambda-1}h \in \mathcal{V}^+(\omega)\Big\}.$$

Therefore, from Lemmas 4.2 and 4.1(1) that problem (1.1) with $\mu = \varepsilon\mu^*$ has at least one positive solution, which contradicts conclusion (2a) of Theorem 3.1.

Assuming $[f(t)]_- \not\equiv 0$, estimate (3.7) can be proved analogously to (3.6). □

Proof of Theorem 3.6. It follows from Theorem 3.1 and Lemmas 4.6 and 4.9 that there exists $\mu_0 \in]0,\infty[$ such that conclusions (1), (2), and (3) of the theorem hold. Since u is a solution to problem (1.1) if and only if $-u$ is a solution to problem (4.10), conclusions (5), (6), and (7) of the theorem hold as well. Finally, conclusion (4) of the theorem follows from Lemma 4.6 and the above-mentioned equivalence. □

Proof of Theorem 3.9. Let the set $\mathcal{A}$ be given by formula (5.1). In view of Lemmas 4.3(2,3) and 4.4(1), there exists $\varepsilon > 0$ such that $]-\varepsilon,\varepsilon[\cap\mathcal{A} \neq \emptyset$. Define the numbers μ_* and μ^* by (5.2). Then, $-\infty \leq \mu_* < 0$, $\mu^* > 0$, and Lemma 4.8 implies that $\mu^* < +\infty$.

Conclusion (1)*:* It follows from (5.1), (5.2), and the condition $\mu^* \in]0,+\infty[$.

Conclusion (2)*:* We first show that

$$\mu^* \in \mathcal{A}. \tag{5.18}$$

Indeed, let $\{\mu_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive numbers such that

$$\mu_n \in \mathcal{A} \quad \text{for } n \in \mathbb{N}, \qquad \lim_{n\to+\infty} \mu_n = \mu^*.$$

Moreover, for any $n \in \mathbb{N}$, let u_n be a positive solution to problem (1.1) with $\mu = \mu_n$. It follows from Lemma 4.8 that condition (4.13) holds. By the standard arguments using in the proof of a well-possedness of the periodic problems for second-order ODEs, one can show that there exists a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ such that (5.6) is satisfied, where $u^* \in AC^1([0,\omega])$ is a solution to problem (1.1) with $\mu = \mu^*$. Since the functions u_{n_k}, $k \in \mathbb{N}$, are positive, it is clear that

$$u^*(t) \geq 0 \quad \text{for } t \in [0,\omega]. \tag{5.19}$$

In view of the hypothesis $(p,f) \in \mathcal{U}(\omega)$ and the positivity of μ^*, problem (4.23) has a unique solution v, which is positive. By (1.1) with $\mu = \mu^*$, (3.11), (4.23), and (5.19), we obtain

$$z''(t) \geq p(t)z(t) \quad \text{for a.e. } t \in [0,\omega], \quad z(0) = z(\omega), \quad z'(0) = z'(\omega),$$

where $z(t) := u^*(t) - \mu^* v(t)$ for $t \in [0,\omega]$. Therefore, the hypothesis $p \in \mathcal{V}^+(\omega)$ yields $z(t) \geq 0$ for $t \in [0,\omega]$. Hence, we have

$$u^*(t) \geq \mu^* v(t) > 0 \quad \text{for } t \in [0,\omega]$$

and, thus condition (5.18) holds.

Since u^* is a positive solution to problem (1.1) with $\mu = \mu^*$, in view of Lemma 4.3(2), to prove conclusion (2) of the theorem, it is sufficient to show that problem (1.1) with $\mu = \mu^*$ does not have more than one positive solution. Suppose on the contrary that problem (1.1) with $\mu = \mu^*$ has a positive solution different from u^*. Then, it follows from Lemma 4.3(2) (with $\alpha(t) := u^*(t)$ and $\mu := \mu^*$) that problem (1.1) with $\mu = \mu^*$ possesses solutions $\tilde{u}_*$, $\tilde{u}^*$ such that

$$\tilde{u}^*(t) > \tilde{u}_*(t) > 0 \quad \text{for } t \in [0,\omega].$$

Therefore, Lemma 4.13 (with $\alpha_1(t) := \tilde{u}^*(t)$ and $\alpha_2(t) := \tilde{u}_*(t)$) guarantees that there exist $\tilde{\mu} > \mu^*$ and a positive function $\tilde{\alpha} \in AC^1([0,\omega])$ satisfying (5.13) and (5.14). Consequently, by Lemma 4.1(1) (with $\alpha(t) := \tilde{\alpha}(t)$ and $\mu := \tilde{\mu}$), we conclude that problem (1.1) with $\mu = \tilde{\mu}$ has at least one positive solution, which contradicts the above-proved conclusion (1).

Conclusions (3)*:* Having a positive solution u^* to problem (1.1) with $\mu = \mu^*$, it is clear that all the hypotheses of Lemma 4.12 (with $\alpha(t) := u^*(t)$) are fulfilled. Consequently, for any $\mu \in]0,\mu^*[$, $(p,\mu f) \in \mathcal{U}(\omega)$ and there exist functions $\alpha_1, \alpha_2 \in AC^1([0,\omega])$ satisfying conditions (4.4), (4.5), and (4.6) and, therefore, conclusion (3) of the theorem follows from Lemma 4.3(3).

Conclusion (4)*:* It follows immediately from [11, Corollary 2.31(2)].

Conclusion (5)*:* Let $\mu_0 \in \mathcal{A} \cap]-\infty, 0[$ and $\mu \in [\mu_0, 0[$ be arbitrary and let u_0 be a positive solution to problem (1.1) wigth $\mu = \mu_0$. Define the function α by (5.3). Clearly, $\alpha(t) > 0$ for $t \in [0,\omega]$. It follows from (1.1) with $\mu = \mu_0$ that α satisfies (4.2) and (5.4), because $\mu_0 \leq \mu < 0$ and (3.11) holds. Therefore, Lemma 4.3(2) yields $\mu \in \mathcal{A}$. Consequently, $]\mu_*, 0[\subseteq \mathcal{A}$ and, thus, conclusion (5) of the theorem follows from Lemma 4.3(1,2).

Conclusion (6)*:* Assume that $\mu_* > -\infty$. Then, it follows immediately from (5.1) and (5.2) that, for any $\mu < \mu_*$, problem (1.1) has no positive solution. □

Proof of Proposition 3.13. By Remark 3.11, it follows from (3.16) that condition (3.12) holds. Let μ_*, μ^* be the numbers appearing in the conclusion of Theorem 3.9.

We first show that μ_* satisfies (3.17), where Δ is defined in Remark 2.5. Suppose on the contrary that (3.17) does not hold, i.e.,

$$\mu_* > -\frac{(\lambda-1)\,[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\lambda\left[\lambda\int_0^\omega h(s)\,\mathrm{d}s\right]^{\frac{1}{\lambda-1}}\int_0^\omega [f(s)]_-\,\mathrm{d}s}\,.$$

Then, $\mu_* \in\,]-\infty, 0[$ and there exists $\varepsilon > 1$ such that

$$0 < \int_0^\omega [\varepsilon\mu_* f(s)]_+\,\mathrm{d}s = -\varepsilon\mu_*\int_0^\omega [f(s)]_-\,\mathrm{d}s \le \frac{(\lambda-1)\,[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\lambda\left[\lambda\int_0^\omega h(s)\,\mathrm{d}s\right]^{\frac{1}{\lambda-1}}}\,.$$

Therefore, it follows from Lemmas 4.4(1) and 4.3(2) that problem (1.1) with $\mu = \varepsilon\mu_*$ has a positive solution, which contradicts conclusion (6) of Theorem 3.9.

Now we show that μ^* satisfies (3.18), where Δ is defined in Remark 2.5. Suppose on the contrary that (3.18) does not hold, i.e.,

$$\mu^* < \frac{(\lambda-1)\,[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\lambda\left[\lambda\int_0^\omega h(s)\,\mathrm{d}s\right]^{\frac{1}{\lambda-1}}\int_0^\omega [f(s)]_+\,\mathrm{d}s}\,. \tag{5.20}$$

By the conditions $(p, f) \in \mathcal{U}(\omega)$ and $\mu^* > 0$, we obtain $(p, \mu^* f) \in \mathcal{U}(\omega)$. Therefore, in view of (5.20), it follows from Lemmas 4.4(2) and 4.3(3) that problem (1.1) with $\mu = \mu^*$ has exactly two positive solutions, which contradicts conclusion (2) of Theorem 3.9.

We finally show that μ^* satisfies (3.19), where Γ is given by (2.3). Suppose on the contrary that (3.19) does not hold, i.e.,

$$\mu^* \ge \frac{(\lambda-1)[\Gamma(p)\int_0^\omega [p(s)]_-\,\mathrm{d}s - \int_0^\omega [p(s)]_+\,\mathrm{d}s]^{\frac{\lambda}{\lambda-1}}}{\lambda\left[\lambda\int_0^\omega h(s)\,\mathrm{d}s\right]^{\frac{1}{\lambda-1}}[\int_0^\omega [f(s)]_+\,\mathrm{d}s - \Gamma(p)\int_0^\omega [f(s)]_-\,\mathrm{d}s]}\,.$$

Then, it follows from Lemma 4.5 that problem (1.1) with $\mu = \mu^*$ has no positive solution, which contradicts conclusion (2) of Theorem 3.9. □

Proof of Theorem 3.14. We first note that, by Remark 3.11, condition (3.12) holds. Therefore, it follows from Theorem 3.9(1,2,3) and Lemmas 4.6 and 4.7 that there exists $\mu_0 \in\,]0, +\infty[$ such that conclusions (1), (2), and (3) of the theorem hold. Since u is a solution to problem (1.1) if and only if $-u$ is a solution to problem (4.10), conclusions (5), (6), and (7) of the theorem hold as well. Finally, the validity of conclusion (4) of the theorem follows immediately from Theorem 3.9(4). □

6. Conclusions

The existence and exact multiplicity of solutions to problem (1.1) was studied depending on the choice of the parameter μ. We extended the conclusions stated in [2, Theorem 1.1] for the case of undamped Duffing equation (1.2) with $c := 0$ and weakened hypotheses (1.3) and (1.4). Our results confirm a conjecture formulated in [2, Remark 3, p. 2502] because, at least in the case of $c = 0$, the conclusions of Theorem 1.1 (except for the asymptotic stability) are still true for d which changes its sign (and belongs to a certain class of functions). We also provided both lower and upper estimates of the "critical" values μ_*, μ^* (resp. μ_0) of the parameter μ appearing in the conclusions of Theorems 3.1 and 3.9 (resp. Theorems 3.6 and 3.14).

The approach used in [2] employs identifying the fold point on bifurcation curves and the continuation method combined with the Sturm's comparison theorem, topological degree, and the maximum principle. We used a slightly different approach; we proved our results by using the method of lower and upper functions only, which was combined with the the maximum and anti-maximum principles. The results obtained substantially generalize the results available in the literature because they are not only specific sufficient conditions. Our general results hold for all the equations of the type studied whose coefficient in the linear part belongs to a certain sufficiently wide class of functions. Such a class is described in terms of the behavior of the corresponding linear periodic problem and does not exclude the so-called resonant cases.

Finally, it is worth mentioning that if the results concerning the maximum and anti-maximum principles are known for the periodic linear problem

$$u'' = p(t)u + g(t)u'; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)$$

with $p, g \in L([0, \omega])$, the parameter-dependent problem

$$u'' = p(t)u + g(t)u' + h(t)|u|^\lambda \operatorname{sgn} u + \mu f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega)$$

might be also studied in a similar way as (1.1). The first steps are already done for the Duffing equation with a constant damping coefficient g (see, e. g., [2, 8]).

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