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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO EIGENVALUE PROBLEMS FOR SCHRÖDINGER-BOPP-PODOLSKY EQUATIONS

LORENA SORIANO HERNANDEZ, GAETANO SICILIANO

Abstract. We study the existence and multiplicity of solutions for the Schrödinger-Bopp-Podolsky system

$$\begin{gathered} -\Delta u+\phi u=\omega u \quad \text{in } \Omega\\ a^2\Delta^2\phi-\Delta\phi=u^2 \quad \text{in } \Omega\\ u=\phi=\Delta\phi=0 \quad \text{on } \partial\Omega\\ \int_\Omega u^2\,dx=1 \end{gathered}$$

where Ω is an open bounded and smooth domain in $\mathbb{R}^3$, $a>0$ is the Bopp-Podolsky parameter. The unknowns are $u,\phi:\Omega\to\mathbb{R}$ and $\omega\in\mathbb{R}$. By using variational methods we show that for any $a>0$ there are infinitely many solutions with diverging energy and divergent in norm. We show that ground states solutions converge to a ground state solution of the related classical Schrödinger-Poisson system, as $a\to 0$.

1. Introduction

In this article we prove the existence of solutions for the Schrödinger-Bopp-Podolsky system

$$\begin{gathered} -\Delta u+\phi u=\omega u \quad \text{in } \Omega\\ a^2\Delta^2\phi-\Delta\phi=u^2 \quad \text{in } \Omega\\ u=\phi=\Delta\phi=0 \quad \text{on } \partial\Omega\\ \int_\Omega u^2\,dx=1 \end{gathered} \tag{1.1}$$

on a bounded and smooth domain $\Omega\subset\mathbb{R}^3$. Also we study the behavior of this system as the parameter a tends to zero.

In (1.1), the first equation is a Schrödinger equation which relates the modulus of the charged wave function $\psi(x,t)=u(x)e^{-i\omega t}$ of a non relativistic particle, its frequency $\omega\in\mathbb{R}$ and the electrostatic potential ϕ generated by its motion. The value of the charge has been settled to one for simplicity. In particular the electrostatic potential obeys to the electromagnetic field theory of generalized electrodynamics

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developed by Bopp and Podolsky. It is evident by the second equation that the source of the electrostatic field is the same wave function. In the above system the unknowns are the real functions $u, \phi : \Omega \to \mathbb{R}$ and the real number ω. To these equations we associate suitable Dirichlet boundary conditions, that roughly speaking, indicate that the particle is constrained to "live" in the bounded region Ω, and the potential vanishes on its boundary. The normalizing condition is quite reasonable from a physical point of view since it is just the L^2 norm of the wave function, that in the physical applications represents the probability of finding the particle in the region Ω.

More details on the physical background as well as the deduction of the equations can be found in [4, Section 2], where for the first time such a system was introduced. After the work [4] this kind of system has been extensively studied. In particular the problem has been addressed in the whole space and in bounded domains, where existence and multiplicity of solutions have been proved by using variational methods and critical point theory. We refer the reader to the recent papers [2, 5, 7, 9, 10, 11, 13].

In a natural way we can associate with (1.1) its "limit" problem, namely when $a = 0$. In particular, the difference is formally in the second equation which now is $-\Delta\phi = u^2$, namely the classical Poisson equation, highlighting the fact that in this case the Maxwell theory of the electromagnetic field has been used. Of course this affects also the first equation, since ϕ is different. This "limit" problem, called Schrödinger-Maxwell (or Schrödinger-Poisson system) is studied in [3], where the authors showed a general reduction method to study similar systems involving the interaction between matter and electromagnetic field. However there are some reasons for which the electromagnetic theory of Bopp-Podolsky is preferable to the Maxwell one, and this is discussed in [4]. In this paper we want to show once more in which sense the Bopp-Podolsky theory is an approximation of the Maxwell theory.

Going back to problem (1.1), our approach is variational. Indeed as usual in these cases, we will see that a suitable energy functional on certain Sobolev spaces can be defined and its critical points are exactly the weak solutions we want, according to the definition given below. We work in the Sobolev spaces $H_0^1(\Omega)$ with the usual norm

$$\|u\| := \Big(\int_\Omega |\nabla u|^2\,dx\Big)^{1/2}$$

and, for a fixed $a > 0$ we consider the space $\mathbb{H} = H_0^1(\Omega) \cap H^2(\Omega)$ endowed with the norm

$$\|\phi\|_a := \Big(a^2 \int_\Omega |\Delta\phi|^2\,dx + \int_\Omega |\nabla\phi|^2\,dx\Big)^{1/2},$$

and the associated scalar product

$$(\phi, \psi)_a = a^2 \int_\Omega \Delta\phi\Delta\psi\,dx + \int_\Omega \nabla\phi\nabla\psi\,dx.$$

Throughout this work we denote by

$$|u|_p := \Big(\int_\Omega |u|^p\,dx\Big)^{1/p}$$

the norm in $L^p(\Omega)$. We define the L^2-sphere in $H_0^1(\Omega)$ by

$$B := \{u \in H_0^1(\Omega) : |u|_2 = 1\}.$$

We recall that for a fixed $a > 0$, the triple $(\omega_a, u_a, \phi_a) \in \mathbb{R} \times B \times \mathbb{H}$ is a weak solution of (1.1) if

$$\int_\Omega \nabla u_a \nabla v\, dx + \int_\Omega \phi_a u_a v\, dx = \omega_a \int_\Omega u_a v\, dx \quad \text{for all } v \in H_0^1(\Omega) \tag{1.2}$$

and

$$a^2 \int_\Omega \Delta\phi_a \Delta v\, dx + \int_\Omega \nabla\phi_a \nabla v\, dx = \int_\Omega u_a^2 v\, dx \quad \text{for all } v \in \mathbb{H}. \tag{1.3}$$

However it worth saying that the weak solutions, if any, are sldo classical, as stated in the the first result.

Theorem 1.1. *If $(\omega_a, u_a, \phi_a) \in \mathbb{R} \times B \times \mathbb{H}$ is a weak solution of (1.1) then $u_a \in C^{2,\lambda}(\overline{\Omega})$ and $\phi_a \in C^{4,\lambda}(\overline{\Omega})$ for some $\lambda \in (0, 1)$.*

Here we are using the classical notation for Hölder spaces $C^{j,\lambda}$, $0 < \lambda \le 1$. The proof of this result involves classical boot-strap arguments. Our main result concerning with the existence of solutions is stated as follows.

Theorem 1.2. *Let $a > 0$. Then there is a sequence $\{(\omega_{a,n}, u_{a,n}, \phi_{a,n})\}_n \subset \mathbb{R} \times B \times \mathbb{H}$ of solutions of (1.1) with*

$$\omega_{a,n} \to \infty, \quad \|u_{a,n}\| \to \infty, \quad \text{as } n \to \infty$$

We will see in the proof, that more precise information can be deduced. For example the energy levels are divergent, and we can assume that $u_{a,1}$ is positive. See the proof of Theorem 1.2. In our approach the frequencies ω will appear as Lagrange multipliers associated to critical points of the energy functional on the constraint B.

Our last result concerns with the asymptotic behavior of the ground state solutions, obtained for $n = 1$ in Theorem 1.2, whenever a tends to zero. By ground state we mean a solution of the system with minimal energy, in the sense specified later. To this aim we consider the "limit" problem

$$\begin{gathered} -\Delta u + \phi u = \omega u \quad \text{in } \Omega \\ -\Delta\phi = u^2 \quad \text{in } \Omega \\ u = \phi = 0 \quad \text{on } \partial\Omega \\ \int_\Omega u^2\, dx = 1 \end{gathered} \tag{1.4}$$

studied by Benci and Fortunato in [3], where existence result of a ground state and even of multiple solutions $\{(\omega_{0,n}, u_{0,n}, \phi_{0,n})\}_n \subset \mathbb{R} \times B \times H_0^1(\Omega)$ is obtained. This system has also been extensively studied in the last decades under different boundary conditions and/or the nonlinearity. For example the case of a Berestycki-Lions type nonlinearity has been studied in [8].

Theorem 1.3. *Let $\{(\omega_{a,1}, u_{a,1}, \phi_{a,1})\}_{a>0} \in \mathbb{R} \times B \times \mathbb{H}$ be ground state solutions of (1.1) found in Theorem 1.2. Then as $a \to 0$, up to subsequences, we have*

$$u_{a,1} \to u_0 \text{ and } \phi_a \to \phi_0 \text{ in } H_0^1(\Omega), \quad \omega_{a,1} \to \omega_0 \text{ in } \mathbb{R} \tag{1.5}$$

where $(\omega_0, u_0, \phi_0) \in \mathbb{R} \times B \times H_0^1(\Omega)$ is a ground state solution of (3.5).

We will see in the proof that there is convergence also of the ground state levels.

This last result corroborates the fact that Schrödinger-Poisson (also called Schrödinger-Maxwell) systems can be seen as limit of Schrödinger-Bopp-Podolsky systems as already seen in [4, 12, 11]. This is essentially due to the fact that the Maxwell theory of electromagnetism is the limit of the generalized Bopp-Podolsky theory of electromagnetism.

We spend few words on our methods. We use critical point theory to show how the solutions can be associated with a critical point of a functional on a suitable manifold in an Hilbert space. In view of the applications of variational methods and to use topological invariants of the Ljusternick-Schnirelmann Theory, some facts like compactness and geometry of the functional have to be shown. We remind that in many problems of this type, the frequency ω of the wave function is fixed. Then the approach in finding solutions is different, in particular the L^2 norm of the solutions u is not given a priori.

In our case, the wave function is completely unknown, so both u and ω are unknowns, and we are looking for solutions with a priori fixed L^2 norm. Let us recall that the L^2 norm is constant in time on the solutions of the evolution problem, so it is constantly equal to the L^2 norm of the initial datum. As a consequence, the unknowns ω related to the solutions will be found as the Lagrange multipliers associated to the critical points on the manifold made by the unit sphere in L^2. For these reasons, we think that it is natural to consider the frequencies of the wave function, ω, as an unknown and the L^2 norm of u fixed, since it is more interesting also from a physical point of view.

The paper is organized as follows. In the subsequent Subsection 1.1 we show once for all that the weak solutions are classical. This is a classical fact which is independent of the variational framework or the way we use to find weak solutions.

Then we focus in proving the existence of solutions. In Section 2 the variational setting is implemented. This will be fundamental in order to define the energy functional and then look for its critical points, characterized as weak solutions of (1.1). In the final Section 3 the proofs of Theorem 1.2 and Theorem 1.3 are given. We use $C, C', \dots$ to denote suitable positive constants whose value may change from line to line and which do not depend on the functions involved in the inequalities.

1.1. **Proof of Theorem 1.1.** This subsection is devoted to show that every weak solution is necessary a classical solution. For the sake of simplicity we omit here the parameter a in the solutions.

Let $(\omega, u, \phi) \in \mathbb{R} \times B \times \mathbb{H}$ be a weak solution of (1.1), then $\psi := -a^2\Delta\phi + \phi$ is a weak solution of the Dirichlet problem

$$\begin{gathered}\Delta\psi = u^2 \quad \text{in } \Omega,\\ \psi = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

Now, if $u \in H_0^1(\Omega)$, then $u \in L^6(\Omega)$ and u^2 belongs to $L^3(\Omega)$. Thus, by [6, Theorem 9.9] we have

$$-a^2\Delta\phi + \phi = \psi \in W^{2,3}(\Omega). \tag{1.6}$$

Recall that Ω is a bounded set. If $\phi \in \mathbb{H}$ is a solution of (1.6) with $\psi \in W^{2,2}(\Omega)$, the interior regularity increases because [6, Theorem 8.10] implies that $\phi \in W^{4,2}(\Omega)$ which leads us to the fact that $\phi \in C^{2,\lambda}(\overline{\Omega})$ with $\lambda \in (0, \frac{1}{2}]$ by the Sobolev embedding [1, Theorem 5.4].

Now, considering the first equation of (1.1),

$$-\Delta u + \phi u = \omega u \quad \text{in } \Omega,$$

we have that $u \in H_0^1(\Omega)$ is the unique solution of $\Delta u = (\phi - \omega)u \in L^2(\Omega)$ because $\phi \in C^{2,\lambda}(\overline{\Omega})$. Then, by [6, Theorem 9.9], it holds

$$\Delta u = (\phi - \omega)u \in H_0^2(\overline{\Omega}).$$

Therefore [6, Theorem 8.10] implies that $\phi \in H_0^4(\overline{\Omega})$ which leads us to the fact that $u \in C^{2,\lambda}(\overline{\Omega})$ with $\lambda \in (0, 1/2]$ by [1, Theorem 5.4, part II]. Since $u \in H_0^1(\Omega)$ and $u \in C^{2,\lambda}(\overline{\Omega})$, $\lambda \in (0, 1/2]$, we obtain

$$-\Delta \psi = u^2 \in H^2(\Omega).$$

By [6, Theorem 8.10] it follows that

$$-a^2 \Delta \phi + \phi = \psi \in H^4(\Omega),$$

and then the interior regularity of ϕ increases by the same Theorem, i.e. $\phi \in H^6(\Omega)$. Finally, by Part II of the Sobolev embedding [1, Theorem 5.4],

$$\phi \in H^6(\Omega) \hookrightarrow C^{4,\lambda}(\overline{\Omega}),$$

where $\lambda \in (0, 1/2]$.

2. Variational setting

To prove the existence of solutions we set the right variational framework. Since the system has a Lagrangian derivation (see [4]), it is natural to look at solutions as critical point of a suitable energy functional. We define the functional on $H_0^1(\Omega) \times \mathbb{H}$ by

$$F_a(u, \phi) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{1}{2} \int_\Omega \phi u^2 \, dx - \frac{a^2}{4} \int_\Omega |\Delta \phi|^2 \, dx - \frac{1}{4} \int_\Omega |\nabla \phi|^2 \, dx. \tag{2.1}$$

Straightforward computations show that F_a is C^1 with derivatives given by

$$\partial_u F_a(u, \phi)[v] = \int_\Omega \nabla u \nabla v \, dx + \int_\Omega uv\phi \, dx, \quad \forall v \in H_0^1(\Omega) \tag{2.2}$$

$$\partial_\phi F_a(u, \phi)[v] = \frac{1}{2} \int_\Omega u^2 v \, dx - \frac{a^2}{2} \int_\Omega \Delta \phi \Delta v \, dx - \frac{1}{2} \int_\Omega \nabla \phi \nabla v \, dx, \quad \forall v \in \mathbb{H}. \tag{2.3}$$

Then we have a first variational principle.

Theorem 2.1. *Let $a > 0$. The triple $(\omega_a, u_a, \phi_a) \in \mathbb{R} \times H_0^1(\Omega) \times \mathbb{H}$ is a weak solution of (1.1) if, and only if, (u_a, ϕ_a) is a critical point of F_a restricted to $B \times \mathbb{H}$ having ω_a as a Lagrange multiplier.*

Proof. An ordered pair $(u_a, \phi_a) \in H_0^1(\Omega) \times \mathbb{H}$ is a critical point of F_a constrained to $B \times \mathbb{H}$ if and only if there exists a Lagrange multiplier $\omega_a \in \mathbb{R}$ such that

$$\partial_u F_a(u_a, \phi_a) = \omega_a u_a \quad \text{and} \quad \partial_\phi F_a(u_a, \phi_a) = 0$$

Taking into account the expressions of the partial derivatives in (2.2) and (2.3) this is equivalent to (1.2) and (1.3), namely to say that $(\omega_a, u_a, \phi_a) \in \mathbb{R} \times H_0^1(\Omega) \times \mathbb{H}$ is a weak solution of system (1.1). □

2.1. **Reduced functional.** The functional F_a in (2.1) is unbounded both from above and below. Then the usual methods of critical point theory cannot be directly applied. To deal with this issue, we shall reduce the functional in (2.1) to the study of another functional depending on the single variable u, following a procedure introduced by Benci and Fortunato in [3] for these kind of problems.

Proposition 2.2. *Given $a > 0$ and $u \in B$, the problem*

$$\begin{gathered} a^2\Delta^2\phi - \Delta\phi = u^2 \quad \text{in } \Omega \\ \Delta\phi = \phi = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.4}$$

has a unique (and non trivial) weak solution $\Phi_a(u) \in \mathbb{H}$. Moreover it minimizes the functional

$$E_a(\phi) = \frac{1}{2}\int_\Omega |\nabla\phi|^2\,dx + \frac{a^2}{2}\int_\Omega |\Delta\phi|^2\,dx - \int_\Omega u^2\phi\,dx.$$

Proof. For every $u \in B$, we define the linear functional

$$L_u : v \in \mathbb{H} \mapsto \int_\Omega u^2 v\,dx \in \mathbb{R}$$

The Hölder inequality and the Sobolev embedding imply, for $v \in \mathbb{H}$, and suitable constants $C, C' > 0$

$$|\int_\Omega u^2 v\,dx| \le |u|_4^2|v|_2 \le C'|u|_4^2|\nabla v|_2 \le C\|u\|^2\|v\|_a. \tag{2.5}$$

Then, the functional L_u is continuous, and by Riesz's Theorem, there exists a unique vector, that we denote with $\Phi_a(u) \in \mathbb{H}$ such that

$$L_u[v] = (\Phi_a(u), v)_a = \int_\Omega \nabla\Phi_a(u)\nabla v\,dx + a^2\int_\Omega \Delta\Phi_a(u)\Delta v\,dx, \quad \forall v \in \mathbb{H}.$$

In other words $\Phi_a(u) \in \mathbb{H}$ is the unique weak solution of (2.4) and satisfies

$$\int_\Omega u^2 v\,dx = a^2\int_\Omega \Delta\Phi_a(u)\Delta v\,dx + \int_\Omega \nabla\Phi_a(u)\nabla v\,dx, \quad \forall v \in \mathbb{H}. \tag{2.6}$$

Finally it is standard to see that $\Phi_a(u)$ is the unique minimizer of E_a. □

In particular from (2.6), by taking $v = \Phi_a(u)$, it follows that

$$\int_\Omega u^2\Phi_a(u)\,dx = a^2\int_\Omega |\Delta\Phi_a(u)|^2\,dx + \int_\Omega |\nabla\Phi_a(u)|^2\,dx = \|\Phi_a(u)\|_a^2. \tag{2.7}$$

Since by (2.5) it holds

$$\int_\Omega u^2\Phi_a(u)\,dx \le C\|u\|^2\|\Phi_a(u)\|_a, \tag{2.8}$$

from (2.7) we have the estimate

$$\|\Phi_a(u)\|_a \le C\|u\|^2. \tag{2.9}$$

Set now

$$\Gamma_a := \{(u,\phi) \in H_0^1(\Omega) \times \mathbb{H} : \partial_\phi F_a(u,\phi) = 0\}.$$

Take the level set $B = \{u \in H_0^1(\Omega) : |u|_2 = 1\}$ and define the map

$$\Phi_a : u \in B \mapsto \Phi_a(u) \in \mathbb{H} \tag{2.10}$$

where $\Phi_a(u)$ is the unique solution given in Proposition 2.2. Actually

$$\Phi_a(u) = (a^2\Delta^2 - \Delta)^{-1}u^2$$

where $(a^2\Delta^2 - \Delta)^{-1} : \mathbb{H}' \to \mathbb{H}$ is the Riesz isomorphism.

Proposition 2.3. *The map Φ_a is C^1 and Γ_a is its graph.*

Proof. By the Sobolev embedding, $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ is continuous and it is easy to see that the map $u \mapsto u^2$ is C^1 from $H_0^1(\Omega)$ into $L^3(\Omega)$ which is continuously embedded into $\mathbb{H}'$. Since the operator $(a^2\Delta^2 - \Delta)^{-1}$ is the Riesz isomorphism it is C^1 and then the map Φ_a, as composition of C^1 maps, is C^1 too.

Finally, the graph of Φ_a is

$$\mathrm{Gr}(\Phi_a) := \{(u, \phi) \in M : (a^2\Delta^2 - \Delta)^{-1}u^2 = \phi\}.$$

Note that $(u, \phi) \in \mathrm{Gr}(\phi_a)$ means that $(a^2\Delta^2 - \Delta)\phi = u^2$ which is equivalent to say that $\partial_\phi F_a(u, \phi) = 0$, which in turn is also equivalent to having $(u, \phi) \in \Gamma_a$. □

We are in a position to define the *reduced functional*

$$J_a(u) := F_a(u, \Phi_a(u)). \tag{2.11}$$

From (2.7) we have

$$\frac{a^2}{4}\int_\Omega |\Delta\Phi_a(u)|^2\,dx + \frac{1}{2}\int_\Omega |\nabla\Phi_a(u)|^2\,dx = \frac{1}{2}\int_\Omega u^2\Phi_a(u)\,dx - \frac{a^2}{4}\int_\Omega |\Delta\Phi_a(u)|^2\,dx$$

and hence the functional J_a takes the form

$$\begin{aligned} J_a(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{a^2}{4}\int_\Omega |\Delta\Phi_a(u)|^2\,dx + \frac{1}{2}\int_\Omega |\nabla\Phi_a(u)|^2\,dx \\ &\quad - \frac{1}{4}\int_\Omega |\nabla\Phi_a(u)|^2\,dx \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{a^2}{4}\int_\Omega |\Delta\Phi_a(u)|^2\,dx + \frac{1}{4}\int_\Omega |\nabla\Phi_a(u)|^2\,dx\,. \end{aligned} \tag{2.12}$$

Note that the dependence of J_a on a, is “explicit” because of the presence of a^2, but also “implicit” via the map Φ_a.

The functional J_a is then bounded from below, by Proposition 2.3, C^1. Then, the Fréchet derivative of J_a at u is given by

$$J_a'(u) = \partial_u F_a(u, \Phi_a(u)) + \partial_\phi F_a(u, \Phi_a(u))\Phi_a'(u) = \partial_u F_a(u, \Phi_a(u)) \tag{2.13}$$

as linear and continuous operators on $H_0^1(\Omega)$. Taking into account (2.2) we obtain

$$J_a'(u)[v] = \int_\Omega \nabla u \nabla v\,dx + \int_\Omega uv\Phi_a(u)\,dx, \quad \forall v \in H_0^1(\Omega). \tag{2.14}$$

Recall by Theorem 2 that we are reduced to find critical points (u_a, ϕ_a) of F_a on $B \times \mathbb{H}$ with the associated Lagrange multiplier ω_a. The following is a second variational principle and describes the relation between critical points of F_a on $B \times \mathbb{H}$ and critical points of J_a restricted to B.

Proposition 2.4. *Let $(u_a, \phi_a) \in B \times \mathbb{H}$ and $\omega_a \in \mathbb{R}$. The following statements are equivalent.*

(i) *The pair (u_a, ϕ_a) is a critical point of F_a constrained to $B \times \mathbb{H}$ having ω_a as Lagrange multiplier.*

(ii) *The function u_a is a critical point of J_a constrained to B having ω_a as Lagrange multiplier and $\phi_a = \Phi_a(u_a)$.*

Proof. Condition (i) means that

$$\partial_u F_a(u_a, \phi_a) = \omega_a u_a \quad \text{and} \quad \partial_\phi F_a(u_a, \phi_a) = 0.$$

But then by Proposition 2.3 it has to be $\phi_a = \Phi_a(u_a)$ and by (2.13), $J_a'(u_a) = \omega_a u_a$. This is exactly (ii).

On the other hand, (ii) implies

$$J_a'(u_a) = \omega_a u_a \quad \text{and} \quad (u_a, \Phi_a(u_a)) \in \mathrm{Gr}(\Phi_a)$$

and then $\partial_\phi F_a(u_a, \phi_a) = 0$. Consequently, again by (2.13), we infer

$$\omega_a u_a = J_a'(u_a) = \partial_u F_a(u_a, \Phi_a(u_a))$$

so (i) is proved. □

In particular the above result says that all the solutions are of type $(\omega_a, u_a, \Phi_a(u))$. In view of the previous result, for brevity we may refer just to the unknown u as a solution of the system (ω and ϕ are then univocally determined), and J_a to its energy.

2.2. **Properties of the functional J_a.** A useful tool in critical point theory to obtain the compactness is the well known Palais-Smale condition that we recall now. We say that J_a satisfies the Palais-Smale condition on the manifold $B \subset H_0^1(\Omega)$ if any sequence $\{w_n\}_n \subset B$ such that

$$\{J_a(w_n)\}_n \text{ is bounded and } J_a'(w_n) \to 0 \text{ in } \mathrm{T}_{w_n}B,$$

called also a Palais-Smale sequence, has a convergent subsequence in the $H_0^1(\Omega)$ norm to some element w (which is then necessarily in B).

Lemma 2.5. *The functional J_a constrained to B satisfies the Palais-Smale condition.*

Proof. Let $\{w_n\}_n \subset B$ be a Palais-Smale sequence for J_a. Then, there exist two sequences $\{\lambda_n\}_n \subset \mathbb{R}$ and $\{\varepsilon_n\}_n \subset H^{-1}(\Omega)$, where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$, such that $\varepsilon_n \to 0$ and, see (2.12),

$$J_a'(w_n) = \lambda_n w_n + \varepsilon_n, \tag{2.15}$$

$$J_a(w_n) = \frac{1}{2}\|w_n\|^2 + \frac{1}{4}\|\Phi_a(w_n)\|_a^2 \to c. \tag{2.16}$$

In particular $\{w_n\}_n$ and $\{\Phi_a(w_n)\}_n$ are bounded in $H_0^1(\Omega)$ and $\mathbb{H}$, respectively.

By (2.14) and (2.15) we obtain

$$\int_\Omega |\nabla w_n|^2\,dx + \int_\Omega w_n^2 \Phi_a(w_n)\,dx = \lambda_n + \varepsilon_n[w_n].$$

Using the boundedness of $\{w_n\}_n$, (2.8) and the fact that $\varepsilon_n \to 0$, we see that also $\{\lambda_n\}_n$ has to be bounded.

Equation (2.15) is rewritten as $-\Delta w_n + w_n \Phi_a(w_n) - \lambda_n w_n = \varepsilon_n$ and applying the inverse Riesz isomorphism $\Delta^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$, we obtain that

$$w_n = \Delta^{-1}(w_n \Phi_a(w_n)) - \lambda_n \Delta^{-1} w_n - \Delta^{-1}\varepsilon_n, \tag{2.17}$$

and $\{\Delta^{-1}\varepsilon_n\}_n$ is a convergent sequence. Now $\{w_n\Phi_a(w_n)\}_n$ is bounded in $L^2(\Omega)$ because of the estimates

$$\int_\Omega |w_n\Phi_a(w_n)|^2\,dx \le |\Phi_a(w_n)|_4^2|w_n|_4^2 \le C\|\Phi_a(w_n)\|_a^2|w_n|_4^2.$$

Then $\{w_n\Phi_a(w_n)\}_n$ is also bounded in $H^{-1}(\Omega)$. Actually since Δ^{-1} is compact, we deduce that (up to subsequences)

$$\big\{\Delta^{-1}(w_n\Phi_a(w_n))\big\}_n, \big\{\lambda_n\Delta^{-1}w_n\big\}_n \quad \text{are convergent.}$$

Going back to (2.17), we infer that $\{w_n\}_n$ is convergent (up to subsequences) in $H_0^1(\Omega)$, and the limit is of course in B. □

Let us recall also some basic facts about Genus Theory. Let A be a closed and symmetric subset A of a Banach space. The set A has genus $n\in\mathbb{N}$, denoted by $\gamma(A)=n$, if there exists an odd map $h\in C(A,\mathbb{R}^n\setminus\{0\})$ and n is the smallest integer having this property. If $A=\emptyset$, we say that $\gamma(A)=0$ and if there is not any integer satisfying the property, we set $\gamma(A)=\infty$.

Lemma 2.6. *For any integer m there exists a compact and symmetric subset* K *of B such that $\gamma(\mathrm{K})=m$.*

Proof. Let $\mathrm{H}_m := \operatorname{span}\{u_1,\dots,u_m\}$ be a m-dimensional subspace of $H_0^1(\Omega)$. Define

$$\mathrm{K} := B\cap \mathrm{H}_m = \{u\in \mathrm{H}_m : |u|_2 = 1\}.$$

We consider the odd homeomorphism $h:\mathrm{K}\to\mathbb{S}^{m-1}$ defined by

$$h(u) = \frac{x}{\|x\|_{\mathbb{R}^m}},$$

where $x=(x_1,\dots,x_m)\in\mathbb{R}^m$. By the genus invariance via odd homeomorphism (see e.g. [14, Proposition 5.4]), we obtain

$$\gamma(\mathrm{K}) = \gamma(\mathbb{S}^{m-1}) = m.$$

The proof is complete. □

The next result is well known in critical point theory, however we revise the argument.

Lemma 2.7. *For any $c\in\mathbb{R}$ the sublevel set*

$$J_a^c := \{u\in B : J_a(u)\le c\}$$

has finite genus.

Proof. Suppose by contradiction that there exists a real number c such that $\gamma(J_a^c)=\infty$. This means that

$$D := \{b\in\mathbb{R} : \gamma(J_a^b)=\infty\}\ne\emptyset.$$

We know that J_a is bounded from below on B, hence

$$-\infty < b^* := \inf D < \infty.$$

We claim that $b^*\notin D$. Indeed, since J_a satisfies the Palais-Smale condition on B (Lemma 2.5), the set

$$K_{b^*} := \{u\in B : J_a(u) = b^*,\ J|_B'(u)=0\}$$

is compact. By properties of the genus (see [14, Proposition 5.4]), there exists a closed symmetric neighborhood Z of K_{b^*} such that $\gamma(Z)<\infty$, then $b^*\notin D$.

By the deformation lemma (see [14, Theorem 3.11]), there exist $\varepsilon > 0$ and an odd homeomorphism η such that $\eta(1, J_a^{b^*+\varepsilon} \setminus Z) \subset J_a^{b^*-\varepsilon}$. Using properties (2), (3) and (5) of [14, Proposition 5.4], we obtain

$$\gamma(J_a^{b^*+\varepsilon}) \le \gamma(J_a^{b^*+\varepsilon} \setminus Z) + \gamma(Z) \le \gamma(J_a^{b^*-\varepsilon}) + \gamma(Z) < \infty,$$

which goes against the fact that b^* is equals to $\inf D$. Therefore for all $c \in \mathbb{R}$ it has to be $\gamma(J_a^c) < \infty$. □

3. Proof of the main results

Proof of Theorem 1.2. We show that for any $a > 0$, the functional J_a restrict to B has infinitely many critical points.

Let n be a positive integer. By Lemma 2.7, there exists a positive integer $k = k(a, n)$ such that

$$\gamma(J_a^n) = k.$$

Now, consider the collection

$$\mathcal{A}_{k+1} := \{A \subset B : A \text{ is symmetric and closed with } \gamma(A) \ge k+1\}. \tag{3.1}$$

By Lemma 2.6, there exists a compact set $\mathrm{K} \subset B$ such that $\mathrm{K} \in \mathcal{A}_{k+1}$, then $\mathcal{A}_{k+1} \neq \emptyset$.

Since by the definition,

$$\gamma(A) > \gamma(J_a^n), \text{ for all } A \in \mathcal{A}_{k+1},$$

by the monotonicity property of genus $A \not\subset J_a^n$, it follows that

$$\sup J_a(A) > n, \text{ for all } A \in \mathcal{A}_{k+1}.$$

Consequently

$$b_{a,n} := \inf\{\sup J_a(A) : A \in \mathcal{A}_{k+1}\} \ge n.$$

We know by Lemma 2.5 that J_a satisfies the Palais-Smale condition on B and it is an even functional. Then it follows from [14, Theorem 5.7] that $b_{a,n}$ is a critical value of J_a on B, achieved on some $u_{a,n} \in B$. By the Lagrange multipliers theorem, for any $n \in \mathbb{N}$ there exist $\omega_{a,n} \in \mathbb{R}$ such that

$$J_a'(u_{a,n}) = \omega_{a,n} u_{a,n} \quad \text{with } J_a(u_{a,n}) = b_{a,n} \ge n.$$

Now evaluating $J_a'(u_{a,n}) = \omega_{a,n} u_{a,n}$ on the same $u_{a,n}$ we find that

$$\frac{1}{2}\int_\Omega |\nabla u_{a,n}|^2\,dx + \frac{1}{2}\int_\Omega \Phi_a(u_{a,n}) u_{a,n}^2\,dx = \frac{1}{2}\omega_{a,n}. \tag{3.2}$$

In particular $\omega_{a,n} > 0$. Replacing the above equation in the functional given by

$$J_a(u_{a,n}) = \frac{1}{2}\int_\Omega |\nabla u_{a,n}|^2\,dx + \frac{a^2}{4}\int_\Omega |\Delta\Phi_a(u_{a,n})|^2\,dx + \frac{1}{4}\int_\Omega |\nabla\Phi_a(u_{a,n})|^2\,dx, \tag{3.3}$$

we have

$$b_{a,n} = J_a(u_{a,n}) = \frac{1}{2}\omega_{a,n} - \frac{1}{4}\int_\Omega |\Delta\Phi_a(u_{a,n})|^2\,dx - \frac{a^2}{4}\int_\Omega |\nabla\Phi_a(u_{a,n})|^2\,dx$$

or

$$\omega_{a,n} = 2b_{a,n} + \frac{a^2}{2}\int_\Omega |\Delta\Phi_a(u_{a,n})|^2\,dx + \frac{1}{2}\int_\Omega |\nabla\Phi_a(u_{a,n})|^2\,dx > 2n \tag{3.4}$$

which shows that $\omega_{a,n} \to \infty$ as $n \to \infty$. We note that (3.4) implies also that

$$\omega_{a,n} \geq \frac{1}{2}\|\Phi_a(u_{a,n})\|_a^2.$$

Recalling (2.9), we rewrite (3.2) as

$$\begin{aligned}\omega_{a,n} &= \int_\Omega |\nabla u_{a,n}|^2\,dx + a^2\int_\Omega |\Delta\Phi_a(u_{a,n})|^2\,dx + \int_\Omega |\nabla\Phi_a(u_{a,n})|^2\,dx\\ &= \|u_{a,n}\|^2 + \|\Phi_a(u_{a,n})\|_a^2\\ &\leq \|u_{a,n}\|^2 + C\|u_{a,n}\|^4\end{aligned}$$

and then $\|u_{a,n}\| \to \infty$ as $n \to \infty$.

Summing up, for any $a > 0$ fixed, we have found, for any $n \in \mathbb{N}$:

$$u_{a,n} \in B \subset H_0^1(\Omega), \quad \phi_{a,n} := \Phi_a(u_{a,n}) \in \mathbb{H}, \quad \omega_{a,n} \in \mathbb{R}$$

solutions of (1.1), proving Theorem 1.2. Furthermore the above computations provide the additional information and estimates on the norm of the solutions and the energy levels of the functional:

(1) $J_a(u_{a,n}) = \frac{1}{2}\|u_{a,n}\|^2 + \frac{1}{4}\|\phi_{a,n}\|_a^2 \geq n$,
(2) $\omega_{a,n} = \|u_{a,n}\|^2 + \|\phi_{a,n}\|_a^2 > 2n$,
(3) $\|\phi_{a,n}\|_a^2 \leq 2\omega_{a,n}$,
(4) $\|\phi_{a,n}\|_a \leq C\|u_{a,n}\|^2$.

It is well known that $u_{a,1}$ is the minimum of J_a, for this reason we say that $(\omega_{a,1}, u_{a,1}, \phi_{a,1})$ is a ground state solution of (1.1). Correspondingly, $b_{a,1}$ is the ground state level. Observe that since $J_a(|u|) = J_a(u)$, the ground state $u_{a,1}$ can be assumed positive. The proof complete. □

Remark 3.1. Besides $b_{a,n}$, the functional J_a may have other critical levels. Hence system (1.1) may have solutions other than the ones we found above. Whenever we need, we use the generic notation $(\omega_a, u_a, \Phi_a(u_a))$ for a solution of (1.1), which is not necessarily at a minimax level $b_{a,n}$, reserving the notation $(\omega_{a,n}, u_{a.n}, \phi_{a,n})$ for the solutionsfound at the minimax energy level $b_{a,n}$. In this case, it is still true that the "generic" solutions satisfy

(1) $J_a(u_a) = \frac{1}{2}\|u_a\|^2 + \frac{1}{4}\|\Phi_a(u_a)\|_a^2 > 0$,
(2) $\omega_a = \|u_a\|^2 + \|\Phi_a(u_a)\|_a^2 > 0$,
(3) $\|\Phi_a(u_a)\|_a^2 \leq 2\omega_a$,
(4) $\|\Phi_a(u_a)\|_a \leq C\|u_a\|^2$.

Being solutions, they of course satisfy $J_a'(u_a) - \omega_a u_a = 0$ in $H^{-1}(\Omega)$. These facts will be used later.

Proof of Theorem 1.3. Let us consider the classical Schrödinger-Poisson system in Ω given by

$$\begin{gathered}-\Delta u + \phi u = \omega u \quad \text{in } \Omega\\ -\Delta\phi = u^2 \quad \text{in } \Omega\\ u = \phi = 0 \quad \text{on } \partial\Omega\\ \int_\Omega u^2\,dx = 1.\end{gathered} \tag{3.5}$$

Note that, when $a = 0$ system (1.1) reduces formally to system (3.5). In this sense (3.5) can be seen as the "limit" problem of (1.1). Benci and Fortunato in [3, Theorem 1], obtained multiple solutions for the Schrödinger-Poisson system. They study the problem by variational methods by considering the functional on $H_0^1(\Omega) \times H_0^1(\Omega)$

$$F_0(u,\phi) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega \phi u^2\,dx - \frac{1}{4}\int_\Omega |\nabla\phi|^2\,dx. \tag{3.6}$$

Denoted by

$$\Phi_0 : u \in B \mapsto \Phi_0(u) \in H_0^1(\Omega)$$

the map which assigns to u the unique solution of the second equation in (3.5) satisfying $\Phi_0(u) = 0$ on $\partial\Omega$, they reduced to find critical points of

$$J_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{4}\int_\Omega |\nabla\Phi_0(u)|^2\,dx$$

on $B = \{u \in H_0^1(\Omega) : |u|_2 = 1\}$. In this case

$$J_0'(u)[v] = \int_\Omega \nabla u\nabla v\,dx + \int_\Omega uv\Phi_0(u)\,dx, \quad \forall v \in H_0^1(\Omega) \tag{3.7}$$

and J_0 satisfies the Palais-Smale condition. Then, by applying the genus index theory, they find infinitely many critical points, denoted hereafter coherently with $\{u_{0,n}\}_n \subset B$. To any $u_{0,n}$ are associated Lagrange multipliers on $\omega_{0,n} \in \mathbb{R}$ and $\phi_{0,n} := \Phi_0(u_{0,n})$ in such a way that $\{(\omega_{0,n}, u_{0,n}, \phi_{0,n})\}_n$ are solution of (3.5), namely $J_0'(u_{0,n}) - \omega_{0,n}u_{0,n} = 0$ in $H^{-1}(\Omega)$, or

$$\int_\Omega \nabla u_{0,n}\nabla v\,dx + \int_\Omega u_{0,n}v\phi_{0,n}\,dx - \omega_{0,n}\int_\Omega u_{0,n}v\,dx = 0, \quad \forall v \in H_0^1(\Omega).$$

Moreover

$$b_{0,n} := J_0(u_{0,n}) \to +\infty, \quad \|u_{0,n}\| \to +\infty, \quad \omega_{0,n} \to +\infty \qquad \text{as } n \to \infty,$$

and the critical values are characterized by

$$b_{0,n} = \inf\{\sup J_0(A) : A \in \mathcal{A}_{k+1}\}, \quad \mathcal{A}_{k+1} \text{ as in (3.1)}.$$

In particular $u_{0,1}$ is the minimum of J_0 on B and $b_{0,1}$ the ground state level. Also in this case the solutions are classical and it follows that $\Delta\phi_{0,n} = 0$ on the boundary $\partial\Omega$ and $\phi_{0,n} \in \mathbb{H} = H_0^1(\Omega) \cap H^2(\Omega)$. For all these facts see [3].

We denoted by $(\omega_{0,n}, u_{0,n}, \phi_{0,n})$ the solutions of (3.5) obtained with the genus index theory, then characterized by the levels $b_{0,n}$ above. Again, as in Remark 3.1, since J_0 may have also other critical levels, we denote with $(\omega_0, u_0, \Phi_0(u_0))$ a generic solution of (3.5), then not necessarily at the minimax level $b_{0,n}$ for J_0. It is obvious now that, if $a > 0$, systems (1.1) and (3.5) can not have the same solutions, then

$$J_0'(u_a) - \omega_0 u_a \neq 0 \quad \text{and} \quad J_a'(u_0) - \omega_a u_0 \neq 0 \quad (\text{as operators on } H_0^1(\Omega)).$$

In particular this happens for the solutions obtained at the minimax levels: $u_{a,n}$ is not a critical point of J_0, as well as $u_{0,n}$ is not a critical point of J_a.

The following result is fundamental for the convergence of the solutions of the second equation of systems (3.5) and (1.1).

Lemma 3.2. *For a fixed $v \in H_0^1(\Omega)$ let $\Phi_0(v)$ and $\Phi_a(v)$ be the unique solutions of*

$$\begin{gathered} -\Delta\phi = v^2 \quad \text{in } \Omega \\ \phi = 0 \quad \text{on } \partial\Omega \end{gathered}$$

and of

$$\begin{gathered} -\Delta\phi + a^2\Delta^2\phi = v^2 \quad \text{in } \Omega \\ \Delta\phi = \phi = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

respectively. Then, as $a \to 0$ we have (up to subsequences)

$$\Phi_a(v) \to \Phi_0(v) \quad \text{in } \ H_0^1(\Omega) \quad \text{and} \quad a\Delta\Phi_a(v) \to 0 \text{ in } L^2(\Omega).$$

Proof. We already know that, since the solutions are classical, $\Phi_a(v), \Phi_0(v) \in \mathbb{H}$. From

$$|\nabla\Phi_a(v)|_2^2 + a^2|\Delta\Phi_a(v)|_2^2 = \int_\Omega v^2\Phi_a(v)\,dx \le C|v|_4^2|\nabla\Phi_a(v)|_2$$

we see that $\{\Phi_a(v)\}_{a\in(0,1]}$ is bounded in $H_0^1(\Omega)$. Then there exists $\overline{\phi} \in H_0^1(\Omega)$ such that $\Phi_a(v) \rightharpoonup \overline{\phi}$ in $H_0^1(\Omega)$ and strongly in $L^p(\Omega), p \in [1,6)$. Going back in the equality above we deduce that $\{|a\Delta\Phi_a(v)|_2\}_{a\in(0,1]}$ is bounded (in fact, convergent). In particular

$$\lim_{a\to 0} a\int_\Omega a\Delta\Phi_a(v)\zeta\,dx = 0 \quad \forall\zeta \in L^2(\Omega). \tag{3.8}$$

Then for every $\xi \in C_0^\infty(\Omega)$, passing to the limit in the equality

$$\int_\Omega \nabla\Phi_a(v)\nabla\xi\,dx + a\int_\Omega a\Delta\Phi_a(v)\Delta\xi\,dx = \int_\Omega v^2\xi\,dx,$$

we infer that

$$\int_\Omega \nabla\overline{\phi}\nabla\xi\,dx = \int_\Omega v^2\xi\,dx$$

and then, by unicity, that $\overline{\phi} = \Phi_0(v)$. Finally,

$$\begin{aligned} &|\nabla\Phi_0(v) - \nabla\Phi_a(v)|_2^2 + |a\Delta\Phi_a(v)|_2^2 \\ &= |\nabla\Phi_0(v)|_2^2 - 2\int_\Omega \nabla\Phi_a(v)\nabla\Phi_0(v)\,dx + |\nabla\Phi_a(v)|_2^2 + |a\Delta\Phi_a(v)|_2^2 \\ &= |\nabla\Phi_0(v)|_2^2 - 2\int_\Omega \nabla\Phi_a(v)\nabla\phi_0(v)\,dx + \int_\Omega v^2\Phi_a(v)\,dx \\ &\to -|\nabla\Phi_0(v)|_2^2 + \int_\Omega v^2\Phi_0(v)\,dx = 0 \end{aligned}$$

which shows that $\Phi_a(v) \to \Phi_0(v)$ in $H_0^1(\Omega)$ and $a\Delta\Phi_a(v) \to 0$ in $L^2(\Omega)$. □

Now we can study the behavior of the generic solutions of (1.1) whenever a tends to zero. Roughly speaking it says that if we have a priori bound, then there is compactness for the solutions.

Proposition 3.3. *Let $\{(\omega_a, u_a, \Phi_a(u_a))\}_{a>0} \in \mathbb{R}\times B\times\mathbb{H}$ be solutions of* (1.1)*. If $\{u_a\}_{a\in(0,1]}$ is bounded in $H_0^1(\Omega)$, then as $a \to 0$ (up to subsequence),*

$$u_a \to u_0 \text{ and } \Phi_a(u_a) \to \Phi_0(u_0) \text{ in } H_0^1(\Omega), \quad \omega_a \to \omega_0 \text{ in } \mathbb{R},$$

where $(\omega_0, u_0, \Phi_0(u_0)) \in \mathbb{R}\times B\times H_0^1(\Omega)$ is a solution of (3.5)*.*

Moreover the following convergences hold:

(i) $a\Delta\Phi_a(u_a) \to 0$ *in* $L^2(\Omega)$,
(ii) $J_a(u_a), J_0(u_a), J_a(u_0) \to J_0(u_0)$,
(iii) $J_a'(u_0) - \omega_a u_0, J_a'(u_a) - \omega_0 u_a, J_a'(u_0) - \omega_0 u_0 \to 0$ *in* $H^{-1}(\Omega)$,
(iv) $J_0'(u_a) - \omega_0 u_a, J_0'(u_a) - \omega_a u_a, J_0'(u_0) - \omega_a u_0 \to 0$ *in* $H^{-1}(\Omega)$.

The limits in (iv) say that mixing the solutions of (1.1) and (3.5), we obtain almost solution of the limit problem: the triples $(\omega_0, u_a, \Phi_0(u_a)), (\omega_a, u_a, \Phi_0(u_a))$ and $(\omega_a, u_0, \Phi_0(u_0))$ are almost solution of (3.5).

Proof. The boundedness of $\{u_a\}_{a\in(0,1]}$ implies from (2.9) the boundedness of the sequence $\{\|\Phi_a(u_a)\|_a\}_{a\in(0,1]}$, then of the sequences $\{|\nabla\Phi_a(u_a)|_2\}_{a\in(0,1]}$ and of the sequence $\{|a\Delta\Phi_a(u_a)|_2\}_{a\in(0,1]}$. Therefore there exists $\overline{u} \in H_0^1(\Omega)$ and $\overline{\phi} \in H_0^1(\Omega)$ such that, as $a \to 0$,

$$u_a \rightharpoonup \overline{u}, \quad \Phi_a(u_a) \rightharpoonup \overline{\phi} \quad \text{in } H_0^1(\Omega). \tag{3.9}$$

It follows that

$$|\nabla\overline{\phi}|_2^2 \le \liminf_{a\to 0} |\nabla\Phi_a(u_a)|_2^2. \tag{3.10}$$

From (3.9), and using the compact Sobolev embeddings, for any $\xi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega u_a^2 \xi\, dx \to \int_\Omega \overline{u}^2 \xi\, dx\,, \quad \int_\Omega \nabla\Phi_a(u_a)\nabla\xi\, dx \to \int_\Omega \nabla\overline{\phi}\nabla\xi\, dx$$

and, for a suitable $C > 0$,

$$\Big|\int_\Omega a\Delta\Phi_a(u_a)\Delta\xi\, dx\Big| \le |\Delta\Phi_a(u_a)|_2 |\Delta\xi|_2 \le C.$$

We conclude, passing to the limit as $a \to 0$ in the equality

$$\int_\Omega \nabla\Phi_a(u_a)\nabla\xi\, dx + a^2\int_\Omega \Delta\Phi_a(u_a)\Delta\xi\, dx = \int_\Omega u_a^2\xi\, dx,$$

that

$$\int_\Omega \nabla\overline{\phi}\nabla\xi\, dx = \int_\Omega \overline{u}^2\xi\, dx. \tag{3.11}$$

Moreover for u_a a solution, using (2.8), we infer

$$0 < \omega_a = |\nabla u_a|_2^2 + \int_\Omega \Phi_a(u_a)u_a^2\, dx \le |\nabla u_a|_2^2 + C\|u_a\|^2\|\Phi_a(u_a)\|_a$$

and then $\{\omega_a\}_{a\in(0,1]}$ is bounded too, and we can assume $\omega_a \to \overline{\omega}$. We know also that for any $\xi \in C_0^\infty(\Omega)$,

$$\int_\Omega \nabla u_a\nabla\xi\, dx + \int_\Omega \Phi_a(u_a)u_a\xi\, dx = \omega_a\int_\Omega u_a\xi\, dx$$

and passing to the limit as $a \to 0$, using that $u_a \to \overline{u}, \Phi_a(u_a) \to \overline{\phi}$ in $L^2(\Omega)$, we obtain

$$\int_\Omega \nabla\overline{u}\nabla\xi\, dx + \int_\Omega \overline{\phi}\overline{u}\xi\, dx = \overline{\omega}\int_\Omega \overline{u}\xi\, dx. \tag{3.12}$$

By density, (3.11), and (3.12) we deduce that $(\overline{\omega}, \overline{u}, \overline{\phi})$ is a solution of the (3.5) system, then we can rename it $(\omega_0, u_0, \Phi_0(u_0))$ and we have proved that

$$u_a \rightharpoonup u_0, \quad \Phi_a(u_a) \rightharpoonup \Phi_0(u_0) \quad \text{in } H_0^1(\Omega) \quad \text{and} \quad \omega_a \to \omega_0.$$

The strong convergence of $\{u_a\}_{a\in(0,1]}$ is actually a consequence of the compactness, because of the boundedness of the domain. Since

$$\int_\Omega |\Phi_a(u_a)u_a|^2\,dx \le |\Phi_a(u_a)|_4^2|u_a|_4^2 \le C\|\Phi_a(u_a)\|_a^2|u_a|_4^2 \le C,$$

from

$$-u_a + \Delta^{-1}(\Phi_a(u_a)u_a) = \omega_a\Delta^{-1}u_a,$$

using the compactness of Δ^{-1}, we see that indeed $\{u_a\}_{a\in(0,1]}$ has to be convergent in $H_0^1(\Omega)$, and the limit is necessarily u_0.

Let us pass to the strong convergence of $\{\Phi_a(u_a)\}_{a\in(0,1]}$ in $H_0^1(\Omega)$. We know that $\Phi_a(u_a)$ minimizes the functional

$$E_a(\phi) = \frac{1}{2}|\nabla\phi|_2^2 + \frac{a^2}{2}|\Delta\phi|_2^2 - \int_\Omega u_a^2\phi\,dx$$

and then if $\{\xi_n\}_n \subset C_0^\infty(\Omega)$ is such that $\xi_n \to \Phi_0(u_0)$ in $H_0^1(\Omega)$ as $n\to\infty$, we obtain $E_a(\Phi_a(u_a)) \le E_a(\xi_n)$, namely

$$\begin{aligned}\frac{1}{2}|\nabla\Phi_a(u_a)|_2^2 &\le \frac{1}{2}|\nabla\Phi_a(u_a)|_2^2 + \frac{a^2}{2}|\Delta\Phi_a(u_a)|_2^2\\ &\le \frac{1}{2}|\nabla\xi_n|_2^2 + \frac{a^2}{2}|\Delta\xi_n|_2^2 - \int_\Omega u_a^2\xi_n\,dx + \int_\Omega u_a^2\Phi_a(u_a)\,dx.\end{aligned} \tag{3.13}$$

Observe that

$$\lim_{a\to0}\int_\Omega u_a^2\xi_n\,dx = \int_\Omega u_0^2\xi_n\,dx \quad\text{and}\quad \lim_{a\to0}\int_\Omega u_a^2\Phi_a(u_a)\,dx = \int_\Omega u_0^2\Phi_0(u_0)\,dx\,.$$

Then from (3.13) we obtain

$$\limsup_{a\to0}\frac{1}{2}|\nabla\Phi_a(u_a)|_2^2 \le \frac{1}{2}|\nabla\xi_n|_2^2 - \int_\Omega u_0^2\xi_n\,dx + \int_\Omega u_0^2\Phi_0(u_0)\,dx.$$

Passing to the limit in n in the above inequality we deduce

$$\limsup_{a\to0}\frac{1}{2}|\nabla\Phi_a(u_a)|_2^2 \le \frac{1}{2}|\nabla\Phi_0(u_0)|_2^2$$

that joint with (3.10) gives $|\nabla\Phi_a(u_a)|_2 \to |\nabla\Phi_0(u_0)|_2$ and so $\Phi_a(u_a)\to\Phi_0(u_0)$ in $H_0^1(\Omega)$. The strong convergence to a solution of the (3.5) system is proved.

As a consequence, as $a\to0$,

$$\begin{aligned}|a\Delta\Phi_a(u_a)|_2^2 &= \int_\Omega u_a^2\Phi_a(u_a)\,dx - |\nabla\Phi_a(u_a)|_2^2\\ &\to \int_\Omega u_0^2\Phi_0(u_0)\,dx - |\nabla\Phi_0(u_0)|_2^2 = 0\end{aligned}$$

proving (i).

Clearly, by (i) and the above strong convergence, it is

$$\begin{aligned}J_a(u_a) &= \frac{1}{2}|\nabla u_a|_2^2 + \frac{a^2}{2}|\Delta\Phi_a(u_a)|_2^2 + \frac{1}{4}|\nabla\Phi_a(u_a)|_2^2\\ &\to \frac{1}{2}|\nabla u_0|_2^2 + \frac{1}{4}|\nabla\Phi_0(u_0)|_2^2 = J_0(u_0).\end{aligned}$$

By using the continuity of the map Φ_0 we obtain

$$J_0(u_a) = \frac{1}{2}|\nabla u_a|_2^2 + \frac{1}{4}|\nabla\Phi_0(u_a)|_2^2 \to \frac{1}{2}|\nabla u_0|_2^2 + \frac{1}{4}|\nabla\Phi_0(u_0)|_2^2 = J_0(u_0).$$

Moreover, by Lemma 3.2 with $v := u_0$ we have

$$\begin{aligned} J_a(u_0) &= \frac{1}{2}|\nabla u_0|_2^2 + \frac{a^2}{2}|\Delta\Phi_a(u_0)|_2^2 + \frac{1}{4}|\nabla\Phi_a(u_0)|_2^2 \\ &\to \frac{1}{2}|\nabla u_0|_2^2 + \frac{1}{4}|\nabla\Phi_0(u_0)|_2^2 = J_0(u_0) \end{aligned}$$

and these last three limits prove (ii).

The proof of the limits in (iii) and (iv) follows the same lines we use until now: just use Lemma 3.2 with $v := u_0$, the strong convergence of the solutions proved above and (i). As an example let us verify just the first limit in (iii).

For any $\xi \in C_0^\infty(\Omega)$, using Lemma 3.2 with $v = u_0$, we have

$$\begin{aligned} J_a'(u_0)[\xi] &= \int_\Omega \nabla u_0 \nabla\xi\,dx + a^2\int_\Omega \Delta\Phi_a(u_0)\Delta\xi\,dx + \int_\Omega \nabla\Phi_a(u_0)\nabla\xi\,dx \\ &\to \int_\Omega \nabla u_0\nabla\xi\,dx + \int_\Omega \nabla\Phi_0(u_0)\nabla\xi\,dx = J_0'(u_0)[\xi]. \end{aligned}$$

By density the convergence is true for any $v \in H_0^1(\Omega)$. Since it is also easy to see that the limit is uniform in v and $\omega_a \to \omega_0$, we have $J_a'(u_0) - \omega_a u_0 \to J_0'(u_0) - \omega_0 u_0 = 0$, being u_0 a critical point of J_0 on B with Lagrange multiplier ω_0. The proof is then complete. □

Remark 3.4. In addition to the convergence $J_a(u_0) \to J_0(u_0)$, we have further information. By (2.1) and (3.6), for any $a > 0$, $u \in H_0^1(\Omega)$, and $\phi \in \mathbb{H}$, we have

$$F_a(u,\phi) < F_0(u,\phi).$$

Then if u_0 is a critical point of J_0, since $\Phi_0(u_0) \in \mathbb{H}$, we infer that

$$J_a(u_0) = F_a(u_0,\Phi_0(u_0)) < F_0(u_0,\Phi_0(u_0)) = J_0(u_0)\,.$$

We stress the fact that in Proposition 3.3 a fundamental assumption has been the a priori bound, namely the boundedness of $\{u_a\}_{a\in(0,1]}$.

In particular Proposition 3.3 and Remark 3.4 hold for the solutions of Theorem 1.2. We state for convenience explicitly the result for n fixed.

Corollary 3.5. *Fixed $n^* \in \mathbb{N}$, let $\{(\omega_{a,n^*}, u_{a,n^*}, \phi_{a,n^*})\}_{a>0} \in \mathbb{R}\times B\times\mathbb{H}$ be solutions of* (1.1) *found in Theorem 1.2. If $\{u_{a,n^*}\}_{a\in(0,1]}$ is bounded in $H_0^1(\Omega)$, then as $a \to 0$ (up to subsequence)*

$$u_{a,n^*} \to u_0 \text{ and } \phi_{a,n^*} \to \Phi_0(u_0) \text{ in } H_0^1(\Omega), \quad \omega_{a,n^*} \to \omega_0 \text{ in } \mathbb{R}.$$

where $(\omega_0, u_0, \Phi_0(u_0)) \in \mathbb{R}\times B\times H_0^1(\Omega)$ is a solution of (3.5)*.*

Moreover the following convergences hold:

- (i) $a\Delta\Phi_{a,n^*} \to 0$ *in* $L^2(\Omega)$,
- (ii) $J_a(u_{a,n^*}), J_0(u_{a,n^*}), J_a(u_0) \to J_0(u_0)$, *and* $J_a(u_0) < J_0(u_0)$,
- (iii) $J_a'(u_0) - \omega_{a,n^*}u_0, J_a'(u_{a,n^*}) - \omega_0 u_{a,n^*}, J_a'(u_0) - \omega_0 u_0 \to 0$ *in* $H^{-1}(\Omega)$,
- (iv) $J_0'(u_{a,n^*}) - \omega_0 u_{a,n^*}, J_0'(u_{a,n^*}) - \omega_{a,n^*}u_{a,n^*}, J_0'(u_0) - \omega_{a,n^*}u_0 \to 0$ *in* $H^{-1}(\Omega)$.

Remark 3.6. By (2.9), we see that the boundedness of $\{u_{a,n^*}\}_{a\in(0,1]}$ in $H_0^1(\Omega)$ is equivalent

- (i) by (3.2), to require that $\{\omega_{a,n^*}\}_{a\in(0,1]}$ be bounded; or
- (ii) by (3.3), to require that $\{J_a(u_{a,n^*})\}_{a\in(0,1]}$ be bounded.

This fact will be important in the proof of Theorem 1.3. An analogous observation can be made for the generic solutions $(\omega_a, u_a, \Phi_a(u_a))$, however we will not use it.

Two natural questions arise from Corollary 3.5:

(1) in which case the solutions $\{u_{a,n^*}\}_{a\in(0,1]}$ are bounded?
(2) Even if they are bounded, then by the limit in (ii) in the Corollary, $b_{a,n^*} \to J_0(u_0)$, can we say that $J_0(u_0) = b_{0,n^*}$? In other words, does the minimax levels converge to the respective minimax levels?

In case $n^* = 1$, namely in case of ground state solutions, we can give a positive answer to both questions: not only the solutions are automatically bounded as a goes to zero, but the limit is a ground state solution of (3.5), i.e. $b_{a,1} \to b_{0,1}$. In fact we can give the proof of Theorem 1.3.

Let $u_{a,1}$ be the ground state of J_a, and $u_{0,1}$ the ground state of J_0. We have

$$J_a(u_{a,1}) \le J_a(u_{0,1}) < J_0(u_{0,1}) = b_{0,1}, \tag{3.14}$$

where the strict inequality is due to Remark 3.4 replacing the generic solution u_0 of (3.5) with the particular one $u_{0,1}$. Then

$$\limsup_{a\to 0} J_a(u_{a,1}) \le b_{0,1} \tag{3.15}$$

and by (ii) of Remark 3.6 we have the boundedness of $\{u_{a,1}\}_{a\in(0,1]}$ in $H_0^1(\Omega)$, the *a priori* bound we were looking for. Corollary 3.5 gives, as $a \to 0$,

$$u_{a,1} \to u_0, \quad \phi_{a,1} \to \Phi_0(u_0), \quad \omega_{a,1} \to \omega_0$$

and $(\omega_0, u_0, \Phi_0(u_0)) \in \mathbb{R} \times B \times H_0^1(\Omega)$ solves (3.5). We do not know yet if u_0 is a minimum of J_0. However by the first limit in (ii) in Corollary 3.5 and (3.15),

$$b_{a,1} = J_a(u_{a,1}) \to J_0(u_0) \le b_{0,1}$$

On the other hand it holds $b_{0,1} \le J_0(u_0)$, so that, as a tends to zero, $b_{a,1} \to b_{0,1}$ and u_0 is a minimum of J_0 on B. The proof of Theorem 1.3 is complete.

We conclude by saying that for the other solutions $\{u_{a,n^*}\}_{a\in(0,1]}$ which are not at the ground state level, namely for $n^* \ne 1$, although it is always true that (see Remark 3.4),

$$J_a(u_{0,n^*}) < J_0(u_{0,n^*}) = b_{0,n^*}, \tag{3.16}$$

we cannot guarantee the first inequality in (3.14), i.e. $J_a(u_{a,n^*}) \le J_a(u_{0,n^*})$, which would give, joint with (3.16), the boundedness $J_a(u_{a,n^*}) < b_{0,n^*}$. We leave this as an interesting open problem.

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Lorena Soriano Hernandez
Departamento de Matemática, Universidade de Brasília - UnB, 70910-900, Campus Universitário Darcy Ribeiro, Asa Norte, Brasília, DF, Brazil
Email address: loresohe199@gmail.com

Gaetano Siciliano
Departamento de Matemática, Universidade de São Paulo, Rua do Matão 1010, 05508-090 São Paulo, SP, Brazil
Email address: sicilian@ime.usp.br