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OSCILLATION CRITERIA OF FOURTH-ORDER NONLINEAR SEMI-NONCANONICAL NEUTRAL DIFFERENTIAL EQUATIONS VIA A CANONICAL TRANSFORM

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Abstract. In this work first we transform the semi-noncanonical fourth order neutral delay differential equations into canonical type. This simplifies the investigations of finding the relationships between the solution and its companion function which plays an important role in the oscillation theory of neutral differential equations. Moreover, we improve these relationships based on the monotonic properties of positive solutions. We present new conditions for the oscillation of all solutions of the corresponding equation which improve the oscillation results already reported in the literature. Examples are provided to illustrate the importance of our main results.

1. Introduction

In recent years, the oscillation theory has expanded and developed greatly since this phenomena take part in different models from real world applications, see, e.g., the papers [7, 8, 21] dealing with biological mechanisms (for models from mathematical biology where oscillation and/or delay actions may be formulated by means of cross-diffusion terms). Moreover, the study of neutral functional differential equations has attracted considerable/significant attention because it arise in many fields such as control theory, communication, mechanical engineering, biodynamics, physics, economics and so on, see [10, 29, 30] and the references therein. In particular, Emden–Fowler differential equations have many applications in mathematical, theoretical and chemical physics; we refer the reader to the papers [18, 19] for more details. In view of the above observations, one can see that the investigation of oscillatory and asymptotic behavior of solutions of delay and neutral type fourth order functional differential equations has received immense interest in recent times; for example, see [1, 2, 3, 4, 5, 6, 12, 14, 20, 22, 23, 24, 26, 27, 28, 31, 32] and the references cited therein. The aim of this study is to establish new oscillation conditions for all solutions of the neutral delay differential equation

$$L_4 z(t) + q(t)x^{\alpha}(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

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where $z(t) = x(t) + a(t)x(\tau(t))$, α is a ratio of odd positive integers, and L_4 is an iterated operator defined as follows:

$$L_0 z = z, L_i z = p_i(L_{i-1}z))' \quad \text{for } i = 1, 2, 3, \text{ and } L_4 z = (L_3 z))'.$$

During this study, we assume the following assumptions:

(A1) $p_i \in C^{(4-i)}([t_0, \infty), (0, \infty))$ for $i = 1, 2, 3$;
(A2) $a, q \in C([t_0, \infty), [0, \infty))$ with $0 \le a(t) < 1$ and q does not vanish eventually;
(A3) $\tau \in C^1([t_0, \infty), \mathbb{R})$ with $\tau'(t) > 0$, $\sigma \in C([t_0, \infty), \mathbb{R})$ is nondecreasing, $\tau(t) \le t$, $\sigma(t) \le t$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$.

We define

$$\Omega_i(t) = \int_t^\infty \frac{1}{p_i(s)} ds \quad \text{for } i = 1, 2, 3,$$

and introduce the classification as in [28]. The equation (1.1) is in semi-noncanonical form if either one of the 3 conditions hold:

$$\Omega_1(t_0) < \infty, \quad \Omega_2(t_0) = \infty, \quad \Omega_3(t_0) < \infty, \tag{1.2}$$

$$\Omega_1(t_0) < \infty, \quad \Omega_2(t_0) < \infty, \quad \Omega_3(t_0) = \infty, \tag{1.3}$$

$$\Omega_1(t_0) = \infty, \quad \Omega_2(t_0) < \infty, \quad \Omega_3(t_0) < \infty. \tag{1.4}$$

By a solution of (1.1), we mean a function $x \in C([t_*, \infty), \mathbb{R})$ for $t_* \ge t_0$, which has the property $L_i z \in C^1([t_0, \infty), \mathbb{R})$ for $i = 1, 2, 3$ and $\sup\{|x(t)| : t \ge t_x\} > 0$ for $t_x \ge t_*$ and x satisfies (1.1) on $[t_*, \infty)$. Such a solution x of (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, it is said to be *nonoscillatory*. The equation itself is called oscillatory if all its solutions oscillate.

Recently in [1, 3, 4, 5, 14, 31], the authors studied the oscillatory properties of solutions of (1.1) in each one of following cases:

$$\Omega_i(t_0) = \infty, \text{ for } i = 1, 2, 3, \text{ i.e., equation (1.1) is in canonical form;}$$

$$\Omega_1(t_0) = \Omega_2(t_0) = \infty, \quad \Omega_3(t_0) < \infty;$$

$$\Omega_1(t_0) < \infty, \quad \Omega_2(t_0) < \infty, \quad \Omega_3(t_0) = \infty;$$

without changing the form of the equation. In [26, 27, 28] the authors studied equation (1.1) when

$$\Omega_i(t_0) < \infty, \quad i = 1, 2, 3, \quad \text{i.e., equation (1.1) is in noncanonical form,}$$

or (1.3) or (1.4) hold, by transforming the equations into canonical form. The main advantage of studying (1.1) in canonical form is that using famous Kiguradze's Lemma (see [13]) to classify the behavior of nonoscillatory solutions results in there exisitng only two types of solutions where as six types for semi-noncanonical equations. Suppose, we keep the equation (1.1) as it is and if x is a positive solution of (1.1), then the companion function z must satisfy six possible cases and it is very difficult to get a relationship between z and x and this is certainly essential to obtain oscillation criteria for the equation (1.1). Further note that if the studied fourth order neutral differential equation is not in canonical form, then the authors proved only that every solution is either oscillatory or tends to zero asymptotically, see [5, 11, 17]. To overcome these difficulties first we transform (1.1) into canonical type, which reduce the classification into two cases and from these one can easily obtain the relation between x and z (see also the paper [9] for more interesting details). Thus, our method not only reduces the number of classification types of non-oscillatory solutions but it is also very helpful in finding a relation between z and x. Hence, the authors believe that the results obtained here form a significant contribution to the oscillation theory of fourth order functional differential equations.

2. Main results

Throughout, and withouth further mention, we assume that (1.2) holds. For this case, we use the notation

$$a_3(t) = p_3(t)\Omega_3^2(t),\ a_2(t) = \frac{p_2(t)}{\Omega_3(t)\Omega_1(t)}, \quad a_1(t) = p_1(t)\Omega_1^2(t), \quad y(t) = \frac{z(t)}{\Omega_1(t)}.$$

Theorem 2.1. *Let*

$$\int_{t_0}^{\infty} \frac{1}{a_2(t)} dt = \infty. \tag{2.1}$$

Then the semi-noncanonical operator $L_4 z$ has the canonical form

$$L_4 z(t) = \frac{1}{\Omega_3(t)} (a_3(t)(a_2(t)(a_1(t)y'(t))')')'. \tag{2.2}$$

Proof. With a simple calculation we observe that

$$\Big(p_1(t)\Omega_1^2(t)\Big(\frac{z(t)}{\Omega_1(t)}\Big)'\Big)' = \big(\Omega_1(t)p_1(t)z'(t) + z(t)\big)' = \Omega_1(t)(p_1(t)z'(t))'.$$

Now,

$$\begin{aligned}
&\Big(p_3(t)\Omega_3^2(t)\Big(\frac{p_2(t)}{\Omega_3(t)\Omega_1(t)}\Big(p_1(t)\Omega_1^2(t)\Big(\frac{z(t)}{\Omega_1(t)}\Big)'\Big)'\Big)'\Big)' \\
&= \Big(p_3(t)\Omega_3^2(t)\Big(\frac{p_2(t)}{\Omega_3(t)}\Big(p_1(t)z'(t)\Big)'\Big)'\Big)' \\
&= [p_3(t)\Omega_3(t)(p_2(t)(p_1(t)z'(t))')' + p_2(t)(p_1(t)z'(t))']' \\
&= \Omega_3(t)(p_3(t)(p_2(t)(p_1(t)z'(t))')')'.
\end{aligned}$$

Therefore,

$$L_4 z(t) = \frac{1}{\Omega_3(t)} (a_3(t)(a_2(t)(a_1(t)y'(t))')')'.$$

To see that (2.2) is in canonical form, note that

$$\begin{gathered}
\int_{t_0}^{\infty} \frac{1}{a_3(t)} dt = \int_{t_0}^{\infty} \frac{1}{p_3(t)\Omega_3^2(t)} dt = \lim_{t\to\infty} \frac{1}{\Omega_3(t)} - \frac{1}{\Omega_3(t_0)} = \infty, \\
\int_{t_0}^{\infty} \frac{1}{a_1(t)} dt = \int_{t_0}^{\infty} \frac{1}{p_1(t)\Omega_1^2(t)} dt = \lim_{t\to\infty} \frac{1}{\Omega_1(t)} - \frac{1}{\Omega_1(t_0)} = \infty, \\
\int_{t_0}^{\infty} \frac{1}{a_2(t)} dt = \infty
\end{gathered}$$

by (2.2). This completes the proof. □

From Theorem 2.1, we see that under condition (2.1), equation (1.1) can be written in the equivalent canonical form

$$\overline{L}_4 y(t) + \Omega_3(t) q(t) x^{\alpha}(\sigma(t)) = 0,$$

where $\overline{L}_0 y = y$, $\overline{L}_i y = a_i(\overline{L}_{i-1} y)'$ for $i = 1, 2, 3$, and $\overline{L}_4 y = (\overline{L}_3 y)'$.

Corollary 2.2. *The semi-noncanonical equation* (1.1) *is oscillatory if and only if the canonical equation*
$$\overline{L}_4 y(t) + \Omega_3(t) q(t) x^{\alpha}(\sigma(t)) = 0 \tag{EC1}$$
is oscillatory.

Lemma 2.3. *Assume that* (2.1) *holds. If $x(t)$ is an eventually positive solution of* (EC1)*, then the companion function $y(t)$ is positive and satisfies either*
$$y(t) \in \mathcal{S}_1 \Leftrightarrow \overline{L}_1 y(t) > 0,\ \overline{L}_2 y(t) < 0,\ \overline{L}_3 y(t) > 0,\ \overline{L}_4 y(t) \le 0,$$
or
$$y(t) \in \mathcal{S}_3 \Leftrightarrow \overline{L}_1 y(t) > 0,\ \overline{L}_2 y(t) > 0,\ \overline{L}_3 y(t) > 0,\ \overline{L}_4 y(t) \le 0.$$
Hence the set $\mathcal{S}$ of all positive solutions of (EC1) *has the decomposition $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_3$.*

For convenience, we denote:
$$f^{[0]}(t) = t, f^{[j]}(t) = f(f^{[j-1]}(t)) \quad \text{for } j = 1, 2, \dots,$$
$$A_j(t) = \int_{t_1}^{t} \frac{1}{a_j(s)} ds, \quad j = 1, 2, 3,$$
$$Q_2(t) = \int_{t_1}^{t} \frac{1}{a_2(s)} A_3(s) ds, \quad Q_3(t) = \int_{t_1}^{t} \frac{1}{a_1(s)} Q_2(s) ds,$$
$$D_1(t) = \Omega_3(t) q(t) B_1^{\alpha}(\sigma(t); m), \quad D_2(t) = \Omega_3(t) q(t) B_2^{\alpha}(\sigma(t); m),$$
$$R_1(t) = \Big(\frac{1}{a_2(t)} \int_t^{\infty} \frac{1}{a_3(s)} \int_s^{\infty} D_1(v) dv ds\Big)\Big(\int_{t_1}^{\sigma(t)} \frac{1}{a_1(s)} ds\Big)^{\alpha},$$
$$R_2(t) = D_2(t)\Big(\int_{t_1}^{\sigma(t)} \frac{1}{a_1(s)} \int_{t_1}^{s} \frac{1}{a_2(v)} \int_{t_1}^{v} \frac{1}{a_3(s_1)} ds_1 dv ds\Big)^{\alpha},$$
and we assume without further mention that
$$a(\tau^{[2r]}(t)) \frac{\Omega_1(\tau^{[2r+1]}(t))}{\Omega_1(\tau^{[2r]}(t))} < 1$$
for every integer $r \ge 0$ and $t \ge t_1$ for some $t_1 \ge t_0$.

Lemma 2.4. *Suppose that x is an eventually positive solution of* (EC1)*. Then, eventually,*
$$x(t) \ge \sum_{r=0}^{m} \Big(\prod_{l=0}^{2r} a\big(\tau^{[l]}(t)\big)\Big)\Big[\frac{\Omega_1(\tau^{[2r]}(t)) y(\tau^{[2r]}(t))}{a(\tau^{[2r]}(t))} - \Omega_1(\tau^{[2r+1]}(t)) y(\tau^{[2r+1]}(t))\Big] \tag{2.3}$$
for each integer $m \ge 0$.

Proof. From the definition of x and z, we have
$$\begin{aligned} x(t) &= z(t) - a(t) x(\tau(t)) \\ &= z(t) - a(t) z(\tau(t)) + a(t) a(\tau(t)) x(\tau^{[2]}(t)) \\ &= z(t) - a(t) z(\tau(t)) + a(t) a(\tau(t)) z(\tau^{[2]}(t)) - a(t) a(\tau(t)) a(\tau^{[2]}(t)) x(\tau^{[3]}(t)) \end{aligned}$$
and so on. Thus,
$$x(t) \ge \sum_{r=0}^{m} (-1)^r \Big(\prod_{l=0}^{r} a\big(\tau^{[l]}(t)\big)\Big) \frac{z(\tau^{[r]}(t))}{a(\tau^{[r]}(t))}$$

for each odd integer $m \geq 0$, or

$$x(t) \geq \sum_{r=0}^{m} \Big(\prod_{l=0}^{2r} a\big(\tau^{[l]}(t)\big)\Big)\Big[\frac{z(\tau^{[2r]}(t))}{a(\tau^{[2r]}(t))} - z(\tau^{[2r+1]}(t))\Big]$$

for each integer $m \geq 0$. Now using $z(t) = \Omega_1(t)y(t)$ we obtain the desired result. □

Lemma 2.5. *Assume that x is an eventually positive solution of* (1.1) *and suppose that* (2.1) *holds. Then*

(i) *if $y(t) \in \mathcal{S}_1$, then $\frac{y(t)}{A_1(t)}$ is decreasing for $t \geq t_1$ for some $t_1 \geq t_0$;*

(ii) *if $y(t) \in \mathcal{S}_3$, then $\frac{y(t)}{Q_3(t)}$ is decreasing and $\overline{L}_1 y(t) \geq Q_2(t)\overline{L}_3 y(t)$ for $t \geq t_1$ for some $t_1 \geq t_0$.*

Proof. Let $x(t)$ be an eventually positive solution of (1.1). Then, $x(t)$ is also an eventually positive solutions of (EC1). Thus, by Lemma 2.3, the companion function $y(t)$ is positive and satisfies either $y(t) \in \mathcal{S}_1$ or $y(t) \in \mathcal{S}_3$. The remainder of the proof is similar to that of [6, Theorem 3.1] and so the details are omitted. □

Lemma 2.6. *Assume that x is an eventually positive solution of* (1.1) *and suppose* (2.1) *holds. If the companion function $y(t) \in \mathcal{S}_1$, then*

$$x(t) \geq B_1(t;m)y(t), \tag{2.4}$$

and if $y(t) \in \mathcal{S}_3$, then

$$x(t) \geq B_2(t;m)y(t), \tag{2.5}$$

where

$$B_1(t;m) = \sum_{r=0}^{m} \Big(\prod_{l=0}^{2r} a\big(\tau^{[l]}(t)\big)\Big)\Omega_1(\tau^{[2r]}(t))\Big[\frac{1}{a(\tau^{[2r]}(t))} - \frac{\Omega_1(\tau^{[2r+1]}(t))}{\Omega_1(\tau^{[2r]}(t))}\Big]\frac{A_1(\tau^{[2r]}(t))}{A_1(t)},$$

and

$$B_2(t;m) = \sum_{r=0}^{m} \Big(\prod_{l=0}^{2r} a\big(\tau^{[l]}(t)\big)\Big)\Omega_1(\tau^{[2r]}(t))\Big[\frac{1}{a(\tau^{[2r]}(t))} - \frac{\Omega_1(\tau^{[2r+1]}(t))}{\Omega_1(\tau^{[2r]}(t))}\Big]\frac{Q_3(\tau^{[2r]}(t))}{Q_3(t)}$$

for all positive integer $m \geq 0$.

Proof. From Lemma 2.4, we have (2.3) holds. Based on the monotonic properties of $y(t) \in \mathcal{S}_1$, we see that $y(\tau^{[2r+1]}(t)) \leq y(\tau^{[2r]}(t))$ for $r = 0, 1, 2, \dots,$ is obtained. Thus, (2.3) becomes

$$x(t) \geq \sum_{r=0}^{m} \Big(\prod_{l=0}^{2r} a\big(\tau^{[l]}(t)\big)\Big)\Big[\frac{\Omega_1(\tau^{[2r]}(t))}{a(\tau^{[2r]}(t))} - \Omega_1(\tau^{[2r+1]}(t))\Big]y(\tau^{[2r]}(t)). \tag{2.6}$$

From Lemma 2.5 (i), we see that

$$y(\tau^{[2r]}(t)) \geq \Big(\frac{A_1(\tau^{[2r]}(t))}{A_1(t)}\Big)y(t). \tag{2.7}$$

Using (2.7) in (2.6), one can obtain (2.4). Again based on the monotonic properties of $y(t) \in \mathcal{S}_3$, we see that

$$y(\tau^{[2r+1]}(t)) \leq y(\tau^{[2r]}(t)), \quad \text{for } r = 0, 1, 2, \dots.$$

Thus again (2.6) holds. Now from Lemma 2.5 (ii), we see that

$$y(\tau^{[2r]}(t)) \geq \Big(\frac{Q_3(\tau^{[2r]}(t))}{Q_3(t)}\Big)y(t). \tag{2.8}$$

Substituting (2.8) in (2.6), we obtain (2.5). The proof of lemma is complete. □

Remark 2.7. It is easy to verify that for $m = 0$, we have

$$B_1(t;0) = B_2(t;0) = \Omega_1(t)\Big(1 - a(t)\frac{\Omega_1(\tau(t))}{\Omega_1(t)}\Big).$$

Thus, the relation (2.4) and (2.5) reduce to

$$x(t) \geq \Omega_1(t)\Big(1 - a(t)\frac{\Omega_1(\tau(t))}{\Omega_1(t)}\Big)y(t).$$

Theorem 2.8. *Let* (2.1) *hold. Suppose that both first-order delay differential equations*

$$w'(t) + R_1(t)w^{\alpha}(\sigma(t)) = 0, \tag{2.9}$$

$$u'(t) + R_2(t)u^{\alpha}(\sigma(t)) = 0 \tag{2.10}$$

are oscillatory. Then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be an eventually positive solution of (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then, $x(t)$ is also an eventually positive solutions of (EC1). Thus, it follows from Lemma 2.3 that either $y(t) \in \mathcal{S}_1$ or $y(t) \in \mathcal{S}_3$ for $t \geq t_1$. First we assume that $y(t) \in \mathcal{S}_1$. From (EC1) and (2.4), we have

$$\overline{L}_4 y(t) + D_1(t)y^{\alpha}(\sigma(t)) \leq 0. \tag{2.11}$$

Since $a_1(t)y'(t)$ is decreasing, we see that

$$y(t) \geq \int_{t_1}^{t} a_1(s)\frac{y'(s)}{a_1(s)}ds \geq a_1(t)y'(t)\int_{t_1}^{t}\frac{1}{a_1(s)}ds. \tag{2.12}$$

Integrating (2.11) from t to ∞, we obtain

$$(a_2(t)(a_1(t)y'(t))')' \geq \frac{y^{\alpha}(\sigma(t))}{a_3(t)}\int_t^{\infty} D_1(s)ds. \tag{2.13}$$

Integrating (2.13) from t to ∞, we obtain

$$-(a_1(t)y'(t))' \geq \frac{y^{\alpha}(\sigma(t))}{a_2(t)}\int_t^{\infty}\frac{1}{a_3(v)}\int_v^{\infty} D_1(s)\,ds\,dv. \tag{2.14}$$

From (2.12) and (2.14), we observe that

$$-(a_1(t)y'(t))' \geq R_1(t)(a_1(\sigma(t))y'(\sigma(t)))^{\alpha}. \tag{2.15}$$

Letting $w(t) = a_1(t)y'(t)$ in (2.15), it follows from (2.15) that w is a positive solution of the differential inequality

$$w'(t) + R_1(t)w^{\alpha}(\sigma(t)) \leq 0.$$

Therefore, by [25, Theorem 1], the associated delay differential equation (2.9) also has a positive solution. This contradiction implies that $\mathcal{S}_1$ is empty.

Next, we shall assume that $y(t) \in \mathcal{S}_3$. From (EC1) and (2.5), we have

$$\overline{L}_4 y(t) + D_2(t)y^{\alpha}(\sigma(t)) \leq 0. \tag{2.16}$$

Noting that $a_3(t)(a_2(t)(a_1(t)y'(t))')'$ is decreasing, we see that

$$\begin{aligned} a_2(t)(a_1(t)y'(t))' &\geq \int_{t_1}^{t} \frac{1}{a_3(s)} a_3(s)(a_2(s)(a_1(s)y'(s))')' ds \\ &\geq a_3(t)(a_2(t)(a_1(t)y'(t))')' \int_{t_1}^{t} \frac{1}{a_3(s)} ds. \end{aligned}$$

Integrating the last inequality, we obtain

$$y'(t) \geq a_3(t)(a_2(t)(a_1(t)y'(t))')' \frac{1}{a_1(t)} \int_{t_1}^{t} \frac{1}{a_2(v)} \int_{t_1}^{v} \frac{1}{a_3(s)}\, ds\, dv.$$

Integrating once more, we see that $u(t) = a_3(t)(a_2(t)(a_1(t)y'(t))')'$ satisfies

$$y(t) \geq u(t) \int_{t_1}^{t} \frac{1}{a_1(s)} \int_{t_1}^{s} \frac{1}{a_2(v)} \int_{t_1}^{v} \frac{1}{a_3(s_1)} ds_1 dv ds.$$

Using the last estimate in (2.16), we see that u is a positive solution of the differential inequality

$$u'(t) + R_2(t) u^{\alpha}(\sigma(t) \leq 0,$$

which, in view of Philos [25, Theorem 1], implies that the corresponding differential equation (2.10) also has a positive solution. This is again a contradiction and so $\mathcal{S}_3$ is empty. The proof of the theorem is complete. □

Applying suitable criteria for the oscillation of (2.9) and (2.10) with $\alpha \in (0, 1]$, we obtain immediately the following conditions for the oscillation of (1.1). The first one is due to [16, Theorem 1], whereas the second one is due to [15, Theorem 2].

Corollary 2.9. *Let $\alpha = 1$ and let (2.1) hold. If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} H(s) ds > \frac{1}{e}, \tag{2.17}$$

where $H(t) = \min\{R_1(t), R_2(t)\}$, then (1.1) is oscillatory.

Corollary 2.10. *Let (2.1) hold and $\alpha \in (0, 1)$. If*

$$\int_{t_0}^{\infty} H(t) dt = \infty, \tag{2.18}$$

then (1.1) is oscillatory.

Lemma 2.11. *Let $x(t)$ be an eventually positive solution of (EC1). Then*

(i) *if $y(t) \in \mathcal{S}_1$, then $y^{\alpha-1}(t) \geq \phi_1(t)$, where*

$$\phi_1(t) = \begin{cases} 1, & \text{if } \alpha = 1, \\ \epsilon_1, & \text{if } \alpha > 1, \\ \epsilon_2 A_1^{\alpha-1}(t), & \text{if } \alpha < 1, \end{cases}$$

and ϵ_1 and ϵ_2 are positive constants for all $t \geq t_1 \geq t_0$;

(ii) *if $y(t) \in \mathcal{S}_3$, then $y^{\alpha-1}(t) \geq \phi_2(t)$, where $\phi_2(t)$ is given by*

$$\phi_2(t) = \begin{cases} 1, & \text{if } \alpha = 1, \\ \epsilon_3, & \text{if } \alpha > 1, \\ \epsilon_4 Q_3^{\alpha-1}(t), & \text{if } \alpha < 1, \end{cases}$$

and ϵ_3 and ϵ_4 are positive constants for all $t \geq t_1 \geq t_0$.

The proof of the above lemma is similar to taht of [27, Lemma 2.10] and it is are omitted here. By using Riccati transformation method we obtain the following result.

Theorem 2.12. *Let* (2.1) *hold. If there are positive functions* $\rho_1, \rho_2 \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \Big(\frac{\rho_1(v)}{a_2(v)} \int_v^\infty \frac{1}{a_3(s)} \int_s^\infty F_1(s_1)ds_1 ds - \frac{a_1(v)(\rho_1'(v))^2}{4\rho_1(v)} \Big) dv = \infty, \tag{2.19}$$

$$\limsup_{t\to\infty} \int_{t_1}^{t} (\rho_2(s)F_2(s) - \frac{a_1(s)(\rho_2'(s))^2}{4\rho_2(s)Q_2(s)}))ds = \infty, \tag{2.20}$$

where

$$F_1(t) = \frac{D_1(t)A_1^\alpha(\sigma(t))}{A_1^\alpha(t)}\phi_1(t) \quad \textit{and} \quad F_2(t) = \frac{D_2(t)Q_3^\alpha(\sigma(t))}{Q_3^\alpha(t)}\phi_2(t)$$

for all $t \ge t_1 \ge t_0$*, then equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be an eventually positive solution of (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. Then, $x(t)$ is also an eventually positive solutions of (EC1). Thus, it follows from Lemma 2.3 that either $y(t) \in \mathcal{S}_1$ or $y(t) \in \mathcal{S}_3$ for $t \ge t_1$.

First we assume that $y(t) \in \mathcal{S}_1$. In this case, from Lemma 2.5 (i) and Lemma 2.11 (i), we observe that

$$y^\alpha(\sigma(t)) \ge \frac{A_1^\alpha(\sigma(t))}{A_1^\alpha(t)}\phi_1(t)y(t).$$

Using the above estimate in (2.11), we see that

$$\overline{L}_4 y(t) + F_1(t)y(t) \le 0.$$

An integration of the latter expression from t to ∞ yields

$$\overline{L}_3 y(t) \ge \int_t^\infty F_1(s)y(s)ds \ge y(t)\int_t^\infty F_1(s)ds. \tag{2.21}$$

Now integrating (2.21) from t to ∞, we have

$$\overline{L}_2 y(t) + \Big(\int_t^\infty \frac{1}{a_3(v)} \int_v^\infty F_1(s)\,ds\,dv \Big) y(t) \le 0. \tag{2.22}$$

Let us define

$$\mu_1(t) = \rho_1(t)\frac{\overline{L}_1 y(t)}{y(t)}, \quad t \ge t_1. \tag{2.23}$$

From (2.22) and (2.23), we observe that

$$\begin{aligned} \mu_1'(t) &= \rho_1'(t)\frac{\overline{L}_1 y(t)}{y(t)} + \frac{\rho_1(t)}{a_2(t)}\frac{\overline{L}_2 y(t)}{y(t)} - \frac{\rho_1(t)y'(t)\overline{L}_1 y(t)}{y^2(t)} \\ &\le -\frac{\rho_1(t)}{a_2(t)}\int_t^\infty \frac{1}{a_3(v)}\int_v^\infty F_1(s)\,ds\,dv + \frac{a_1(t)(\rho_1'(t))^2}{4\rho_1(t)}. \end{aligned} \tag{2.24}$$

Integrating (2.24) from t_1 to t yields

$$\int_{t_1}^{t} \Big(\frac{\rho_1(v)}{a_2(v)} \int_v^\infty \frac{1}{a_3(s)} \int_s^\infty F_1(s_1)ds_1 ds - \frac{a_1(v)(\rho_1'(v))^2}{4\rho_1(v)} \Big) dv \le \mu_1(t_1),$$

which contradicts (2.19) as $t \to \infty$.

Next assume that $y(t) \in \mathcal{S}_3$. Then from Lemma 2.5 (ii) and Lemma 2.11 (ii), we see that

$$y^\alpha(\sigma(t)) \geq \frac{Q_3^\alpha(\sigma(t))}{Q_3^\alpha(t)} \phi_2(t) y(t), \tag{2.25}$$

$$\overline{L}_1 y(t) \geq Q_2(t) \overline{L}_3 y(t). \tag{2.26}$$

Using (2.25) in (2.16), we obtain

$$\overline{L}_4 y(t) + F_2(t) y(t) \leq 0. \tag{2.27}$$

We define

$$\mu_2(t) = \rho_2(t) \frac{\overline{L}_3 y(t)}{y(t)}, \quad t \geq t_2. \tag{2.28}$$

From (2.27)-(2.28), we obtain, for $t \geq t_2$,

$$\begin{aligned} \mu_2'(t) &= \rho_2'(t) \frac{\overline{L}_3 y(t)}{y(t)} + \frac{\rho_2(t) \overline{L}_4 y(t)}{y(t)} - \frac{\rho_2(t) \overline{L}_3 y(t) y'(t)}{y^2(t)} \\ &\leq -\rho_2(t) F_2(t) + \frac{\rho_2'(t)}{\rho_2(t)} \mu_2(t) - \frac{Q_2(t) \mu_2^2(t)}{\rho_2(t) a_1(t)} \\ &\leq -\rho_2(t) F_2(t) + \frac{a_1(t) (\rho_2'(t))^2}{4 \rho_2(t) Q_2(t)}. \end{aligned} \tag{2.29}$$

Integrating (2.29) from t_2 to t yields

$$\int_{t_2}^t (\rho_2(s) F_2(s) - \frac{a_1(s)(\rho_2'(s))^2}{4\rho_2(s) Q_2(s)})) ds \leq \mu_2(t_2),$$

which contradicts (2.20) as $t \to \infty$. The proof is complete. $\square$

Letting $\rho_1(t) = A_1(t)$, $\rho_2(t) = Q_3(t)$ and $\alpha = 1$, one can immediately get the following result.

Corollary 2.13. *Let $\alpha = 1$. If*

$$\limsup_{t\to\infty} \int_{t_1}^t \Big(\frac{A_1(v)}{a_2(v)} \int_v^\infty \frac{1}{a_3(s)} \int_s^\infty D_1(s_1) \frac{A_1(\sigma(s_1))}{A_1(s_1)} ds_1 ds - \frac{1}{4a_1(v) A_1(v)} \Big) dv = \infty$$

and

$$\limsup_{t\to\infty} \int_{t_1}^t \Big(Q_3(\sigma(s)) D_2(s) - \frac{Q_2(s)}{4a_1(s) Q_3(s)} \Big) ds = \infty \tag{2.30}$$

for all $t_1 \geq t_0$, then (1.1) is oscillatory.

3. Examples

In this section, we provide two examples to show the importance of our results.

Example 3.1. Consider the semi-noncanonical neutral delay differential equation

$$\Big(t^2 \Big(\frac{1}{t^2} (t^2 z'(t))' \Big)' \Big)' + \frac{h}{t^2} x(\lambda t) = 0, \quad t \geq 1, \tag{3.1}$$

where $z(t) = x(t) + \frac{1}{4} x(\frac{t}{2})$, $h > 0$ is a constant, and $\lambda \in (0,1)$. A simple computation shows that

$$\Omega_3(t) = \Omega_1(t) = \frac{1}{t}, a_1(t) = a_2(t) = a_3(t) = 1 \text{ and } y(t) = t\Big(x(t) + \frac{1}{4} x(\frac{t}{2}) \Big).$$

The transformed equation is

$$y^{(4)}(t) + \frac{h}{t^3} x(\lambda t) = 0, \quad t \geq 1,$$

which is clearly in canonical form. Now we see that, for $m = 0$,

$$\begin{gathered} B_1(t;0) = B_2(t;0) = \frac{1}{2t}, \\ D_1(t) = D_2(t) = \frac{h}{2\lambda t^4}, \quad R_1(t) \approx \frac{h}{12t}, \\ R_2(t) \approx \frac{h\lambda^2}{12t}, \quad H(t) = \frac{h\lambda^2}{12t}. \end{gathered}$$

Clearly (2.1) holds. Condition (2.17) becomes

$$\liminf_{t\to\infty} \int_{\lambda t}^{t} \frac{h\lambda^2}{12s} ds = \frac{h\lambda^2}{12} \ln \frac{1}{\lambda} > \frac{1}{e}.$$

Hence by Corollary 2.9, equation (3.1) is oscillatory if $h > \frac{12}{\lambda^2 e \ln \frac{1}{\lambda}}$.

Note that using Corollary 2.13, we see that equation (3.1) is oscillatory if $h > 9/\lambda^2$ and therefore Corollary 2.9 gives better condition than Corollary 2.13.

Example 3.2. Consider the semi-noncanonical nonlinear neutral differential equation

$$\Big(t^2\Big(\frac{1}{t^2}(t^2 z'(t))'\Big)'\Big)' + htx^3(\lambda t) = 0, \quad t \geq 1, \tag{3.2}$$

where $z(t) = x(t) + \frac{1}{4}x(\frac{t}{2})$, $h > 0$ is a constant, and $\lambda \in (0,1)$. The transformed equation is

$$y^{(4)}(t) + hx^3(\lambda t) = 0$$

and it is clearly of canonical type. A simple calculation shows that

$$\begin{gathered} B_1(t;0) = B_2(t;0) = \frac{1}{2t}, \quad D_1(t) = D_2(t) = \frac{h}{8\lambda^3 t^3}, \\ A_1(t) \approx t, \quad A_2(t) \approx t, \quad A_3(t) \approx t, \quad Q_2(t) \approx \frac{t^2}{2}, \\ Q_3(t) \approx \frac{t^3}{6}, \quad \phi_1(t) = \epsilon_1, \quad \phi_2(t) = \epsilon_3, \quad F_1(t) \approx \frac{h\epsilon_1}{8t^3}, \quad F_2(t) \approx \frac{h\lambda^6\epsilon_3}{8t^3}. \end{gathered}$$

Choose $\rho_1(t) = 1$ and $\rho_2(t) = t^2$, we see that conditions (2.19) and (2.20) become

$$\begin{gathered} \limsup_{t\to\infty} \int_1^t \frac{h\epsilon_1}{16s} ds = \lim_{t\to\infty} \frac{h\epsilon_1}{16} \ln t = \infty, \\ \limsup_{t\to\infty} \int_1^t \Big(\frac{h\lambda^6\epsilon_3}{8s} - \frac{2}{s^2}\Big) ds = \infty. \end{gathered}$$

That is, conditions (2.19) and (2.20) are satisfied. Hence, by Theorem 2.12, equation (3.2) is oscillatory.

Remark 3.3. Note that none of the results reported in the literature [4, 5, 6, 14, 19, 22, 23, 27] applied to (3.1) and (3.2) to get any conclusion.

4. Conclusions

In this paper, by transforming the semi-noncanonical equation to canonical type equation, we establish oscillation criteria using comparison and Riccati transformation methods. The oscillation criteria presented in this paper are new in the sense that it gives all solutions are oscillatory instead of every solution is either oscillatory or tends to zero asymptotically.

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